

**Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig**

**Existence for VT-harmonic maps from
compact manifolds with boundaries**

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Preprint no.: 89

2020



EXISTENCE FOR VT -HARMONIC MAPS FROM COMPACT MANIFOLDS WITH BOUNDARIES

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ABSTRACT. In the paper, we consider a kind of generalized harmonic maps, namely, the VT -harmonic maps. We will prove an existence theorem for Dirichlet problems of VT -harmonic maps from compact manifolds with boundaries.

Keywords and phrases: Dirichlet problem; Existence; VT -Harmonic map.

MSC 2010: 58E15; 58E20; 53C27

1. INTRODUCTION

For a map u from a Riemannian manifold (M^m, g) to a Riemannian manifold (N^n, h) , the energy functional is defined as

$$E(u) = \int_M \frac{|du|^2}{2} d\nu_g,$$

its critical point is called *harmonic map*. The Euler-Lagrange equation is:

$$(1.1) \quad \tau(u) = 0,$$

where $\tau(u) = \text{tr}_g(\nabla du)$ is the tension field of u . Another way to define harmonic map is: consider the second fundamental form ∇du of the map u defined using the Levi-Civita connections on M and N , taking the trace of it with respect to the metric g on M , one obtains the tension field $\tau(u)$ and then the equation (1.1).

Harmonic map has many generalizations, an important one is Hermitian harmonic map introduced by Jost and Yau ([10]), which are maps from Hermitian manifolds to Riemannian manifolds satisfying a nonlinear elliptic system. Along this line, affine harmonic map ([9]) and V-harmonic map [2] were investigated. Recently, in [1] the authors introduced the following generalized harmonic maps:

Definition 1 (VT -harmonic map, c.f. [1]). Let (M, g) be a compact manifold with boundary, (N, h) a compact Riemannian manifold. A map $u : (M, g) \rightarrow (N, h)$ is called

The research is partially supported by NSFC (Grant No. 11971358). The second author thanks the Max-Planck Institute for Mathematics in the Sciences for good working conditions when part of this work was carried out.

a VT -harmonic map iff u satisfies

$$(1.2) \quad \tau_V u + Tr_g T(du, du) = 0,$$

where $\tau_V u = \tau(u) + du(V)$, $\tau(u) = Tr_g(\nabla du)$, $V \in \Gamma(TM)$, $T \in \Gamma(\otimes^{1,2}TN)$.

In particular, if $T \equiv 0$, u is called a V -harmonic map ([2]).

Recall that when considering harmonic maps from Hermitian manifolds (affine manifolds resp.) M into Riemannian manifolds N , one obtains Hermitian harmonic maps (affine harmonic maps resp.), they are both V -harmonic maps with V being the difference between the underlying connections and the Levi-Civita ones on M (c.f. [2], [9]). More generally, if we consider harmonic maps between two Hermitian manifolds (affine manifolds resp.) M and N , then we will have VT -harmonic maps, with V , T being the differences of the underlying connections and the Levi-Civita ones on M , N respectively. It is expected that VT -harmonic maps will have interesting applications in the geometry of the underlying manifolds equipped with natural connections rather than the Levi-Civita ones (see e.g. [10]).

Existence is a fundamental problem for harmonic maps. Eells and Sampson [3] used the heat flow method to obtain the existence of harmonic map from compact Riemannian manifold without boundary to nonpositively curved Riemannian manifold. Also using the heat flow method, Hamilton [5] obtained the existence result for Dirichlet problems and Neuman problems of harmonic maps from manifolds with boundary to nonpositively curved Riemannian manifolds with convex boundaries. When the curvature of the target manifold is positive, Hildebrandt, Kaul and Widman (see e.g. [6, 7, 8]) established existence results for Dirichlet problems by assuming the image of the map is contained in a suitable geodesic ball in the target manifold.

In this paper, we will consider existence of solutions for the following Dirichlet problem for VT -harmonic maps:

$$(1.3) \quad \begin{cases} \tau_V u + Tr_g T(du, du) = 0, & \text{on } M, \\ u = \Phi, & \text{on } \partial M, \end{cases}$$

where $\Phi \in C^{1+\alpha}(\partial M, N)$.

In local coordinates, the above equation can be written as

$$(1.4) \quad \Delta_V u^l + (\Gamma_{ik}^l + T_{ik}^j)g^{\alpha\beta} D_\alpha u^i D_\beta u^k = 0, \quad 1 \leq l \leq n, \quad \forall t \in [0, 1].$$

where, operator $\Delta_V u := \Delta_{M,g} u + \langle V, \nabla u \rangle_g$, $u \in C^2(M)$, $V \in \Gamma(TM)$.

In general, the equation of VT -harmonic maps has no variational structure, this makes the existence problem more difficult than that of the usual harmonic maps.

Jost and Yau [10] investigated the existence of Hermitian harmonic maps from Hermitian manifolds into compact nonpositively curved Riemannian manifolds through the heat flow method.

By using maximum principle to establish a priori estimates, in [2] the authors obtained the existence of V -harmonic maps from Riemannian manifold with boundary into geodesic ball in targeted manifold. This was extended to the case of VT -harmonic maps in [1]:

Theorem 1 (Theorem 3 in [1]). *Let (M^m, g) be a compact Riemannian manifold with nonempty boundary ∂M and (N^n, h) a complete Riemannian manifold without boundary. Let $d : N \times N \rightarrow \mathbb{R}$ be the distance function on N and $B_{(1+\sigma)R}(p) := \{q \in N : d(p, q) \leq (1 + \sigma)R\}$ a regular ball in N , that is, disjoint from the cut locus of its center p and of radius $(1 + \sigma)R < \frac{\pi}{2\sqrt{\kappa}}$, where $\kappa = \max\{0, \sup_{B_{(1+\sigma)R}(p)} K_N\}$ and $\sup_{B_{(1+\sigma)R}(p)} K_N$ is an upper bound of the sectional curvature K of N on $B_{(1+\sigma)R}(p)$, and $\sigma > 0$ is any given constant.*

Suppose $u_0 \in H^{2,q}(M, N)$ ($q > m$) with $u_0(M) \subset B_R(p)$. For appropriate σ and R , there exists a constant C_0 depending only on κ, σ, R and the geometry of N , such that if

$$\max |\nabla T| + \max |T| \leq C_0,$$

then the initial-boundary value problem

$$(1.5) \quad \begin{cases} \partial_t u = \tau(u) + du(V) + \text{Tr}_g T(du, du), \\ u - u_0 \in H_0^{2,q}(M, N), \quad u(0) = u_0, \quad u(M \times [0, \infty)) \subset B_R(p), \end{cases}$$

admits a unique global solution u which subconverges to a unique solution $u \in H^{2,q}(M, N)$ of the Dirichlet problem

$$(1.6) \quad \begin{cases} \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0, \\ u - u_0 \in H_0^{2,q}(M, N), \end{cases}$$

such that $u(M) \subset B_R(p)$.

In this paper, by using the Leray-Schauder theory as in [6] we will derive the existence of VT -harmonic maps from Riemannian manifold with boundary into geodesic ball in targeted manifold, which relaxes the smallness conditions on radius of the geodesic ball $B_{(1+\sigma)R}(p)$ and the tensor T in the result of [1] by giving explicit upper bounds for them.

Before stating our main result, we first give some definitions.

Definition 2 (c.f. [7]). A set $A \subset N$ is within normal range of a point $q \in N$ iff its distance to the cut locus of q is positive, and A is within normal range of a closed set

$B \subset N$ iff its distance to the cut locus of every point of B is positive. The q -star hull of a set $A \subset N$ which is within normal range of the point q is the union of all shortest geodesic arcs connecting q and the points in A .

The function $a_\nu(t)$ in our theorems is defined as (c.f. [7])

$$a_\nu(t) = \begin{cases} t\sqrt{\nu} \cot(t\sqrt{\nu}), & \text{if } \nu > 0, \quad 0 \leq t < \frac{\pi}{\sqrt{\nu}}, \\ t\sqrt{-\nu} \coth(t\sqrt{-\nu}), & \text{if } \nu \leq 0, \quad 0 \leq t < \infty. \end{cases}$$

For $\nu > 0$, the function is decreasing, which belongs to $(-\infty, 1]$, as $t \rightarrow \frac{\pi}{\sqrt{\nu}}$, $a \rightarrow -\infty$, when $t = 0$, it is equal to 1. For $\nu < 0$, the function is increasing, which belongs to $[1, \infty)$, as $t \rightarrow \infty$, $a \rightarrow \infty$, when $t = 0$, it is equal to 1.

Now we can state our main result:

Theorem 2. *Let (M^m, g) be a compact manifold with boundary, (N^n, h) a complete Riemannian manifold without boundary. Fix a point $q \in N$, suppose the map $\Phi : \partial M \rightarrow B_R(q) \subset N$ which belongs to class $C^{1+\alpha}(\partial M, N)$, $\alpha \in (0, 1)$, where $R = \sup_{\partial M} \sqrt{d_\Phi^q} < \frac{\pi}{4\sqrt{\kappa}}$, where $d_\Phi^q(p) := d^N(q, \Phi(p))$, $p \in \partial M$. In addition, $B_R(q)$ is in the normal range of the q -star hull of $\Phi(\partial M)$. Assume the tensor T satisfy the following assumption:*

$$(1.7) \quad \max_N \{ \|T\|_{L^\infty}, \|\nabla T\|_{L^\infty} \} < \min \left\{ \sqrt{\kappa}, \left(\frac{2\kappa(4s_0 - \pi s_0 + \pi)}{(7\pi + 8\sqrt{\kappa})s_0} \right)^{\frac{1}{2}} \right\}.$$

where the constant $\kappa > 0$ is the upper bound of the sectional curvature of N , $s_0 = \min\{m, n\}$.

Then there exists a constant $V_0 > 0$ such that when $\|V\|_{C^0(M)} < V_0$, the Dirichlet problem (1.3) admits a solution, namely, a VT -harmonic map U in $C^{1+\alpha}(M, N) \cap C^3(\overset{\circ}{M}, N)$ satisfying $U(M) \subset B_R(q)$.

Remark 1. The bound of $\max_N \{ \|T\|_{L^\infty}, \|\nabla T\|_{L^\infty} \}$ depends on our choices of ϵ_2 and ϵ_3 in (3.2).

Remark 2 (c.f. pp 4-5 in [8]). One can verify the assumption that $B_R(q)$ is in the normal range of all of its points if one of the following four condition holds:

- N is simply connected and $K^N \leq 0$ ([4, p. 201])
- N is compact, connected and non-orientable, n is even, $0 < K \leq \kappa$ and $R < \frac{\pi}{4\sqrt{\kappa}}$ ([4, pp. 227-228], [11, pp. 3-4]).
- N is connected and orientable, n is even, $0 < K \leq \kappa$ and $R < \frac{\pi}{4\sqrt{\kappa}}$ ([4, pp. 229-230]).
- N is simply connected, n is even, $\frac{\kappa}{4} < K \leq \kappa$ and $R < \frac{\pi}{2\sqrt{\kappa}}$ ([4, p. 254]).

Remark 3. The q -star hull of $\Phi(\partial M)$ belongs to $B_R(q)$ by the condition of R . The normal range condition implies that Φ_t (see Theorem 3) can be represented in normal coordinates around any point of $\Phi_t(\partial M)$.

2. SOME LEMMAS

Before proving our main theorem, we first give some lemmas.

Lemma 1 (c.f. [1], page 90). *Let (M, g) be a compact manifold with boundary, (N, h) a complete Riemannian manifold without boundary.. Suppose $u : (M, g) \rightarrow (N, h)$ is a VT-harmonic map. Then*

$$(2.1) \quad \frac{1}{2} \Delta_V e(u) \geq -A_1 e(u) - A_2(\kappa, T, \nabla T) e(u)^2,$$

where $A_1 = (1 - \epsilon_3) \|V\|_{L^\infty}^2 + A + \epsilon_2$, $A_2(\kappa, T, \nabla T) = (1 - \epsilon_3 + \frac{1}{\epsilon_3}) \|T\|_{L^\infty}^2 + \kappa(1 - \frac{1}{s_0}) + \frac{1}{4\epsilon_2} \|\nabla T\|_{L^\infty}^2$, $s_0 = \min\{m, n\}$. Constant $A > 0$ depends only on V and Ric_M , ϵ_2, ϵ_3 are arbitrary small positive constants.

Remark 4. In fact, by the computation in [1], page 90, one has

$$(2.2) \quad \frac{1}{2} \Delta_V e(u) \geq -A_1 e(u) - A_2(\kappa, T, \nabla T) e(u)^2 + (1 - \epsilon_3) \left(1 + \frac{1}{m}\right) |\nabla \sqrt{e(u)}|^2.$$

We can choose suitable constants ϵ_2, ϵ_3 to get the inequality in the lemma while keeping $A_1, A_2 \geq 0$.

Lemma 2. *Assume the conditions in Theorem 2 hold and let $U \in C^2(M, N)$ be a map whose image lies in $B_R(q)$, which solves the following equation written in normal coordinates centered at q :*

$$(2.3) \quad \Delta_V u^l + t(\Gamma_{ik}^l + T_{ik}^l) g^{\alpha\beta} D_\alpha u^i D_\beta u^k = 0, \quad 1 \leq l \leq n, \quad \forall t \in [0, 1].$$

where u is the representation of the map U in this normal coordinates. Then we have

$$\Delta_V |u|^2 \geq 2t(a_\kappa(R) - \|T\|_\infty R) e(u).$$

In particular, if $u \in C^2(M, N)$ is a VT-harmonic map, then for a fixed $p \in B_R(q)$, we have

$$(2.4) \quad \Delta_V d^p \geq 2(a_\kappa(R) - \|T\|_\infty R) e(u),$$

where $d^p(x) := [\text{dist}(u(x), p)]^2$, for $x \in M$.

Proof. The conditions of Theorem 2 imply that we can compute in a fixed normal coordinates in N .

$$\Delta_V |u|^2 = 2u^j \Delta_V u^j + 2g^{\alpha\beta} D_\alpha u^j D_\beta u^j$$

$$\begin{aligned}
&= 2t \left(\delta_{ik} - \Gamma_{ik}^l(u)u^l - T_{ik}^l u^l \right) g^{\alpha\beta} D_\alpha u^i D_\beta u^k + 2(1-t) g^{\alpha\beta} D_\alpha u^j D_\beta u^j \\
&\geq 2t \left(h_{ik}(u) + \Gamma_{ikj}(u)u^j \right) g^{\alpha\beta} D_\alpha u^i D_\beta u^k - 2t T_{ik}^l u^l g^{\alpha\beta} D_\alpha u^i D_\beta u^k \\
&\geq 2t a_\kappa(R) e(u) - 2t T_{ik}^l u^l g^{\alpha\beta} D_\alpha u^i D_\beta u^k \\
&\geq 2t e(a_\kappa(R) - \|T\|_\infty R),
\end{aligned}$$

where in the second equality, we have used the VT -harmonic map equation, and in the second inequality, we have used the curvature upper bound on N , i.e., $K^N \leq \kappa$, thus [7, Lemma 1, (3.1)] implies that

$$a_\kappa(R) \xi^i \xi^k \leq h_{ik}(u) + \Gamma_{ikj} u^j \xi^i \xi^k, \forall \xi \in R^n,$$

where R is the constant in $\sup_M |u| = R < \frac{\pi}{\sqrt{\kappa}}$. The second part follows, noting that VT -harmonic map satisfies equation (2.3) for $t = 1$ and $d^p = |u|^2$ using normal coordinates centered at point $p \in N$. □

Theorem 3. *Let (M^m, g) be a compact manifold with boundary, (N^n, h) a complete Riemannian manifold without boundary. Fix a point $q \in N$, suppose the map $\Phi : \partial M \rightarrow B_R(q) \subset N$ which belongs to class $C^{1+\alpha}(\partial M, N)$, $\alpha \in (0, 1)$, where $R = \sup_{\partial M} \sqrt{d_\Phi^q} < \frac{\pi}{4\sqrt{\kappa}}$, where $d_\Phi^q(p) := d^N(q, \Phi(p))$, $p \in \partial M$. Denote ϕ the representation of Φ with respect to any normal chart around q . In addition, $B_R(q)$ is in the normal range of the q -star hull of $\phi(\partial M)$. Assume the tensor T satisfy the following assumption:*

$$(2.5) \quad \|T\|_{L^\infty} R < a_\kappa(R),$$

where the constant $\kappa \geq 0$ is the upper bound of the sectional curvature of N .

Set $\phi_t = t\phi$, $t \in [0, 1]$, let ϕ_t be the representation of Φ_t . If $U_t : M \rightarrow N$ is a VT -harmonic map in the space $C^{1+\alpha}(M, N) \cap C^3(\overset{\circ}{M}, N)$ satisfying $U(M) \subset B_R(q)$, $U_t|_{\partial M} = \Phi_t$ for some $t \in [0, 1]$, then there exists a constant $C_1 > 0$ such that when $\|V\|_\infty < C_1$,

$$(2.6) \quad \sup_{\partial M} e(U_t) \leq \Omega.$$

where Ω is a constant depending on $R, \alpha, q, \Phi, C_1, \|T\|_\infty, N$.

Proof. We choose normal coordinates $U = (u^1, u^2, \dots, u^n)$ around $U_t(p)$ which belongs to $U_t(\partial M)$. There exists a constant $\Lambda_1(\alpha, q, \Phi)$ such that

$$\|\Phi_{t, U_t(p)}\|_{C^{1+\alpha}(\partial M, R^n)} \leq \Lambda_1.$$

for $t \in [0, 1]$.

Let $h = h_{t,p}$ solves the Dirchlet problem

$$\begin{cases} \Delta_V h = 0, & \text{on } \mathring{M} \\ h = \Phi_{t,u(p)}, & \text{on } \partial M \end{cases}$$

for $t \in [0, 1], p \in \partial M$.

The Schauder theory implies that

$$\|h_{t,p}\|_{C^{1+\alpha}(M)} \leq C_2(\|\Phi_{t,u(p)}\|_{C^{1+\alpha}(\partial M)} + \|\langle V, \nabla h_{t,p} \rangle\|_{C^0(M)} + \|h\|_{C^0(M)}).$$

By the maximum principle and Schauder theory, there exists a constant $C_3 > 0$ depending only on m, n, R, φ and a constant $\Lambda_2(\alpha, q, \Phi, C_3, R)$ such that such that when $\|V\|_{C^0(M)} < C_3$, we have

$$\Delta_M |h_{t,p}|^2 + \|h_{t,p}\|_{C^1(M, R^n)} \leq \Lambda_2.$$

Since u is VT-harmonic map,

$$(2.7) \quad \Delta_V u^l + (\Gamma_{ik}^l + T_{ik}^l) g^{\alpha\beta} D_\alpha u^i D_\beta u^k = 0, \quad 1 \leq l \leq n.$$

There exists a constant $\Lambda_3(\alpha, q, \Phi, \|T\|_\infty, S)$ such that

$$\sum_{l=1}^n |\Delta_V u^l|^2 \leq (\Lambda_3 e(U_t))^2.$$

Let $w \in C^2(M, R)$ solves the Dirichlet problem

$$\begin{cases} \Delta_V w = c\Lambda_2, & \text{on } \mathring{M} \\ w = 0, & \text{on } \partial M. \end{cases}$$

where $c = \frac{\Lambda_3}{2a_\kappa(R)^{-2}\|T\|_\infty R}$. Thus, there exists a constant $C_4 > 0$ depending only on m, n, R, φ such that such that when $\|V\|_{C^0(M)} < C_4$, we have

$$\sup_M |\nabla w| \leq \Lambda_4.$$

for some constant $\Lambda_4(\alpha, q, u, R, C_4, \|T\|_\infty, N)$.

Let u_t be the representation of U_t in terms of normal coordinates u^1, u^2, \dots, u^n around $U_t(p)$. Let $p \in \partial M$, through coordinates change, without loss of generality, we choose unit vector $v \in T_p M$ pointing outwards such that at p :

$$\left\| \frac{\partial U_t}{\partial v} \right\|_N = \frac{\partial u_t^1}{\partial v}.$$

Using the auxillary function above, we can consider the function defined by

$$g = c|u_t|^2 - c|h_{t,p}|^2 - u_t^1 + h_{t,p}^1 + w.$$

By the assumption and Lemma 2, we have

$$c\Delta_V|u_t|^2 \geq 2c(a_\kappa(R) - \|T\|_\infty R)e = \Lambda_3 e_{u_t}.$$

Consequently, we have

$$\Delta_V g = c\Delta_V|u_t|^2 - c\Delta_V|h_{t,p}|^2 - \Delta_V u_t^1 + \Delta_V h_{t,p}^1 + \Delta_V w \geq 0.$$

The strong maximum principle and Hopf maximum principle implies that

$$-\frac{\partial g}{\partial v} \leq 0,$$

considering the facts that $u_t^i(p) = h_{t,p}^i = 0, p \in \partial M, 1 \leq i \leq n, g|_{\partial M} = 0$. Therefore, we have at p :

$$\left\| \frac{\partial U_t}{\partial v} \right\|_N = \frac{\partial u_t^1}{\partial v} = -\frac{\partial g}{\partial v} + \frac{\partial w}{\partial v} + \frac{\partial h_{t,p}^1}{\partial v} \leq \Lambda_2 + \Lambda_4.$$

Therefore,

$$e_{U_t}(p) \leq m \sup_v \left\| \frac{\partial U_t}{\partial v}(p) \right\|_N^2 \leq m(\Lambda_2 + \Lambda_4)^2 = \Omega(\alpha, q, \Phi, \|T\|_\infty, C_3, C_4, R).$$

□

Lemma 3. *Let A_1, A_2 be constants as in Lemma 1, and R, M_1 be positive numbers where $R < \frac{\pi}{4\sqrt{\kappa}}$, let $U \in C^1(M, N) \cap C^3(M, N)$ be a VT-harmonic map whose image lies in $B_R(q)$ satisfying*

$$\sup_M d_U^q \leq R^2, \quad \sup_M e(u) \leq M_1$$

and the tensor T satisfy the following assumption:

$$(2.8) \quad \|T\|_{L^\infty} R < a_\kappa(R),$$

where the constant $\kappa \geq 0$ is the upper bound of the sectional curvature of N . Then

$$(2.9) \quad \Delta_V \left(e(u) + \frac{A_1 + A_2 M_1}{a_\kappa(R) - \|T\|_\infty R} d^q(u) \right) \geq 0, \quad \text{on } M.$$

Proof. By (2.2) and $A_2 \geq 0$, we have

$$\begin{aligned} \frac{1}{2} \Delta_V e(u) &\geq -A_1 e(u) - A_2(\kappa, T, \nabla T) e(u)^2 \\ &\geq -A_1 e(u) - A_2(\kappa, T, \nabla T) M_1 e(u) \end{aligned}$$

By (2.4), we have

$$\Delta_V d^q \geq 2(a_\kappa(R) - \|T\|_\infty R)e.$$

Thus

$$\Delta_V \left(e(u) + \frac{A_1 + A_2 M_1}{a_\kappa(R) - \|T\|_\infty R} d^q(u) \right) \geq 0.$$

□

3. PROOF OF THEOREM 2

Proof of Theorem 2. We choose normal coordinates $U = (u^1, u^2, \dots, u^n)$ around $U_t(p)$ in the q -star hull of $u(\partial M)$. As usual, the representations of mappings U, Φ etc. in these coordinates we shall denote by u, ϕ , etc. We define $R = \sup_{\partial M} \sqrt{d_\Phi^q} < R_0 < \frac{\pi}{4\sqrt{\kappa}}$.

Firstly, we introduce two auxillary equations:

$$\begin{cases} \Delta_V \psi^l = -(\Gamma_{jk}^l + T_{jk}^l) \gamma^{\alpha\beta} D_\alpha u^j D_\beta u^k, & 1 \leq l \leq n \\ \psi^l = 0, & \text{on } \partial M \end{cases}$$

and

$$\begin{cases} \Delta_V h = 0, & \text{in } M \\ h = \phi, & \text{on } \partial M \end{cases}$$

Next, we introduce a compact map

$$\begin{cases} \Theta : C^1(M, \mathbb{R}^n) \rightarrow C^1(M, \mathbb{R}^n) \\ \Theta : u \mapsto \psi. \end{cases}$$

It is enough to find a solution of

$$(3.1) \quad u = \Theta(u) + h, \quad u \in C^1(M, \mathbb{R}^n).$$

By the Leray-Schauder theory, it is enough to prove firstly $\deg(F_{t,0}) = 1$, then $\deg(F_{1,s}) = 1$ for the map $F_{t,s} = Id - t\Theta - sh$, where $0 \leq t, s \leq 1$. In order to apply Lemma 2 and Theorem 3, we introduce a set $A \subset C^1(M, \mathbb{R}^n)$ which is defined by

$$A = A(R_0, M_1) = \{u \in C^1(M, \mathbb{R}^n); \quad \|d_u^q\|_{C^0(M, \mathbb{R}^N)} < R_0, \|e(u)\|_{C^0(M, \mathbb{R}^N)} < M_1\},$$

whose boundary is

$$\partial A = \{u \in C^1(M, \mathbb{R}^n); \quad \|d_u^q\|_{C^0(M, \mathbb{R}^N)} = R_0 \quad \text{or} \quad \|e(u)\|_{C^0(M, \mathbb{R}^N)} = M_1\},$$

where R_0, M_1 are positive constants, $d_u^q(p) := d^N(q, u(p)), p \in M$.

If $0 \in F_{t,0}(\partial A)$, which amounts to the fact that there exists $u \in \partial A$ such that $F_{t,0}(u) = 0$, i.e.

$$\begin{cases} u = t\Theta(u); & \text{on } M \\ u = 0, & \text{on } \partial M \end{cases}$$

But in this case, u solves equation (2.3), then Lemma 2 with the maximum principle implies that $u = 0$, which contradicts to the fact that $u \in \partial A$. Then homotopy invariance of topology degree implies that $\deg(F_{t,0}) = 1$, since $\deg(F_{0,0}) = 1$.

Next we prove $\deg(F_{1,s}) = 1$ by the similar argument. To this end, we choose

$$\sup_{\partial M} |\phi|^2 < R_0 < \frac{\pi}{4\sqrt{\kappa}}; \quad M_1 > \Omega + \frac{A_1 + A_2 M_1}{a_\kappa(R_0) - \|T\|_\infty R_0} R_0^2.$$

The latter inequality is equivalent to

$$(a_\kappa(R_0) - \|T\|_\infty R_0 - A_2 R_0^2) M_1 > (a_\kappa(R_0) - \|T\|_\infty R_0) \Omega + A_1 R_0^2.$$

Recall

$$A_2(\kappa, T, \nabla T) = (1 - \epsilon_3 + \frac{1}{\epsilon_3}) \|T\|_{L^\infty}^2 + \kappa(1 - \frac{1}{s_0}) + \frac{1}{4\epsilon_2} \|\nabla T\|_{L^\infty}^2.$$

We expect to prove the following inequality holds by our assumption:

$$(3.2) \quad \left(a_\kappa(R_0) - \|T\|_\infty R_0 - (1 - \epsilon_3 + \frac{1}{\epsilon_3}) \|T\|_{L^\infty}^2 R_0^2 - \kappa(1 - \frac{1}{s_0}) R_0^2 - \frac{1}{4\epsilon_2} \|\nabla T\|_{L^\infty}^2 R_0^2 \right) M_1 > (a_\kappa(R_0) - \|T\|_\infty R_0) \Omega + A_1 R_0^2.$$

The assumption in the theorem implies that

$$(3.3) \quad a_\kappa(R_0) - \|T\|_\infty R_0 > 0.$$

We choose $\epsilon_2 = \frac{1}{4}$, $\epsilon_3 = \frac{1}{2}$ in inequality (3.2), it is enough to require that

$$(3.4) \quad \kappa(1 - \frac{1}{s_0}) R_0^2 + \frac{5}{2} \|T\|_{L^\infty}^2 R_0^2 + \|\nabla T\|_{L^\infty}^2 R_0^2 < a_\kappa(R_0) - \|T\|_\infty R_0.$$

which is equivalent to

$$(3.5) \quad \kappa(1 - \frac{1}{s_0}) + \frac{5}{2} \|T\|_{L^\infty}^2 + \|\nabla T\|_{L^\infty}^2 < \frac{a_\kappa(R_0) - \|T\|_\infty R_0}{R_0^2}.$$

By the decreasing property of $a_\kappa(t)$ for $0 \leq t < \frac{\pi}{4\sqrt{\kappa}}$, we have

$$\inf_{0 \leq R < \frac{\pi}{4\sqrt{\kappa}}} \frac{a_\kappa(R)}{R} = \sqrt{\kappa}.$$

It suffice to require that

$$(3.6) \quad \kappa(1 - \frac{1}{s_0}) + \frac{5}{2} \|T\|_{L^\infty}^2 + \|\nabla T\|_{L^\infty}^2 < \frac{4\sqrt{\kappa}}{\pi} (\sqrt{\kappa} - \|T\|_\infty).$$

By the assumption,

$$(3.7) \quad \max_N \{ \|T\|_{L^\infty}, \|\nabla T\|_{L^\infty} \} < \left(\frac{\kappa(\frac{4}{\pi} - 1 + \frac{1}{s_0})}{\frac{7}{2} + \frac{4\sqrt{\kappa}}{\pi}} \right)^{\frac{1}{2}} = \left(\frac{2\kappa(4s_0 - \pi s_0 + \pi)}{(7\pi + 8\sqrt{\kappa})s_0} \right)^{\frac{1}{2}}.$$

Thus, the inequality (3.6) can be solved to make sure that the coefficient of M_1 in (3.2) is positive. Then we can choose M_1 to satisfy (3.2).

If $0 \in F_{1,s}(\partial A)$, which amounts to the fact that $\exists u \in \partial A$ such that $F_{1,s}(u) = 0$, which is

$$\begin{cases} u = \Theta(u) + sh; & \text{on } M \\ u = s\phi, & \text{on } \partial M \end{cases}$$

However, in this case, u is a VT-harmonic map with boundary value $s\phi$. By Theorem 3, we have

$$\begin{aligned} \sup_M e &\leq \sup_M \left(e + \frac{A_1 + A_2 M_1}{a_\kappa(R) - \|T\|_\infty R} d^q \right) \leq \sup_{\partial M} \left(e + \frac{A_1 + A_2 M_1}{a_\kappa(R) - \|T\|_\infty R} d^q \right) \\ &\leq \sup_{\partial M} (e) + \frac{A_1 + A_2 M_1}{a_\kappa(R_0) - \|T\|_\infty R_0} R_0^2 \leq \Omega + \frac{A_1 + A_2 M_1}{a_\kappa(R_0) - \|T\|_\infty R_0} R_0^2 < M_1. \end{aligned}$$

this again contradicts to the fact that $u \in \partial A$. Thus, we have proved $\deg(F_{1,s}) = 1$ which implies the existence. □

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