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A PERELMAN-TYPE NO SHRINKING BREATHER THEOREM FOR NONCOMPACT HARMONIC RICCI FLOW

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Abstract. In this paper, we prove a no shrinking breather theorem for noncompact harmonic Ricci flow under the condition that Sic (see (1.4)) is bounded from below, which extends a recent result of Cheng and Zhang for noncompact Ricci flow in [1].

1. Introduction and main result

The Ricci flows on a manifold M can be regarded as orbits in the space

$$\mathrm{Met}(M)/(\mathrm{Diff} \oplus \mathrm{Scal}),$$

where $\mathrm{Met}(M)$ represents the space of all the Remannian metrics on M and $\mathrm{Diff} \oplus \mathrm{Scal}$ represents the group of self-diffeomorphisms of M and scalings in $\mathrm{Met}(M)$. The breathers are the periodic orbits in this space, "no breather" means that the periodic orbits must be static. In the compact case, Perelman [7] proved the first no breather theorem by utilizing the powerful entropy formulas. Lu and Zheng in [4] provided a new proof of no shrinking breather theorem by constructing an ancient solution from a given shrinking breather.

In the general noncompact case, Q.S. Zhang in [11] proved a no breather theorem under suitable conditions at infinity and assuming boundedness of curvature tensor, and further proved a no shrinking breather theorem for asymptotically flat manifolds with positive scalar curvature. By applying the method of Lu and Zheng in association with a reduced distance estimate, Y.J. Zhang in [12] proved a no shrinking breather theorem under the condition of bounded sectional curvature. Cheng and Zhang in [1] extend this no shrinking breather theorem to the fullest generality, that is, assuming only a lower bound for the Ricci curvature. Recently, they obtained a no expanding breather theorem for noncompact Ricci flow in [2]. Inspired by [1], the purpose of this paper is to prove a no shrinking breather theorem for noncompact harmonic Ricci flow.

Let us first recall some basic definitions of harmonic Ricci flow.

Definition 1.1. *Let $g(t)$ be a family of Riemannian metrics on a complete manifold M and let $\phi(t)$ be a family of smooth maps from M to a closed target manifold N. Given a positive coupling constant α, we*

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call a smooth family $(g(t), \phi(t))$ **a solution to the harmonic Ricci flow** *if it satisfies*

$$\begin{cases} \dfrac{\partial g(t)}{\partial t} = -2\mathrm{Ric}_{g(t)} + 2\alpha \nabla\phi(t) \otimes \nabla\phi(t), \\ \dfrac{\partial \phi(t)}{\partial t} = \bar{\tau}_{g(t)}(\phi(t)), \end{cases} \tag{1.1}$$

where $\mathrm{Ric}_{g(t)}$ *denotes the Ricci curvature of* $(M, g(t))$, $\bar{\tau}_{g(t)}(\phi(t))$ *denotes the tension field of the map* $\phi(t)$ *with respect to the evolving metric* $g(t)$.

Definition 1.2. *A solution* $(g(t), \phi(t))_{t\in[0,T]}$ *of harmonic Ricci flow is called a breather if there exists* $0 \le t_1 < t_2 \le T$, *a constant* $b > 0$ *and a self-diffeomorphism* $\psi : M \to M$ *satisfying*

$$\begin{cases} g(t_2) = b\psi^* g(t_1) \\ \phi(t_2) = \psi^* \phi(t_1). \end{cases} \tag{1.2}$$

The cases $b < 1$, $b = 1$ *and* $b > 1$ *correspond to* **shrinking, steady and expanding breathers** *respectively.*

Definition 1.3. *A solution* $(g(t), \phi(t))$ *of harmonic Ricci flow is called a gradient harmonic Ricci soliton if there exists a smooth potential function* $f : M \to \mathbb{R}$ *and a constant* ε *such that*

$$\begin{cases} \mathrm{Ric} - \alpha\nabla\phi \otimes \nabla\phi + \mathrm{Hess}(f) + \varepsilon g = 0 \\ \bar{\tau}_g(\phi) - \langle \nabla\phi, \nabla f \rangle = 0, \end{cases} \tag{1.3}$$

where $\mathrm{Hess}(f)$ *is the Hessian of* f. *The cases* $\varepsilon < 0$, $= 0$, *and* > 0 *is called* **shrinking, steady and expanding gradient solitons** *respectively.*

Remark 1.4. *Gradient solitons generate self-similar solutions to the harmonic Ricci flow, called the canonical forms. Let the positive function* $c(t)$, *the one-parameter family of diffeomorphisms* $\psi_t : M \to M$ *with* $\psi_0 = id_M$, *the evolving metric* $g(t)$ *and map* $\phi(t)$ *be defined as follows*

$$c(t) = 1 + 2\varepsilon t,$$

$$\frac{\partial}{\partial t}\psi_t = \frac{\nabla f}{c(t)} \circ \psi_t,$$

$$g(t) = c(t)\psi_t^* g,$$

$$\phi(t) = \psi_t^* \phi.$$

Then $(g(t), \phi(t))$ *is a solution of harmonic Ricci flow* (1.1), *which is called the canonical forms.*

Müller in [5] proved that any periodic orbits in $\mathrm{Met}(M)/(\mathrm{Diff} \oplus \mathrm{Scal})$ must be static for closed M:

Theorem 1.5. (see [5] **Theorem 7.5)** *A steady, expanding, or shrinking breather on a closed manifold is a steady, expanding, or shrinking gradient soliton, respectively. Moreover, in the steady case, the soliton is stationary and in the expanding case,* $g(t)$ *changes by scaling and* $\phi(t)$ *is harmonic.*

We extend the no shrinking breather theorem to the complete noncompact case. For simplicity, we introduce the (0,2)-tensor

$$\mathrm{Sic} := \mathrm{Ric} - \alpha \nabla\phi \otimes \nabla\phi, \tag{1.4}$$

which is denoted as S_{ij} in local coordinates, and its trace

$$S := R - \alpha|\nabla\phi|^2.$$

Theorem 1.6. *Any complete noncompact shrinking breather with* $\mathrm{Sic} \geq -kg$ *is (the canonical form of) a shrinking gradient soliton, where* k *is a nonnegative constant.*

The paper is organized as follows. In section 2, we introduce some basic properties about reduced distance (also called l-distance) for the backward harmonic Ricci flow. In section 3, we will use a given shrinking breather to construct an ancient solution, then we choose a sequence of points on this ancient solution and estimate the l-distance at these points. We verify that the ancient solution constructed from a given shrinking breather is locally uniformly Type I along the sequence of points, furthermore, after rescaling such flows, we have locally uniform estimates for both the l-function and its gradient. In section 4, we obtain that a locally uniformly Type I ancient solution has an asymptotic shrinking gradient harmonic Ricci soliton, and then we give a proof of Theorem 1.6.

2. Preliminaries

2.1. **Reduced Distance for Harmonic Ricci Flows.** Let M be a manifold with a family of complete time-dependent Riemannian metrics g_τ. Assume that $(M, g_\tau)_{\tau\in[0,T]}$ evolves by the backward harmonic Ricci flow

$$\frac{\partial}{\partial\tau} g_\tau = 2\mathrm{Sic}_{g_\tau}, \tag{2.1}$$

where τ is the backward time. In analogy to Perelman's $\mathcal{L}$-distance for the Ricci flow in [7] (see also e.g. [10]), we will introduce some basic properties about reduced distance functions for flow (2.1). In compact case, similar results have been obtained by Müller in [5].

Definition 2.1. (see [6]**)** *Let* $\tau \in (0, T]$*, Perelman's* $\mathcal{L}$*-energy of a piecewise* C^1 *curve* $\gamma(s) : [0, \tau] \to M$ *for flow* (2.1) *is defined by*

$$\mathcal{L}(\gamma(s)) = \int_0^\tau \sqrt{s}\left(S_{(g(s),\phi(s))}(\gamma(s)) + \left|\frac{\partial(\gamma(s))}{\partial s}\right|^2_{g(s)}\right) ds. \tag{2.2}$$

For a fixed point $p \in M$, for any $(q, \tau) \in M \times (0, T]$, define

$$L(q, \tau) = \inf_{\gamma\in\Gamma} \mathcal{L}(\gamma(s)), \tag{2.3}$$

where $\Gamma = \{\gamma : [0,\tau] \to M \mid \gamma(0) = p, \gamma(\tau) = q\}$. The minimizer is called $\mathcal{L}$-geodesic, the existence of such curves will be discussed in the following lemma. Perelman's l-function for flow (2.1), also known as the reduced distance function, is defined as

$$l(q,\tau) = \frac{L(q,\tau)}{2\sqrt{\tau}}. \tag{2.4}$$

An easy computation for l function leads to the following lemma.

Lemma 2.2. *The l-function is invariant under the scaling $g(\tau) \to g_c(\tau) := c^{-1}g(c\tau)$ and $\phi(\tau) \to \phi_c(\tau) := \phi(c\tau)$, i.e.*

$$l_{(g_c,\phi_c)}(q,\tau) = l_{(g,\phi)}(q,c\tau). \tag{2.5}$$

Proof Let $\gamma(s) : [0,\tau] \to M$ be the minimal $\mathcal{L}$-geodesic from p to $q = \gamma(\tau)$ for flow $(g_c(\tau), \phi_c(\tau))$, let $u = cs$, then from Theorem 2.3 we know that $\gamma(u)$ is also the minimal $\mathcal{L}$-geodesic for flow $(g(c\tau), \phi(c\tau))$ because $\mathrm{Sic}_{(g_c(\tau),\phi_c(\tau))} = \mathrm{Sic}_{(g(c\tau),\phi(c\tau))}$. Combine with (2.2) and (2.3) we have

$$\begin{aligned}
L_{(g_c,\phi_c)}(q,\tau) &= \int_0^\tau \sqrt{s}\left(S_{(g_c(s),\phi_c(s))} + \left|\frac{\partial(\gamma(s))}{\partial s}\right|^2_{g_c(s)}\right)ds \\
&= \int_0^\tau \sqrt{s}\left(R_{g_c(s)} - \alpha|\nabla\phi_c(s)|^2_{g_c(s)} + \left|\frac{\partial(\gamma(s))}{\partial s}\right|^2_{g_c(s)}\right)ds \\
&= \int_0^\tau \sqrt{s}\left[c\left(R_{g(cs)} - \alpha|\nabla\phi(cs)|^2_{g(cs)}\right) + \frac{1}{c}\left|\frac{\partial(\gamma(cs))}{\partial s}\right|^2_{g(cs)}\right]ds \\
&= \int_0^{c\tau} \frac{\sqrt{u}}{\sqrt{c}}\left(S_{(g(u),\phi(u))} + \left|\frac{\partial(\gamma(u))}{\partial u}\right|^2_{g(u)}\right)du \\
&= \frac{L_{(g,\phi)}(q,c\tau)}{\sqrt{c}},
\end{aligned}$$

hence from (2.4) we get

$$l_{(g_c,\phi_c)}(q,\tau) = \frac{L_{(g_c,\phi_c)}(q,\tau)}{2\sqrt{\tau}} = \frac{L_{(g,\phi)}(q,c\tau)}{2\sqrt{c\tau}} = l_{(g,\phi)}(q,c\tau).$$

□

The reduced volume for flow (2.1) is defined as

$$\mathcal{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-l(\cdot,\tau)} dg(\tau). \tag{2.6}$$

A basic property of $\mathcal{V}(\tau)$ is its invariance under the scaling $g(\tau) \to g_c(\tau)$ and $\phi(\tau) \to \phi_c(\tau)$.

For $v \in T_pM$, let γ_v denote the $\mathcal{L}$-geodesic such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$, the variational vector field along the geodesic variation of γ_v is the $\mathcal{L}$-Jacobi field. If γ_v exists on $[0,\tau]$, then the $\mathcal{L}$-exponential map $\mathcal{L}exp_p^\tau : T_pM \to M$ at time τ is defined as

$$\mathcal{L}exp_p^\tau(v) = \gamma_v(\tau).$$

When we say $\mathcal{L}exp_p^\tau(v) = q$ is conjugate to p along $\mathcal{L}$-geodesic γ_v we mean that there is a nontrivial $\mathcal{L}$-Jacobi field J along γ_v with $J(0) = 0$ and $J(\tau) = 0$. Let $U(\tau) \subset T_pM$ denote the maximal domain of $\mathcal{L}exp_p^\tau$, the injectivity domain at time τ is defined as

$$\begin{aligned}\Omega^{T_pM}(\tau) =\{&v \in U(\tau) \mid \gamma_v : [0,\tau] \to M \text{ is the unique minimal } \mathcal{L}-\text{geodesic from } p \text{ to } \gamma_v(\tau),\\ &\gamma_v(\tau) \text{ is not conjugate to } p \text{ alog } \gamma_v.\},\end{aligned}$$

correspondingly we define

$$\begin{aligned}\Omega(\tau) =\{&q \in M \mid \text{ There is a unique minimal } \mathcal{L}-\text{geodesic } \gamma : [0,\tau] \to M\\ &\text{with } \gamma(0) = p, \gamma(\tau) = q, q \text{ is not conjugate to } p \text{ alog } \gamma.\}.\end{aligned}$$

We have

$$\Omega(\tau) = \mathcal{L}exp_p^\tau(\Omega^{T_pM}(\tau)).$$

The the cut-locus is defined as

$$C(\tau) = M \backslash \Omega(\tau).$$

Assuming a lower bound for the tensor Sic at each time slice, we have the following theorems:

Theorem 2.3. *If* $\mathrm{Sic}_{g(\tau)} \geq -kg(\tau)$ *on* $[0,T]$ *for a nonnegative constant k, here* $g(\tau)$ *is give by* (2.1)*, then the following hold:*
(1) For any $(q,\tau) \in M \times (0,T]$*, there exists a minimal* $\mathcal{L}$*-geodesic connecting* $(p,0)$ *and* (q,τ)*, that is,* $\mathcal{L}exp_p^\tau$ *is onto.*
(2) L is locally Lipschitz in space-time.
(3) For each $\tau \in (0,T)$*,* $C(\tau) \subset M$ *is a closed set of zero Remannian measure. Consequently* $\cup_{0<\tau<T} C(\tau) \times \{\tau\}$ *is a closed set of zero Remannian measure in* $M \times (0,T)$*.*
(4) For any nonnegative Lipschitz function η *with compact support, we have*

$$-\int_M \nabla l \cdot \nabla \eta dg \leq \int_{*M} \eta \Delta l dg, \tag{2.7}$$

where the integral $\int_{*M}$ *means* $\liminf_{\epsilon \to 0} \int_{M-U_\epsilon}$*,* $U_\epsilon = U_\epsilon(C(\tau))$ *is the open* ϵ*-tubular neighborhood of* $C(\tau)$ *with respect to* $g(\tau)$*.*

Proof Using the flow equation (2.1) and the condition $\mathrm{Sic}_{g(\tau)} \geq -kg(\tau)$ we have:

$$e^{-2k\tau} g(0) \leq g(\tau) \leq e^{2k(T-\tau)} g(T)$$

for each τ, therefore we can obtain estimates for the $\mathcal{L}$-distance like [10] Lemma 2.3. One can refer to [10] Proposition 2.8, 2.14, and 2.16 and Lemma 2.21 for more details. □

Theorem 2.4. (see [5]**)** *Let l be the reduced distance function defined in* (2.4)*. If* $\mathrm{Sic}_{g(\tau)} \geq -kg(\tau)$ *on* $[0,T]$ *for a nonnegative constant k, then on* $\cup_{0<\tau<T} \Omega(\tau) \times \{\tau\}$ *it holds that*

$$2\frac{\partial l}{\partial \tau} + |\nabla l|^2 - S + \frac{l}{\tau} = 0, \tag{2.8}$$

$$\frac{\partial l}{\partial \tau} - \Delta l + |\nabla l|^2 - S + \frac{n}{2\tau} \geq 0, \tag{2.9}$$

$$2\Delta l - |\nabla l|^2 + S + \frac{l-n}{\tau} \leq 0. \tag{2.10}$$

Furthermore, a combination of (2.7), (2.9) and (2.10) shows that for any $0 < \tau_1 < \tau_2 < T$ and any nonnegative Lipschitz function η compactly supported on $M \times [\tau_1, \tau_2]$, it holds that

$$\int_{\tau_1}^{\tau_2} \int_M \left(\nabla l \cdot \nabla \eta + \left(\frac{\partial l}{\partial \tau} + |\nabla l|^2 - S + \frac{n}{2\tau} \right) \eta \right) dg(\tau) d\tau \geq 0, \tag{2.11}$$

and

$$\int_M \left(-2\nabla l \cdot \nabla \eta + \left(-|\nabla l|^2 + S + \frac{l-n}{\tau} \right) \eta \right) dg(\tau) \leq 0 \text{ for any } \tau \in (0,T). \tag{2.12}$$

Lemma 2.5. *Let γ be an $\mathcal{L}$-geodesic starting from $(p,0)$. For any $\tau \in (0,T)$ and $\gamma(\tau) \in \Omega(\tau)$ it holds that*

$$\nabla l(\gamma(\tau), \tau) = \gamma'(\tau). \tag{2.13}$$

Proof The lemma comes from the first variation formula of $\mathcal{L}$-energy, one can find a proof in [5]. □

Theorem 2.6. *Let $\mathcal{V}(\tau)$ be the reduced volume defined in* (2.6)*. If* $\mathrm{Sic}_{g(\tau)} \geq -kg(\tau)$ *on* $[0,T]$ *for a nonnegative constant k, then*
(1) $\mathcal{V}(\tau) \leq 1$ for all $\tau \in (0,T]$,
(2) $\mathcal{V}(\tau)$ is non-increasing in τ.

Proof Analogous to the case of Ricci flow in [10], the monotonicity of the reduced volume mainly comes from the monotonicity of the Jacobian of the $\mathcal{L}$-exponential map, one can find a proof on page 80 in [5]. □

2.2. **Cheeger-Gromov Convergence and Compactness of Harmonic Ricci Flows.** Let $(M_i, g_i(\tau), \phi_i(\tau), x_i)$ be a sequence of pointed, complete harmonic Ricci flows for $\tau \in (a,b)$, we introduce the definition of Cheeger-Gromov convergence and the compactness theorem.

Definition 2.7. (see [9] **Definition 3.1)** *A sequence $\{(M_i, g_i(\tau), \phi_i(\tau), x_i)\}$ of pointed, complete harmonic Ricci flows converges to $(M_\infty, g_\infty(\tau), \phi_\infty(\tau), x_\infty)$ for $\tau \in (a,b)$ if there exists an exhaustion $\{U_i\}$ of M_∞ by open sets with $x_\infty \in U_i$ for all i, and a family of pointed diffeomorphisms $\{\Phi_i : (U_i, x_\infty) \to (V_i, x_i) \subset M_i\}$ such that*

$$(U_i \times (a,b), \Phi_i^*(g_i(\tau)|_{V_i} + d\tau^2), \Phi_i^*(\phi_i(\tau)|_{V_i}))$$

converges uniformly in C^∞ on compact sets to

$$(M_\infty \times (a,b), g_\infty(\tau) + d\tau^2, \phi_\infty(\tau)),$$

where $d\tau^2$ is the standard metric on (a,b).

Theorem 2.8. (see [3] **Theorem 2.9)** *Let $\{(M_i, g_i(\tau), \phi_i(\tau), x_i)\}$ be a given sequence of pointed, complete harmonic Ricci flows, with fixed target manifold N, all defined for $\tau \in (a, b)$, where $-\infty \le a < 0 < b \le +\infty$, with non-increasing coupling functions $\alpha_i(\tau) \in [\underline{\alpha}, \overline{\alpha}]$, where $0 < \underline{\alpha} \le \overline{\alpha} < \infty$. Let $B_{g_i(0)}(x_i, r)$ be a geodesic ball of radius r, under $g_i(0)$, centered at x_i. Assume further that*
(1) $\forall r > 0$ there exists a constant $M = M(r)$ such that $\forall \tau \in (a, b)$ and $\forall i$

$$\sup_{B_{g_i(0)}(x_i,r)} |Rm_{g_i(\tau)}|_{g_i(\tau)} \le M,$$

(2) $\inf_i(\mathrm{inj}(M_i, g_i(0), x_i)) > 0$.
Then there exist a manifold M_∞, a harmonic Ricci flow $(g_\infty(\tau), \phi_\infty(\tau))$ on $M_\infty \times (a, b)$ with target manifold N, a non-increasing coupling function $\alpha_\infty(\tau)$, and a point $x_\infty \in M_\infty$, such that $(M_\infty, g_\infty(0))$ is complete and

$$(M_i, g_i(\tau), \phi_i(\tau), x_i) \to (M_\infty, g_\infty(\tau), \phi_\infty(\tau), x_\infty),$$

and $\alpha_i(\tau) \to \alpha_\infty(\tau)$ pointwise in (a, b).

2.3. **Gradient shrinking soliton equation.** In order to finish the proof of Theorem 1.6, we need to find a sequence of smooth functions f_i such that the scaled sequence of backward harmonic Ricci flows

$$\{(M, g_i(\tau), \phi_i(\tau), x_i, f_i)_{\tau \in [\tau_1, \tau_2]}\}_{i=1}^{\infty},$$

after passing to a subsequence, converges to a shrinking gradient harmonic Ricci soliton. For this purpose, we introduce the following theorem which is similar to the arguement for Ricci flow in [7].

Theorem 2.9. (see [13] **Corollary 9)** *Let $(g(\tau), \phi(\tau))_{\tau \in [\tau_1, \tau_2]}$ be a smooth solution of the harmonic Ricci flow with constant $\alpha > 0$, and let $f : M \times [\tau_1, \tau_2] \to \mathbb{R}$ be a smooth function. Define*

$$u := (4\pi\tau)^{-\frac{n}{2}} e^{-f},$$

$$v := (\tau(2\Delta l - |\nabla l|^2 + S) + f - n)u.$$

If u satisfies the conjugate heat equation

$$\Box^* u = \frac{\partial u}{\partial \tau} - \Delta u + S u = 0, \tag{2.14}$$

then we have

$$\Box^* v = -2\tau[|\mathrm{Sic} + \mathrm{Hess}(f) - \frac{1}{2\tau} g|^2 + \alpha|\tau_g(\phi) - \langle \nabla\phi, \nabla f\rangle|^2]u. \tag{2.15}$$

Remark 2.10. *Note that if $v = 0$, we can get a gardient shrinking harmonic Ricci soliton with f being the potential function for $u > 0$.*

3. The Ancient Solution Constructed from the Breather

In this section, we first use a given shrinking breather to construct an ancient solution, this method was first applied by Lu and Zheng in [4] to Ricci flow in the compact case and then by Cheng and Zhang in [1] in the noncompact case. Then we verify the so-called locally uniformly Type I property for this ancient solution and locally uniformly estimates for l distances, which was a key tool introduced in [1] to deal with the noncompact Ricci flow case.

3.1. **Construction of Ancient solution.** After rescaling and translating in time, from Definition 1.2 we know a solution $(g_0(\tau), \phi_0(\tau))_{\tau\in[0,1]}$ of backward harmonic Ricci flow is called a shrinking breather if there exists a constant $b \in (0, 1)$ and a diffeomorphism $\psi : M \to M$ such that

$$\begin{cases} bg_0(1) = \psi^* g_0(0) \\ \phi_0(1) = \psi^* \phi_0(0). \end{cases} \tag{3.1}$$

Here we assume that $(g_0(\tau), \phi_0(\tau))$ is smooth on $[0, 1]$. For $i \geq 0$, let

$$\tau_i = \sum_{k=0}^{i} b^{-k}. \tag{3.2}$$

There exists $C_0 > 1$ depending only on b such that

$$b^{-i} \leq \tau_i \leq C_0 b^{-i} \tag{3.3}$$

for each $i \geq 0$. For any $\tau \in [\tau_{i-1}, \tau_i]$, let

$$\begin{cases} g_i(\tau) = b^{-i}(\psi^i)^* g_0(b^i(\tau - \tau_{i-1})), \\ \phi_i(\tau) = (\psi^i)^* \phi_0(b^i(\tau - \tau_{i-1})), \end{cases} \tag{3.4}$$

and then we define

$$(g(\tau), \phi(\tau)) = \begin{cases} (g_0(\tau), \phi_0(\tau)), & \tau \in [0, \tau_0], \\ (g_i(\tau), \phi_i(\tau)), & \tau \in [\tau_{i-1}, \tau_i]. \end{cases} \tag{3.5}$$

Lemma 3.1. (see [5]**)** *Let $(g(\tau), \phi(\tau))$ be a smooth solution of the backward harmonic Ricci flow with positive coupling constant α, then we have the following evolution equations*

$$\begin{aligned} \frac{\partial}{\partial\tau}\Gamma_{ij}^{p} &= g^{pq}(\nabla_i R_{jq} + \nabla_j R_{iq} - \nabla_q R_{ij} - 2\alpha\nabla_i\nabla_j\phi\nabla_q\phi), \\ \frac{\partial}{\partial\tau}S_{ij} &= -\Delta_L S_{ij} - 2\alpha\bar{\tau}_g(\phi)\nabla_i\nabla_j\phi, \\ \nabla_\tau(\nabla_i\nabla_j\phi) &= \nabla_i\nabla_j\frac{\partial}{\partial\tau}\phi - (\frac{\partial}{\partial\tau}\Gamma_{ij}^{p})\nabla_p\phi +^N Rm(\frac{\partial}{\partial\tau}\phi, \nabla_i\phi)\nabla_j\phi, \end{aligned}$$

where Γ_{ij}^{p} denotes the Christoffel symbols, ∇_τ is a covariant time derivative induced by $\frac{\partial}{\partial\tau}$ and Δ_L denotes the Lichnerowicz Laplacian that is defined on symmetric two-tensors t_{ij} by

$$\Delta_L t_{ij} := \Delta t_{ij} + 2R_{ipjq}t_{pq} - R_{ip}t_{pj} - R_{jp}t_{pi}.$$

Lemma 3.2. *Let $(g(\tau), \phi(\tau))$ be a smooth solution of the backward harmonic Ricci flow with positive coupling constant α, define*

$$\mathcal{W}(g(\tau), \phi(\tau)) := \frac{\partial}{\partial \tau}(\mathrm{Sic}(\tau)) = \frac{\partial}{\partial \tau}(\mathrm{Ric}(g(\tau)) - \alpha \nabla\phi(\tau) \otimes \nabla\phi(\tau)),$$
$$\mathcal{T}(g(\tau), \phi(\tau)) := \frac{\partial}{\partial \tau}(\bar{\tau}_{g(\tau)}(\phi(\tau))),$$

then under the scaling $g \to \bar{g} := cg$ we have

$$\mathcal{W}(\bar{g}(\tau), \phi(\tau)) = c^{-1}\mathcal{W}(g(\tau), \phi(\tau)),$$
$$\mathcal{T}(\bar{g}(\tau), \phi(\tau)) = c^{-2}\mathcal{T}(g(\tau), \phi(\tau)).$$

Proof A direct calculation shows $\frac{\partial}{\partial \tau} g^{ij} = -g^{jl}g^{ik}\frac{\partial}{\partial \tau} g_{lk}$. Combined with the Lemma 3.1, our proof is completed directly. □

Theorem 3.3. *$(g(\tau), \phi(\tau))$ defined in (3.5) is a smooth ancient solution to the backward harmonic Ricci flow.*

Proof First we show for any $\tau \in [\tau_{i-1}, \tau_i]$, $(g_i(\tau), \phi_i(\tau))$ is a solution of the backward harmonic Ricci flow. Using the backward harmonic Ricci flow equation and (3.4) we know

$$\begin{aligned}\frac{\partial}{\partial \tau} g_i(\tau) &= 2(\psi^i)^* \mathrm{Sic}(g_0(b^i(\tau - \tau_{i-1})), \phi_0(b^i(\tau - \tau_{i-1})))\\ &= 2\mathrm{Sic}(b^{-i}(\psi^i)^* g_0(b^i(\tau - \tau_{i-1})), (\psi^i)^*\phi_0(b^i(\tau - \tau_{i-1})))\\ &= 2\mathrm{Sic}(g_i(\tau), \phi_i(\tau)),\\ \frac{\partial}{\partial \tau}\phi_i(\tau) &= -b^i(\psi^i)^*\bar{\tau}_{g_0(b^i(\tau-\tau_{i-1}))}\phi_0(b^i(\tau - \tau_{i-1}))\\ &= -\bar{\tau}_{b^{-i}(\psi^i)^* g_0(b^i(\tau-\tau_{i-1}))}(\psi^i)^*\phi_0(b^i(\tau - \tau_{i-1}))\\ &= -\bar{\tau}_{g_i(\tau)}(\phi_i(\tau)).\end{aligned}$$

Next we need to check the smoothness at each τ_i. Applying (3.1) we have

$$g_i(\tau_{i-1}) = b^{-i}(\psi^i)^* g_0(0) = b^{-(i-1)}(\psi^{i-1})^* g_0(1) = b^{-(i-1)}(\psi^{i-1})^* g_0(b^{i-1}(\tau_{i-1} - \tau_{i-2})) = g_{i-1}(\tau_{i-1}),$$

$$\phi_i(\tau_{i-1}) = (\psi^i)^*\phi_0(0) = (\psi^{i-1})^*\phi_0(1) = (\psi^{i-1})^*\phi_0(b^{i-1}(\tau_{i-1} - \tau_{i-2})) = \phi_{i-1}(\tau_{i-1}).$$

Let $\frac{\partial_+}{\partial \tau}$ and $\frac{\partial_-}{\partial \tau}$ stand for the right and the left derivatives, respectively, then from (3.1) and (3.4) we have

$$\frac{\partial_+}{\partial \tau} g_i(\tau_{i-1}) = (\psi^i)^* \frac{\partial_+}{\partial \tau} g_0(0) = 2(\psi^i)^*\mathrm{Sic}(g_0(0), \phi_0(0)) = 2(\psi^{i-1})^*\mathrm{Sic}(g_0(1), \phi_0(1)) = \frac{\partial_-}{\partial \tau} g_{i-1}(\tau_{i-1}),$$

$$\frac{\partial_+}{\partial \tau}\phi_i(\tau_{i-1}) = b^i(\psi^i)^* \frac{\partial_+}{\partial \tau}\phi_0(0) = b^i(\psi^i)^*\bar{\tau}_{g_0(0)}\phi_0(0)) = b^{i-1}(\psi^{i-1})^*\bar{\tau}_{g_0(1)}\phi_0(1)) = \frac{\partial_-}{\partial \tau}\phi_{i-1}(\tau_{i-1}).$$

Form Lemma 3.2 we know

$$\begin{aligned}
\frac{\partial_+^2}{\partial\tau^2}g_i(\tau_{i-1}) &= 2b^i(\psi^i)^*(\frac{\partial_+}{\partial\tau}\mathrm{Sic})(g_0(0),\phi_0(0)) = 2b^i(\psi^i)^*\mathcal{W}(g_0(0),\phi_0(0))\\
&= 2b^{i-1}(\psi^{i-1})^*\mathcal{W}(g_0(1),\phi_0(1)) = \frac{\partial_-^2}{\partial\tau^2}g_{i-1}(\tau_{i-1}),\\
\frac{\partial_+^2}{\partial\tau^2}\phi_i(\tau_{i-1}) &= b^ib^i(\psi^i)^*(\frac{\partial_+}{\partial\tau}\overline{\tau}_{g_0(0)}(\phi_0(0))) = b^ib^i(\psi^i)^*\mathcal{T}(g_0(0),\phi_0(0))\\
&= b^{i-1}b^{i-1}(\psi^{i-1})^*\mathcal{T}(g_0(1),\phi_0(1)) = \frac{\partial_-^2}{\partial\tau^2}\phi_{i-1}(\tau_{i-1}).
\end{aligned}$$

In the case $k \geq 3$, similarly we have

$$\begin{aligned}
\frac{\partial_+^k}{\partial\tau^k}g_i(\tau_{i-1}) &= \frac{\partial_-^k}{\partial\tau^k}g_{i-1}(\tau_{i-1}),\\
\frac{\partial_+^k}{\partial\tau^k}\phi_i(\tau_{i-1}) &= \frac{\partial_-^k}{\partial\tau^k}\phi_{i-1}(\tau_{i-1}).
\end{aligned}$$

From (3.2) we know $\tau_i \to \infty$ as $i \to \infty$, hence $(g(\tau), \phi(\tau))$ is a smooth ancient solution to the backward harmonic Ricci flow. □

3.2. **Locally Uniformly Type I Ancient Solution and Estimates for l Distances.** Fix a point $p_0 \in M$, for each $i \geq 0$ define

(3.6)
$$x_i = \psi^{-(i+1)}(p_0).$$

Let $\sigma : [0,1] \to M$ be a smooth curve with $\sigma(0) = p_0$ and $\sigma(1) = x_0$. Define

(3.7)
$$\sigma_i(\tau) = \psi^{-(i+1)} \circ \sigma(b^{i+1}(\tau - \tau_i)), \quad \forall \tau \in [\tau_i, \tau_{i+1}]$$

and $\gamma_i : [0, \tau_{i+1}] \to M$ as

(3.8)
$$\gamma_i(\tau) = \begin{cases} \sigma(\tau), & \tau \in [0,1],\\ \sigma_j(\tau), & \tau \in [\tau_j, \tau_{j+1}],\ 0 \leq j \leq i. \end{cases}$$

Then $\gamma_i(\tau)$ is a continuous and piecewise smooth curve with $\gamma_i(0) = p_0$ and $\gamma_i(\tau_{i+1}) = x_{i+1}$ since

$$\gamma_i(\tau_i) = \psi^{-(i+1)} \circ \sigma(0) = \psi^{-i} \circ \sigma(1) = \psi^{-i} \circ \sigma(b^i(\tau_i - \tau_{i-1})) = \gamma_{i-1}(\tau_i).$$

Claim: There exists a constant $C > 0$ independent of i such that $l(x_{i+1}, \tau_{i+1}) \leq C, \forall i$.

The proof of the claim is similar to [1] Section 3. For any $\tau \in [\tau_j, \tau_{j+1}]$, we have

$$\begin{aligned}
&S(\sigma_j(\tau), \tau) = S_{g_{j+1}(\tau)}(\sigma_j(\tau))\\
=&S_{b^{-(j+1)}(\psi^{j+1})^*g_0(b^{j+1}(\tau-\tau_j))}(\psi^{-(j+1)} \circ \sigma(b^{j+1}(\tau - \tau_j)))\\
=&b^{j+1}S_{g_0(b^{j+1}(\tau-\tau_j))}(\sigma(b^{j+1}(\tau - \tau_j)))\\
\leq&C_1b^{j+1} \leq \tau_{j+1}b^{j+1}\frac{C_1}{\tau} \leq \frac{B}{\tau},
\end{aligned}$$

where we used (3.3) in the last line and $C_1 = \max_{\tau\in[0,1]} S_{g_0(\tau)}(\sigma(\tau))$. Then we have

$$\begin{aligned}\mathcal{L}(\gamma_i) &= \mathcal{L}(\sigma) + \sum_{j=0}^{i} \int_{\tau_j}^{\tau_{j+1}} \sqrt{\tau}\left(S(\sigma_j(\tau),\tau) + \left|\frac{\partial\sigma_j(\tau)}{\partial\tau}\right|^2_{g_{j+1}(\tau)}\right) d\tau \\ &\leq C_2 + \sum_{j=0}^{i} \int_{\tau_j}^{\tau_{j+1}} \sqrt{\tau}(\frac{B}{\tau} + C_3 b^{j+1}) d\tau \\ &\leq C_2 + C_4 \sum_{j=0}^{i} b^{-\frac{j+1}{2}},\end{aligned}$$

where we use C_2, a constant independent of i, to represent $\mathcal{L}(\sigma)$, $C_3 = \max_{\tau\in[0,1]} |\frac{\partial\sigma(\tau)}{\partial\tau}|^2_{g_0(\tau)}$. Hence

$$l(x_{i+1},\tau_{i+1}) \leq \frac{\mathcal{L}(\gamma_i)}{2\sqrt{\tau_{i+1}}} \leq C, \tag{3.9}$$

where C is a constant independent of i.

Definition 3.4. *Let $(g(\tau),\phi(\tau))_{\tau\in[0,\infty)}$ be an ancient solution to the backward harmonic Ricci flow. Fix $p_0 \in M$ and let $\{(x_i,\tau_i)\}_{i=1}^{\infty} \subset M\times(0,\infty)$ be a sequence of space-time points with $\tau_i \to \infty$. $(g(\tau),\phi(\tau))$ is called* **locally uniformly Type I along the space-time sequence** $\{(x_i,\tau_i)\}_{i=1}^{\infty}$ *if the following hold*

$$\limsup_{i\to\infty} l(x_i,\tau_i) < \infty, \tag{3.10}$$

where l is the reduced distance function based at $(p_0, 0)$. For all $A > 0$ and $i_0 \in \mathbb{N}$, depending on A, it holds that

$$\sup_{B_{g(\tau_i)}(x_i,\sqrt{\tau_i}r)\times[\tau_i,2\tau_i]} |Rm_{g(\tau)}|_{g(\tau)} \leq \frac{C(r)}{\tau_i} \ for\ all\ i \geq i_0\ and\ r \leq A, \tag{3.11}$$

where $C : (0,\infty) \to (0,\infty)$ is a positive function.

$$\mathrm{Sic}_{g(\tau)} \geq -K(\tau)g(\tau)\ for\ all\ \tau \in [0,\infty), \tag{3.12}$$

where $K : [0,\infty) \to (0,\infty)$ is a positive function.

$$\lim_{i\to\infty} \inf((\tau_i)^{-\frac{1}{2}} \mathrm{inj}(g(\tau_i), x_i)) > 0, \tag{3.13}$$

where $\mathrm{inj}(g,x)$ is the injectivity radius of the Riemannian metric g at x.

Theorem 3.5. *Let $(g_0(\tau),\phi_0(\tau))_{\tau\in[0,1]}$ be a shrinking breather as defined in (3.1) with $\mathrm{Sic} \geq -kg$, where k is a nonnegative constant, and let $(g(\tau),\phi(\tau))_{\tau\in[0,\infty]}$ be the ancient solution as defined in (3.5). Let $p_0 \in M$ be a fixed point, $x_i = \psi^{-(i+1)}(p_0)$ and $\tau_i = \sum_{j=0}^{i} b^{-j}$. Then $(g(\tau),\phi(\tau))_{\tau\in[0,\infty]}$ is a locally uniformly Type I ancient solution along the space-time sequence $\{(x_i,\tau_i)\}_{i=1}^{\infty}$.*

Proof From (3.9) we know (3.10) is true. (3.11) is true because for any $\tau \in [\tau_i, \tau_{i+1}]$, a direct computation from (3.3), (3.4) and (3.6) shows that there exists $i_0 \in \mathbb{N}$, depending on $A > 0$, for all $i \geq i_0$ and $r \leq A$ we have

$$|Rm_{g_{i+1}(\tau)}|_{g_{i+1}(\tau)} \leq \frac{1}{\tau_i} \max_{\tau\in[0,1]} |Rm_{g_0(\tau)}|_{g_0(\tau)},$$

and

$$B_{g_{i+1}(\tau_i)}(x_i, \sqrt{\tau_i} r) \subset B_{g_0(0)}(p_0, r).$$

(3.12) is true because for any $\tau \in [\tau_i, \tau_{i+1}]$, from (3.4) we have

$$\begin{aligned}\mathrm{Sic}_{g_{i+1}(\tau)} &= (\psi^{i+1})^* \mathrm{Sic}_{b^{-(i+1)}g_0(b^{i+1}(\tau-\tau_i))} = (\psi^{i+1})^* \mathrm{Sic}_{g_0(b^{i+1}(\tau-\tau_i))} \\ &\geq -kb^{i+1}b^{-(i+1)}(\psi^{i+1})^* g_0(b^{i+1}(\tau-\tau_i)) \geq -K(\tau) g_{i+1}(\tau).\end{aligned}$$

(3.13) is true because for any $\tau \in [\tau_i, \tau_{i+1}]$, from (3.4) and (3.6) we have

$$(\tau_i)^{-\frac{1}{2}} \mathrm{inj}(g_{i+1}(\tau_i), x_i) > \mathrm{inj}(g_0(0), p_0).$$

□

Let $(g(\tau), \phi(\tau))_{\tau\in[0,\infty)}$, $p_0 \in M$ and $\{(x_i, \tau_i)\}_{i=1}^{\infty} \subset M \times (0, \infty)$ be as in the Definition 3.4. Now we consider the following rescaled backward harmonic Ricci flows:

$$(M, g_i(\tau), \phi_i(\tau), (x_i, 1), l_i(\tau))_{\tau\in[1,2]}, \tag{3.14}$$

where $g_i(\tau) = \tau_i^{-1} g(\tau\tau_i)$, $\phi_i(\tau) = \phi(\tau\tau_i)$ and

$$l_i(\tau) = l(\tau\tau_i). \tag{3.15}$$

Indeed, from Lemma 2.2 we know that l_i is the reduced distance from $(p_0, 0)$ of the flow $(g_i(\tau), \phi_i(\tau))$. From the Definition 3.4, (3.10)-(3.13) becomes the following under the scaled flows (3.14).

$$l_i(x_i, 1) = l(x_i, \tau_i) \leq C \ for\ all\ i \geq 0, \tag{3.16}$$

where C is a constant independent of i.

$$|Rm_{g_i(\tau)}|_{g_i(\tau)} \leq C_0(r)\ on\ B_{g_i(1)}(x_i, r) \times [1, 2]\ for\ all\ i\ large\ enough, \tag{3.17}$$

where $C_0 : (0, \infty) \to (0, \infty)$ is a positive function.

$$\mathrm{Sic}_{g_i(\tau)} \geq -K_i g_i(\tau)\ for\ all\ i\ and\ for\ all\ \tau \in [1, 2], \tag{3.18}$$

where $\{K_i\}_{i=1}^{\infty}$ is a positive sequence.

$$\lim_{i\to\infty} \inf(\mathrm{inj}(g_i(1), x_i)) > 0. \tag{3.19}$$

We have the following estimates for both the l_i-distance and its gradient with respect to the flow $(g_i(\tau), \phi_i(\tau))$ by utilizing (3.16) and (3.17).

Theorem 3.6. *Consider the sequence of backward harmonic Ricci flows* (3.14) *satisfying* (3.16) *and* (3.17). *Then for any $\epsilon \in (0, \frac{1}{4})$, there exists a positive function $C(\epsilon, \cdot) : (0, \infty) \to (0, \infty)$, such that for any $r > 0$ it holds that*
(1) $0 \le l_i(x, \tau) \le C(\epsilon, r)$ for $(x, \tau) \in B_{g_i(1)}(x_i, r) \times [1 + \epsilon, 2]$ for all i large enough,
(2) $|\frac{\partial l_i}{\partial \tau}(x, \tau)|_{g_i(\tau)} + |\nabla l_i(x, \tau)|_{g_i(\tau)} \le C(\epsilon, r)$ for $(x, \tau) \in (B_{g_i(1)}(x_i, r) \times [1 + \epsilon, 2 - \epsilon]) \cap (\cup_{\tau \in [1+\epsilon, 2-\epsilon]} \Omega(\tau) \times \{\tau\})$ and for all i large enough.

Proof The positivity of l_i-distance follows from the positivity of S on ancient solutions, see [8] Theorem 1.4. The upper bound of l_i-distance follows from the locally uniform curvature bound (3.16) and (3.17). The second result follows from Theorem 2.4 and Lemma 2.5. One can refer to [1] Proposition 5.1 for more details. □

4. Asymptotic Gradient Shrinking Soliton

In this section, we will prove that a locally uniformly Type I ancient solution always has an asymptotic shrinking gradient harmonic Ricci soliton. This lead to the proof of the main result Theorem 1.6.

Theorem 4.1. *Let $(g(\tau), \phi(\tau))_{\tau \in [0,\infty)}$ be an ancient solution to the backward harmonic Ricci flow. Let $p_0 \in M$ and $\{(x_i, \tau_i)\}_{i=1}^{\infty}$ be a space time sequence such that $\tau_i \to \infty$. Suppose $(g(\tau), \phi(\tau))$ is locally uniformly Type I along $\{(x_i, \tau_i)\}_{i=1}^{\infty}$. Let $g_i(\tau) = \tau_i^{-1} g(\tau\tau_i)$, $\phi_i(\tau) = \phi(\tau\tau_i)$, $l_i(\tau) = l(\tau\tau_i)$, and l is the reduced distance based at $(p_0, 0)$. Then the sequence of tuples*

$$\{(M, g_i(\tau), \phi_i(\tau), x_i, l_i)_{\tau \in [1,2]}\}_{i=1}^{\infty}, \tag{4.1}$$

after passing to a subsequence, converges to the canonical form of a shrinking gradient harmonic Ricci soliton

$$(M_\infty, g_\infty(\tau), \phi_\infty(\tau), x_\infty, l_\infty)_{\tau \in (1,2)} \tag{4.2}$$

with l_∞ being the potential function, that is

$$\begin{cases} \mathrm{Sic}_{g_\infty} + \mathrm{Hess}(l_\infty) - \dfrac{1}{2\tau} g_\infty = 0, \\ \bar{\tau}_{g_\infty} \phi_\infty - \langle \nabla \phi_\infty, \nabla l_\infty \rangle = 0, \end{cases} \tag{4.3}$$

for all $\tau \in (1, 2)$. Here $(M, g_i(\tau), \phi_i(\tau), x_i)_{\tau \in [1,2]}$ converge in the smooth Cheeger-Gromov sense and the functions l_i converge in the weak $*W^{1,2}_{loc}$ sense.

Remark 4.2. *For simplicity, we still write $l_i \circ \Phi_i$ as l_i. Here $l_i \circ \Phi_i$ are functions on $M_\infty \times [1, 2]$ and Φ_i was introduced in Definition 2.7.*

Proof Using (3.17) and (3.19), the Cheeger-Gromov convergence of the sequence of backward harmonic Ricci flows

$$\{(M, g_i(\tau), \phi_i(\tau), (x_i, 1))_{\tau \in [1,2]}\}_{i=1}^{\infty}$$

follows from Theorem 2.8. A direct consequence of Theorem 3.6 shows that after passing to a subsequence, $\{l_i\}_{i=1}^{\infty}$ converges in the weak $*W^{1,2}_{loc}(M_\infty \times (1,2))$ sense to a locally Lipschitz function l_∞. By passing to a further subsequence, we may also assume that $(4\pi\tau)^{-\frac{n}{2}}e^{-l_i} \to (4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty}$ in the same sense.

Next we will use Theorem 2.9 to prove that (4.2) is a shrinking gradient harmonic Ricci soliton with l_∞ being the potential function, the core argument is to show that $(4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty}$ is a smooth solution to the conjugate heat equation $\Box^* u = 0$ on $M_\infty \times (1,2)$. Since the method of proof is similar to the case of Ricci flow in [1] Proposition 6.4, we only give a brief sketh here.

Step 1. From (2.12), we have

$$\int_M (-2\nabla l_i \cdot \nabla\eta - |\nabla l_i|^2_{g_i(\tau)}\eta)dg_i(\tau) \leq -\int_M (S_{(g_i,\phi_i)} + \frac{l_i - n}{\tau})\eta dg_i(\tau)$$
$$\leq \frac{n}{\tau}\int_M \eta dg_i(\tau)$$

for all $\tau \in [1,2]$ and any smooth nonnegative function η compactly supported on M with $\sup\eta \leq 1$, here we used the facts that $S_{(g_i,\phi_i)} \geq 0$ and $l_i \geq 0$ introduced in Theorem 3.6. By (3.18) we can choose a time-independent cut-off function, in combination with the upper bound of the reduced volume introduced in Theorem 2.6, we obtain that there exists a constant C_0 independent of i such that

$$\int_M |\nabla l_i|^2_{g_i(\tau)}(4\pi\tau)^{-\frac{n}{2}}e^{-l_i}dg_i(\tau) \leq C_0 \tag{4.4}$$

for all $\tau \in [1,2]$ and for all i.

Step 2. Replacing η by $\eta(4\pi\tau)^{-\frac{n}{2}}e^{-l_i}$ in (2.11), we have

$$\int_{\tau_1}^{\tau_2}\int_M (\eta\frac{\partial}{\partial\tau}((4\pi\tau)^{-\frac{n}{2}}e^{-l_i}) + \langle\nabla\eta, \nabla((4\pi\tau)^{-\frac{n}{2}}e^{-l_i})\rangle$$
$$+\eta S_{(g_i,\phi_i)}(4\pi\tau)^{-\frac{n}{2}}e^{-l_i})dg_i(\tau)d\tau \leq 0,$$

where η is a smooth nonnegative function and compactly supported on $M \times [\tau_1, \tau_2]$ with $\sup\eta \leq 1$. Taking the cut-off function mentioned in step 1 and using the upper bound of the reduced volume and (4.4) we obtain

$$\begin{aligned}\mathcal{V}_i(\tau_2) - \mathcal{V}_i(\tau_1) \leq \int_{\tau_1}^{\tau_2}\int_M &(\eta\frac{\partial}{\partial\tau}((4\pi\tau)^{-\frac{n}{2}}e^{-l_i}) + \langle\nabla\eta, \nabla((4\pi\tau)^{-\frac{n}{2}}e^{-l_i})\rangle \\ &+\eta S_{(g_i,\phi_i)}(4\pi\tau)^{-\frac{n}{2}}e^{-l_i})dg_i(\tau)d\tau \leq 0.\end{aligned} \tag{4.5}$$

Step 3. By the monotonicity property of the reduced volume introduced in Theorem 2.6, we have

$$\lim_{i\to\infty}\mathcal{V}_i(\tau) = \mathcal{V}_\infty$$

for all $\tau \in [1,2]$, here $\mathcal{V}_\infty \in [0,1]$ is a constant. Let us fix $1 < \tau_1 < \tau_2 < 2$ and an arbitrary smooth nonnegative function η compactly supported on $M_\infty \times [\tau_1, \tau_2]$ with $\sup\eta \leq 1$, then $\eta \circ \Phi_i^{-1}$ satisfies (4.5), take a limit for (4.5), we obtain

$$\int_{\tau_1}^{\tau_2}\int_{M_\infty}(\eta\frac{\partial}{\partial\tau}((4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty})+\langle\nabla\eta,\nabla((4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty})\rangle$$
$$+\eta S_{(g_\infty,\phi_\infty)}(4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty})dg_\infty(\tau)d\tau=0.$$

Then $(4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty}$ is a weak solution to the conjugate heat equation $\frac{\partial u}{\partial\tau}-\Delta u+Su=0$ on $M_\infty\times(1,2)$. From the standard local regularity theory of linear parabolic equations, l_∞ is smooth and $(4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty}$ is a classical solution to the conjugate heat equation.

Now, let us define

$$u:=(4\pi\tau)^{-\frac{n}{2}}e^{-l_\infty},$$
$$v:=(\tau(2\Delta l_\infty-|\nabla l_\infty|^2+S_{(g_\infty,\phi_\infty)})+l_\infty-n)u.$$

Then $\Box^* u=0$ generates

$$-\frac{\partial l_\infty}{\partial\tau}-|\nabla l_\infty|^2+S_{(g_\infty,\phi_\infty)}+\Delta l_\infty-\frac{n}{2\tau}=0 \tag{4.6}$$

on $M_\infty\times(1,2)$. On the other hand, from (2.8) we have

$$2\frac{\partial l_\infty}{\partial\tau}+|\nabla l_\infty|^2-S_{(g_\infty,\phi_\infty)}+\frac{l_\infty}{\tau}=0, \tag{4.7}$$

on $M_\infty\times(1,2)$, here l_i satisfies (4.7) in the sense of distribution. Combining (4.6) and (4.7) we obtain

$$v\equiv 0.$$

Using Theorem 2.9, we prove that a locally uniformly Type I ancient solution always has an asymptotic shrinking gradient harmonic Ricci soliton. □

Proof of Theorem 1.6 On the one hand, (3.4) implies that the rescaled flows $(M,g_i(\tau),\phi_i(\tau),(x_i,1))_{\tau\in[1,\frac{\tau_{i+1}}{\tau_i}]}$ differ the original shrinking breather $(M,g_0(\tau-1),\phi_0(\tau-1),(p_0,1))_{\tau\in[1,2]}$ only by a scaling factor $\tau_i^{-1}b^{-(i+1)}$ and a base-point-preserving diffeomorphism $\psi^{-(i+1)}$ bacause

$$g_i(\tau)=\tau_i^{-1}g(\tau\tau_i)=\tau_i^{-1}b^{-(i+1)}(\psi^{i+1})^*g_0(b^{(i+1)}(\tau_i\tau-\tau_i)),$$
$$\phi_i(\tau)=\phi(\tau\tau_i)=(\psi^{i+1})^*\phi_0(b^{i+1}(\tau_i\tau-\tau_i)).$$

On the other hand, Theorem 4.1 implies

$$(M,g_i(\tau),\phi_i(\tau),x_i,l_i)_{\tau\in[1,2]}\to(M_\infty,g_\infty(\tau),\phi_\infty(\tau),x_\infty,l_\infty)_{\tau\in(1,2)},$$

where the latter is the canonical form of a shrinking gradient harmonic Ricci soliton. Since $\tau_i^{-1}b^{-(i+1)}\to 1$ and $\frac{\tau_{i+1}}{\tau_i}\to b^{-1}$, we have

$$(M,g_0(\tau-1),\phi_0(\tau-1),(p_0,1))_{\tau\in[1,b^*]}\cong(M_\infty,g_\infty(\tau),\phi_\infty(\tau),(x_\infty,1))_{\tau\in[1,b^*]},$$

where $b^*=\max\{2,b^{-1}\}$, this completes the proof.

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