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by

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Abstract

A tensor $\mathbf{v}$ is the sum of at least $\mathrm{rank}(\mathbf{v})$ elementary tensors. In addition, a 'border rank' is defined: $\underline{\mathrm{rank}}(\mathbf{w}) = r$ holds if $\mathbf{w}$ is a limit of rank-r tensors. Usually, the set of rank-r tensors is not closed, i.e., tensors with $r = \underline{\mathrm{rank}}(\mathbf{w}) < \mathrm{rank}(\mathbf{w})$ may exist. It is easy to see that in such a case the representation of rank-r tensors $\mathbf{v}$ contains diverging elementary tensors as $\mathbf{v}$ approaches $\mathbf{w}$. In a first part we recall results about the uniform strength of the divergence in the case of general nonclosed tensor formats (restricted to finite dimensions). The second part discusses the r-term format for infinite-dimensional tensor spaces. It is shown that the general situation is very similar to the behaviour of finite-dimensional model spaces. The third part contains the main result: it is proved that in the case of $\underline{\mathrm{rank}}(\mathbf{w}) = 2 < \mathrm{rank}(\mathbf{w})$ the divergence strength is $\gtrsim \varepsilon^{-1/2}$, i.e., if $\|\mathbf{v} - \mathbf{w}\| < \varepsilon$ and $\mathrm{rank}(\mathbf{v}) \le 2$, the parameters of $\mathbf{v}$ increase at least proportionally to $\varepsilon^{-1/2}$.

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1 Introduction

The essential tool for the numerical treatment of tensors are appropriate sparse representations, i.e., the elements of the huge tensor spaces are represented by parameters or moderate size. There are several representations (also called 'formats') with different properties. One of the properties is whether they are closed or not. As explained below, nonclosed formats can lead to the unfavourable occurrence of an instability, known as cancellation effect from numerical differentiation.

To be concrete, we describe two examples of nonclosed formats. Let

$$\mathbf{V} = \bigotimes_{j=1}^{d} V_j \qquad (V_j \text{ vector spaces}, d \ge 3) \tag{1.1}$$

be the tensor space. The underlying field is $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For fixed integer r, the r-term format (also called 'canonical representation', 'CP', etc.) consists of all tensors of the form

$$\mathbf{v} = \sum_{\nu=1}^{r} \bigotimes_{j=1}^{d} v_\nu^{(j)} \qquad \text{with} \quad v_\nu^{(j)} \in V_j. \tag{1.2}$$

We denote this set by $\mathcal{R}_r$ (cf. [4, §7, p. 233ff]). It is easy to see that, e.g., $\mathcal{R}_2$ is not closed (cf. De Silva–Lim [2]). Choose $V_j = \mathbb{K}^2$, $a = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $b = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Then $\mathbf{v}(t) = \frac{1}{2t}\left[\otimes^d (a + tb) - \otimes^d (a - tb)\right]$ belongs to $\mathcal{R}_2$. It is the centred divided difference tending to the derivative

$$\frac{\mathsf{d}v(t)}{\mathsf{d}t}|_{t=0} = b \otimes a \otimes \ldots \otimes a \;+\; a \otimes b \otimes \ldots \otimes a \;+\; \ldots \;+\; a \otimes a \otimes \ldots \otimes b. \tag{1.3}$$

It is known that the latter tensor has rank d (cf. [4, Lemma 3.46, p. 72]; note that $d \ge 3$); hence, it does not belong to $\mathcal{R}_2$ but to its closure. For later purpose we notice that the approximation error of the centred difference is $\varepsilon \sim t^2$, while the two terms of $\mathbf{v}(t)$ are of the size $\frac{1}{2t} + O(1) \sim \varepsilon^{-1/2}$.

Another source of nonclosed representations are graph-based formats involving a graph not being a tree (therefore containing a cycle). Such representations are often used in physics (cf. [7]). In general, these formats are not closed (cf. Landsberg [6, Theorem 14.1.2.2]). The simplest example corresponds to a cycle and is called the 'cyclic matrix product representation'. Let $V_j = \mathbb{K}^{n_j}$ and fix integers $\rho_j \geq 2$. Then matrices $A_i^{(j)} \in \mathbb{K}^{\rho_{j-1} \times \rho_j}$ $(1 \leq i \leq n_j)$ define the components

$$\mathbf{v}[i_1, \ldots, i_d] = \operatorname{trace}\left(A_{i_1}^{(1)} A_{i_2}^{(2)} \cdots A_{i_d}^{(d)}\right)$$

of a tensor $\mathbf{v} \in \mathbf{V}$. A concrete example of a tensor not being in this format but in its closure, is given in [4, Theorem 12.11, p. 469]. For more examples see Czapliński–Michałek–Seynnaeve [1].

In §2 we give a survey of the results about the unstable behaviour of nonclosed formats. The first example from above exhibits divergence of the representing parameters like $\varepsilon^{-1/2}$. Possible questions are: Do other tensors exist with weaker divergence, can the divergence be arbitrarily weak, are there uniform estimates?

The analysis in §2 is restricted to the finite-dimensional case, since otherwise the compactness arguments do not apply. The infinite-dimensional case $\dim V_j = \infty$ of the r-term format $\mathcal{R}_r$ in a general Banach tensor space will be considered in §3. It will turn out that the behaviour is almost equal to the finite-dimensional one. This proves that we can restrict the study to the model spaces $\mathbf{V} = \bigotimes_{j=1}^{d} \mathbb{K}^{n_j}$.

In the case of the format $\mathcal{R}_2$, the relevant model space is $\bigotimes_{j=1}^{d} \mathbb{K}^2$. In §4 we prove that the divergence strength $\varepsilon^{-1/2}$ is the minimal one, i.e., the approximation of all tensors of border rank 2 but rank > 2 requires parameters diverging at least like $c\varepsilon^{-1/2}$. Concerning the possible dependence of the constant c on the tensor we refer to §2.

2 General Divergence Behaviour of Nonclosed Formats

The following statements are described in detail and proved in [5] and [4, §§9.5.3–9.5.6, p. 312ff]. We consider a format satisfying the following simple conditions. It is described by a continuous mapping

$$\rho : \mathcal{D} \subset \mathcal{P} \to \mathbf{V} \tag{2.1}$$

with $0 \in \mathcal{D}$ and $\rho(0) = 0$, where P is a normed vector space and D a closed subset (usually $\mathcal{D} = \mathcal{P}$).[1] The format is defined by

$$\mathcal{F} = \operatorname{range}(\rho).$$

A two-sided cone condition is required: $\mathbf{v} \in \mathcal{F}$ implies $\lambda \mathbf{v} \in \mathcal{F}$ for all $\lambda \in \mathbb{K}$. In order to apply compactness arguments for bounded sets, we require $\dim(P) < \infty$ and $\dim \mathbf{V} < \infty$. The norms on P and $\mathbf{V}$ are denoted by $\|\cdot\|$. Finally, $\mathcal{F}$ is assumed to be nonclosed, i.e., there is a disjoint set $\mathcal{B}$ of 'border tensors' such that

$$\overline{\mathcal{F}} = \mathcal{F} \dot{\cup} \mathcal{B}. \tag{2.2}$$

Since, in general, the representation of $\mathbf{v} \in \mathcal{F}$ is not unique, we introduce the minimal[2] bound of the corresponding parameter by

$$\sigma(\mathbf{v}) := \inf\{\|p\| : \mathbf{v} = \rho(p)\}. \tag{2.3}$$

A natural task is to approximate some $\mathbf{w} \in \mathcal{B}$ by tensors $\mathbf{v} \in \mathcal{F}$ with $\|\mathbf{w} - \mathbf{v}\| < \varepsilon$. The smallest parameter size is given by

$$\delta(\mathbf{w}, \varepsilon) := \inf\{\sigma(\mathbf{v}) : \mathbf{v} \in \mathcal{F},\ \|\mathbf{w} - \mathbf{v}\| < \varepsilon\}. \tag{2.4}$$

If $\mathbf{w} \in \mathcal{B}$, weakly monotone divergence $\delta(\mathbf{w}, \varepsilon) \nearrow \infty$ holds as $\varepsilon \searrow 0$. The proof is based on the following lemma (cf. [2, Proposition 4.8])

Lemma 2.1 *Let $\mathbf{v}_i \in \mathcal{F}$ with $\mathbf{v}_i := \rho(p_i)$ be a sequence converging to $\mathbf{w}$. Then $\sup_i \|p_i\| < \infty$ implies $\mathbf{w} \in \mathcal{F}$.*

[1] In the case of $\mathcal{F} = \mathcal{R}_r$, we may choose $\mathcal{D} = \mathcal{P}$ as the product space $(V_1 \times \ldots \times V_d)^r$ containing the vectors $v_\nu^{(j)}$ from (1.2). Alternatively, $\mathcal{D}$ may be the subset of $\mathcal{P} = \mathbf{V}$ containing all r-tuples of elementary tensors.

[2] The right-hand side in (2.3) can be replaced by $\min\{\ldots\}$.

The proof uses a convergent subsequence $p_i \to p^*$ so that $\mathbf{w} = \lim \rho(p_i) = \rho(p^*)$ proves $\mathbf{w} \in \mathcal{F}$. Correspondingly, $\delta(\mathbf{w}, \varepsilon) \nearrow \infty$ follows by an indirect proof. It ensures the existence of the diverging quantity $\delta(\mathbf{w}, \varepsilon)$, but does not describe the strength of divergence quantitatively. In particular, the divergence might be different for different $\mathbf{w} \in \mathcal{B}$.

Aiming at a *uniform* statement, the strongest formulation would be

$$\delta(\mathbf{w}, \varepsilon) \geq c\, \delta_0(\varepsilon) \quad \text{with } \delta_0(\varepsilon) \nearrow \infty \text{ as } \varepsilon \searrow 0 \text{ and } c > 0 \qquad \text{for all } \mathbf{w} \in \mathcal{B} \text{ with } \|\mathbf{w}\| = 1. \tag{2.5}$$

It turns out that (2.5) holds if and only if $\mathcal{B} \cup \{0\}$ (or $\{\mathbf{w} \in \mathcal{B},\, \|\mathbf{w}\| = 1\}$) is closed. However, in the case of $\mathcal{F} = \mathcal{R}_r$ the set $\mathcal{B} \cup \{0\}$ is not closed. The counterexample is $\mathbf{v}(t) := (a + tb) \otimes a \otimes a + a \otimes b \otimes a + a \otimes a \otimes b$ with linearly independent $a, b \in \mathbb{K}^2$, since $\mathbf{v}(0) \in \mathcal{R}_2$ and $\mathbf{v}(t) \in \mathcal{R}_3$ for $t \neq 0$. Define the exceptional set $\partial\mathcal{B}$ by

$$\overline{\mathcal{B}} = \mathcal{B} \,\dot\cup\, \partial\mathcal{B} \qquad \text{(disjoint union)}.$$

Replacing c in (2.5) by a factor $\operatorname{dist}(\mathbf{w}, \partial\mathcal{B})$, we obtain the following generally valid statement.

Theorem 2.2 *There is a function δ_1 with $\delta_1(\varepsilon) \nearrow \infty$ as $\varepsilon \searrow 0$ such that*

$$\delta(\mathbf{w}, \varepsilon) \geq \operatorname{dist}(\mathbf{w}, \partial\mathcal{B})\, \delta_1(\varepsilon) \quad \textit{for all } \mathbf{w} \in \mathcal{B} \textit{ with } \|\mathbf{w}\| = 1.$$

This inequality ensures the existence of a minimal divergence strength δ_1, but the indirect proof does not describe the concrete nature of δ_1.

Since $\partial\mathcal{B} \subset \mathcal{F}$, closedness of $\partial\mathcal{B}$ implies $\operatorname{dist}(\mathbf{w}, \partial\mathcal{B}) > 0$. In fact, for $\mathcal{F} = \mathcal{R}_2$ the set $\partial\mathcal{B}$ is closed.

3 Infinite Dimensions

Now the vector spaces V_j in (1.1) may be infinite dimensional. We restrict our considerations to $\mathcal{F} = \mathcal{R}_r$. Note that the cyclic matrix product format cannot be extended to $\dim V_j = \infty$, since it would require infinitely many matrices.

We recall that $\mathbf{v} \in \mathcal{R}_r$ corresponds to a representation $\mathbf{v} = \sum_{i=1}^{r} \bigotimes_{j=1}^{d} v_i^{(j)}$. We may define the mapping ρ in (2.1) by $\rho : p = (e_i)_{i=1,\ldots,r} \mapsto \mathbf{v} = \sum_{i=1}^{r} e_i$ with elementary tensors e_i (cf. Footnote 1). This is a natural choice since ρ is linear. The norm of p is chosen as

$$\|p\| = \sqrt{\sum\nolimits_i \|e_i\|^2}. \tag{3.1}$$

Remark 3.1 *The vectors $v_i^{(j)}$ in $e_i = \bigotimes_{j=1}^{d} v_i^{(j)}$ can be scaled equally: $\|v_i^{(j)}\|_j = \|v_i^{(k)}\|_k$. Then the later inequality (3.3) implies $\|v_i^{(j)}\|_j \leq (C_\otimes^* \|p\|)^{1/d}$.*

The next statements mention another format: the 'tensor subspace representation' $\mathcal{T}_\mathbf{r}$ (also called Tucker representation). Let $\mathbf{r} := (r_1, \ldots, r_d)$ a d-tuple of integers. Then $\mathcal{T}_\mathbf{r}$ consists of all tensors of the form $\mathbf{v} \in \bigotimes_{j=1}^{d} U_j$ with subspaces U_j subject to $\dim U_j \leq r_j$ (cf. [4, §8, p. 257ff]).

Different from the finite-dimensional case, the norm of the topological tensor space plays an important role. We require two properties:

(a) The tensor product must be a continuous map from $V_1 \times \ldots \times V_d$ into $\mathbf{V}$, i.e.,

$$\left\| \bigotimes_{j=1}^{d} v^{(j)} \right\| \leq C_\otimes \prod_{j=1}^{d} \|v^{(j)}\|_j, \qquad \text{for all } v^{(j)} \in V_j, \tag{3.2}$$

where $\|\cdot\|_j$ is the norm on V_j, while $\|\cdot\|$ is the norm on $\mathbf{V}$. An equivalent statement is $\|\cdot\| \lesssim \|\cdot\|_\wedge$, where $\|\cdot\|_\wedge$ is the projective crossnorm (cf. [4, §4.2.3, p. 116ff and §4.3.1.2, p. 138]).

(b) An analogous estimate corresponding to the Banach spaces V_j^* of the continuous linear functionals is valid:

$$\left\| \bigotimes_{j=1}^{d} \varphi^{(j)} \right\|^* \leq C_\otimes^* \prod_{j=1}^{d} \|\varphi^{(j)}\|_j^* \qquad \text{for all } \varphi^{(j)} \in V_j^*, \tag{3.3}$$

where $\|\cdot\|^*$ and $\|\cdot\|_j^*$ are the related dual norms. An equivalent statement is $\|\cdot\| \gtrsim \|\cdot\|_\vee$, where $\|\cdot\|_\vee$ is the injective crossnorm. Note that $\|\cdot\|_\vee$ is the weakest possible reasonable crossnorm (cf. [4, §4.3.1.3, p. 139]).

Under these conditions the representation $\mathcal{T}_{\mathbf{r}}$ is closed.[3] This properties leads to an estimate of the rank by means of the border rank (denoted by $\underline{\text{rank}}$).

Remark 3.2 *A tensor with* $\underline{\text{rank}}(\mathbf{v}) = r$ *belongs to* $\mathcal{T}_{\mathbf{r}}$ *with* $\mathbf{r} := (r, \dots, r)$. *In particular, we have*[4]

$$\text{rank}(\mathbf{v}) \le \underline{\text{rank}}(\mathbf{v})^d.$$

Proof. There is a sequence $\mathbf{v}_i \to \mathbf{v}$ with $\text{rank}(\mathbf{v}_i) \le r$. Since $\mathbf{v}_i \in \mathcal{R}_r \subset \mathcal{T}_{\mathbf{r}}$ and the set $\mathcal{T}_{\mathbf{r}}$ is closed, we conclude that $\mathbf{v} \in \mathcal{T}_{\mathbf{r}}$. Now use $\text{rank}(\mathbf{v}) \le \prod_{j=1}^d r_j$ for $\mathbf{v} \in \mathcal{T}_{(r_1,\dots,r_d)}$. ∎

Let $\mathbf{w}$ be the limit of $\{\mathbf{v}_i\}$. There are unique minimal subspaces $U_j^{\min}(\mathbf{w})$ with the property

$$\mathbf{w} \in \mathbf{U}(\mathbf{w}) := \bigotimes_{j=1}^d U_j^{\min}(\mathbf{w}) \subset \mathbf{V} \tag{3.4}$$

(cf. [4, §6, p. 201ff]). Let $r_j := \dim(U_j^{\min}(\mathbf{w}))$. Remark 3.2 implies $r_j \le r = \underline{\text{rank}}(\mathbf{w})$. We introduce the model space

$$\mathbf{U}_{\text{mod}} := \bigotimes_{j=1}^d \mathbb{K}^{r_j}$$

equipped with the Euclidean norm $\|\cdot\|_{\text{mod}}$. Note that §2 applies to $\mathbf{U}_{\text{mod}}$ and the format $\mathcal{F} = \mathcal{R}_r$. In the following we shall show that the divergence behaviour $\delta(\mathbf{w}, \varepsilon)$ is equal to the divergence behaviour in the model space $\mathbf{U}_{\text{mod}}$ up to constants for which explicit bounds can be given.

Next we need the following result whose proof is postponed to the end of this section.

Lemma 3.3 *There are projections*[5] $P_j : V_j \to U_j^{\min}(\mathbf{w})$ *with* $\|P_j\| \le r_j$ *such that* $\mathbf{P} := \bigotimes_{j=1}^d P_j$ *is a projection onto* $\mathbf{U}(\mathbf{w})$ *with the bound* $\|\mathbf{P}\| \le C_\otimes C_\otimes^* \dim(\mathbf{U}(\mathbf{w}))$. *Furthermore,* $\mathbf{P}$ *maps* $\mathcal{R}_r$ *into* $\mathcal{R}_r$.

$\mathbf{U}(\mathbf{w})$ and $\mathbf{U}_{\text{mod}}$ are isomorphic since $\dim(U_j^{\min}(\mathbf{w})) = \dim(\mathbb{K}^{r_j})$. Hence, the norms (restriction of $\|\cdot\|$ to $\mathbf{U}(\mathbf{w})$ and $\|\cdot\|_{\text{mod}}$) are equivalent. However, the constants in the equivalence inequalities are still undetermined.

The definition of $\delta(\mathbf{w}, \varepsilon) = \inf\{\sigma(\mathbf{v}) : \mathbf{v} \in \mathcal{F},\ \|\mathbf{w} - \mathbf{v}\| < \varepsilon\}$ also involves tensors $\mathbf{v}$ from outside of $\mathbf{U}(\mathbf{w})$. Now we introduce

$$\delta_U(\mathbf{w}, \varepsilon) = \inf\{\sigma(\mathbf{u}) : \mathbf{u} \in \mathcal{F} \cap \mathbf{U}(\mathbf{w}),\ \|\mathbf{w} - \mathbf{u}\| < \varepsilon\}. \tag{3.5}$$

Proposition 3.4 $\delta_U(\mathbf{w}, \varepsilon)$ *and* $\delta(\mathbf{w}, \varepsilon)$ *satisfy the following inequalities:*

$$\delta_U(\mathbf{w}, \varepsilon) \ge \delta(\mathbf{w}, \varepsilon) \ge \delta_U(\mathbf{w}, \varepsilon \|\mathbf{P}\|) / \|\mathbf{P}\|$$

with $\|\mathbf{P}\| \le C_\otimes C_\otimes^* \dim(\mathbf{U}(\mathbf{w}))$ *and* $\dim(\mathbf{U}(\mathbf{w})) = \dim(\mathbf{U}_{\text{mod}}) \le \underline{\text{rank}}(\mathbf{w})^d$.

Proof. $\delta_U(\mathbf{w}, \varepsilon) \ge \delta(\mathbf{w}, \varepsilon)$ is the trivial estimate. Fix some $\mathbf{v} \in \mathcal{F}$, $\|\mathbf{w} - \mathbf{v}\| < \varepsilon$, from the right-hand side in (2.4). Set $\hat{\mathbf{u}} := \mathbf{P}\mathbf{v} \in \mathcal{F} \cap \mathbf{U}(\mathbf{w})$ with $\mathbf{P}$ from Lemma 3.3 and note that $\|\mathbf{w} - \hat{\mathbf{u}}\| = \|\mathbf{P}(\mathbf{w} - \mathbf{v})\| \le \|\mathbf{P}\| \|\mathbf{w} - \mathbf{v}\| < \varepsilon \|\mathbf{P}\|$. Hence, $\hat{\mathbf{u}}$ appears in the right-hand side of (3.5) for ε replaced by $\varepsilon \|\mathbf{P}\|$ and leads to the parameter size $\sigma(\hat{\mathbf{u}}) \le \|\mathbf{P}\| \sigma(\mathbf{v})$ (cf. (3.1)). This proves $\sigma(\mathbf{v}) \ge \sigma(\hat{\mathbf{u}}) / \|\mathbf{P}\| \ge \delta_U(\mathbf{w}, \varepsilon \|\mathbf{P}\|) / \|\mathbf{P}\|$. Forming the infimum over all $\mathbf{v} \in \mathcal{F}$, $\|\mathbf{w} - \mathbf{v}\| < \varepsilon$, we obtain the second inequality. ∎

Now we compare $\mathbf{U}(\mathbf{w})$ and $\mathbf{U}_{\text{mod}}$. For $U_j^{\min}(\mathbf{w})$ we choose a basis $\{b_i^{(j)}\}$ and functionals $\{\varphi_i^{(j)}\}$ according to the following lemma of Auerbach (cf. [4, Lemma 4.20, p. 104]).

Lemma 3.5 *For any* n*-dimensional subspace of a Banach space, there exists a basis* $\{b_\nu : 1 \le \nu \le n\}$ *and a dual system* $\{\varphi_\nu : 1 \le \nu \le n\}$ *(i.e.,* $\varphi_\nu(b_\mu) = \delta_{\nu\mu}$*) with* $\|b_\nu\| = \|\varphi_\nu\|^* = 1$.

[3] It is even 'weakly closed', i.e., if $\mathbf{v}_i \in T_{\mathbf{r}}$ has a weak limit, this limit belongs to $T_{\mathbf{r}}$.

[4] The estimate can be improved using better bounds of the maximal rank (cf. [4, §3.2.6.5, p. 71f]).

[5] The estimates can be improved if $\|\cdot\|$ is a uniform crossnorm (cf. [4, §4.2.8, p. 133]): $\|P_j\| \le \sqrt{r_j}$ and $\|\mathbf{P}\| \le \sqrt{\dim(\mathbf{U}(\mathbf{w}))}$ (cf. [4, Theorem 4.16, p. 103]).

Define the isomorphism $\Phi : \mathbf{U}(\mathbf{w}) \to \mathbf{U}_{\rm mod}$ by $\Phi = \bigotimes_{j=1}^d \phi_j$ with $\phi_j(b_i^{(j)}) := e_i^{(j)}$, where $e_i^{(j)}$ are the standard unit vectors of $\mathbb{K}^{r_j}$. Let $\mathcal{F}$ and $\mathcal{B}$ correspond to $\mathbf{U}(\mathbf{w}) \subset \mathbf{V}$, while $\mathcal{F}_{\rm mod}$ and $\mathcal{B}_{\rm mod}$ correspond to the model space $\mathbf{U}_{\rm mod}$.

Remark 3.6 *Φ maps $\mathcal{F}$ into $\mathcal{F}_{\rm mod}$ and $\mathcal{B}$ into $\mathcal{B}_{\rm mod}$. The following estimates hold:*

$$\|\Phi\| \le C_\otimes^* \sqrt{\dim(\mathbf{U}(\mathbf{w}))}, \quad \left\|\Phi^{-1}\right\| \le C_\otimes \sqrt{\dim(\mathbf{U}(\mathbf{w}))}, \quad \left\|\phi_j\right\|, \|\phi_j^{-1}\| \le \sqrt{r_j} \le \sqrt{\underline{\rm rank}(\mathbf{w})}.$$

Proof. (a) $\Phi : \mathcal{F} \to \mathcal{F}_{\rm mod}$ and $\Phi : \mathcal{B} \to \mathcal{B}_{\rm mod}$ follow from the fact that isomorphisms do not change the rank and border rank.

(b) Set $\mathbf{i} := (i_1, i_2, \dots, i_d)$. $\mathbf{x} \in \mathbf{U}(\mathbf{w})$ has a representation $\sum_{\mathbf{i}} \alpha_{\mathbf{i}} \bigotimes_{j=1}^d b_{i_j}^{(j)}$ with $\alpha_{\mathbf{i}} = \left(\bigotimes_{j=1}^d \varphi_{i_j}^{(j)}\right) \mathbf{x}$. $\|\Phi\mathbf{x}\| = \sqrt{\sum_{\mathbf{i}} |\alpha_{\mathbf{i}}|^2}$ and $|\alpha_{\mathbf{i}}| \le \left\|\bigotimes_{j=1}^d \varphi_{i_j}^{(j)}\right\|^* \|\mathbf{x}\| \le C_\otimes^* \prod_{j=1}^d \|\varphi_{i_j}^{(j)}\|^* \|\mathbf{x}\| = C_\otimes^* \|\mathbf{x}\|$ prove $\|\Phi\| \le C_\otimes^* \sqrt{\dim(\mathbf{U}(\mathbf{w}))}$.

(c) Let $\mathbf{y} := \sum_{\mathbf{i}} \alpha_{\mathbf{i}} \bigotimes_{j=1}^d e_{i_j}^{(j)}$ be an element of $\mathbf{U}_{\rm mod}$. Application of $\Phi^{-1} = \bigotimes_{j=1}^d \phi_j^{-1}$ yields $\Phi^{-1}\mathbf{y} = \sum_{\mathbf{i}} \alpha_{\mathbf{i}} \bigotimes_{j=1}^d b_{i_j}^{(j)} \in \mathbf{U}(\mathbf{w})$. Then

$$\begin{aligned}\left\|\Phi^{-1}\mathbf{y}\right\| &\le \sum\nolimits_{\mathbf{i}} |\alpha_{\mathbf{i}}| \left\|\bigotimes\nolimits_{j=1}^d b_{i_j}^{(j)}\right\| \le C_\otimes \sum\nolimits_{\mathbf{i}} |\alpha_{\mathbf{i}}| \prod\nolimits_{j=1}^d \|b_{i_j}^{(j)}\| \\ &= C_\otimes \sum\nolimits_{\mathbf{i}} |\alpha_{\mathbf{i}}| \le C_\otimes \sqrt{\dim(\mathbf{U}(\mathbf{w}))} \sqrt{\sum\nolimits_{\mathbf{i}} |\alpha_{\mathbf{i}}|^2} = C_\otimes \sqrt{\dim(\mathbf{U}(\mathbf{w}))}\, \|\mathbf{y}\|\end{aligned}$$

proves the second inequality.

(d) The estimates of $\left\|\phi_j\right\|$ and $\|\phi_j^{-1}\|$ follow by the same lines. ∎

Let

$$\delta_{\rm mod}(\mathbf{x}, \varepsilon) = \inf\{\sigma(\mathbf{y}) : \mathbf{y} \in \mathcal{F}_{\rm mod} \cap \mathbf{U}_{\rm mod}, \|\mathbf{x} - \mathbf{y}\| < \varepsilon\} \tag{3.6}$$

describe the divergence behaviour of $\mathbf{x} \in \mathcal{B}_{\rm mod}$ in $\mathbf{U}_{\rm mod}$. $\mathbf{w} \in \mathcal{B}$ corresponds to $\mathbf{x} = \Phi\mathbf{w}$. The next result shows that the divergence behaviour of $\mathbf{w}$ and $\mathbf{x}$ is equal up the controlled constants. The proof follows by the same lines as for Proposition 3.4.

Proposition 3.7 *The following estimates hold with Φ defined in Remark 3.6:*

$$\delta_U(\mathbf{w}, \varepsilon) \ge \delta_{\rm mod}(\Phi\mathbf{w}, \varepsilon \|\Phi\|) / \left\|\Phi^{-1}\right\|, \quad \delta_{\rm mod}(\Phi\mathbf{w}, \varepsilon) \ge \delta_U(\mathbf{w}, \varepsilon \left\|\Phi^{-1}\right\|) / \|\Phi\|.$$

Concerning the inequality in Theorem 2.2, we remark that Φ maps $\partial\mathcal{B}$ onto $\partial\mathcal{B}_{\rm mod}$ and

$$\|\Phi\|^{-1} \operatorname{dist}(\Phi\mathbf{w}, \partial\mathcal{B}_{\rm mod}) \le \operatorname{dist}(\mathbf{w}, \partial\mathcal{B}) \le \left\|\Phi^{-1}\right\| \operatorname{dist}(\Phi\mathbf{w}, \partial\mathcal{B}_{\rm mod}).$$

Remark 3.8 *If $\delta_{\rm mod}(\cdot, \varepsilon)$ is of the form $c\varepsilon^{-\kappa}$, a factor in the second argument can be moved outside: $\delta_{\rm mod}(\cdot, \varepsilon C) = C^{-\kappa} \delta_{\rm mod}(\cdot, \varepsilon)$.*

Now we give the proof of Lemma 3.3.

Proof. Let $\{b_i^{(j)} : 1 \le i \le r_j\}$ be an Auerbach basis of $U_j^{\min}(\mathbf{w})$ with corresponding dual functionals $\{\varphi_i^{(j)}\}$. Set $P_j := \sum_{i=1}^{r_j} b_i^{(j)} \varphi_i^{(j)}$. $\|P_j\| \le r_j$ is immediate. Obviously, the image $\mathbf{P}\mathbf{x} = \bigotimes_{j=1}^d P_j x^{(j)}$ of an elementary tensor $\mathbf{x} = \bigotimes_{j=1}^d x^{(j)}$ is again elementary. Therefore, $\mathbf{P}\mathbf{x}$ with $\mathbf{x} \in \mathcal{R}_r$ consists at most of r elementary tensors.

$\mathbf{P}$ equals $\sum_{i_1, i_2, \dots, i_d} \left(\bigotimes_{j=1}^d b_i^{(j)}\right) \left(\bigotimes_{j=1}^d \varphi_i^{(j)}\right)$. For any $\mathbf{x} \in \mathbf{V}$ we have

$$\begin{aligned}&\left\|\left(\bigotimes\nolimits_{j=1}^d b_i^{(j)}\right) \left(\bigotimes\nolimits_{j=1}^d \varphi_i^{(j)}\right) \mathbf{x}\right\| \le \left\|\bigotimes\nolimits_{j=1}^d b_i^{(j)}\right\| \left\|\bigotimes\nolimits_{j=1}^d \varphi_i^{(j)}\right\|^* \|\mathbf{x}\| \\ &\le C_\otimes C_\otimes^* \prod\nolimits_{j=1}^d \|b_i^{(j)}\| \prod\nolimits_{j=1}^d \|\varphi_i^{(j)}\|_j^* \|\mathbf{x}\| = C_\otimes C_\otimes^* \|\mathbf{x}\|.\end{aligned}$$

Summation over all terms yields $\|\mathbf{P}\| \le C_\otimes C_\otimes^* \prod_{j=1}^d r_j = C_\otimes C_\otimes^* \dim(\mathbf{U}(\mathbf{w}))$. ∎

Finally, we add two remarks illustrating the "finite-dimensional nature" of algebraic tensors. The first property is also mentioned by Fernández–Unzueta [3]. We recall that the topological tensor space is the completion of the algebraic tensor space $\mathbf{V}_{\mathsf{alg}} := \bigotimes_{j=1}^d V_j$ with respect to some norm.

Remark 3.9 *The closure of $\mathcal{R}_r \subset \mathbf{V}$ is independent of the norm of $\mathbf{V}$.*

Proof. Let $\mathbf{V}_I$ and $\mathbf{V}_{II}$ be two topological tensor spaces with respect to two different norms $\|\cdot\|_I$ and $\|\cdot\|_{II}$. Let $\mathbf{w} \in \mathbf{V}_I$ be the limit of a sequence $\mathbf{v}_i \in \mathcal{R}_r \subset \mathbf{V}_{\mathsf{alg}} \subset \mathbf{V}_I$. As seen above, the projected sequence $\hat{\mathbf{v}}_i = \mathbf{P}\mathbf{v}_i$ satisfies $\hat{\mathbf{v}}_i \in \mathbf{U}(\mathbf{w})$ and $\hat{\mathbf{v}}_i \to \mathbf{w}$ with respect to the norm $\|\cdot\|_I$. However, the restrictions of the norms $\|\cdot\|_I$ and $\|\cdot\|_{II}$ to the finite-dimensional subspace $\mathbf{U}(\mathbf{w}) \subset \mathbf{V}_I \cap \mathbf{V}_{II}$ are equivalent so that $\hat{\mathbf{v}}_i \to \mathbf{w}$ also holds with respect to $\|\cdot\|_{II}$. ∎

For the solution of optimisation problems it is desired that the infimum of a cost function $F(\mathbf{v})$, $\mathbf{v} \in \mathcal{F}$, is also a minimum. Assuming a reflexive Banach space $\mathbf{V}$, the minimum is taken by a *weak* limit $\mathbf{w}$ of some sequence $\mathbf{v}_i \in \mathcal{F}$. If $\mathcal{F}$ is weakly closed (cf. Footnote 3), the minimiser $\mathbf{w}$ belongs to $\mathcal{F}$. For infinite-dimensional spaces weak convergence and strong convergence (standard convergence) must be distinguished. In finite dimensions both kinds of convergence coincide.

Remark 3.10 *If $\mathbf{w}$ is a weak limit of $\mathbf{v}_i \in \mathcal{R}_r$, it is also a strong limit of (possibly other) $\hat{\mathbf{v}}_i \in \mathcal{R}_r$. Hence, the weak closure and the standard closure of $\mathcal{R}_r$ coincide.*

Proof. Assume $\mathbf{v}_i \in \mathbf{V}$ and $\mathbf{v}_i \rightharpoonup \mathbf{w}$ with $\operatorname{rank}(\mathbf{w}) < \infty$. Note that $\mathbf{w} \in \mathbf{U}(\mathbf{w})$ (cf. (3.4)). Let $\mathbf{P}$ be a bounded projection of $\mathbf{V}$ onto $\mathbf{U}(\mathbf{w})$ (cf. Lemma 3.3). Then also $\hat{\mathbf{v}}_i := \mathbf{P}\mathbf{v}_i \in \mathbf{U}(\mathbf{w})$ satisfies $\hat{\mathbf{v}}_i \rightharpoonup \mathbf{w}$. However, inside of the finite-dimensional subspace $\mathbf{U}(\mathbf{w})$, weak convergence implies strong convergence, i.e., $\hat{\mathbf{v}}_i \to \mathbf{w}$. For the proof of $\hat{\mathbf{v}}_i \rightharpoonup \mathbf{w}$ let $\varphi \in \mathbf{V}^*$ be a functional. Then $\varphi(\hat{\mathbf{v}}_i - \mathbf{w}) = \varphi(\mathbf{P}\mathbf{v}_i - \mathbf{w}) = \varphi(\mathbf{P}\mathbf{v}_i - \mathbf{P}\mathbf{w}) = (\varphi\mathbf{P})(\mathbf{v}_i - \mathbf{w}) \to 0$, since also $\varphi\mathbf{P} \in \mathbf{V}^*$. ∎

4 Tensors of Border Rank 2

4.1 Result

As seen in Section 3, it suffices to study the behaviour of the model spaces $\bigotimes_{j=1}^d \mathbb{K}^{r_j}$. In the case of $\mathcal{F} = \mathcal{R}_2$ the model space is

$$\mathbf{V} = \otimes^d \mathbb{K}^2 \qquad \text{with } d \geq 3$$

endowed with the Euclidean norm. The set $\mathcal{B}$ from (2.2) consists of all tensors $\mathbf{w} \in \mathbf{V}$ with $\underline{\operatorname{rank}}(\mathbf{w}) = 2$ but $\operatorname{rank}(\mathbf{w}) > 2$. The example of the tensor in (1.3) and its approximation by centred divided differences shows that $\delta(\mathbf{w}, \varepsilon) \leq c\,\varepsilon^{-1/2}$ (notation: $\delta(\mathbf{w}, \varepsilon) \lesssim \varepsilon^{-1/2}$). Below we prove the opposite inequality: $\delta(\mathbf{w}, \varepsilon) \gtrsim \varepsilon^{-1/2}$. This proves

$$\delta(\mathbf{w}, \varepsilon) \sim \varepsilon^{-1/2}.$$

The same equivalence holds for $\delta_1(\varepsilon)$ in Theorem 2.2.

For the proof of $\delta(\mathbf{w}, \varepsilon) \gtrsim \varepsilon^{-1/2}$ we show that a limit $\mathbf{w} = \lim \mathbf{v}_i$ of a sequence with weaker divergence must belong to $\mathcal{F}$. The following result can be considered as a stronger form of Lemma 2.1.

Theorem 4.1 *$\delta(\mathbf{w}, \varepsilon) = o(\varepsilon^{-1/2})$ for $\mathbf{w} \in \mathbf{V}$ with $\underline{\operatorname{rank}}(w) \leq 2$ implies $\mathbf{w} \in \mathcal{R}_2$.*

The parameter p in $\mathbf{v} = \rho(p) \in \mathcal{R}_2$ is a pair of elementary tensors: $p = (p^{(1)}, p^{(2)})$ with $p^{(i)} = \bigotimes_{j=1}^d v_i^{(j)}$ and $\rho(p) = p^{(1)} + p^{(2)}$. The norm $\|p\|$ is defined in (3.1). The statement $\delta(\mathbf{w}, \varepsilon) = o(\varepsilon^{-1/2})$ implies that there is a sequence $\mathbf{v}_\mu$ represented by parameters p_μ such that

$$\mathbf{v}_\mu = \rho(p_\mu) \in \mathcal{R}_2 \quad \text{with} \quad \varepsilon_\mu := \|\mathbf{v}_\mu - \mathbf{w}\| \to 0,\ \|p_\mu\| = o(\varepsilon_\mu^{-1/2}). \tag{4.1}$$

4.2 Choice of Subsequences

Let $\mathbf{v}_\mu \in \mathcal{R}_2$ in (4.1) be represented in the form

$$\mathbf{v}_\mu = \bigotimes_{j=1}^d v_{1,\mu}^{(j)} - \bigotimes_{j=1}^d v_{2,\mu}^{(j)} =: \rho\left(\left(v_{i,\mu}^{(j)}\right)\right) \qquad (v_{1,\mu}^{(j)}, v_{2,\mu}^{(j)} \in \mathbb{K}^2)$$

using the parameters $p_\mu = (p_{1,\mu}, p_{2.\mu})$ with $p_{i,\mu} = \bigotimes_{j=1}^d v_{i,\mu}^{(j)}$. According to Remark 3.1, we use an equal scaling of the factors:

$$\|v_{i,\mu}^{(j)}\| = \left\| \bigotimes_{j=1}^d v_{i,\mu}^{(j)} \right\|^{1/d} = \|p_{i,\mu}\|^{1/d} \qquad \text{for } i = 1, 2. \tag{4.2}$$

If the sequence of the Euclidean norms $\|p_\mu\|$ (cf. (3.1)) is bounded, the assertion $\mathbf{w} \in \mathcal{R}_2$ follows from Lemma 2.1. Hence, we may assume that there is a subsequence[6] with $\|p_\mu\| \to \infty$. One of the quantities $\|p_{1,\mu}\|$ and $\|p_{2,\mu}\|$ must be unbounded. W.l.o.g. let $\limsup \|p_{1,\mu}\| = \infty$. Passing to a subsequence, we get

$$\delta_\mu := \|p_{1,\mu}\| \to \infty. \tag{4.3}$$

The scaled quantities $x_{1,\mu}^{(j)} := \delta_\mu^{-1/d} v_{1,\mu}^{(j)}$ are bounded: $|x_{1,\mu}^{(j)}| \le \left(\delta_\mu^{-1}\|p_{1,\mu}\|\right)^{1/d} \le 1$ (cf. (4.2)). We choose a subsequence such that $x_{1,\mu}^{(j)} \to x_1^{(j)}$ for $j = 1, \dots, d$.

Convergence $\mathbf{v}_\mu \to \mathbf{w}$ implies that $\sup \|\mathbf{v}_\mu\| < \infty$ and $\mathbf{x}_\mu := \frac{1}{\delta_\mu}\mathbf{v}_\mu \to 0$. Note that $\mathbf{x}_\mu = \mathbf{x}_{1,\mu} - \mathbf{x}_{2,\mu}$ with $\mathbf{x}_{1,\mu} := \bigotimes_{j=1}^d x_{1,\mu}^{(j)} \to \mathbf{x}_1 := \bigotimes_{j=1}^d x_1^{(j)}$. Hence, $\mathbf{x}_\mu \to 0$ implies $\mathbf{x}_{2,\mu} \to \mathbf{x}_2 = \bigotimes_{j=1}^d x_2^{(j)}$ with $\mathbf{x}_2 = \mathbf{x}_1$. The scaling (4.2) fixes the factors $x_2^{(j)}$ up to a (complex) sign. W.l.o.g. we may choose the signs of $v_{2,\mu}^{(j)}$ such that $x_1^{(j)} = x_2^{(j)}$.

After these preparations we obtain the representation

$$\mathbf{v}_\mu = \delta_\mu[\mathbf{x}_{1,\mu} - \mathbf{x}_{2,\mu}] \quad \text{with } \mathbf{x}_{i,\mu} = \bigotimes_{j=1}^d x_{i,\mu}^{(j)} \text{ and } x_{i,\mu}^{(j)} \to x_1^{(j)} \ (i = 1, 2) \text{ as } \mu \to \infty.$$

(4.2) and (4.3) imply $\|\mathbf{x}_{i,\mu}\| = \|x_{i,\mu}^{(j)}\| = 1$ $(i = 1, 2)$ and $\|x_1^{(j)}\| = 1$ for $1 \le j \le d$.

There are orthogonal transformations $Q_j : \mathbb{K}^2 \to \mathbb{K}^2$ mapping $x_1^{(j)}$ into $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. The statement of the theorem does not change when we apply the product $\mathbf{Q} := \bigotimes_{j=1}^d Q_j$ and replace $\mathbf{v}_\mu$ by $\mathbf{Q}\mathbf{v}_\mu$ (norms are unchanged, $\mathbf{w} \in \mathcal{R}_2$ if and only if $\mathbf{Q}\mathbf{w} \in \mathcal{R}_2$). In the following we assume $x_1^{(j)} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. $x_{1,\mu}^{(j)}$ and $x_{2,\mu}^{(j)}$ are perturbations of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ tending to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. We denote the terms of $\mathbf{v}_\mu = \delta_\mu \left[\bigotimes_{j=1}^d x_{1,\mu}^{(j)} - \bigotimes_{j=1}^d x_{2,\mu}^{(j)}\right]$ by

$$\bigotimes_{j=1}^d x_{1,\mu}^{(j)} = \left(1 + \eta'_\mu\right) \cdot \bigotimes_{j=1}^d \begin{bmatrix} 1 \\ \beta'_{j,\mu} \end{bmatrix}, \quad \bigotimes_{j=1}^d x_{2,\mu}^{(j)} = \left(1 + \eta''_\mu\right) \cdot \bigotimes_{j=1}^d \begin{bmatrix} 1 \\ -\beta''_{j,\mu} \end{bmatrix}$$
$$\text{with} \quad \eta'_\mu, \eta''_\mu, \beta'_{j,\mu}, \beta''_{j,\mu} \to 0 \qquad \text{as } \mu \to 0.$$

The above representation of $\mathbf{v}_\mu$ corresponds to the parameters p_μ with $\|p_\mu\| = 2\delta_\mu\left(1 + o(1)\right)$. Therefore the asymptotic behaviour $\|p_\mu\| = o(\varepsilon_\mu^{-1/2})$ in (4.1) becomes

$$\delta_\mu = o(\varepsilon_\mu^{-1/2}) \quad \text{as } \varepsilon_\mu := \|\mathbf{v}_\mu - \mathbf{w}\| \to 0. \tag{4.4}$$

In the following we omit the index μ to simplify the notation. Instead of $\mathbf{v}_\mu \to \mathbf{w}$, $\delta_\mu = o(\varepsilon_\mu^{-1/2})$, etc. we write $\mathbf{v} \to \mathbf{w}$, $\delta = o(\varepsilon^{-1/2})$, etc.:

$$\mathbf{v} = \delta \left\{ (1 + \eta') \cdot \bigotimes_{j=1}^d \begin{bmatrix} 1 \\ \beta'_j \end{bmatrix} - (1 + \eta'') \bigotimes_{j=1}^d \begin{bmatrix} 1 \\ -\beta''_j \end{bmatrix} \right\} \to \mathbf{w} \tag{4.5}$$
$$\text{with} \quad \delta = o(\varepsilon^{-1/2}) \to \infty, \quad \eta', \eta'', \beta'_j, \beta''_j \to 0.$$

In the first step we restrict the proof to the real field $\mathbb{K} = \mathbb{R}$. The complex case will be discussed in §4.10.

[6] A subsequence of $\{p_\mu : \mu \in \mathbb{N}\}$ can be understood as $\{p_\mu : \mu \in \mathbb{N}'\}$ with some subset $\mathbb{N}' \subset \mathbb{N}$ of infinite cardinality. This allows us to use the unchanged notation p_μ. Choosing a further subsequence, we replace $\mathbb{N}'$ by another infinite subset $\mathbb{N}'' \subset \mathbb{N}'$.

4.3 Notations

The tensor $\mathbf{v} \in \mathbf{V}$ has 2^d components $\mathbf{v}[i_1, i_2, \ldots, i_d]$. The following notation indicates the position of the indices $i_j = 2$. For instance,

$$\begin{aligned} v_{<k>} &:= \mathbf{v}[i_1, i_2, \ldots, i_d] \quad \text{with } i_k = 2 \text{ and } i_j = 1 \text{ otherwise,} \\ v_{<k,\ell>} &:= \mathbf{v}[i_1, i_2, \ldots, i_d] \quad \text{with } i_k = i_\ell = 2 \text{ and } i_j = 1 \text{ otherwise.} \end{aligned}$$

The meaning of $v_{<k,\ell,m>}$ etc. is obvious. The empty case is

$$v_{<>} = \mathbf{v}[1, 1, \ldots, 1].$$

The number of indices in the bracket $< \ldots >$ corresponds to the number of factors β'_j or β''_j in the representation of $v_{<\ldots>}$. Therefore $v_{<k,\ell>}$ and $v_{<k,\ell,\ldots>}$ are called 'higher order components'.

4.4 Distinction of Cases

The asymptotic behaviour of $\mathbf{v}$ is known, but we need to know how the involved quantities β'_j, β''_j behave. In the following, the index $j \in \{1, \ldots, d\}$ is fixed.

If $\beta'_j = \beta''_j = 0$ for almost all members of the sequence, the limit $\mathbf{w}$ must be a multiple of $\bigotimes_{j=1}^{d} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathcal{R}_1 \subset \mathcal{R}_2$. Otherwise, we restrict to a subsequence with $|\beta'_j| + |\beta''_j| > 0$. Consider the quotient $q_j := |\beta''_j / \beta'_j|$ (set $q_j = \infty$ if $\beta'_j = 0$). If $\{q_j\}$ has an accumulation point 0, we can extract a subsequence with $q_j \to 0$, leading to the first statement in case (4.6):

$$\beta''_j = o(\beta'_j) \qquad \text{or} \qquad (\beta''_j = O(\beta'_j) \text{ and } \beta'_j \beta''_j \geq 0). \tag{4.6}$$

Otherwise, if $\{q_j\}$ has the improper accumulation point ∞, the first statement in case (4.7) applies to a suitable subsequence::

$$\beta'_j = o(\beta''_j) \qquad \text{or} \qquad (\beta'_j = O(\beta''_j) \text{ and } \beta'_j \beta''_j \geq 0). \tag{4.7}$$

In the remaining case there exists an accumulation point $q \in (0, \infty)$. Choose a subsequence with $q_j \to q$. The subsequence can be selected in such a way that all members β'_j of the sequence have the same sign and all β''_j have the same sign. In this case $\beta'_j \sim \beta''_j$ holds and we can distinguish two subcases: $\beta'_j \beta''_j \geq 0$ is added to (4.6) and (4.7), whereas $\beta'_j \beta''_j < 0$ defines the last case (4.8):

$$\beta'_j \sim -\beta''_j \qquad \text{and} \qquad \beta'_j \beta''_j < 0. \tag{4.8}$$

Hence, for each $j \in \{1, \ldots, d\}$, one of the conditions (4.6), (4.7), (4.8) applies. Conditions (4.6) and (4.7) may apply simultaneously.

Remark 4.2 *The idea of the following proof is to show that the tensor* $\mathbf{w}$ *from (4.1) has*

$$\textit{at most } \text{three } \textit{nonzero components } w_{<>}, w_{<i>}, w_{<j>} \ (i \neq j). \tag{4.9}$$

W.l.o.g. we may assume $i = 1$, $j = 2$. *Then the tensor is of the form*

$$\begin{bmatrix} w_{<>} \\ w_{<1>} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ + \ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ w_{<2>} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This proves $\operatorname{rank}(\mathbf{w}) \leq 2$ *and yields* $\mathbf{w} \in \mathcal{F} = \mathcal{R}_2$.

In the sequel, we have to discuss the various combinations of (4.6), (4.7), (4.8) for $j \in \{1, \ldots, d\}$.

4.5 Case A: Condition (4.6)

The component $v_{<j>}$ is of the form

$$v_{<j>} = \delta \left[(1+\eta')\,\beta'_j + (1+\eta'')\,\beta''_j \right] \tag{4.10}$$

(cf. (4.5)). Under assumption (4.6), $\delta\beta'_j$ is the leading term, i.e., $v_{<j>} \sim \delta\beta'_j$.

Remark 4.3 *Assume that (4.6) holds for j in (4.10). Then there are two cases (a) and (b).*
Case (a): $v_{<j>} \to w_{<j>} \neq 0$ implies

$$\beta''_j \lesssim \beta'_j \sim 1/\delta. \tag{4.11}$$

Case (b): $v_{<j>} \to w_{<j>} = 0$ implies $\beta'_j, \beta''_j \lesssim \varepsilon/\delta$.
In both cases we have

$$\beta'_j, \beta''_j \lesssim 1/\delta. \tag{4.12}$$

Proof. In Case (a), $\delta\,(1+\eta')\,\beta'_j$ has a nonvanishing limit, i.e. $\delta\beta'_j \to c \neq 0$, proving $\beta'_j \sim 1/\delta$. $\beta''_j \lesssim \beta'_j$ follows from (4.6).

In Case (b), condition (4.4) requires[7] $\left|\delta\,(1+\eta')\,\beta'_j\right| \lesssim |v_{<j>}| = |v_{<j>} - w_{<j>}| \leq \varepsilon$, proving $\beta'_j \lesssim \varepsilon/\delta$. The same argument holds for β''_j. ∎

Conclusion 4.4 *Let (4.6) holds for j. Then all higher order components $w_{<j,k>}$, $w_{<j,k,\ell,\ldots>}$ containing the index j vanish.*

Proof. The component $v_{<j,k>}$ is of the form

$$v_{<j,k>} = \delta \left[(1+\eta')\,\beta'_j\beta'_k - (1+\eta'')\,\beta''_j\beta''_k \right] \qquad (j \neq k) \tag{4.13}$$

(cf. (4.5)). $\delta\beta'_j\beta'_k \lesssim \beta'_k \to 0$ and $\delta\beta''_j\beta''_k \lesssim \beta''_k \to 0$ follow from (4.12) and $\beta'_k, \beta''_k \to 0$. Hence, $v_{<j,k>} \to w_{<j,k>} = 0$. Components of higher order like, e.g., $v_{<j,k,\ell>}$ contain even more β-factors and must tend to zero. ∎

The crucial statement is the next one.

Lemma 4.5 *(4.6) and $w_{<j>} \neq 0$ hold at most for* two *different indices j_1 and j_2.*

Proof. For an indirect proof assume that these properties hold for three indices. For ease of a simple notation, assume that the corresponding indices are $j = 1, 2, 3$. The component $v_{<1,2,3>}$ is of the form

$$v_{<1,2,3>} = \delta \left[(1+\eta')\,\beta'_1\beta'_2\beta'_3 + (1+\eta'')\,\beta''_1\beta''_2\beta''_3 \right]. \tag{4.14}$$

(4.6) implies $v_{<1,2,3>} \sim \delta\beta'_1\beta'_2\beta'_3$. Since $v_{<1,2,3>} \to w_{<1,2,3>} = 0$ (cf. Conclusion 4.4), condition (4.4) requires $\left|\delta\beta'_1\beta'_2\beta'_3\right| \sim |v_{<1,2,3>}| \leq \varepsilon$. (4.11) states that $\beta'_j \sim 1/\delta$ for $j = 1, 2, 3$. Combination with the previous inequality yields $\delta^{-2} \lesssim \varepsilon$, i.e., $\delta \gtrsim \varepsilon^{-1/2}$ in contradiction to $\delta = o(\varepsilon^{-1/2})$ (cf. (4.4)). ∎

4.6 Case B: Conditions (4.7)

Remark 4.3 holds with interchanged roles of β'_j and β''_j. Also Conclusion 4.4 and Lemma 4.5 hold with condition (4.6) replaced by (4.7).

[7] $\varepsilon := \|\mathbf{v} - \mathbf{w}\|$ in (4.4) implies $|v_{<\ldots>} - w_{<\ldots>}| \leq \varepsilon$ for all components.

4.7 Case C: Conditions (4.6) and (4.7)

Lemma 4.5 in Case A and the modified Lemma 4.5 in Case B cover the cases that condition (4.6) applies to three indices or that (4.7) applies to three indices. Now we consider the mixed case, i.e.,
(a) (4.6) holds for two indices and (4.7) holds for one index or
(b) (4.6) holds for one and (4.7) for two indices.

By symmetry, it is sufficient to consider case (a).

Lemma 4.6 *It is impossible that (4.6) and* $w_{<j>} \neq 0$ *hold for* two *different indices* j_1 *and* j_2, *while (4.7) and* $w_{<j_3>} \neq 0$ *hold for a third index* j_3 *(*$j_1 \neq j_3 \neq j_2$*).*

Proof. W.l.o.g. we may assume $j_1 = 1$, $j_2 = 2$, $j_3 = 3$. Consider

$$v_{<1,2>} = \delta \left[(1+\eta')\,\beta'_1\beta'_2 - (1+\eta'')\,\beta''_1\beta''_2 \right]$$

(cf. (4.13)). First, we assume that the two terms in $v_{<1,2>}$ are of the same order: $\delta\beta'_1\beta'_2 \sim \delta\beta''_1\beta''_2$ (note that in this case the terms may cancel). Because of (4.6) we conclude that $\beta'_1 \sim \beta''_1$ and $\beta'_2 \sim \beta''_2$, implying property (4.6) for $j = 3$. Hence, Lemma 4.5 can be applied and yields the contradiction.

Otherwise, $\delta\beta'_1\beta'_2$ is the only leading term in $v_{<1,2>} \to w_{<1,2>} = 0$ (cf. Conclusion 4.4). The requirement $|v_{<1,2>} - w_{<1,2>}| = |v_{<1,2>}| \sim \left|\delta\beta'_1\beta'_2\right| \lesssim \varepsilon$ together with $\beta'_1, \beta'_2 \sim 1/\delta$ (cf. (4.11)) implies $\left|\delta\beta'_1\beta'_2\right| \sim 1/\delta \lesssim \varepsilon$ and $\delta \gtrsim \varepsilon^{-1}$ in contradiction to (4.4). ∎

4.8 Case D: Condition (4.8)

Now we assume that condition (4.8) holds for all $j \in \{1, \ldots, d\}$. Because of $\beta'_j \sim \beta''_j$ and their opposite signs, the terms in (4.10) may cancel. It is not possible to estimate $\delta\beta'_j$ or $\delta\beta''_j$ by means of $v_{<j>}$. For this purpose we introduce the notation

$$\beta''_j = -\,(1+\alpha_j)\,\beta'_j \tag{4.15}$$

with a bounded α_j. Note that $\alpha_j \to 0$ may occur. Inserting (4.15) into (4.10) we obtain

$$\begin{aligned} v_{<j>} &= \delta\beta'_j \left[(1+\eta') - (1+\eta'')\,(1+\alpha_j) \right] = \delta\beta'_j \left[\eta' - \eta'' - (1+\eta'')\alpha_j \right] \\ &= \beta'_j v_{<>} - (1+\eta'')\delta\beta'_j\alpha_j. \end{aligned} \tag{4.16}$$

Remark 4.7 *(a) Assume condition (4.8) and (4.15) for an index j. Then* $|\delta\beta'_j\alpha_j| \leq O(1)$. *If in addition* $w_{<j>} \neq 0$, *then*

$$\delta\beta'_j\alpha_j \sim 1.$$

(b) Let (4.8) hold for two indices j and k. Then $v_{<j,k>} \to w_{<j,k>} = 0$ *and* $w_{<j,k,\ldots>} = 0$ *for all higher order components containing j and k.*

Proof. (a) Since $\beta'_j v_{<>} \to 0$, the term $(1+\eta'')\delta\beta'_j\alpha_j \to w_{<j>}$ in (4.16) must be bounded. Furthermore, $\delta\beta'_j\alpha_j \to w_{<j>} \neq 0$ implies $\delta\beta'_j\alpha_j \sim 1$.
(b) $v_{<j,k>}$ has the representation

$$v_{<j,k>} = \beta'_j\beta'_k v_{<>} - (1+\eta'')\delta\beta'_j\beta'_k(\alpha_j + \alpha_k + \alpha_j\alpha_k). \tag{4.17}$$

Then $\beta'_j\beta'_k v_{<>} \to 0$ and part (a) yield $\delta\beta'_j\beta'_k\alpha_j = \left(\delta\beta'_j\alpha_j\right)\beta'_k \lesssim \beta'_k \to 0$ as well as $\delta\beta'_j\beta'_k\alpha_k = \left(\delta\beta'_k\alpha_k\right)\beta'_j \lesssim \beta'_j \to 0$. This proves $v_{<j,k>} \to w_{<j,k>} = 0$. ∎

Now we consider the behaviour of $\delta\alpha_j$. Either $|\delta\alpha_j| \leq O(1)$ holds or $|\delta\alpha_j| \to \infty$ (for a subsequence). The first case is the harmless one.

Remark 4.8 *Condition (4.8) together with* $|\delta\alpha_j| \leq O(1)$ *yields* $v_{<j>} \to w_{<j>} = 0$.

The next lemma is the analogue of Lemma 4.5.

Lemma 4.9 *Condition (4.8) cannot hold for three indices* j_1, j_2, j_3 *with* $w_{<j_i>} \neq 0$ $(1 \leq i \leq 3)$.

Proof. For an indirect proof assume (4.8) with $w_{<j>} \neq 0$ for three indices j. W.l.o.g. we assume $j_1 = 1$, $j_2 = 2$, $j_3 = 3$. Thanks to Remark 4.8, $|\delta\alpha_j| \to \infty$ must hold implying $\delta\beta'_j\alpha_j \sim 1$ (cf. Remark 4.7a).

Two of the three values α_1, α_2, α_3 must have the same sign – say $|\alpha_1 + \alpha_2| = |\alpha_1| + |\alpha_2|$. $\delta\alpha_j \to \infty$ implies that the second term $(1+\eta'')\delta\beta'_1\beta'_2(\alpha_1 + \alpha_2 + \alpha_1\alpha_2) \sim \delta\beta'_1\beta'_2(\alpha_1 + \alpha_2)$ in $v_{<1,2>}$ (cf. (4.17)) is the leading one. From (4.4) and $\delta\beta'_j\alpha_j \sim 1$ we conclude that

$$\left|\delta\beta'_1\beta'_2\,(\alpha_1+\alpha_2)\right| = \left|\delta\beta'_1\beta'_2\right|(|\alpha_1|+|\alpha_2|) = \left|\delta\beta'_1\alpha_1\right|\left|\beta'_2\right| + \left|\delta\beta'_2\alpha_2\right|\left|\beta'_1\right| \sim \left|\beta'_1\right| + \left|\beta'_2\right| \lesssim \varepsilon.$$

Then $\delta \lesssim o(\varepsilon^{-1/2})$ from (4.4) yields $\delta\beta'_1, \delta\beta'_2 \lesssim o(\varepsilon^{+1/2})$, which implies $v_{<1>} \to w_{<1>} = 0$ as well as $w_{<2>} = 0$ in contradiction to our assumption. ∎

Assuming condition (4.8) for all $j \in \{1, \ldots, d\}$, we conclude from Lemma 4.9 that at most two components $w_{<j>}$ may be nonzero, while all higher order terms vanish because of Remark 4.7b. Hence Remark 4.2 proves $\mathbf{w} \in \mathcal{R}_2$.

Next we discuss the mixed situation when indices with condition (4.6) or (4.7) as well as indices with condition (4.8) are present.

4.9 Case E: Mixed Situation

Lemma 4.10 *Let j be an index for which $w_{<j>} \neq 0$ and either (4.6) or (4.7) are valid. Then $w_{<k>} = 0$ holds for all k subject to condition (4.8).*

Proof. (4.6) and (4.7) yield completely symmetric situations. W.l.o.g. assume (4.6). Conclusion 4.4 states that $v_{<j,k>} \to w_{<j,k>} = 0$. $v_{<j,k>}$ is of the form

$$v_{<j,k>} = \delta\beta'_k\left[(1+\eta')\,\beta'_j + (1+\eta'')\,(1+\alpha_k)\,\beta''_j\right]$$

(note that both terms have the same sign, no cancellation!). $\delta\beta'_j \sim 1$ from (4.11) implies

$$\left|\beta'_k\right| \sim \left|\delta\beta'_k\,(1+\eta')\,\beta'_j\right| \leq |v_{<j,k>}| \leq \varepsilon$$

because of (4.4). Together with $\delta \lesssim o(\varepsilon^{-1/2})$, we obtain $\delta\beta'_k\alpha_k \lesssim o(\varepsilon^{1/2}) \to 0$ so that $v_{<k>} \to w_{<k>} = 0$. ∎

Let condition (4.6) or (4.7) hold for $j \in D_1 \neq \emptyset$, while (4.8) holds for $j \in D_2 := \{1, \ldots, d\}\backslash D_1 \neq \emptyset$. For all $k \in D_2$ we have $w_{<k>} = 0$ as stated in Lemma 4.10. Among D_1 there can be at most two indices with $w_{<j>} \neq 0$ (cf. Lemmata 4.5 and 4.6).

Remark 4.7b states that $w_{<j,k>} = w_{<j,k,\ldots>} = 0$ for higher order components with $j, k \in D_2$. Otherwise, one of the indices must belong to D_1. Then $w_{<j,k>} = w_{<j,k,\ldots>} = 0$ follows from Conclusion 4.4. This proves the following result, so that again Remark 4.2 can be applied.

Remark 4.11 *Also in the mixed case, $w_{<j>} \neq 0$ occurs for at most two indices j, while all higher order components vanish: $w_{<j,k>} = w_{<j,k,\ldots>} = 0$.*

4.10 The Complex Case

Now we discuss the modifications for the complex tensor space $\otimes^d\mathbb{C}^2$.

The quantities β'_j and β''_j are of the form

$$\beta'_j = \rho'_j \exp(\mathrm{i}\omega'_j) \quad \text{and} \quad \beta''_j = \rho''_j \exp(\mathrm{i}\omega''_j) \qquad (\,\rho'_j, \rho''_j \geq 0,\ \omega'_j, \omega''_j \in [0, 2\pi)\,).$$

Taking a subsequence, we can ensure that[8]

$$\omega'_j \to \omega^*_j \in [0, 2\pi) \qquad \text{and} \qquad \omega''_j \to \omega^{**}_j \in [0, 2\pi).$$

The condition $\beta'_j\beta''_j \geq 0$ in (4.6) and (4.7) has to be replaced by

$$\left|\omega^*_j - \omega^{**}_j\right| \neq \pi,$$

[8] Convergence *modulo* 2π.

i.e., we have to avoid that in the limit the complex signs of β_j' and β_j'' are opposite. Then the same conclusions follow as in the real case.

The inequality $\beta_j'\beta_j'' < 0$ in (4.8) becomes

$$\left|\omega_j^* - \omega_j^{**}\right| = \pi.$$

Again we set $\beta_j'' = -\beta_j'(1+\alpha_j)$ with bounded α_j. For the proof of Lemma 4.9 we used the fact that at least two of the three quantities α_1, α_2, α_3 have the same sign, so that, e.g., $|\alpha_1+\alpha_2| = |\alpha_1| + |\alpha_2|$. Now we use that among three complex values α_1, α_2, α_3 there are at least two – say α_1 and α_2 – with $|\alpha_1+\alpha_2| \le \frac{1}{2}(|\alpha_1| + |\alpha_2|)$. This leads to the same results.

With these modifications the previous proof can be repeated for the complex case.

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