Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Local quantum Fisher information and one-way quantum deficit in spin-$\frac{1}{2}$ XX Heisenberg chain with three-spin interaction

by

Biao-Liang Ye, Bo Li, Xiao-Bin Liang, and Shao-Ming Fei

Preprint no.: 93 2020

Local quantum Fisher information and one-way quantum deficit in spin-$\frac{1}{2}$ XX Heisenberg chain with three-spin interaction

Biao-Liang Ye,[1,*] Bo Li,[1,2] Xiao-Bin Liang,[1,2] and Shao-Ming Fei[3,4,†]

[1] *Quantum Information Research Center, Shangrao Normal University, Shangrao 334001, China*

[2] *School of Mathematics & Computer Science, Shangrao Normal University, Shangrao 334001, China*

[3] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

[4] *Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

(Dated: November 4, 2019)

We explore quantum phase transitions in the spin-1/2 XX chain with three-spin interaction in terms of local quantum Fisher information and one-way quantum deficit, together with the demonstration of quantum fluctuations. Analytical results are derived and analyzed in details.

I. INTRODUCTION

Quantum entanglement plays a vital role in quantum information processing [1]. As an important resource, quantum entangled states have been used in quantum teleportation [2], remote state preparation [3], secure quantum-communications network [4], etc. Besides quantum entanglement, quantum discord characterizes non-classical correlations [5]. The one-way quantum deficit [6] is another key measure to describe quantum correlation [7]. While the quantum Fisher information [8, 9] is important in the estimation accuracy scenarios.

On the other hand, the quantum phase transitions have received much attention in condensed matter physics [10]. The quantum fluctuations are able to be illustrated by quantum correlations. In Ref. [11] the role of entanglement played in phase transition and theory of

*Electronic address: biaoliangye@gmail.com

†Electronic address: feishm@cnu.edu.cn

critical phenomena in XY system has been investigated. The quantum discord and entanglement between the nearest-neighbor qubits in an infinite (spin-1/2) chain described by the Heisenberg model (XXZ Hamiltonian) were investigated in [12], and the critical points associated with quantum phase transitions at finite temperature had been analyzed. In Ref.[13], the authors studied the effects of Dzyaloshinskii-Moriya interaction on pairwise quantum discord, entanglement, and classical correlation in the anisotropic XY spin-half chain. It has been shown that the quantum discord can be useful to highlight the quantum phase transition, especially for the long-distance spins, while entanglement decays rapidly. The quantum discord has been also used to show the quantum phase transition in XX model in [14]. In Ref. [15] the authors connected the local quantum coherence based on Wigner-Yanase skew information and the quantum phase transitions. The local quantum coherence and its derivatives are used effectively in detecting different types of quantum phase transitions in different spin systems. In Ref. [16] the authors introduced a coherence susceptibility approach in identifying quantum phase transitions induced by quantum fluctuations.

Although different measures of quantum correlations have been used to characterize quantum phase transitions in different spin chain systems, both local quantum Fisher information and one-way quantum deficit have not been adopted to study quantum phase transition in Hersenberg XX models. In this paper, we investigate the quantum phase transitions of the XX chain with three spin interaction. We explore the quantum fluctuation via both local quantum Fisher information and one-way quantum deficit to investigate the quantum phase transition. We review the basic definitions of local quantum Fisher information and one-way quantum deficit in Sec. II. In Sec.III, the Hersenberg spin-$\frac{1}{2}$ is introduced and the main results are presented. Conclusions are given in Sec. IV.

II. PRELIMINARIES

We first recall the basic definitions of local quantum Fisher information and one-way quantum deficit.

Local quantum Fisher information Quantum Fisher information (QFI) is recognized as the most widely used quantity for characterizing the ultimate accuracy in parameter estimation scenarios. Recently, many efforts have been made toward evaluating the dynamics of QFI to establish the relevance of quantum entanglement in quantum metrology. It has been demonstrated that quantum entanglement leads to a notable improvement of the accuracy of parameter estimation. It is natural to ask whether quantum correlations beyond quantum entanglement can be related to the precision in quantum metrology protocols.

For an arbitrary quantum state ρ_θ that depends on a parameter θ, the QFI is given by [17],

$$F(\rho_\theta) = \frac{1}{4}\mathrm{Tr}[\rho_\theta L_\theta^2], \tag{1}$$

where the symmetric logarithmic derivative L_θ is defined as the solution of the following equation

$$\frac{\partial \rho_\theta}{\partial \theta} = \frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta). \tag{2}$$

The parametric states ρ_θ can be obtained from an initial probe state ρ subjected to a unitary transformation $U_\theta = e^{-iH\theta}$ which is dependent on θ and a Hermitian operator H, i.e., $\rho_\theta = U_\theta^\dagger \rho U_\theta$. In this case $F(\rho_\theta)$ is given by

$$F(\rho, H) = \frac{1}{2}\sum_{i \neq j} \frac{(p_i - p_j)^2}{p_i + p_j} |\langle \psi_i | H | \psi_j \rangle|^2, \tag{3}$$

where $\rho = \sum_{i=1} p_i |\psi_i\rangle\langle\psi_i|$ is spectral decomposition of ρ, $p_i \geq 0$ and $\sum_{i=1} p_i = 1$.

Now consider a bipartite quantum state ρ_{AB} in the Hilbert space $H = H_A \otimes H_B$. We assume that the dynamics of the first subsystem is governed by the local phase shift transformation $e^{-i\theta H_A}$, with $H_A = H_a \otimes I_B$ the local Hamiltonian. In this case QFI reduces to local quantum Fisher information (LQFI),

$$F(\rho, H_A) = \mathrm{Tr}(\rho H_A^2) - \sum_{i \neq j} \frac{2 p_i p_j}{p_i + p_j} |\langle \psi_i | H_A | \psi_j \rangle|^2. \tag{4}$$

Local quantum Fisher information was introduced to deal with pairwise quantum measures of discord type. The local quantum Fisher information $Q(\rho)$ is defined as the minimum quantum Fisher information over all local Hamiltonians H_A acting on the subsystem A [18],

$$Q(\rho) = \min_{H_A} F(\rho, H_A). \tag{5}$$

For local Hamiltonian $H_A = \vec{\sigma} \cdot \vec{r}$, with $|\vec{r}| = 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices, one has $\mathrm{Tr}(\rho H_A^2) = 1$, and the second term in (4) can be written as

$$\sum_{i \neq j} \frac{2 p_i p_j}{p_i + p_j} |\langle \psi_i | H_A | \psi_j \rangle|^2 = \sum_{i \neq j} \sum_{l,k=1}^{3} \frac{2 p_i p_j}{p_i + p_j} \langle \psi_i | \sigma_l \otimes I_B | \psi_j \rangle \langle \psi_j | \sigma_k \otimes I_B | \psi_i \rangle = \vec{r}^\dagger \cdot T \cdot \vec{r}, \tag{6}$$

where the elements of the 3×3 symmetric matrix T are given by

$$T_{lk} = \sum_{i \neq j} \frac{2p_i p_j}{p_i + p_j} \langle \psi_i | \sigma_l \otimes I_B | \psi_j \rangle \langle \psi_j | \sigma_k \otimes I_B | \psi_i \rangle. \tag{7}$$

To minimize $F(\rho, H_A)$, it is necessary to maximize the quantity $\vec{r}^\dagger \cdot T \cdot \vec{r}$ over all unit vectors $\vec{r}$. The maximum value coincides with the maximum eigenvalue of T. Hence, the minimal value of local quantum Fisher information $Q(\rho)$ is

$$Q(\rho) = 1 - \lambda_{\max}(T), \tag{8}$$

where $\lambda_{\max}$ denotes the maximal eigenvalue of the symmetric matrix T defined by (7).

One-way quantum deficit (OWQD) The one-way quantum deficit is defined as the difference of the von Neumann entropy of a bipartite state, ρ_{AB}, before and after a measurement performed on, without a loss of generality, subsystem B [7],

$$\Delta = \min_{\Pi_A^i} S(\sum_i \Pi_k^B(\rho_{AB})) - S(\rho_{AB}), \tag{9}$$

where Π_k^B is the measurement on subsystem B and $S(\rho_{AB}) = -\mathrm{Tr}\rho_{AB} \log \rho_{AB}$ is the von Neumann entropy. Throughout the article, log is in base 2. The minimum is taken over all local measurements Π_k^B.

The states after the projective measurement can be expressed as

$$\widetilde{\rho}_{AB} = \sum_k (I \otimes \Pi_k) \rho_{AB} (I \otimes \Pi_k)^\dagger. \tag{10}$$

The post-measurement states is given by

$$\rho_{AB}^k = \frac{1}{p_k} (I \otimes \Pi_k) \rho_{AB} (I \otimes \Pi_k)^\dagger, \tag{11}$$

where

$$p_k = \mathrm{Tr}[(I \otimes \Pi_k) \rho_{AB} (I \otimes \Pi_k)^\dagger]. \tag{12}$$

Here Π_k $(k = 0, 1)$ are the general orthogonal projectors

$$\Pi_k = V|k\rangle\langle k|V^\dagger, \tag{13}$$

where V belongs to the special unitary group $SU(2)$. The rotations V may be parametrized by two parameters θ and ϕ, respectively,

$$V = \begin{pmatrix} \cos(\theta/2) & -e^{-i\phi}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{14}$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$.

III. QUANTUM PHASE TRANSITION OF THE HEISENBERG-$\frac{1}{2}$ XX SPIN CHAIN MODEL WITH THREE SPIN INTERACTION

The Hamiltonian of the Heisenberg spin-$\frac{1}{2}$ XX chain can be expressed as [19],

$$\begin{aligned} H &= \sum_{l=1}^{N} -J(S_l^x S_{l+1}^x + S_l^y S_{l+1}^y + \Delta S_l^z S_{l+1}^z) - J'\{(S_{l-1}^x S_l^z S_{l+1}^y - S_{l-1}^y S_l^z S_{l+1}^x) \\ &+ \Delta(S_{l-1}^y S_l^x S_{l+1}^z - S_{l-1}^z S_l^x S_{l+1}^y) + \Delta(S_{l-1}^z S_l^y S_{l+1}^x - S_{l-1}^x S_l^y S_{l+1}^z)\}, \end{aligned} \tag{15}$$

where S_l^q $(q = x, y, z)$ are spin operators of $S = 1/2$-spin on site l, N is the total number of spins, J is the nearest-neighbor Heisenberg exchange coupling, J' is the strength of three spin interaction, and Δ represents the anisotropy parameter. The model exhibits several quantum phases depending on the parameters J'/J and Δ. The same Hamiltonian is used in the study of current-carrying states for the system with only the nearest-neighbor interactions, where the three-spin terms play the role of the Lagrange multiplier.

If $\Delta = 0$, the Hamiltonian (15) reduces to a free spinless fermion model,

$$H = \sum_{l=1}^{N} -J(S_l^x S_{l+1}^x + S_l^y S_{l+1}^y) - J'\{(S_{l-1}^x S_l^z S_{l+1}^y - S_{l-1}^y S_l^z S_{l+1}^x)\}. \tag{16}$$

Applying the Jordan-Wigner transformation,

$$\begin{aligned} S_l^x &= 1/2\Pi_{n=1}^{l-1}(1 - 2c_n^\dagger c_n)(c_l^\dagger + c_l); \\ S_l^y &= 1/2i\Pi_{n=1}^{l-1}(1 - 2c_n^\dagger c_n)(c_l^\dagger - c_l); \\ S_l^z &= c_l^\dagger c_l - 1/2, \end{aligned} \tag{17}$$

H can be rewritten as

$$H = \sum_{l=1}^{N}[-J/2(c_l^\dagger c_{l+1} + h.c.) + J'/4i(c_l^\dagger c_{l+2} - h.c.)], \tag{18}$$

which can be diagonalized by means of the Fourier transformation,

$$H = \sum_k \epsilon(k) c_k^\dagger c_k, \tag{19}$$

where the energy dispersion

$$\epsilon(k) = -J[\cos k - \alpha/2 \sin(2k)], \tag{20}$$

with $\alpha = J'/J$.

The matrix form of the Hamiltonian form can be denoted as

$$\rho = \begin{pmatrix} u_{ij} & 0 & 0 & 0 \\ 0 & \omega_{ij} & y_{ij} & 0 \\ 0 & y_{ij} & \omega_{ij} & 0 \\ 0 & 0 & 0 & u_{ij} \end{pmatrix}, \tag{21}$$

where all the elements of the matrix can be written in terms of spin-spin correlation functions,

$$\begin{aligned} u_{ij} &= \frac{1}{4} + \langle S_i^z S_j^z \rangle, \\ \omega_{ij} &= \frac{1}{4} - \langle S_i^z S_j^z \rangle, \\ y_{ij} &= \langle S_i^x S_j^x \rangle + \langle S_i^y S_j^y \rangle, \end{aligned}$$

with $\langle S_i^q S_j^q \rangle$ $(q = x, y, z)$ the two-point spin-spin functions at sites i and j, and the expectation value taken over all the quantum states.

Using the method proposed by Lieb, Schultz, and Mattis [20], we can also calculate the spin-spin correlation functions,

$$\langle S_l^x S_{l+m}^x \rangle = \langle S_l^y S_{l+m}^y \rangle = \frac{1}{4} \begin{vmatrix} G_{l,l+1} & G_{l,l+2} & \cdots & G_{l,l+m} \\ G_{l,l} & G_{l,l+1} & \cdots & G_{l,l+m-1} \\ \cdots & \cdots & \ddots & \cdots \\ G_{l,l-m+2} & G_{l,l-m+3} & \cdots & G_{l,l+1} \end{vmatrix}, \tag{22}$$

and

$$\langle S_l^z S_{l+m}^z \rangle = -1/4 (G_{l,l+m})^2, \tag{23}$$

with

$$G_{l,l+m} = \begin{cases} \frac{2}{m\pi} \sin(m\frac{\pi}{2}) & \alpha < 1, \\ \frac{1}{m\pi}[1 - (-1)^m] \sin(m \arcsin(1/\alpha)) & \alpha \geq 1. \end{cases} \tag{24}$$

Let $t_1 = 4\langle S_l^x S_{l+m}^x \rangle$, $t_2 = 4\langle S_l^y S_{l+m}^y \rangle$ and $t_3 = 4\langle S_l^z S_{l+m}^z \rangle$. The 3×3 matrix T is of the form,

$$T = \begin{pmatrix} \frac{(t_3+1)\left(2t_1^2+t_3-1\right)}{t_1^2-1} & 0 & 0 \\ 0 & \frac{(t_3+1)\left(2t_1^2+t_3-1\right)}{t_1^2-1} & 0 \\ 0 & 0 & \frac{2t_1^2+t_3-1}{t_3-1} \end{pmatrix}, \tag{25}$$

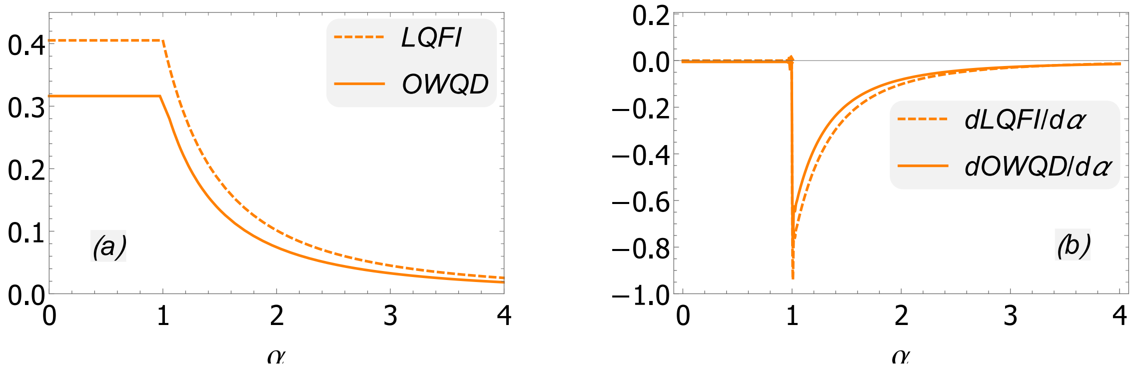

FIG. 1: $m = 1$: (a) LQFI and OWQD with respect to α. The orange dashed line denotes LQFI. The orange solid line shows OWQD. (b) The derivatives of LQFI (dashed orange line) and OWQD (solid orange line) with respect to α, respectively.

with eigenvalues

$$\left\{\frac{2t_1^2+t_3-1}{t_3-1}, \frac{(t_3+1)\,(2t_1^2+t_3-1)}{(t_1-1)(t_1+1)}, \frac{(t_3+1)\,(2t_1^2+t_3-1)}{(t_1-1)(t_1+1)}\right\}. \tag{26}$$

From (8) we have the LQFI,

$$LQFI = 1 - \max[\frac{(t_3+1)\,(2t_1^2+t_3-1)}{t_1^2-1}, \frac{2t_1^2+t_3-1}{t_3-1}]. \tag{27}$$

By tedious calculation, we can also work out the OWQD. The analytical expression of OWDQ for the Heisenberg XX spin chain is given by

$$\begin{aligned}\Delta \quad &= -2\,(1+2t_1)\log(1+2t_1) - 2\,(1-2t_1)\log(1-2t_1)\\ &+\frac{1}{4}[(1-t_3+2t_1)\log(1-t_3+2t_1)\\ &+(1-t_3-2t_1)\log(1-t_3-2t_1)\\ &+2(1+t_3)\log(1+t_3)],\end{aligned} \tag{28}$$

with the optimal value attained at $\phi = 0$ and $\theta = \pi/4$.

From the above analytical expressions, we can show the quantum fluctuations in the Heisenberg spin-$\frac{1}{2}$ XX spin chain system.

Fig. 1 shows the LQFI, OWQD and their derivatives with respect to α. In Fig. 1(a), the dashed line denotes LQFI, while the solid line stands for OWQD. One sees that in region $[0, 1]$, both of them are in a fixed value. LQFI and OWQD decrease quickly at first and

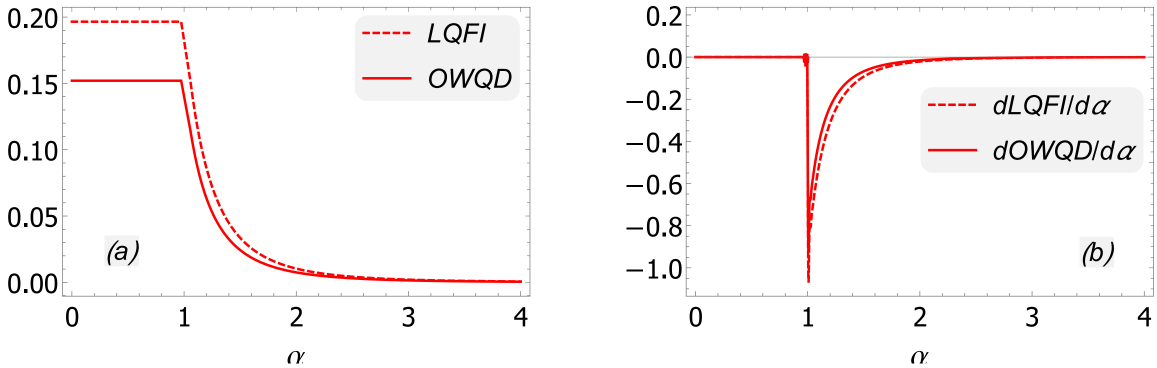

FIG. 2: $m = 2$: (a) LQFI and OWQD with respect to α. The orange dashed (solid) line denotes LQFI (OWQD). (b) Dashed (solid) orange line denotes the derivative of LQFI (OWQD) with respect to α.

then slowly in the region $\alpha > 1$. They have the same trends. However, LQFI is greater than OWQD. They approach zero when α goes to infinity.

The derivatives of LQFI and OWQD with respect to α are shown in Fig. 1(b), from which we see that the quantum phase transition happens at $\alpha = 1$. In the region $\alpha \in [0, 1]$, both derivatives of LQFI and OWQD are zero. For $\alpha > 1$, the derivatives of OWQD is greater than that of LQFI, meaning that the slope related to LQFI is steeper than to OWQD.

Fig.2 shows LQFI and OWQD, and their derivatives with respect to α when $m = 2$. The LQFI and OWQD for $m = 2$ are smaller than that for $m = 1$ in Fig.2(a), respectively. Around the region $\alpha \in [1, 2]$, the slope of the lines is more than the ones for $m = 1$. When α gets larger, the LQFI is approximately coincident with the OWQD. Both quantum correlation measures LQFI and OWQD show quantum phase transition in Fig. 2(b).

Fig. 3 shows the behavior of LQFI vs α (dashed blue line). Fig. 4 shows the behavior of OWQD vs α (solid blue line). The insets show the quantum phase transition related to their derivatives. One can see that when m increases, the slopes of the lines get larger. However, both of them show quantum phase transition of the Heisenberg XX model by the first derivatives at $\alpha = 1$.

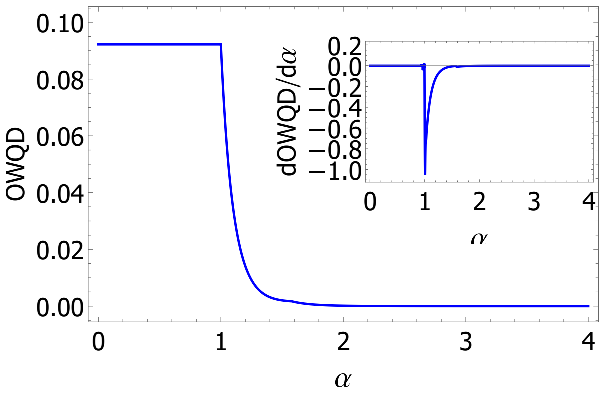

FIG. 3: LQFI and its derivative (inset) with respect to α for $m = 3$.

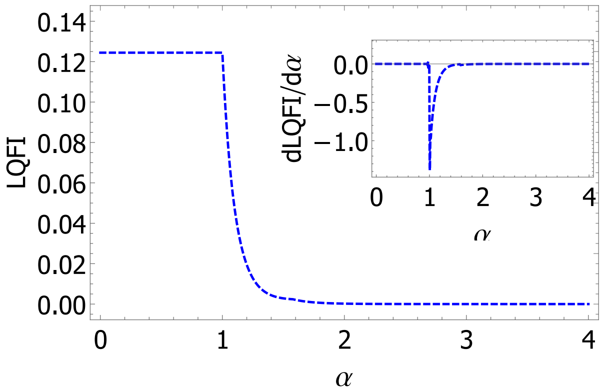

FIG. 4: OWQD and its derivative (inset) with respect to α for $m = 3$. The inset shows the quantum phase transition related to its derivatives.

IV. CONCLUSIONS

We have studied the quantum phase transitions in Heisenberg spin-$\frac{1}{2}$ XX spin model, showing that the quantum phase transition happens at $\alpha = 1$. Both quantum measures, local quantum Fihser information and one-way quantum deficit, are able to show the quantum fluctuation and the quantum phase transition for the Heisenberg spin-$\frac{1}{2}$ XX spin system. Our results may highlight the corresponding experimental demonstrations of the quantum fluctuation and the quantum phase transition in the Heisenberg spin-$\frac{1}{2}$ XX spin systems.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11675113, 11765016, 11847108, and 11905131), the Natural Science Foundation of Beijing (Grant No. KZ201810028042), the Beijing Natural Science Foundation (Z190005), the Natural Science Foundation of Jiangxi Province (Grant No. 20192BAB212005), and Jiangxi Education Department Fund (Grant No. KJLD14088).

---

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).

[3] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Remote state preparation, Phys. Rev. Lett. **87**, 077902 (2001).

[4] J. Yin, Y. Cao, Y.-H. Li, S.-K. Liao, L. Zhang, J.-G. Ren, W.-Q. Cai, W.-Y. Liu, B. Li, H. Dai, G.-B. Li, Q.-M. Lu, Y.-H. Gong, Y. Xu, S.-L. Li, F.-Z. Li, Y.-Y. Yin, Z.-Q. Jiang, M. Li, J.-J. Jia, G. Ren, D. He, Y.-L. Zhou, X.-X. Zhang, N. Wang, X. Chang, Z.-C. Zhu, N.-L. Liu, Y.-A. Chen, C.-Y. Lu, R. Shu, C.-Z. Peng, J.-Y. Wang, and J.-W. Pan, Satellite-based entanglement distribution over 1200 kilometers, Science **356**, 1140 (2017), https://science.sciencemag.org/content/356/6343/1140.full.pdf .

[5] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, The classical-quantum boundary for correlations: Discord and related measures, Rev. Mod. Phys. **84**, 1655 (2012).

[6] B.-L. Ye, Y.-K. Wang, and S.-M. Fei, One-way quantum deficit and decoherence for two-qubit X states, Int. J. Theor. Phys. **55**, 2237 (2016).

[7] A. Streltsov, H. Kampermann, and D. Bruß, Linking quantum discord to entanglement in a measurement, Phys. Rev. Lett. **106**, 160401 (2011).

[8] D. Petz, Covariance and Fisher information in quantum mechanics, Journal of Physics A: Mathematical and General **35**, 929 (2002).

[9] B.-L. Ye, B. Li, Z.-X. Wang, X. Li-Jost, and S.-M. Fei, Quantum Fisher information and

coherence in one-dimensional XY spin models with Dzyaloshinsky-Moriya interactions, Sci. China-Phys. Mech. Astron. **61**, 110312 (2018).

[10] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press , Cambridge, 1999).

[11] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Scaling of entanglement close to a quantum phase transition, Nature **416**, 608 (2002).

[12] T. Werlang, C. Trippe, G. A. P. Ribeiro, and G. Rigolin, Quantum correlations in spin chains at finite temperatures and quantum phase transitions, Phys. Rev. Lett. **105**, 095702 (2010).

[13] B.-Q. Liu, B. Shao, J.-G. Li, J. Zou, and L.-A. Wu, Quantum and classical correlations in the one-dimensional XY model with dzyaloshinskii-moriya interaction, Phys. Rev. A **83**, 052112 (2011).

[14] B.-Q. Liu, B. Shao, and J. Zou, Quantum and classical correlations in isotropic XY chain with three-site interaction, Commun. Theor. Phys. **56**, 46 (2011).

[15] Y.-C. Li and H.-Q. Lin, Quantum coherence and quantum phase transitions, Sci. Rep. **6**, 26365 (2016).

[16] J.-J. Chen, J. Cui, Y.-R. Zhang, and H. Fan, Coherence susceptibility as a probe of quantum phase transitions, Phys. Rev. A **94**, 022112 (2016).

[17] A. Slaoui, L. Bakmou, M. Daoud, and R. A. Laamara, A comparative study of local quantum Fisher information and local quantum uncertainty in Heisenberg XY model, Phys. Lett. A **383**, 2241 (2019).

[18] S. Kim, L. Li, A. Kumar, and J. Wu, Characterizing nonclassical correlations via local quantum Fisher information, Phys. Rev. A **97**, 032326 (2018).

[19] P. Lou, W.-C. Wu, and M.-C. Chang, Quantum phase transition in spin-$\frac{1}{2}$ XX Heisenberg chain with three-spin interaction, Phys. Rev. B **70**, 064405 (2004).

[20] E. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain, Ann. Phys. **16**, 407 (1961).