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by

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Preprint no.: 83 2020

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August 7, 2020

Abstract

We present a new approach to compute selected eigenvalues and eigenvectors of the two-parameter eigenvalue problem. Our method requires computing generalized eigenvalue problems of the same size as the matrices of the initial two-parameter eigenvalue problem. The method is applicable for a class of Helmholtz equations when separation of variables is applied.

1 Introduction

In this work we consider the *two-parameter eigenvalue problem*

$$\begin{aligned}(A_1 + \lambda_1 B_1 + \lambda_2 C_1)u = 0,\\ (A_2 + \lambda_1 B_2 + \lambda_2 C_2)v = 0\end{aligned} \tag{1}$$

with matrices $A_i, B_i, C_i \in \mathbb{R}^{n_i \times n_i}$ for $i = 1, 2$. A solution to this problem is given by possibly complex $(\lambda_1, \lambda_2, u, v)$ if they fulfill (1) and $u, v \neq 0$. We call the pair (λ_1, λ_2) an *eigenvalue of the two-parameter eigenvalue problem* if $u, v \neq 0$ exist such that $(\lambda_1, \lambda_2, u, v)$ satisfies (1) and we call the tensor product $u \otimes v$ an *eigenvector of the two parameter eigenvalue problem* (in the literature the pair (u, v) is often referred to as the eigenvector; the terminology here is simply for convenience).

Two-parameter eigenvalue problems naturally arise in mathematical physics when separation of variables is applied [1, 16]. Consider for example the Helmholtz equation

$$\begin{aligned}\Delta u + \lambda u = 0 \quad &\text{in } \Omega,\\ u = 0 \quad &\text{on } \partial\Omega\end{aligned}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1, y > 0\}$ is half of an open ellipse. Using elliptical coordinates, the problem can be reformulated into

$$\begin{aligned}v''(r) + (\lambda c^2 \sinh^2(r) + \mu)v(r) = 0, \quad v(0) = 0 = v(R)\\ w''(\varphi) + (\lambda c^2 \sin^2(\varphi) - \mu)w(\varphi) = 0, \quad w(0) = 0 = w(\pi)\end{aligned} \tag{2}$$

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with $u(c\cosh(r)\cos(\varphi), c\sinh(r)\sin(\varphi)) = v(r)w(\varphi)$ and $\mu \in \mathbb{R}$ (this will be explained in Section 3). This is of the form (1) after discretization. There are also many other interesting applications, including delay differential equations [11] and optimization [15].

There are several numerical methods for solving this problem. One possibility is Jacobi-Davidson type methods discussed in [5,7]. These methods work well to find eigenvalues close to a given search value. Another possibility is based on homotopy continutation, for example discussed in [3, 14]. These aim to find all eigenvalues.

We present a new algorithm that can be seen as an *alternating method* as they are for example considered in [8] for tensors. This aims to reduce the complexity of finding one solution to the complexity of solving a generalized eigenvalue problem with matrices of size $n \times n$ while solving the two parameter eigenvalue problem in a conventional way requires solving an eigenvalue problem of size $n^2 \times n^2$. Hence, the number of operations is in $o(n^3)$ for one solution, compared to $o(n^6)$ for all solutions using a traditional approach (which solves (4) via the QR or QZ algorithm). While we only establish convergence for extremal eigenpairs, empirically our algorithm also finds all solutions, with $o(n^5)$ cost. It will turn out that this method can find eigenvalues based on their *index*. The index of an eigenvalue of multiparameter eigenvalue problems is a generalisation of ordering real eigenvalues of ordinary eigenvalue problems [16, Ch. 1]. We will need the following assumptions to hold:

A.1 All matrices are symmetric;

A.2 The matrices B_1, B_2, and C_1 are positive definite and C_2 is negative definite.

Notice that in example (2) the second derivative is indeed a symmetric operator, and $\sinh^2(r) > 0$ and $\sin^2(\varphi) > 0$ almost everywhere. A discrete version of these equations hence leads to a problem of the type (1) that satisfies the assumptions **A.1** and **A.2**.

This article is organized as follows: In Section 2, we first motivate our method and state useful results from multiparameter eigenvalue theory. Afterwards we discuss fixed point properties of the algorithm and prove convergence for extremal eigenvalues. Finally, we examine the time complexity of our method briefly. In Section 3, we show that a class of boundary value problems satisfies our assumptions if properly discretized. Finally, we exhibit results of numerical experiments in Section 4.

2 An alternating algorithm for the two-parameter problem

In this section, we derive our algorithm and prove convergence for extremal eigenvalues. First, let us fix our notation. By $\otimes$ we denote the *tensor product*. For vectors this can be seen seen as the outer product, i.e., $u \otimes v$ corresponds to the rank-one matrix $uv^\top$. The tensor product of matrices then corresponds to a linear operator acting on matrices, i.e., the product $A \otimes B$ acts on the matrix X by $AXB^\top$. This means that $(A \otimes B)(u \otimes v) = (Au \otimes Bv)$. It can also be seen as the Kronecker product for matrices [4, Ch. 12]. Then the product of two vectors $u \otimes v$ can be reshaped into a rank-one matrix.

2.1 Derivation of the algorithm

The two-parameter eigenvalue problem (1) can be reduced to two generalized eigenvalue problems [1]. For this define the operators acting on $n_1 \times n_2$ matrices

$$\begin{aligned} M_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ M_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ M_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned} \tag{3}$$

A straightforward computation shows that a solution of (1) satisfies

$$\begin{aligned} M_1(u \otimes v) + \lambda_1 M_0(u \otimes v) = 0, \\ M_2(u \otimes v) + \lambda_2 M_0(u \otimes v) = 0. \end{aligned}$$

Hence, solutions of (1) are also rank-one solutions X of the system of generalized matrix eigenvalue problems

$$\begin{aligned} M_1(X) + \lambda_1 M_0(X) = 0, \\ M_2(X) + \lambda_2 M_0(X) = 0, \end{aligned} \tag{4}$$

which are of size $n_1 n_2 \times n_1 n_2$. Note that the eigenvectors are shared between the two generalized eigenvalue problems. Also under our assumptions solutions of (4) lead to solutions of (1). This is a straightforward consequence of classical theory [1, 16].

Lemma 1. *Under the assumptions **A.1** and **A.2**, the eigenvectors of the generalized eigenvalue problem $M_2(X) + \lambda_2 M_0(X) = 0$ are either rank one or are a linear combination of rank-one eigenvectors. A rank-one eigenvector is then also an eigenvector of the corresponding two-parameter eigenvalue problem.*

Proof. The assumption **A.2** ensures that the eigenspaces of each subproblem in (4) is spanned by rank-one matrices. To see this, first note that the operator M_0 is negative definite, and M_1 and M_2 are symmetric. Therefore all eigenvalues of (4) are real. Second, we can without loss of generality assume B_1 and B_2 to be identity matrices, else we just transform the matrices to $\tilde{A_1} = B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}}$, $\tilde{C_1} = B_1^{-\frac{1}{2}} C_1 B_1^{-\frac{1}{2}}$, etc., and $\tilde{u} = B_1^{\frac{1}{2}} u$, $\tilde{v} = B_2^{\frac{1}{2}} v$. Then M_0 and M_2 are operators in the form of a *Sylvester equation*, i.e., they have the form

$$\begin{aligned} M_0 &= I_{n_1} \otimes C_2 - C_1 \otimes I_{n_1}, \\ M_2 &= I_{n_1} \otimes A_2 - A_1 \otimes I_{n_2}. \end{aligned}$$

A solution to (4) then satisfies

$$\left(I_{n_1} \otimes (A_2 + \lambda_2 C_2) - (A_1 + \lambda_2 C_1) \otimes I_{n_2}\right)(X) = 0,$$

i.e., X is a zero solution of a Sylvester equation. Such operators possess an orthonormal basis of rank-one eigenvectors, namely $u \otimes v$, where u is an eigenvector of $(A_1 + \lambda_2 C_1)$ and v of $(A_2 + \lambda_2 C_2)$ [9, Theorem 4.4.5], both of which are real symmetric and therefore possess an orthonormal basis of eigenvectors. It follows that X is a sum of rank-one eigenvectors corresponding to the eigenvalue 0. The dimension equals that of the null space of $M_2(X) + \lambda_2 M_0(X)$, i.e., the geometric multiplicity of λ_2.

We next show that a rank-one solution to just one of the eigenvalue problems in (4) suffices to get a solution of the initial problem (1). Indeed, assume that $X = u \otimes v$ solves the second eigenvalue problem in (4). We then get

$$\begin{aligned} 0 =& M_2(u \otimes v) + \lambda_2 M_0(u \otimes v) \\ =& B_1 u \otimes A_2 v - A_1 u \otimes B_2 v + \lambda_2 B_1 u \otimes C_2 v - \lambda_2 C_1 u \otimes B_2 v \\ =& B_1 u \otimes (A_2 + \lambda_2 C_2) v - (A_1 + \lambda_2 C_1) u \otimes B_2 v, \end{aligned}$$

which holds true if and only if there is a μ such that

$$\begin{aligned} \mu B_1 u &= (A_1 + \lambda_2 C_1) u, \\ \mu B_2 v &= (A_2 + \lambda_2 C_2) v. \end{aligned}$$

This implies (1) with $\lambda_1 = -\mu$. □

We can conclude that (1) has essentialy $n_1 n_2$ solutions which can be obtained by computing the rank-one solutions of one of the eigenvalue problems in (4). In the following, we select the first of these eigenvalue problems.

The assumptions imply that the operators M_0 and M_1 are symmetric and M_0 is negative definite. Thus, the solution of

$$M_1(X) + \lambda_1 M_0(X) = 0 \tag{5}$$

with *maximal* eigenvalue λ_1 can be obtained by maximizing the *Rayleigh quotient*

$$\mathfrak{R}(X) = \frac{\langle X, M_1(X) \rangle}{\langle X, -M_0(X) \rangle}, \tag{6}$$

and since the solution is a rank-one matrix, we can just maximize $\mathfrak{R}(u \otimes v)$ over u and v. For convenience, define the functions

$$\begin{aligned} a_1(u) &= u^\top A_1 u, & b_1(u) &= u^\top B_1 u, & c_1(u) &= u^\top C_1 u, \\ a_2(v) &= v^\top A_2 v, & b_2(v) &= v^\top B_2 v, & c_2(v) &= v^\top C_2 v. \end{aligned}$$

The assumptions assure $b_1, b_2, c_1 > 0$ and $c_2 < 0$. Then we can write

$$\mathfrak{R}(u \otimes v) = -\frac{u^\top (c_2(v) A_1 - a_2(v) C_1) u}{u^\top (c_2(v) B_1 - b_2(v) C_1) u}.$$

For fixed v the matrix in the denominator is negative definite. Hence, the maximal value is given by the maximal eigenvalue of the generalized eigenvalue problem

$$(c_2(v) A_1 - a_2(v) C_1)\, u = -\lambda (c_2(v) B_1 - b_2(v) C_1) u, \tag{7}$$

and respectively fixing u, the maximal value of

$$\mathfrak{R}(u \otimes v) = -\frac{v^\top (a_1(u) C_2 - c_1(u) A_2) v}{v^\top (b_1(u) C_2 - c_1(u) B_2) v}$$

Algorithm 1: Alternating Algorithm for solving two-parameter eigenvalue problems.

Input : Matrices A_i, B_i, C_i for $i = 1, 2$ satisfying **A.1** and **A.2** and index (j_1, j_2).
Output: Eigenvalue (λ_1, λ_2) of index (j_1, j_2).

select random nonzero $u \in \mathbb{R}^{n_1}$;
while *not converged* **do**
  $a_1 := a_1(u); \quad b_1 := b_1(u); \quad c_1 := c_1(u);$
  compute the eigenvector v corresponding to the j_2-th largest eigenvalue of the symmetric right definite generalized eigenvalue problem

$$(a_1(u)C_2 - c_1(u)A_2)\, v = -\lambda(b_1(u)C_2 - c_1(u)B_2)v;$$

  $a_2 := a_2(v); \quad b_2 := b_2(v); \quad c_2 := c_2(v);$
  compute the eigenpair (u, λ) corresponding to the j_1-th largest eigenvalue of of the symmetric right definite generalized eigenvalue problem

$$(c_2(v)A_1 - a_2(v)C_1)\, u = -\lambda(c_2(v)B_1 - b_2(v)C_1)u;$$

end
$\lambda_1 := \lambda;$
$\lambda_2 := -\frac{a_2 + \lambda_2 b_2}{c_2};$
return λ_1, λ_2.

is given by the maximal eigenvalue of

$$(a_1(u)C_2 - c_1(u)A_2)\, v = -\lambda(b_1(u)C_2 - c_1(u)B_2)v. \tag{8}$$

These are generalized eigenvalue problems with matrices of size $n_1 \times n_1$ and $n_2 \times n_2$, while (5) is a generalized eigenvalue problem of size $n_1 n_2 \times n_1 n_2$.

A similar alternating procedure is of course obtained when minimizing the Rayleigh quotient in (6), i.e., when aiming at the smallest eigenvalue of (5). In general nothing even prevents us from updating u and v with non-extremal eigenpairs of the subproblems (7) and (8) in the hope of finding non-extremal eigenpairs of (5). For instance, we can solve (7) for some j_1-th eigenvalue and (8) for some j_2-th eigenvalue. This idea motivates Algorithm 1.

We will call a pair of eigenvectors u and v a *fixed point of Algorithm 1* if it simultaneously solves the eigenvalue problems (7) and (8). Then choosing the corresponding index (j_1, j_2) the algorithm will not change u and v anymore, provided eigenvalues are simple.

2.2 Fixed point properties

Our aim in this section is to show that in principle we can find all rank-one eigenpairs of the problem (5), and hence of (1), by finding fixed points of the subproblems (7) and (8) solved in Algortihm 1 for all possible input indices $(j_1, j_2) \in \{1, \ldots, n_1\} \times \{1, \ldots, n_2\}$. The first step in this direction is the following lemma, which shows that a fixed point of the algorithm indeed provides a solution to (5) and (1).

Lemma 2. *Let u and v simultaneously solve* (7) *and* (8). *Under the assumptions* **A.1** *and* **A.2**

both eigenvalues coincide. Moreover, setting $\lambda_1 = \lambda$ *and* $\lambda_2 := -\frac{a_2(v)+\lambda_1 b_2(v)}{c_2(v)}$, $(u \otimes v, \lambda_1)$ *is an eigenpair of* (5) *and* $(u, v, \lambda_1, \lambda_2)$ *is a solution for the two-parameter problem* (1).

Proof. Denote the eigenvalue in (7) and (8) by σ and μ, respectively. Then multiplying (7) by $u^\top$ and (8) by $v^\top$, we get

$$c_2(v)a_1(u) - a_2(v)c_1(u) = -\sigma(c_2(v)b_1(u) - b_2(v)c_1(u))$$

and

$$c_2(v)a_1(u) - a_2(v)c_1(u) = -\mu(c_2(v)b_1(u) - b_2(v)c_1(u)).$$

Since **A.2** implies $c_2(v)b_1(u) - b_2(v)c_1(u) < 0$, we have $\sigma = \mu =: \lambda_1$. Collecting terms in (7) and (8) gives

$$\begin{aligned} c_2(v)A_1u + c_2\lambda_1 B_1u - (a_2 + \lambda_1 b_2(v))C_1u = 0, \\ -c_1(u)A_2v - c_1\lambda_1 B_2v + (a_1 + \lambda_1 b_1(u))C_2v = 0. \end{aligned}$$

Dividing the first equation by $c_2(v) < 0$ and the second by $-c_1(u) < 0$, we get (1) with $\lambda_2 = -\frac{a_1(u)+\lambda_2 b_1(u)}{c_1(u)}$, if

$$\frac{a_2(v) + \lambda_1 b_2(v)}{c_2(v)} = \frac{a_1(u) + \lambda_1 b_1(u)}{c_1(u)}.$$

This equation is however just a consequence of

$$c_2(v)a_1(u) - a_2(v)c_1(u) = -\lambda_1(c_2(v)b_1(u) - b_2(v)c_1(u)).$$

By the considerations in Section 2.1 $(u \otimes v, \lambda_1)$ is also a solution of (5). □

The previous lemma does not yet let us conclude that all solutions of (1) occur as fixed points of Algorithm 1 when varying the input index (j_1, j_2). To show this, we need to introduce the notion of the *index of an eigenvalue* (λ_1, λ_2) of the two-parameter eigenvalue problem.

Definition 3. *An eigenvalue* (λ_1, λ_2) *of the two-parameter eigenvalue problem has the index* (j_1, j_2) *if* 0 *is the* j_i*-th smallest eigenvalue of* $A_i + \lambda_1 B_i + \lambda_2 C_i$ *for* $i = 1, 2$.

If 0 is a multiple eigenvalue of $A_i + \lambda_1 B_i + \lambda_2 C_i$, then the corresponding eigenvalue of the two-parameter eigenvalue problem has multiple indices as well. Under the assumption **A.1** every real valued eigenvalue of the two-parameter eigenvalue problem has an index since the matrices $A_i + \lambda_1 B_i + \lambda_2 C_i$ is symmetric. More important for us is the following result, which immediately follows from [16, Theorem 1.4.1].

Theorem 4. *Under the assumptions* **A.1** *and* **A.2** *there is a unique eigenvalue* (λ_1, λ_2) *to every index* $(j_1, j_2) \in \{1, \dots, n_1\} \times \{1, \dots, n_2\}$.

The idea is now to show that the index in the sense of Definition 3 of the eigenvalue solution provided by a fixed point of Algorithm 1 coincides with the given input index (j_1, j_2). Together with Theorem 4 this then implies that all solutions can be obtained this way. For this we need the following lemma.

Lemma 5. *Let A, B be symmetric matrices and B be positive definite. Then λ_i is the i-th largest eigenvalue of A if and only if it is the i-th largest eigenvalue of the generalized eigenvalue problem*

$$(A + \lambda_i B)u = \lambda(I + B)u.$$

Proof. Consider the matrices $M_1 = A - \lambda_i I$ and $M_2 = (I+B)^{-\frac{1}{2}}(A - \lambda_i I)(I+B)^{-\frac{1}{2}}$. The matrix M_2 is well defined since B is positive definite and since $(I+B)^{-\frac{1}{2}} = ((I+B)^{-\frac{1}{2}})^T)$ is invertible, M_1 and M_2 are congruent and therefore have the same number of positive and respectively negative eigenvalues by Sylvester's law of inertia [10, Theorem 4.5.8].

We can rewrite M_2 in the following way:

$$\begin{aligned} M_2 &= (I+B)^{-\frac{1}{2}}(A - \lambda_i I)(I+B)^{-\frac{1}{2}} \\ &= (I+B)^{-\frac{1}{2}}(A + \lambda_i B - \lambda_i B - \lambda_i I)(I+B)^{-\frac{1}{2}} \\ &= (I+B)^{-\frac{1}{2}}(A + \lambda_i B)(I+B)^{-\frac{1}{2}} - \lambda_i I. \end{aligned}$$

This implies that A and $(I+B)^{-\frac{1}{2}}(A + \lambda_i B)(I+B)^{-\frac{1}{2}}$ have the same number of eigenvalues that are smaller or respectively larger than λ_i. Since the eigenvalues of $(I+B)^{-\frac{1}{2}}(A + \lambda_i B)(I+B)^{-\frac{1}{2}}$ are the same as the eigenvalues of the generalized eigenvalue problem

$$(A + \lambda_i B)u = \lambda(I + B)u$$

the claim is proven. □

We are now in a position to prove the main result.

Theorem 6. *Let (j_1, j_2) be the input index of Algorithm 1 and let (u, v, λ, μ) be a fixed point of Algorithm 1. Then $\mu = \lambda$ and the output (λ_1, λ_2) is the eigenvalue of problem (1) with index (j_1, j_2) in the sense of Defintion 3 and $u \otimes v$ is the corresponding eigenvector.*

Proof. Without loss of generality assume that $B_1 = I_{n_1}$ and $B_2 = I_{n_2}$. Let (u, v) be a fixed point of Algorithm 1 and let (λ_1, λ_2) be the corresponding eigenvalue, i.e., λ_1 is the j_1-th largest eigenvalue of

$$\left(c_2(v)A_1 - a_2(v)C_1\right)u = -\lambda(c_2(v)I_{n_1} - b_2(v)C_1)u$$

and λ_2 satisifies $a_2(v) = -\lambda_1 b_2(v) - \lambda_2 c_2(v)$. We can now substitute this into the eigenvalue equation to obtain

$$-\left(c_2(v)A_1 + (b_2(v)\lambda_1 + c_2(v)\lambda_2)C_1)\right)u = \lambda(c_2(v)I_{n_1} - b_2(v)C_1)u$$

Dividing by $c_2(v) < 0$ and using Lemma 5, we see that λ_1 is also the j_1-th largest eigenvalue of $-A_1 - \lambda_2 C_1$ since $-\frac{b_2(v)}{c_2(v)}C_1$ is positive definite. This is equivalent to 0 being the j_1-th smallest eigenvalue of $A_1 + \lambda_1 I_{n_1} + \lambda_2 C_1$. For the second part of the index, we have that λ_1 is the j_2-th largest eigenvalue of

$$\left(a_1(u)C_2 - c_1(u)A_2\right)v = -\lambda(b_1(u)C_2 - c_1(u)I_{n_2})v.$$

We can substitute $a_1(u) = -\lambda_1 b_1(u) - \lambda_2 c_1(u)$ to obtain

$$-\left(c_1(u)A_2 + (\lambda_1 b_1(u) + \lambda_2 c_1(u))C_2\right)v = \lambda(c_1(u)I_{n_2} - b_1(u)C_2)v.$$

We can again divide by $c_1(u)$ and use Lemma 5 to get that λ_1 is the j_2-th largest eigenvalue of $-A_2 - \lambda_1 B_2$ since $-\frac{b_1(u}{c_1(u)}C_2$ is positive definite. This is equivalent to 0 being the j_2-th smallest eigenvalue of $A_2 + \lambda_1 I_{n_2} + \lambda_2 C_2$. □

This result implies that if Algorithm 1 converges, then it computes a solution to the two-parameter eigenvalue problem (1).

Corollary 7. *Let $(u^n, v^n, \lambda^n, \mu^n)$ be a sequence generated by Algorithm 1. If u^n and v^n converge in the projective sense to u and v, i.e., convergence is up to sign flip if we choose normalized u^n and v^n in each step, then λ^n and μ^n converge to the same value λ and u, v, λ are fixed by Algorithm 1. Therefore the output (λ_1, λ_2) is the eigenvalue of problem (1) with index (j_1, j_2) in the sense of Defintion 3 and $u \otimes v$ is the corresponding eigenvector.*

Proof. Let $X^{2n} = u^n \otimes v^n$ and $X^{2n+1} = u^{n+1} \otimes v^n$. Since u^n and v^n converge to u and v respectively X^n converges to $u \otimes v$. Notice that $\lambda^n = \mathfrak{R}(X^{2n+1})$ and $\mu^n = \mathfrak{R}(X^{2n})$. By continuity of the Rayleigh quotient $\mathfrak{R}$, we get $\lim_{n\to\infty} \lambda^n = \lim_{n\to\infty} \mu^n = \lambda$. By continuity of the functions a_i, b_i, c_i for $i = 1, 2$ and continuity of eigenvalues of a nonsingular generalized eigenvalue problem, we get that u and v are eigenvectors of the eigenvalue problems (7) and (8) with eigenvalue λ and continuity also ensures that the eigenvalue is still the j_1-th or respectively j_2-th largest. Hence, (u, v, λ, λ) satisfies the conditions of Theorem 6. □

2.3 Convergence for extremal eigenpairs

So far, we showed that if the algorithm converges, it returns the eigenvalue of the two-parameter eigenvalue problem with the desired index. For extremal indices, we can actually prove convergence.

Theorem 8. *The sequences u^n and v^n generated by Algorithm 1 converge (up to normalization and sign flip) to a solution of (1) if the input index is either $(1, 1)$ or (n_1, n_2) and under the assumption that the respective eigenvalue is simple.*

To prove this, we need the following lemma for proving convergence when each step is at least as good as a line search.

Lemma 9. *Let $f : D \to \mathbb{R}$ be continuously differentiable, where $D \subset \mathbb{R}^n$ is open. Let $\{x^k\}_{k\in\mathbb{N}} \subset K \subset D$ and $g^k = \nabla f(x^k)$ satisfy*

$$f(x^{k+1}) \leq f(x^k - \sigma g^k)$$

for every $\sigma \in \mathbb{R}$ such that $x^k - \sigma g^k \in D$, where K is compact. Then g^k converges to zero as $k \to \infty$.

Proof of Lemma 9. First note that $f(x^{k+1}) \leq f(x^k - \sigma g^k)$ for all σ implies that $f(x^k)$ is monotonically nonincreasing and $\{x^k\}_{k\in\mathbb{N}} \subset K$ implies that $f(x_k)$ is bounded from below. Hence, $f(x^k)$ converges to some value $\tilde{f}$ and since x^k lies in a compact set, the gradients g^k are bounded as well.

Now towards a contradiction assume that $\|g^k\| \geq 2\epsilon > 0$ for some $\epsilon > 0$ and every k. Else either g^k converges to zero or we can choose a subsequence $g^{k'}$ such that $\|g^{k'}\| \geq 2\epsilon > 0$. Since g^k are bounded, there exists $\delta_1 > 0$ such that $x^k - \sigma g^k \in \tilde{K} \subset D$ for every $\sigma \in [0, \delta_1]$, where $\tilde{K}$ is also compact. Since f is continuously differentiable, ∇f is uniformly continuous on $\tilde{K}$. Therefore, there is $\delta_2 \in (0, \delta_1]$ such that

$$\|\nabla f(x^k - \xi g^k) - g^k\| < \epsilon.$$

for all $\xi \in [0, \delta_2]$. By the mean value theorem, Cauchy-Schwarz inequality, and the assumption $\|g^k\| \geq 2\epsilon$, we have

$$\begin{aligned} f(x^k) - f(x^k - \delta_2 g^k) &= \delta_2 \nabla f(x^k - \xi g^k)^\top g^k \\ &\geq \delta_2 \|g^k\|^2 - \delta_2 \|\nabla f(x^k - \xi g^k) - g^k\| \|g^k\| \\ &\geq \delta_2 \|g^k\|^2 - \delta_2 \|g^k\| \epsilon \\ &\geq \frac{\delta_2}{2} \|g^k\|^2 \geq 2\delta_2 \epsilon^2. \end{aligned}$$

This yields the contradiction

$$\infty > f(x^0) - \tilde{f} = \sum_{k=0}^{\infty} f(x^k) - f(x^{k+1}) \geq \sum_{k=0}^{\infty} f(x^k) - f(x^k - \delta_2 g^k) \geq \sum_{k=0}^{\infty} 2\delta_2 \epsilon^2 = \infty,$$

therefore g^k converges to zero. □

Proof of Theorem 8. Without loss of generality, assume that $C_1 = I_{n_1}$ and $C_2 = -I_{n_2}$ for $i = 1, 2$. Finding the vectors u and v for the indices $(1, 1)$ and (n_1, n_2) corresponds to either maximizing or minimizing the Rayleigh quotient

$$\mathfrak{R}(X) = -\frac{\langle X, M_1(X) \rangle}{\langle X, M_0(X) \rangle}.$$

Its gradient is given by

$$\nabla \mathfrak{R}(X) = -\frac{2}{\langle X, M_0(X) \rangle} \left(M_1(X) + \mathfrak{R}(X) M_0(X) \right)$$

and if $X = u \otimes v$, then since $C_1 = I_{n_1}$ and $C_2 = -I_{n_2}$, we have

$$\nabla \mathfrak{R}(u \otimes v) = \frac{2}{\langle u \otimes v, M_0(u \otimes v) \rangle} \left(u \otimes (A_2 + \mathfrak{R}(u \otimes v) B_2) v + (A_1 + \mathfrak{R}(u \otimes v) B_1) u \otimes v \right).$$

This shows that the gradient is an element in the tangent space of the rank-one matrix manifold at $u \otimes v$, which is given by

$$T_{u \otimes v} \mathcal{M}_1 = \{ x \otimes v + u \otimes y : x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2} \}.$$

Now assume every iterate of u and v are unit vectors. Then $c_1(u) = 1 = -c_2(v)$ and the Rayleigh quotient reads

$$\mathfrak{R}(u \otimes v) = \frac{a_2(v) + a_1(u)}{b_1(u) + b_2(v)}.$$

Now let (λ, v) be an eigenpair of

$$-\left(a_1(u) I_{n_2} + A_2\right) v = \lambda (b_1(u) I_{n_2} + B_2) v,$$

which is one step of Algorithm 1. It follows that $\lambda = \mathfrak{R}(u \otimes v)$ and

$$-\left(A_2 + \lambda B_2\right) v = (a_1(u) + \lambda b_1(u)) v.$$

The gradient is then

$$\nabla\mathfrak{R}(u\otimes v)=\frac{2}{\langle u\otimes v,M_0(u\otimes v)\rangle}\left(-u\otimes(a_1(u)+\lambda b_1(u))v+(A_1+\lambda B_1)u\otimes v\right),$$

and is orthogonal to the subspace $\{u\otimes y : y\in\mathbb{R}^{n_2}\}\subset T_{u\otimes v}\mathcal{M}_1$ as

$$\langle\nabla\mathfrak{R}(u\otimes v),u\otimes y\rangle=\frac{2}{\langle u\otimes v,M_0(u\otimes v)\rangle}\left(-(a_1(u)+\lambda b_1(u))v^\top y+(a_1(u)+\lambda b_1(u))v^\top y\right)=0.$$

This was to be expected, as in each step we optimize the Rayleigh quotient with respect to the subspace $\{u\otimes y : y\in\mathbb{R}^{n_2}\}$ or respectively $\{x\otimes v : x\in\mathbb{R}^{n_1}\}$, which makes the gradient orthogonal to that respective space. This, together with $\nabla\mathfrak{R}(u\otimes v)\in T_{u\otimes v}\mathcal{M}_1$, implies that the gradient lies in $\{u\otimes y : y\in\mathbb{R}^{n_2}\}$ or respectively $\{x\otimes v : x\in\mathbb{R}^{n_1}\}$ after each half step.

We now only consider the case of minimizing the Rayleigh quotient, as maximizing can be done analogously. As we can choose our iterates u^k and v^k to be normalized, the iterates $X^k=u^k\otimes v^k$ lie in a compact set. As

$$v^{k+1}=\operatorname*{arg\,min}_{\|v\|=1}\mathfrak{R}(u^k\otimes v),$$

we have $\mathfrak{R}(X^{k+1})\leq\mathfrak{R}(X^k-\sigma\nabla\mathfrak{R}(X^k))$ since $X^k-\sigma\nabla\mathfrak{R}(X^k)$ is of the form $u^k\otimes v$, and the $\mathfrak{R}$ is scaling invariant. We can thus use Lemma 9 and therefore $\nabla\mathfrak{R}(X^k)$ converges to zero, implying that any convergent subsequence of X^k converges to an eigenvector of $M_1(X)+\lambda M_0(X)=0$. Corollary 7 implies that this eigenvector corresponds to the eigenvalue of the correct index. Hence, since this eigenvalue is simple, the sequence (X^k) only has one accumulation point up to sign flip. □

2.4 Complexity

We discuss time complexity of Algorithm 1. We assume $n_1=n_2=n$. Computing the eigendecomposition of a generalized eigenvalue problem needs $o(n^{\omega+\gamma})<o(n^3)$ operations, where ω is the exponent of complexity of matrix multiplication and $\gamma>0$ [2]. This implies that the complexity for computing one eigenvalue of the two-parameter eigenvalue problem is $o(n^3k)$, where k is the number of iterations in Algorithm 1 until a sufficient accuracy is achieved. However, a full eigenvalue decomposition is not always necessary. For extremal eigenvalues, we only need to compute the eigenvector corresponding to the largest or smallest eigenvalue, which is typically possible in $o(n^2)$ operations, for example using LOBPCG [12]. Similarly, if computing the i-th largest eigenvalue can be computed in $o(n^2i)$ operations. In summary, computing an eigenvalue with index (i_1,i_2) needs $o(n^2k\max(\min(i_1,n-i_1+1),\min(i_2,n-i_2+1))$ operations, where k is the number of iterations necessary for the desired accuracy. If the matrices allow for fast matrix vector multiplication, for example if they are sparse, the complexity can be reduced further up to $o(nk\max(\min(i_1,n-i_1+1),\min(i_2,n-i_2+1))$ operations.

3 A class of PDE eigenvalue problems

We now present a class of PDE eigenvalue problems, which can be separated into an appropriate two-paramter eigenvalue problem. Let $\phi:(a,b)\times(c,d)\to U\subset\mathbb{R}^2$ be a diffeomorphism, with

$$g(x,y):=(D\phi(x,y))^\top D\phi(x,y)=\begin{pmatrix}g_1(x)+g_2(y) & 0\\ 0 & g_1(x)+g_2(y)\end{pmatrix},$$

and choose $g_1, g_2 > 0$. This is always possible for such a function g as $g(x,y)$ is positive definite and therefore $g_1(x)+g_2(y) > 0$. Now let $\Omega = \phi((a,b)\times(c,d))$. We consider the Helmholtz equation

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{on } \Omega, \\ u &= 0 \quad \text{at } \partial\Omega. \end{aligned}$$

We can now write Δu in coordinates given by ϕ. Then

$$\begin{aligned} (\Delta u)(\phi(x,y)) =& \frac{1}{\sqrt{\det(g(x,y))}} \nabla \cdot \Big(\sqrt{\det(g(x,y))} g^{-1}(x,y) \nabla (u(\phi(x,y)) \Big) \\ =& \frac{1}{g_1(x)+g_2(y)} \Delta(u(\phi(x,y))). \end{aligned}$$

Making an ansatz $u(x,y) = v(x)w(y)$, we get

$$\begin{aligned} 0 &= v''(x)w(y) + v(x)w''(y) + \lambda g_1(x) v(x) w(y) + \lambda g_2(y) v(x) w(y) \\ &= v(x)\big(w''(y) + \lambda g_1(x) w(y) - \mu w(y)\big) + w(y)\big(v''(x) + \lambda g_2(y) v(x) + \mu v(x)\big). \end{aligned}$$

Now let v and w satisfy

$$v''(x) + \lambda g_1(x) v(x) + \mu v(x) = 0 \tag{9}$$

$$w''(y) + \lambda g_2(y) w(y) - \mu w(y), \tag{10}$$

and $0 = v(a) = v(b) = w(c) = w(d)$. Then $u(x,y) = v(x)w(y)$ solves the original eigenvalue problem, and on a given discretrization (9) satisfies **A.1** and **A.2**.

Example 10. *The complex function $\phi : \mathbb{C} \to \mathbb{C}, z \mapsto z^2$. Then $D\phi(z) = 2z$. Therefore with $z = x + iy$, we have*

$$g(x,y) = \begin{pmatrix} 2x & 2y \\ -2y & 2x \end{pmatrix} \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = \begin{pmatrix} 4x^2+4y^2 & 0 \\ 0 & 4x^2+4y^2 \end{pmatrix}.$$

Example 11. *The complex function $\phi : \mathbb{C} \to \mathbb{C}, z \mapsto e^z$. Then $D\phi(z) = e^z$. Therefore with $z = x + iy$, we have*

$$g(x,y) = \begin{pmatrix} e^x\cos(y) & e^x\sin(y) \\ -e^x\sin(y) & e^x\cos(y) \end{pmatrix} \begin{pmatrix} e^x\cos(y) & -e^x\sin(y) \\ e^x\sin(y) & e^x\cos(y) \end{pmatrix} = \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix}.$$

These coordinates describe an annulus.

Example 12. *The complex function $\phi : \mathbb{C} \to \mathbb{C}, z \mapsto \cosh(z)$. Then $D\phi(z) = \sinh(z)$. Therefore with $z = x + iy$, we have*

$$\begin{aligned} g(x,y) &= \begin{pmatrix} \sinh(x)\cos(y) & \cosh(x)\sin(y) \\ -\cosh(x)\sin(y) & \sinh(x)\cos(y) \end{pmatrix} \begin{pmatrix} \sinh(x)\cos(y) & -\cosh(x)\sin(y) \\ \cosh(x)\sin(y) & \sinh(x)\cos(y) \end{pmatrix} \\ &= \begin{pmatrix} \sinh^2(x)+\sin^2(y) & 0 \\ 0 & \sinh^2(x)+\sin^2(y) \end{pmatrix}. \end{aligned}$$

These are elliptical coordinates.

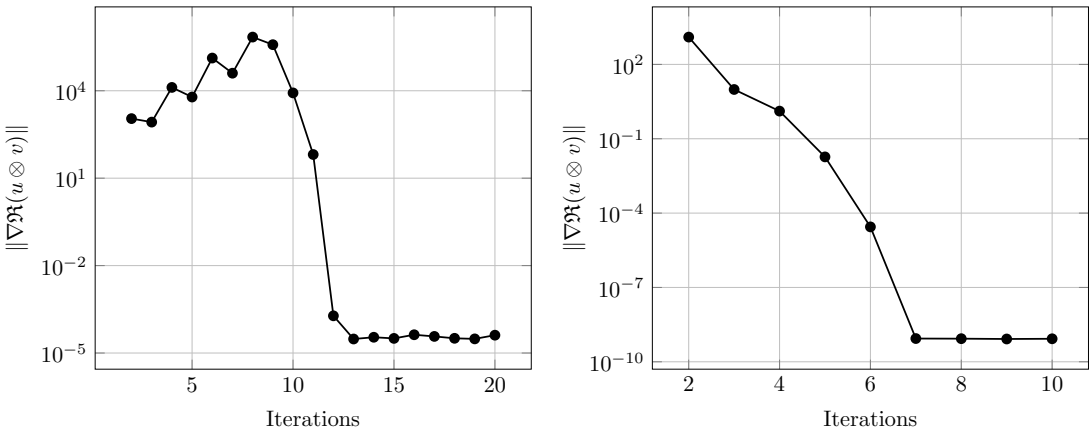


Figure 1: Testing Algorithm 1 with input index $(1,1)$. Figure (a) depicts results for randomly generated 1000×1000 matrices fulfilling Assumptions **A.1** and **A.2** and Figure (b) depicts results for a discretization of (2) on a 1000×1000 grid.

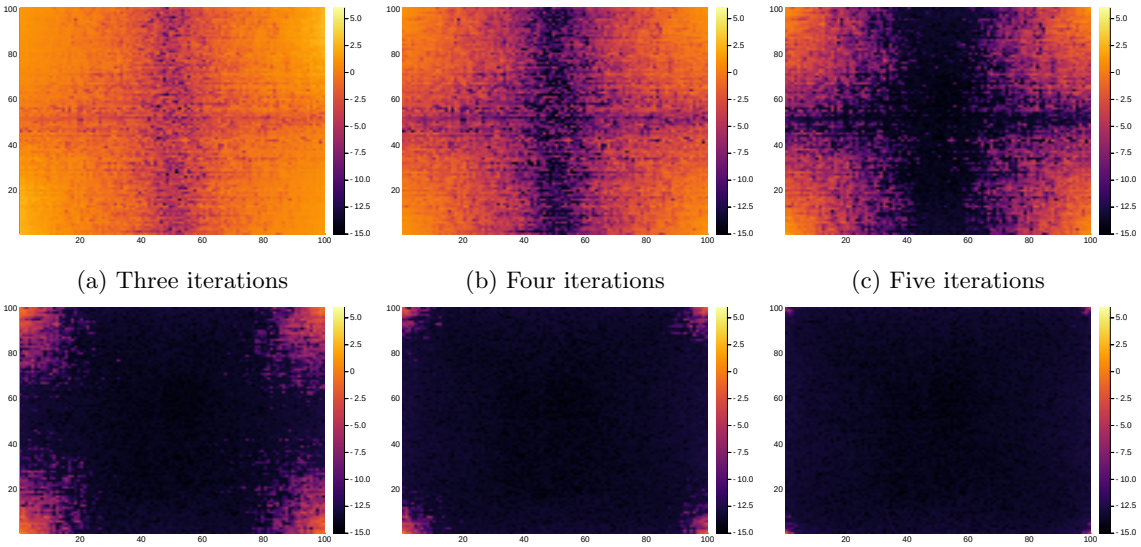
4 Numerical Experiments

In this section, we present results from numerical experiments to test the performance of Algorithm 1. In Figure 1 we depict the norm of the gradient of the Rayleigh quotient after each iteration. For Figure 2 and Figure 3 we computed the j_1-th smallest eigenvalue of $A_1 + \lambda_1 B_1 + \lambda_2 C_1$ and the j_2-smallest eigenvalue of $A_1 + \lambda_1 B_1 + \lambda_2 C_1$, where λ_1 and λ_2 were generated by Algorithm 1. We then depict the sum of absolute values of these eigenvalues. This is zero only for an eigenvalue (λ_1, λ_2) of the corresponding index. The colorscale in Figure 2 for the error is logarithmic and its range is from 10^{-15} to 10^{6}, while the x- and y-axes stand for the corresponding index of the respective eigenvalue. Every picture shows the error after solving one eigenvalue problem in Algorithm 1.

In a first experiment, we generated matrices randomly satisfying the assumptions. Here A_1 and A_2 are the symmetric part of matrices with random Gaussian entries, and for each of the matrices B_1, B_2, C_1, and C_2 we take a matrix M with Gaussian entries and setting the matrices as $\pm MM^\top$. The results are depicted in Figure 1(a) for the smallest eigenvalue with 1000×1000 matrices and in Figure 2 for all eigenvalues with 100×100 matrices. For 100×100 matrices the measured error for every index in Figure 2 was in the range of machine precision after solving nine eigenvalue problems.

Although our convergence result applies only to extremal eigenvalues, we observe quite fast geometric convergence for every eigenvalue pair. Indeed, for every pair the error was in range of machine precision after at most nine iterations. One remarkable observation is that for the non-extremal eigenvalues convergence seems to be even faster.

As a second example, we discretized the Helmholtz equation on half of an ellipse as in the example in Section 1 on a 1000×1000 grid in Figure 1(b) for the smallest eigenvalue λ_1 and for all eigenvalues on a 100×100 grid in Figure 3. In this more structured eigenvalue problem, convergence was slightly faster, as the error for every index was in range of machine precision after solving only six eigenvalue problems as seen in Figure 3 and the gradient is converging faster than for randomly



(a) Three iterations (b) Four iterations (c) Five iterations

(d) Six iterations (e) Seven iterations (f) Eight iterations

Figure 2: Testing Algorithm 1 with random 100×100 matrices fulfilling the assumptions. Every picture is taken after solving one eigenvalue problem in line 4 or 6 of the algorithm. The axis label the indices of the computed eigenvalue and the colorscale shows the sum of the absolute values of the i-th and j-th eigenvalue in (1) on a logarithmic scale.

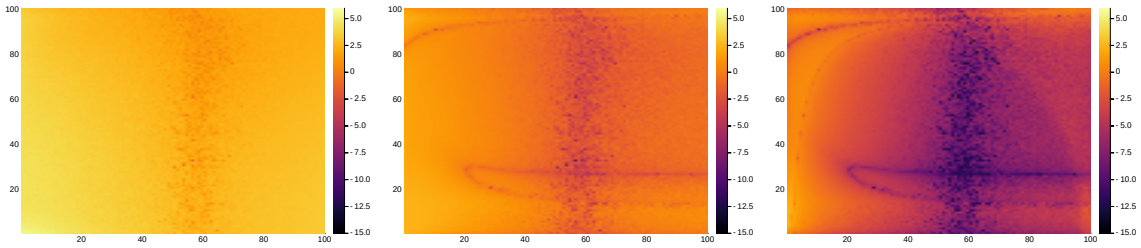
generated matrices as seen in Figure 1(b). Again convergence seems to be faster for non-extremal eigenvalues, however in this case some more structure is visible, which at the moment we cannot explain.

5 Conclusion and outlook

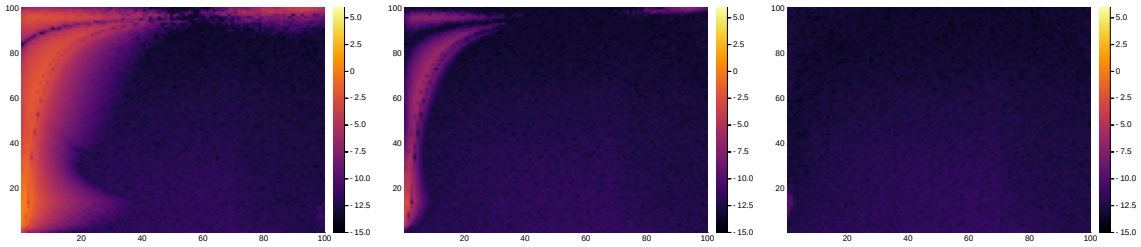
We presented a new method for computing eigenvalues of the two-parameter eigenvalue problem. Our approach only requires solving generalized eigenvalue problems of the size of the matrices of the two-parameter problem and can therefore reduce the complexity compared to conventional methods. Our method also uses a search for the eigenvalues by index.

So far the technique of our proof only established convergence for extremal eigenvalues and under restrictive definiteness assumptions. The numerical experiments however indicated convergence for every eigenvalue. Making this precise is left for future work. Although there are classes of two-parameter eigenvalue problems that satisfy the assumptions, many other interesting applications do not. Therefore it would be important to investigate if a generalization to right definite problems, i.e., the operator M_0 is positive or negative definite, as in [7] or even to non-singular problems, i.e. the operator M_0 is invertible, as in [5] or even to the general case as in [6].

This paper relies heavily on the assumptions **A.1** and **A.2**. A natural question is to relax these conditions, in particular the strong hyperbolicity condition **A.2**. Indeed under weaker assumptions (such as when the matrices are *almost* definite, with a few eigenvalues of the opposite sign), using



(a) One iteration (b) Two iterations (c) Three iterations



(d) Four iterations (e) Five iterations (f) Six iterations

Figure 3: Testing Algorithm 1 with a discretization of (2) on a 100×100 grid. Every picture is taken after solving one eigenvalue problem in line 4 or 6 of the algorithm. The axis label the indices of the computed eigenvalue and the colorscale shows the sum of the absolute values of the i-th and j-th eigenvalue in (1) on a logarithmic scale.

inertia laws [13] one can show that the generalized eigenvalue problem (4), and hence also (1), has many real eigenvalues. It would be of interest to investigate the applicability of the results here in such situations.

Finally, another interesting generalization that could be considered is to multiparameter eigenvalue problems with more than 2 parameters. The eigenvectors then form rank-one tensors of eigenvalues similar to (4) [1]. Again an alternating approach as in [8] can be used, however our proof technique will not work and in practice the generalization will not easily assure convergence even for extremal indices.

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