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**What Lie algebras can tell us about
Jordan algebras**

by




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What Lie algebras can tell us about Jordan algebras

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Abstract

Inspired by Kirillov's theory of coadjoint orbits, we develop a structure theory for finite dimensional Jordan algebras. Given a Jordan algebra \mathcal{J} , we define a generalized distribution $\mathcal{H}^{\mathcal{J}}$ on its dual space \mathcal{J}^* which is canonically determined by the Jordan product in \mathcal{J} , is invariant under the action of what we call the *structure group of \mathcal{J}* , and carries a naturally-defined pseudo-Riemannian bilinear form \mathcal{G}_{ξ} at each point. We then turn to the case of *positive* Jordan algebras and classify the orbits of \mathcal{J}^* under the structure group action. We show that the only orbits which are also leaves of $\mathcal{H}^{\mathcal{J}}$ are those in the closure of the cone of squares or its negative, and these are the only orbits where this pseudo-Riemannian bilinear form determines a Riemannian metric tensor \mathcal{G} .

We discuss applications of our construction to both classical and quantum information geometry by showing that, for appropriate choices of \mathcal{J} , the Riemannian metric tensor \mathcal{G} coincides with the Fisher-Rao metric on non-normalized probability distributions on a finite sample space, or with the Bures-Helstrom metric for non-normalized, faithful quantum states of a finite-level quantum system.

Keywords: Jordan algebras; Lie algebras; Kirillov orbit method; generalized distributions; Peirce decomposition; Fisher-Rao metric; Bures-Helstrom metric.

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1 Introduction

Our guiding question is to what extent we can develop a structure theory for Jordan algebras that is analogous to that of Lie algebras, at least in finite dimensions. The arguably most important finite-dimensional cases are matrix algebras, with the anti-commutator as a product for Jordan algebras, and the commutator for Lie algebras. In the latter case, we have the fundamental observation of Kirillov [23, 24] that a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ of the Lie group G carries a natural homogeneous symplectic structure. Here, \mathfrak{g} and \mathfrak{g}^* denote the Lie algebra of G and its dual. Kirillov’s observation relates algebraic structures to differential geometry and mathematical physics in a deep and very productive way, further explored by Kostant [28], Souriau [35] and many others. This led to spectacular results in representation theory, classical and quantum mechanics, and it is closely related to geometric quantization [25].

On the other hand, Jordan algebras play an important role in the formulation of quantum theories [1, 13, 15, 22], in mathematical physics [4, 44], in color perception theory [6, 32, 33], and in quantum information theory [9]. Of course, the purely mathematical investigation of Jordan algebras and their algebraic and geometrical properties is a well-established subject [7, 16, 27, 41]. However, as far as we know, no analogue of Kirillov’s theory for Lie algebras has been investigated in the case of Jordan algebras, and the purpose of this work is precisely fill this gap.

Obviously, Lie algebras are special, not only because they come from Lie groups, but also because the structure algebra of a Lie algebra, i.e., the algebra generated by left multiplication, is a Lie algebra itself. To understand this picture from a more abstract perspective, it is useful to first consider a general, finite-dimensional algebra. Thus, we consider a finite-dimensional algebra \mathcal{A} , i.e., a finite-dimensional (real or complex) vector space with a bilinear product $\bullet : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. At this moment, no further condition, like associativity or identities of Jacobi, Jordan or any other type is assumed. We denote with \mathcal{A}^* the dual space of \mathcal{A} , and the corresponding pairing is denoted by $\langle \cdot, \cdot \rangle$. Due to the finite-dimensionality, we have the identification $\mathcal{A}^{**} \cong \mathcal{A}$, and for the tangent and cotangent spaces of \mathcal{A}^* , we then have $T_\xi \mathcal{A}^* \cong \mathcal{A}^*$ and $T_\xi^* \mathcal{A}^* \cong \mathcal{A}^{**} \cong \mathcal{A}$. We may then represent the product \bullet via

$$\langle \xi, a \bullet b \rangle \quad \text{for } \xi \in \mathcal{A}^*, a, b \in \mathcal{A}. \tag{1}$$

Furthermore, this induces a multiplication on $C^\infty(\mathcal{A}^*)$ via

$$\{f, g\}_{\mathcal{A}}(\xi) := \langle \xi, d_\xi f \bullet d_\xi g \rangle, \tag{2}$$

and for $f \in C^\infty(\mathcal{A}^*)$, we may define its \mathcal{A} -dual vector field of f as

$$(\nabla_f^{\mathcal{A}})_\xi(g) := \{f, g\}_{\mathcal{A}}(\xi). \tag{3}$$

Note that, in general, this is not a gradient (because \bullet need not be symmetric), but we use the symbol ∇ here because it satisfies many of the formal identities of gradients.

The multiplication in equation (2) is of course compatible with the product \bullet in the sense that, identifying $a \in \mathcal{A}$ with the linear functional $f_a(\xi) := \langle \xi, a \rangle$, the inclusion $\mathcal{A} \hookrightarrow C^\infty(\mathcal{A})$ is an algebra monomorphism

$$\{f_a, f_b\}_{\mathcal{A}} = f_{a \bullet b}.$$

Now, the automorphisms of \mathcal{A} , i.e., the linear isomorphisms $g : \mathcal{A} \rightarrow \mathcal{A}$ with $g(a \bullet b) = (ga) \bullet (gb)$, form a Lie group. The Lie algebra of that group consists of the derivations, i.e., the linear maps $d \in \mathfrak{gl}(\mathcal{A})$ with $d(a \bullet b) = (da) \bullet b + a \bullet (db)$. Moreover, we have the structure Lie group $G(\mathcal{A})$ whose Lie algebra is generated by left multiplications, i.e., by the maps $l_a : (b \mapsto a \bullet b) \in \mathfrak{gl}(\mathcal{A})$.

Therefore, even though we do not assume \mathcal{A} to be a Lie algebra, there is a Lie algebra that is naturally associated to \mathcal{A} , and we may hope to use its theory to gain insight about \mathcal{A} itself. This works to some extent, but a problem arises from the fact that the \mathcal{A} -dual distribution $\mathcal{H}^{\mathcal{A}}$ on \mathcal{A}^* by

$$\mathcal{H}_\xi^{\mathcal{A}} := \{(\nabla_f^{\mathcal{A}})_\xi \mid f \in C^\infty(\mathcal{A}^*)\} \subset T_\xi \mathcal{A}^*. \quad (4)$$

in general is not integrable. We recall that, in the Lie algebra case, $\mathcal{H}^{\mathcal{A}}$ is integrable, and its leaves are precisely the coadjoint orbits which carry a symplectic structure induced by equation (2). Because of this fact, for general algebras, we cannot expect a theory that is as powerful as Kirillov's theory for Lie algebras.

The theory becomes more powerful, however, and the results become stronger, if we also assume some additional structure on \mathcal{A} . Associativity already gives us some leverage, but the more specific case that we are interested in here is when \mathcal{A} is a Jordan algebra. Our strategy then is to combine this Jordan structure, and the identities resulting from it, with the Lie algebra structure that we just have identified.

Thus, we consider a finite-dimensional real Jordan algebra $\mathcal{A} = \mathcal{J}$. In this case, following Koecher's work [27, chapter IV], we can also define an extended structure Lie algebra $\hat{\mathfrak{g}}(\mathcal{J})$ that maps surjectively onto $\mathfrak{g}(\mathcal{J})$. This algebra is the direct sum $\mathcal{D}er_0(\mathcal{J}) \oplus \mathcal{J}$ of the inner derivations (those generated by left multiplications l_x with algebra elements x) and \mathcal{J} itself. The Lie bracket on $\mathcal{D}er_0(\mathcal{J})$ is the commutator, while, for $x, y \in \mathcal{J}$, it is $[x, y] := [l_x, l_y] \in \mathcal{D}er_0(\mathcal{J})$, and, for $d \in \mathcal{D}er_0(\mathcal{J})$ and $x \in \mathcal{J}$, it is $[d, x] = -[x, d] := d(x) \in \mathcal{J}$. Thus, putting $\mathfrak{k} = \mathcal{D}er_0(\mathcal{J})$, $\mathfrak{m}_{\mathcal{J}} = \mathcal{J}$, we have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}_{\mathcal{J}}] \subset \mathfrak{m}_{\mathcal{J}}$, $[\mathfrak{m}_{\mathcal{J}}, \mathfrak{m}_{\mathcal{J}}] = \mathfrak{k}$, i.e., we have a transvective symmetric pair (see Def. 3) which provides us with further structure to work with.

The generalized distribution $\mathcal{H}^{\mathcal{J}}$ on \mathcal{J}^* is still not integrable in general. However, in the case of Jordan algebras, the bilinear form \mathcal{G}_ξ induced by equation (1) on $\mathcal{H}_\xi^{\mathcal{J}}$ is symmetric, and therefore it defines a pseudo-Riemannian metric on the $\mathfrak{m}_{\mathcal{J}}$ -regular part of each $G(\mathcal{J})$ -orbit $\mathcal{O}_{\mathfrak{m}_{\mathcal{J}}}^{reg} \subset \mathcal{J}^*$ (see Def. 2).

According to [27, p. 59], on \mathcal{J} there is the symmetric, bilinear form $\tau(x, y) := \text{tr } l_{\{x, y\}}$, which is also associative with respect to the Jordan product, and we can use it to decompose \mathcal{J} . Specifically, in the case of *positive Jordan algebras*, we can derive further properties from the decomposition, and, being τ positive definite, there is a canonical identification $\mathcal{J} \cong \mathcal{J}^*$. Each $\xi \in \mathcal{J}^*$ has a *spectral decomposition* associating with ξ its *spectral coefficients* $(\lambda_i)_{i=1}^r \in \mathbb{R}^r$, where r denotes the rank of \mathcal{J} . The pair (n_+, n_-) counting the number of positive and negative spectral coefficients is called the *spectral signature* of ξ . We then show the following:

Theorem A (cf. Theorem 1) *If \mathcal{J} is a positive simple real Jordan algebra, then the orbits of the structure group $G(\mathcal{J})$ consist of all elements with the same spectral signature.*

We also characterize the *regular* points (i.e., those where the generalized distribution $\mathcal{H}^{\mathcal{J}}$ is integrable) in such a $G(\mathcal{J})$ -orbit in Theorem 2 and describe the pseudo-Riemannian metric \mathcal{G}_ξ at each regular point $\xi \in \mathcal{J}^*$ in Proposition 5. Specifically, let $\Omega_{\mathcal{J}}$ denote the *cone of squares of \mathcal{J}* , i.e., the interior of the set $\{x^2 \mid x \in \mathcal{J}\}$. Then $\Omega_{\mathcal{J}}$ is the $G(\mathcal{J})$ -orbit of the identity $\mathbf{1}_{\mathcal{J}}$. The characterizations in Theorem 2, Proposition 1, and Proposition 5 lead to the following remarkable description:

Theorem B *Let \mathcal{J} be a positive simple real Jordan algebra. Then all points of a $G(\mathcal{J})$ -orbit $\mathcal{O} \subset \mathcal{J}^*$ are $\mathfrak{m}_{\mathcal{J}}$ -regular iff $\mathcal{O} \subset \overline{\Omega}_{\mathcal{J}}$ or $\mathcal{O} \subset -\overline{\Omega}_{\mathcal{J}}$. The form \mathcal{G} on \mathcal{O} is positive definite in the first and negative definite in the second case, thus defining a Riemannian metric on \mathcal{O} which is invariant with respect to the action of the automorphism group of \mathcal{J} .*

For all regular $\xi \notin \pm\overline{\Omega}_{\mathcal{J}}$ the form \mathcal{G}_ξ is indefinite, so the definiteness of \mathcal{G}_ξ gives a new characterization of $\Omega_{\mathcal{J}}$.

We provide descriptions of the orbits \mathcal{O} and the metric \mathcal{G} for the standard examples of positive simple real algebras. Moreover, the above results easily generalize to the case of non-simple positive Jordan algebras, as these algebras are direct sums of positive simple Jordan algebras.

One example which is particularly interesting in this context is the *associative real Jordan algebra* $\mathcal{J} := \mathbb{R}^n$ whose algebraic operations are defined in a component-wise way. Then, $\Omega_{\mathcal{J}} = \mathbb{R}_+^n$ is the first orthant, which may be regarded as the space of positive finite measures on a sample space \mathcal{X}_n with n elements. What we show is that, on $\Omega_{\mathcal{J}}$, the metric \mathcal{G} is such that its pullback to the submanifold of strictly positive probability distributions on \mathcal{X}_n (i.e., the open interior of the unit simplex inside \mathbb{R}_+^n) coincides with the *Fisher-Rao-metric tensor* which naturally occurs in Classical Information Geometry [2, 3]. This instance shows that we may look at the non-normalized Fisher-Rao metric tensor on $\Omega_{\mathcal{J}} = \mathbb{R}_+^n$ as the analogue of the homogeneous symplectic form on co-adjoint orbits in the case of Lie algebras.

A similar suggestive instance also manifests itself with respect to Quantum Information Geometry. Indeed, the space of observables of a finite-level quantum system may be identified with the Jordan algebra $\mathcal{J} = M_n^{sa}(\mathbb{C})$, so that the automorphism group of \mathcal{J} is the unitary group. Then, $\Omega_{\mathcal{J}}$ can be identified with the space of (non-normalized) faithful quantum states [9, 17], and the metric \mathcal{G} is such that its pullback to the submanifold of faithful quantum states, determined by the condition $\text{Tr}(A) = 1$, coincides with the so-called Bures-Helstrom metric tensor [5, 10, 11, 18, 19, 20, 21, 34, 40, 42, 43] whose relevance in quantum metrology is difficult to overestimate [30, 31, 37, 38]. Also, if we consider the $\mathfrak{m}_{\mathcal{J}}$ -regular orbit $\mathcal{O} \subset \overline{\Omega}_{\mathcal{J}}$ of positive matrices of rank one, it turns out that, essentially because of its unitary invariance, \mathcal{G} is such that its pullback to the submanifold of pure quantum states, determined by the condition $\text{Tr}(A) = 1$, coincides with the Fubini-Study metric tensor [5, 12], which may be thought of as a quantum generalization of the Fisher-Rao metric tensor to the case of pure quantum states [14, 45].

It is worth noting that the above mentioned links between Jordan algebras and Classical and Quantum Information Geometry was already put forward in [8, 9], where, however, only those Jordan algebras associated with von Neumann algebras were considered, and where the complete geometrical picture described in full detail here was only hinted at.

This article is structured as follows. In section 2 we set up our notation and recall some standard results on generalized distributions and group actions. The discussion of the structure

group and the generalized distributions on arbitrary algebras \mathcal{A} follows in Section 3, while in Section 4 we apply these results to Jordan algebras, in particular showing Theorems A and B.

2 Preliminaries

2.1 Notational conventions

For a finite-dimensional (real or complex) vector space V , we write

$$\begin{aligned}\mathrm{Gl}(V) &:= \{\text{linear isomorphisms } g : V \rightarrow V\}, \\ \mathfrak{gl}(V) &:= \{\text{linear endomorphisms } x : V \rightarrow V\}.\end{aligned}$$

The choice of a basis $(e_i)_{i=1}^n$ of V identifies $\mathrm{Gl}(V)$ and $\mathfrak{gl}(V)$ with $\mathrm{Gl}(n, \mathbb{K})$ and $\mathfrak{gl}(n, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}), respectively, motivating this notation. This means that $\mathrm{Gl}(V)$ is a Lie group with Lie algebra $\mathfrak{gl}(V)$.

We denote by V^* the dual space of V . For finite-dimensional real vector spaces V , W and U , we define the contraction map

$$\langle \cdot, \cdot \rangle_V : V^* \otimes V \otimes U \longrightarrow U, \quad \langle \alpha, v \otimes u \rangle_V := \alpha(v)u. \quad (5)$$

We shall often simply write $\langle \cdot, \cdot \rangle$ without the subscript if this causes no ambiguity. Equation (5) includes the case where $U = \mathbb{R}$, i.e., the evaluation map

$$\langle \cdot, \cdot \rangle_V : V^* \otimes V \longrightarrow \mathbb{R}, \quad \langle \alpha, v \rangle_V := \alpha(v). \quad (6)$$

Denoting with $S^2(V)$ and $\Lambda^2 V$, respectively, the symmetric and antisymmetric part of $V \otimes V$, and setting $U := V$ in equation (5), the restriction of $\langle \cdot, \cdot \rangle_V$ to either of the spaces $V^* \otimes V \otimes V$ or $V^* \otimes S^2(V)$ or $V^* \otimes \Lambda^2 V$ is denoted by \lrcorner . Specifically, we have

$$\begin{cases} \theta \lrcorner (v \otimes w) := \langle \theta, v \rangle_V w \\ \theta \lrcorner (v \circ w) := \frac{1}{2} (\langle \theta, v \rangle_V w + \langle \theta, w \rangle_V v) \\ \theta \lrcorner (v \wedge w) := \frac{1}{2} (\langle \theta, v \rangle_V w - \langle \theta, w \rangle_V v) \end{cases} . \quad (7)$$

All these definitions extend to tensor products as well, i.e., we have

$$\begin{aligned} \langle \cdot, \cdot \rangle_{V \otimes W} : (V \otimes W)^* \otimes (V \otimes W \otimes U) &\longrightarrow U, \\ \langle \alpha \otimes \theta, v \otimes w \otimes u \rangle_{V \otimes W} &:= \langle \alpha, v \rangle_V \langle \theta, w \rangle_W u, \end{aligned} \quad (8)$$

again including $U = \mathbb{R}$ as a special case. In particular, the pairing for symmetric tensors is given by

$$\langle \alpha \circ \beta, v \circ w \rangle_{S^2(V)} = \frac{1}{2} (\langle \alpha, v \rangle_V \langle \beta, w \rangle_V + \langle \beta, v \rangle_V \langle \alpha, w \rangle_V), \quad (9)$$

as is easily verified from equation (8) and the definition of the symmetric product $\alpha \circ \beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$ and $v \circ w = \frac{1}{2}(v \otimes w + w \otimes v)$.

If $\phi : V \rightarrow W$ is a linear map, its dual map is denoted by

$$\phi^* : W^* \longrightarrow V^*, \quad \langle \theta, \phi(v) \rangle_W = \langle \phi^* \theta, v \rangle_V. \quad (10)$$

The map induced by ϕ on the tensor algebras of V and W , respectively, also will be denoted by ϕ , i.e.,

$$\phi : \bigotimes^k V \longrightarrow \bigotimes^k W, \quad \phi(v_1 \otimes \cdots \otimes v_k) := \phi(v_1) \otimes \cdots \otimes \phi(v_k). \quad (11)$$

In particular, this applies to the subspaces $S^k(V) \subset \bigotimes^k V$ and $S^k(W) \subset \bigotimes^k W$:

$$\phi : S^k V \longrightarrow S^k W, \quad \phi(v_1 \circ \cdots \circ v_k) := \phi(v_1) \circ \cdots \circ \phi(v_k). \quad (12)$$

Finally, if we define for $\rho \in S^2(V)$ the subspace $\mathcal{D}(\rho) := \{\theta \lrcorner \rho \mid \theta \in V^*\} \subset V$, then

$$\rho \in S^2(\mathcal{D}(\rho)) \subset S^2(V), \quad \text{and} \quad \mathcal{D}(\phi(\rho)) \subset \phi(\mathcal{D}(\rho)). \quad (13)$$

Indeed, picking a basis $(e_i)_{i=1}^n$ of V with dual basis $(e^i)_{i=1}^n$ such that $\mathcal{D}(\rho)$ is spanned by $(e_\nu)_{\nu=1}^m$, then $e^i \lrcorner \rho = 0$ for $i > m$, whence

$$\rho = c^{\nu\mu} e_\nu \circ e_\mu \in S^2(\mathcal{D}(\rho)), \quad \phi(\rho) = c^{\nu\mu} \phi(e_\nu) \circ \phi(e_\mu) \in S^2(\phi(\mathcal{D}(\rho))),$$

so that equation (13) follows.

2.2 Generalized distributions

Let M be a finite-dimensional, real smooth manifold. A *generalized distribution on M* is a family $\mathcal{D} = (\mathcal{D}_p)_{p \in M}$ of subspaces $\mathcal{D}_p \subset T_p M$. We let $\Gamma(\mathcal{D})$ be the set of vector fields X on M with $X_p \in \mathcal{D}_p$ for all p , and we call \mathcal{D} *smooth* if for each $v \in \mathcal{D}_p$ there is a vector field $X \in \Gamma(\mathcal{D})$ with $X_p = v$.

Given a smooth generalized distribution \mathcal{D} , we define the *Frobenius tensor \mathcal{F}_p* at $p \in M$ as

$$\mathcal{F}_p : \Lambda^2 \mathcal{D}_p \longrightarrow T_p M / \mathcal{D}_p, \quad (X_p, Y_p) \longmapsto [X, Y]_p \pmod{\mathcal{D}_p}$$

for $X, Y \in \Gamma(\mathcal{D})$. Since $[X, fY]_p = f(p)[X, Y]_p + (Xf)_p Y_p = f(p)[X, Y]_p \pmod{\mathcal{D}_p}$, it follows that $\mathcal{F}_p(X, Y)$ depends on X_p and Y_p only, i.e., $\mathcal{F} = (\mathcal{F}_p)_{p \in M}$ is a well defined tensor field. We also define the generalized distribution

$$[\mathcal{D}, \mathcal{D}]_p := \mathcal{D}_p + \{[X, Y]_p \mid X, Y \in \Gamma(\mathcal{D})\}, \quad (14)$$

so that the image of \mathcal{F}_p is $[\mathcal{D}, \mathcal{D}]_p / \mathcal{D}_p$.

Definition 1. We call a smooth generalized distribution \mathcal{D} *involutive* at $p \in M$, if $\mathcal{F}_p = 0$ or, equivalently, if $[\mathcal{D}, \mathcal{D}]_p = \mathcal{D}_p$, and we call it *involutive*, if this holds for every p .

Furthermore, an (immersed) submanifold $N \subset M$ with $T_p N = \mathcal{D}_p$ for all $p \in N$ is called an *integral leaf* of \mathcal{D} . If there is an integral leaf of \mathcal{D} containing p , then we call \mathcal{D} *integrable* at $p \in M$ and call p an *integral point* of \mathcal{D} ; if this is the case for each $p \in M$, then we call \mathcal{D} *integrable*.

Clearly, if \mathcal{D} is integrable (at p), then it is also involutive (at p); according to Frobenius' theorem, the converse of this statement holds if \mathcal{D} has constant rank. However, if the rank of \mathcal{D} is non-constant, then the converse may fail to hold [36].

2.3 G -manifolds

Let G be a finite-dimensional, real Lie group with identity element e , Lie algebra $\mathfrak{g} \cong T_e G$, and let M be a G -manifold, i.e., a finite-dimensional, real smooth manifold with a smooth left action $\pi : G \times M \rightarrow M$, $(g, p) \mapsto g \cdot p$. For $p \in M$ we define the *stabilizer of p* to be the subgroup

$$H_p := \{g \in G \mid g \cdot p = p\} \subset G \quad \text{with Lie algebra} \quad \mathfrak{h}_p \subset \mathfrak{g}.$$

Evidently, $H_{g \cdot p} = gHg^{-1}$, and $\mathfrak{h}_{g \cdot p} = Ad_g(\mathfrak{h}_p)$, so that the stabilizer on each G -orbit is unique up to conjugation.

For $X \in \mathfrak{g}$, the *action field of X* is the vector field X_* on M , given by

$$(X_*)_p := \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot p. \quad (15)$$

We define the *orbit distribution on M* by

$$\mathcal{D}^{\mathfrak{g}} := \{(X_*)_p \mid X \in \mathfrak{g}\}. \quad (16)$$

Evidently, $\mathcal{D}^{\mathfrak{g}}$ is integrable, as $\mathcal{D}_p^{\mathfrak{g}}$ is the tangent space of the G -orbit $G \cdot p \subset M$, so that the G -orbits are the integral leaves of $\mathcal{D}^{\mathfrak{g}}$. As the action field X_* and the *right invariant* vector field $(d_{e r_g} X, 0_p)_{(g,p) \in G \times M}$ on $G \times M$ are π -related, we have

$$[X, Y]_* = -[X_*, Y_*] \quad (17)$$

(confirming the involutivity of $\mathcal{D}^{\mathfrak{g}}$), and

$$X \in \mathfrak{h}_p \Leftrightarrow (X_*)_p = 0. \quad (18)$$

For any linear subspace $\mathfrak{m} \subset \mathfrak{g}$, we define the smooth generalized distribution $\mathcal{D}^{\mathfrak{m}}$ by

$$\mathcal{D}_p^{\mathfrak{m}} := \{(X_*)_p \mid X \in \mathfrak{m}\} \subset \mathcal{D}_p^{\mathfrak{g}} = T_p(G \cdot p), \quad (19)$$

and we assert that

$$(X_*)_p \in \mathcal{D}_p^{\mathfrak{m}} \Leftrightarrow X \in \mathfrak{m} + \mathfrak{h}_p. \quad (20)$$

Namely, for equation (20), observe that $(X_*)_p \in \mathcal{D}_p^{\mathfrak{m}}$ iff there is a $Y \in \mathfrak{m}$ with $(X_*)_p = (Y_*)_p$, i.e., $((X - Y)_*)_p = 0$ which by equation (18) is equivalent to $X - Y \in \mathfrak{h}_p$.

Lemma 1. *Let $\mathfrak{m} \subset \mathfrak{g}$ be a linear subspace. Then the following are equivalent:*

1. $\mathcal{D}^{\mathfrak{m}}$ is involutive at $p \in \mathcal{O}$,
2. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m} + \mathfrak{h}_p$,
3. $\mathcal{D}_p^{[\mathfrak{m}, \mathfrak{m}]} \subset \mathcal{D}_p^{\mathfrak{m}}$.

Proof. For $X, Y \in \mathfrak{m}$, $\mathcal{F}_p(X_p, Y_p) = 0$ iff $[X_*, Y_*]_p \in \mathcal{D}_p^{\mathfrak{m}}$, which, by equation (17), is the case iff $([X, Y]_*)_p \in \mathcal{D}_p^{\mathfrak{m}}$, and, by equation (20), this is the case iff $[X, Y] \in \mathfrak{m} + \mathfrak{h}_p$, showing the equivalence of the first two conditions.

The second condition is equivalent to saying that for each $X \in [\mathfrak{m}, \mathfrak{m}]$ there is a $Y \in \mathfrak{m}$ such that $X - Y \in \mathfrak{h}_p$ or, equivalently, that for each $X \in [\mathfrak{m}, \mathfrak{m}]$ there is a $Y \in \mathfrak{m}$ such that $(X_*)_p = (Y_*)_p$, and this is evidently equivalent to the third condition. \square

Definition 2. Let M be a G -manifold and $\mathfrak{m} \subset \mathfrak{g}$ a linear subspace. We call $p \in M$ an \mathfrak{m} -regular point if $\mathcal{D}_p^{\mathfrak{m}} = T_p\mathcal{O}$, where $\mathcal{O} \subset M$ is the G -orbit of p . The subset of \mathfrak{m} -regular points in \mathcal{O} is denoted by

$$\mathcal{O}_{\mathfrak{m}}^{\text{reg}} \subset \mathcal{O}.$$

As the rank of $\mathcal{D}_p^{\mathfrak{m}}$ is a lower semicontinuous function in p , $\mathcal{O}_{\mathfrak{m}}^{\text{reg}} \subset \mathcal{O}$ is open (but possibly empty). As we shall see in later sections, $\mathcal{O}_{\mathfrak{m}}^{\text{reg}}$ may be a proper subset of \mathcal{O} and is not necessarily connected.

Corollary 1. Suppose that $\mathfrak{m} \subset \mathfrak{g}$ is a linear subspace such that

$$\mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]. \quad (21)$$

Then for each $p \in M$ the following are equivalent:

1. p is involutive,
2. p is integrable,
3. p is an \mathfrak{m} -regular point.

In this case, the maximal integral leaf through p is the connected component of p in $\mathcal{O}_{\mathfrak{m}}^{\text{reg}} \subset \mathcal{O} = G \cdot p$.

Proof. By Lemma 1, p is integrable iff $\mathcal{D}_p^{[\mathfrak{m}, \mathfrak{m}]} \subset \mathcal{D}_p^{\mathfrak{m}}$, and as $\mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$, this is the case iff $\mathcal{D}_p^{\mathfrak{m}} = \mathcal{D}_p^{\mathfrak{g}}$, i.e., iff p is \mathfrak{m} -regular. It follows that any integral leaf through p must be (an open subset of) $\mathcal{O}_{\mathfrak{m}}^{\text{reg}} \subset \mathcal{O}$, whence the maximal (connected) leaf through p is its path component in $\mathcal{O}_{\mathfrak{m}}^{\text{reg}}$. \square

Definition 3. A symmetric pair is a pair $(\mathfrak{g}, \mathfrak{k})$ of Lie algebras with a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ satisfying

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \quad (22)$$

We call this pair transvective, if \mathfrak{g} is generated by \mathfrak{m} as a Lie algebra, i.e., if $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$.

Clearly, equation (22) is equivalent to saying that the involution $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ with \mathfrak{k} and \mathfrak{m} as the (+1)- and (-1)-eigenspace, respectively, is a Lie algebra automorphism.

3 Structure groups and canonical distributions on duals of algebras

In this section, we shall consider a finite-dimensional algebra \mathcal{A} , by which we simply mean a finite-dimensional (real or complex) vector space with a bilinear product $\bullet : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. We do not assume any further conditions on this multiplication such as associativity, Jacobi- or Jordan identities, but we shall later discuss the general definitions in each of these cases.

We look at \mathcal{A} as a finite-dimensional, real smooth manifold, and we may regard the multiplication \bullet as a tensor $\mathcal{R}^{\mathcal{A}} \in \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}$, and since $T_{\xi}\mathcal{A}^* \cong \mathcal{A}^*$ as \mathcal{A} is finite-dimensional, we may regard $\mathcal{R}^{\mathcal{A}}$ as a linear bivector field on \mathcal{A}^* , denoted by the same symbol

$$\mathcal{R}^{\mathcal{A}} \in \Gamma(\mathcal{A}^*, T\mathcal{A}^* \otimes T\mathcal{A}^*), \quad (\mathcal{R}^{\mathcal{A}})_{\xi}(a, b) := \langle \xi, a \bullet b \rangle \quad (23)$$

for all $a, b \in \mathcal{A} \cong \mathcal{A}^{**} \cong T^*(T_\xi^* \mathcal{A}^*)$. Therefore, there is an induced multiplication $\{\cdot, \cdot\}_{\mathcal{A}}$ on the space $C^\infty(\mathcal{A}^*)$ of real-valued, smooth functions on \mathcal{A}^* given by

$$\{f, g\}_{\mathcal{A}}(\xi) := (\mathcal{R}^{\mathcal{A}})_\xi(d_\xi f, d_\xi g) = \langle \xi, d_\xi f \bullet d_\xi g \rangle \quad (24)$$

with the canonical identification $d_\xi f, d_\xi g \in T_\xi^* \mathcal{A}^* \cong \mathcal{A}^{**} \cong \mathcal{A}$. Regarding $\mathcal{R}^{\mathcal{A}}$ as a section of $T\mathcal{A}^* \otimes T\mathcal{A}^*$ as in equation (23), contraction in the first entry yields a linear map

$$\sharp_\xi : T_\xi^* \mathcal{A}^* \longrightarrow T_\xi \mathcal{A}^*, \quad \theta \longmapsto \theta \lrcorner \mathcal{R}_\xi^{\mathcal{A}}. \quad (25)$$

Then, for a smooth function $f \in C^\infty(\mathcal{A}^*)$ we define the \mathcal{A} -dual vector field of f as

$$\nabla_f^{\mathcal{A}} := \sharp df \in \mathfrak{X}(\mathcal{A}^*). \quad (26)$$

Unwinding the definitions, we have

$$\begin{aligned} (\nabla_f^{\mathcal{A}}(g))(\xi) &= \langle \sharp d_\xi f, d_\xi g \rangle = \langle d_\xi f \lrcorner \mathcal{R}_\xi^{\mathcal{A}}, d_\xi g \rangle = (\mathcal{R}_\xi^{\mathcal{A}})(d_\xi f, d_\xi g) \\ &\stackrel{(24)}{=} \{f, g\}_{\mathcal{A}}(\xi) \end{aligned}$$

for $f, g \in C^\infty(\mathcal{A}^*)$, so that

$$\nabla_f^{\mathcal{A}}(g) = \{f, g\}_{\mathcal{A}}. \quad (27)$$

We define the *structure constants* of (\mathcal{A}, \bullet) with respect to a basis (e_i) of \mathcal{A} with dual basis (e^i) of \mathcal{A}^* by

$$e_i \bullet e_j = c_{ij}^k e_k, \quad (28)$$

so that from equation (23) we obtain

$$\mathcal{R}^{\mathcal{A}} = c_{ij}^k e^i \otimes e^j \otimes e_k, \quad (29)$$

using the Einstein summation convention. Then, the bivector field on \mathcal{A}^* is given as

$$\mathcal{R}_\xi^{\mathcal{A}} = c_{ij}^k \xi_k e^i \otimes e^j, \quad \text{where } \xi = \xi_k e^k \in \mathcal{A}^*, \quad (30)$$

so that the components of this tensor are linear functions on \mathcal{A}^* . Thus, if we regard $\mathcal{A} \subset C^\infty(\mathcal{A}^*)$ as the set of linear functions with the identification

$$a \longmapsto f_a \in C^\infty(\mathcal{A}^*), \quad f_a(\xi) := \langle \xi, a \rangle, \quad (31)$$

then equation (24) implies

$$\{f_a, f_b\}_{\mathcal{A}} = f_{a \bullet b},$$

so that the inclusion $\mathcal{A} \hookrightarrow C^\infty(\mathcal{A}^*)$ is an algebra monomorphism. Furthermore, in this case, equation (27) reads

$$(\nabla_{f_a}^{\mathcal{A}})_\xi(f_b) = \langle a \lrcorner \mathcal{R}_\xi^{\mathcal{A}}, b \rangle = \langle \xi, a \bullet b \rangle = f_{a \bullet b}(\xi). \quad (32)$$

Remark 1. *If the multiplication \bullet is symmetric (e.g. if \mathcal{A} is a Jordan algebra), the dual vector field $\nabla_f^{\mathcal{A}}$ is usually referred to as the gradient vector field of f , while in the case of an anti-symmetric multiplication \bullet (e.g. if \mathcal{A} is a Lie algebra), it is called the Hamiltonian vector field of f . That is, the term \mathcal{A} -dual vector field subsumes both cases.*

We wish to caution the reader that in case of a skew-symmetric multiplication \bullet the notation $\nabla_f^{\mathcal{A}}$ for the Hamiltonian vector does not match the standard convention. We use it nevertheless to unify our notation.

For a finite-dimensional algebra (\mathcal{A}, \bullet) we define the \mathcal{A} -dual distribution $\mathcal{H}^{\mathcal{A}}$ on \mathcal{A}^* by

$$\mathcal{H}_\xi^{\mathcal{A}} := \{(\nabla_f^{\mathcal{A}})_\xi \mid f \in C^\infty(\mathcal{A}^*)\} = \sharp_\xi(T_\xi^* \mathcal{A}^*) \subset T_\xi \mathcal{A}^*. \quad (33)$$

Definition 4. For a finite-dimensional algebra (\mathcal{A}, \bullet) we define the following:

1. For $a \in \mathcal{A}$ we let $l_a \in \mathfrak{gl}(\mathcal{A})$ be the map $(b \mapsto a \bullet b) \in \mathfrak{gl}(\mathcal{A})$.
2. The structure Lie algebra of \mathcal{A} is the Lie subalgebra $\mathfrak{g}(\mathcal{A}) \subset \mathfrak{gl}(\mathcal{A})$ generated by $\mathfrak{m}_{\mathcal{A}} := \{l_a \mid a \in \mathcal{A}\}$.
3. The structure Lie group of \mathcal{A} is the connected Lie subgroup $G(\mathcal{A}) \subset \text{Gl}(\mathcal{A})$ with Lie algebra $\mathfrak{g}(\mathcal{A})$.
4. A derivation of \mathcal{A} is a linear map $d \in \mathfrak{gl}(\mathcal{A})$ with $d(a \bullet b) = (da) \bullet b + a \bullet (db)$.
5. An automorphism of \mathcal{A} is a linear isomorphism $g : \mathcal{A} \rightarrow \mathcal{A}$ with $g(a \bullet b) = (ga) \bullet (gb)$.

It is straightforward to verify that the automorphisms and derivations form a regular Lie subgroup and a Lie subalgebra $\mathcal{A}ut(\mathcal{A}) \subset \text{Gl}(\mathcal{A})$ and $\mathcal{D}er(\mathcal{A}) \subset \mathfrak{gl}(\mathcal{A})$, respectively, called the *automorphism group* and *derivation algebra* of \mathcal{A} , respectively. In fact, $\mathcal{D}er(\mathcal{A})$ is the Lie algebra of $\mathcal{A}ut(\mathcal{A})$. Moreover, $g \in \mathcal{A}ut(\mathcal{A})$ and $d \in \mathcal{D}er(\mathcal{A})$ iff for all $a \in \mathcal{A}$ we have

$$gl_a g^{-1} = l_{ga}, \quad [d, l_a] = l_{da}. \quad (34)$$

That is, the adjoint action of $\mathcal{A}ut(\mathcal{A})$ and $\mathcal{D}er(\mathcal{A})$ on $\mathfrak{gl}(\mathcal{A})$ preserves the subspace $\mathfrak{m}_{\mathcal{A}}$ and hence the structure Lie algebra $\mathfrak{g}(\mathcal{A})$ and structure Lie group $G(\mathcal{A})$.

Actually, it would be more accurate to call $\mathfrak{g}(\mathcal{A})$ and $G(\mathcal{A})$ the *left-structure* Lie algebra and group, respectively, and to define the *right-structure* Lie algebra and group analogously. However, for simplicity we shall restrict ourselves to the left-structure case, as the right-structure case can be treated in complete analogy.

For $f \in \mathfrak{gl}(\mathcal{A})$ we define its dual $f^* \in \mathfrak{gl}(\mathcal{A}^*)$ by

$$\langle f^*(\xi), a \rangle = \langle \xi, f(a) \rangle \quad (35)$$

for $\xi \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Since then evidently $(fg)^* = g^* f^*$, there are a canonical Lie group and Lie algebra isomorphisms

$$\begin{aligned} \iota : \text{Gl}(\mathcal{A}) &\longrightarrow \text{Gl}(\mathcal{A}^*), & g &\longmapsto (g^{-1})^* \\ \iota_* : \mathfrak{gl}(\mathcal{A}) &\longrightarrow \mathfrak{gl}(\mathcal{A}^*), & f &\longmapsto -f^*. \end{aligned} \quad (36)$$

Indeed, ι_* is the differential of ι at the identity $e \in \text{Gl}(\mathcal{A})$. By definition, we have for $\xi \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$:

$$\langle l_a^*(\xi), b \rangle = \langle \xi, l_a(b) \rangle = \langle \xi, a \bullet b \rangle \stackrel{(32)}{=} (\nabla_{f_a}^{\mathcal{A}})_\xi(fb) \stackrel{(31)}{=} \langle (\nabla_{f_a}^{\mathcal{A}})_\xi, b \rangle.$$

It follows that $l_a^*(\xi) = (\nabla_{f_a}^{\mathcal{A}})_\xi$, so that

$$\mathcal{H}_\xi^{\mathcal{A}} = \mathcal{D}_\xi^{\mathfrak{m}_{\mathcal{A}}} \quad (37)$$

with $\mathcal{H}_\xi^{\mathcal{A}}$ from equation (33) and $\mathfrak{m}_{\mathcal{A}}$ from Definition 4, regarding \mathcal{A}^* as a $G(\mathcal{A})$ -manifold via the representation $\iota : G(\mathcal{A}) \rightarrow \text{Gl}(\mathcal{A}^*)$.

If $g \in \text{Aut}(\mathcal{A})$ is an automorphism, then by equation (34) the action of g^* on \mathcal{A}^* preserves the distribution $\mathcal{H}^{\mathcal{A}}$ and hence permutes integral leaves of equal dimensions, preserving $\mathfrak{m}_{\mathcal{A}}$ -regular points.

As it turns out, if the product \bullet is symmetric or skew-symmetric, then there is a canonical bilinear pairing on $\mathcal{H}^{\mathcal{A}}$.

Proposition 1. *Let (\mathcal{A}, \bullet) be a finite-dimensional real algebra such that \bullet is symmetric (skew-symmetric, respectively). Then on $\mathcal{H}_{\xi}^{\mathcal{A}} = \mathcal{D}_{\xi}^{\mathfrak{m}_{\mathcal{A}}} \subset T_{\xi}\mathcal{A}^*$ there is a canonical non-degenerate symmetric (skew-symmetric, respectively) bilinear form, given by*

$$\mathcal{G}_{\xi}(l_a^*(\xi), l_b^*(\xi)) := \langle \xi, a \bullet b \rangle. \quad (38)$$

Furthermore, \mathcal{G} is preserved by the action of the automorphism group $\text{Aut}(\mathcal{A})$.

Proof. By equation (35), $\langle \xi, a \bullet b \rangle = \langle l_a^*(\xi), b \rangle = \pm \langle l_b^*(\xi), a \rangle$, where the sign \pm depends on the symmetry or skew-symmetry of \bullet . This shows that \mathcal{G} is indeed well defined and non-degenerate.

Finally, if $g \in \text{Aut}(\mathcal{A})$ is an automorphism, then equation (34) together with equation (36) implies that

$$g^* l_a^*(g^{-1})^* = l_{g^{-1}a}^*$$

so that

$$\begin{aligned} \mathcal{G}_{g^*\xi}(g^*(l_a^*(\xi)), g^*(l_b^*(\xi))) &= \mathcal{G}_{g^*\xi}(l_{g^{-1}a}^*(g^*\xi), l_{g^{-1}b}^*(g^*\xi)) \\ &= \langle g^*\xi, (g^{-1}a) \bullet (g^{-1}b) \rangle = \langle \xi, g((g^{-1}a) \bullet (g^{-1}b)) \rangle \\ &= \langle \xi, a \bullet b \rangle = \mathcal{G}_{\xi}(l_a^*(\xi), l_b^*(\xi)), \end{aligned}$$

showing the invariance under the action of the automorphism group. \square

Definition 5. *Let \mathcal{A} be an algebra. A $G(\mathcal{A})$ -orbit $\mathcal{O} \subset \mathcal{A}^*$ is called $\mathfrak{m}_{\mathcal{A}}$ -regular if $\mathcal{O}_{\mathfrak{m}_{\mathcal{A}}}^{\text{reg}} = \mathcal{O}$.*

We shall now give classes of examples of these notions.

1. Lie algebras.

Let $(\mathcal{A}, \bullet) = (\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra. Then the induced section $\Lambda := \mathcal{R}^{\mathfrak{g}} \in \Gamma(\mathfrak{g}^*, \Lambda^2 T\mathfrak{g}^*)$ from equation (23) is a skew-symmetric bi-vector field, and the Jacobi identity implies that the Schouten bracket $[\Lambda, \Lambda] \in \Gamma(\mathfrak{g}^*, \Lambda^3 T\mathfrak{g}^*)$ vanishes [29, 39], so that Λ defines a linear Poisson structure $\{\cdot, \cdot\}$ on \mathfrak{g}^* , also known as the *Kostant-Kirillov-Souriau structure* [26].

Comparing our notions with those established for Poisson manifolds, we observe that for a function $f \in C^{\infty}(\mathfrak{g}^*)$, the \mathfrak{g} -gradient vector field $\nabla_f^{\mathfrak{g}}$ corresponds to the *Hamiltonian vector field* X_f for Poisson manifolds, so that the dual distribution $\mathcal{H}_{\xi}^{\mathcal{A}}$ from equation (33) is the Hamiltonian distribution of the Poisson manifold. It is integrable, as the Hamiltonian vector fields satisfy the identity

$$[X_f, X_g] = -X_{\{f, g\}}.$$

The Jacobi identity implies that $l_a = ad_a$ satisfies $[l_a, l_b] = l_{[a, b]}$, so that $\mathfrak{m}_{\mathcal{A}}$ is closed under the commutator bracket and therefore, $\mathfrak{g}(\mathcal{A}) = \mathfrak{m}_{\mathcal{A}} \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. That is, the action of the structure group is induced by the coadjoint action of G on \mathfrak{g}^* , and the skew-symmetric non-degenerate bilinear form \mathcal{G} on $\mathcal{H}^{\mathcal{A}}$ from equation (38) coincides with the symplectic form

on each coadjoint orbit in \mathfrak{g}^* . By the Jacobi identity, this action consists of automorphisms of the Lie algebra structure, whence this symplectic form is preserved under the coadjoint action.

Therefore, the integral leaves of $\mathfrak{m}_{\mathcal{A}}$ are the coadjoint orbits of \mathfrak{g}^* , equipped with their canonical symplectic form, and hence, each orbit is regular in the sense of our definition.

2. **Associative algebras.** The associativity of the product \bullet is equivalent to saying that $l_a l_b = l_{a \bullet b}$, so that $\{l_a \mid a \in \mathcal{A}\} \subset \mathfrak{gl}(\mathcal{A})$ is a subalgebra. That is, the structure algebra $\mathfrak{g}(\mathcal{A})$ equals $\mathfrak{m}_{\mathcal{A}}$ with the Lie bracket being the commutator.

Thus, if we regard \mathcal{A} as a Lie algebra with the Lie bracket $[a, b] := a \bullet b - b \bullet a$, then the $G(\mathcal{A})$ -orbits are the coadjoint orbits on \mathcal{A}^* , regarded as the dual of a Lie algebra and thus described in the preceding paragraph.

Note that by Proposition 1 the bilinear form \mathcal{G} on these orbits only exists if \bullet is symmetric or anti-symmetric.

If \mathcal{A} is a *commutative* and associative algebra, then $G(\mathcal{A})$ and $\mathfrak{g}(\mathcal{A})$ are *abelian* Lie groups, respectively. In this case, the $G(\mathcal{A})$ -orbits of \mathcal{A}^* are diffeomorphic to the direct product of a torus and Euclidean space.

In the two preceding cases, $\mathfrak{m}_{\mathcal{A}}$ is closed under Lie brackets, so that it coincides with the structure algebra $\mathfrak{g}(\mathcal{A})$. This implies that, by the very definition, $\mathfrak{m}_{\mathcal{A}}$ is integrable having the $G(\mathcal{A})$ -orbits in \mathcal{A}^* as leaves. In particular, all orbits are $\mathfrak{m}_{\mathcal{A}}$ -regular.

In contrast, for a *Jordan algebra* \mathcal{J} , it is no longer true that $\mathfrak{m}_{\mathcal{J}}$ is a Lie algebra, so that not all $G(\mathcal{J})$ -orbits on the dual \mathcal{J}^* are $\mathfrak{m}_{\mathcal{J}}$ -regular in our sense. Since the Jordan product is symmetric, the non-degenerate form \mathcal{G} from (38) defines a pseudo-Riemannian metric on the regular part $\mathcal{O}_{\mathfrak{m}_{\mathcal{J}}}^{reg}$ of each orbit.

We shall describe these structures on the $G(\mathcal{J})$ -orbits on \mathcal{J}^* and the pseudo-Riemannian metric \mathcal{G} in more details, and we will see how, for some specific type of positive Jordan algebras, and suitable orbits, \mathcal{G} is intimately connected with either the Fisher-Rao metric tensor or with the Bures-Helstrom metric tensor used in Classical and Quantum Information Geometry, respectively. This result strengthens the connection between Jordan algebras and Information Geometry initially hinted at in [8, 9].

4 Jordan algebras and Jordan distributions

Let \mathcal{J} be a real, finite-dimensional Jordan algebra, that is, a real vector space endowed with a bilinear symmetric product $\{\cdot, \cdot\}$, satisfying for $x, y \in \mathcal{J}$ the *Jordan identity*

$$\{\{x, y\}, \{x, x\}\} = \{x, \{y, \{x, x\}\}\} \quad (39)$$

By the notions established in the preceding section, we may associate with a Jordan algebra the symmetric bivector field $\mathcal{R}^{\mathcal{J}} \in \Gamma(\mathcal{J}^*, S^2(T\mathcal{J}^*))$ from (23), the musical operator $\#_{\mathcal{J}} : T^*\mathcal{J}^* \rightarrow T\mathcal{J}^*$ from equation (25), the \mathcal{J} -dual vector field $\nabla_{\mathcal{J}}^{\mathcal{J}} = \#df \in \mathfrak{X}(\mathcal{J}^*)$ from equation (26), and the induced \mathcal{J} -dual distribution $\mathcal{H}^{\mathcal{J}} \subset T\mathcal{J}^*$ from equation (33).

In particular, we have $\mathcal{H}^{\mathcal{J}} = \mathcal{D}^{\mathfrak{m}_{\mathcal{J}}}$ by equation (37), where

$$\mathfrak{m}_{\mathcal{J}} = \text{span}\{l_x^* \mid x \in \mathcal{J}\} \subset \mathfrak{gl}(\mathcal{J}^*)$$

is the space of (left-)multiplications with elements $x \in \mathcal{J}$, acting on the dual space \mathcal{J}^* . For every $\mathfrak{m}_{\mathcal{J}}$ -regular point ξ , the vector space $\mathcal{H}_{\xi}^{\mathcal{J}} = \mathcal{D}_{\xi}^{\mathfrak{m}_{\mathcal{J}}}$ carries the non-degenerate symmetric bilinear form \mathcal{G}_{ξ} defined in equation (38).

As we pointed out before, the space $\mathfrak{m}_{\mathcal{J}}$ of (left-)multiplication in \mathcal{J} is not closed under Lie brackets in general. However, the following is known.

Lemma 2. (cf. [27, Lemma IV.7]) For $x, y \in \mathcal{J}$, the commutator $[l_x, l_y]$ is a derivation of \mathcal{J} , and for each $d \in \mathcal{D}er(\mathcal{J})$

$$[d, [l_x, l_y]] = [l_{dx}, l_y] + [l_x, l_{dy}]. \quad (40)$$

We denote by $\mathcal{D}er_0(\mathcal{J})$ the span of all elements of the form $[l_x, l_y]$ for $x, y \in \mathcal{J}$. By equation (40), $\mathcal{D}er_0(\mathcal{J}) \subset \mathcal{D}er(\mathcal{J})$ is an ideal whose elements are called *inner derivations of \mathcal{J}* . This fact can be used to describe the structure Lie algebra of \mathcal{J} .

Definition 6. We define the extended structure Lie algebra $\hat{\mathfrak{g}}(\mathcal{J})$ of \mathcal{J} as follows. As a vector space, $\hat{\mathfrak{g}}(\mathcal{J})$ is defined by

$$\hat{\mathfrak{g}}(\mathcal{J}) = \mathcal{D}er_0(\mathcal{J}) \oplus \mathcal{J}. \quad (41)$$

The Lie bracket on $\hat{\mathfrak{g}}(\mathcal{J})$ is defined as follows:

- on $\mathcal{D}er_0(\mathcal{J}) \subset \mathcal{D}er(\mathcal{J}) \subset \mathfrak{gl}(\mathcal{J})$, the Lie bracket is just the commutator between linear maps;
- for $d \in \mathcal{D}er_0(\mathcal{J})$ and $x \in \mathcal{J}$, $[d, x] = -[x, d] := d(x) \in \mathcal{J}$;
- for $x, y \in \mathcal{J}$ we set $[x, y] := [l_x, l_y] \in \mathcal{D}er_0(\mathcal{J})$.

In fact, the Jacobi identity for this bracket is easily verified using the definitions and equation (40). By the definition of this Lie bracket, it follows that $(\hat{\mathfrak{g}}(\mathcal{J}), \mathcal{D}er_0(\mathcal{J}))$ is a transvective symmetric pair in the sense of Definition 3.

There is a canonical Lie algebra representation of $\hat{\mathfrak{g}}(\mathcal{J})$ on \mathcal{J} , called the *standard representation*, defined by

$$\phi : \hat{\mathfrak{g}}(\mathcal{J}) \longrightarrow \mathfrak{gl}(\mathcal{J}), \quad \begin{cases} \phi(d) := d & \text{for } d \in \mathcal{D}er_0(\mathcal{J}) \subset \mathfrak{gl}(\mathcal{J}) \\ \phi(x) := l_x & \text{for } x \in \mathcal{J}. \end{cases} \quad (42)$$

Indeed, this defines a Lie algebra homomorphism by the definition of the Lie bracket on $\hat{\mathfrak{g}}(\mathcal{J})$ and by equation (34).

Observe that the image $\phi(\hat{\mathfrak{g}}(\mathcal{J})) \subset \mathfrak{gl}(\mathcal{J})$ is generated by all l_x , $x \in \mathcal{J}$, whence equals the structure Lie algebra $\mathfrak{g}(\mathcal{J})$ from Definition 4. Thus, there is a surjective Lie group homomorphism $\hat{G}(\mathcal{J}) \rightarrow G(\mathcal{J})$ with differential ϕ , where $G(\mathcal{J}) \subset \text{Gl}(\mathcal{J})$ is the structure group from Definition 4.¹

In general, ϕ may fail to be injective (the kernel of ϕ contains the center of $\mathfrak{z}(\mathcal{J}) \subset \mathcal{J} \subset \mathfrak{g}(\mathcal{J})$), so that the structure algebra $\mathfrak{g}(\mathcal{J})$ and the extended structure algebra $\hat{\mathfrak{g}}(\mathcal{J})$ may not be isomorphic.

Then we obtain the following integrability criterion.

Proposition 2. Let \mathcal{J} be a Jordan algebra. Then, for the distribution $\mathcal{H}^{\mathcal{J}}$ from equation (33), the following assertions are equivalent.

¹Definition 4 for a Jordan algebra \mathcal{J} coincides with the definition of the structure Lie group and the structure Lie algebra of a Jordan algebra e.g. in [27, chapter IV].

1. $\mathcal{H}_\xi^{\mathcal{J}}$ is involutive at $\xi \in \mathcal{J}^*$,
2. $\mathcal{H}_\xi^{\mathcal{J}}$ is integrable at $\xi \in \mathcal{J}^*$,
3. $\text{Der}_0(\mathcal{J}) \cdot \xi \subset \mathcal{J} \cdot \xi$, where the multiplication refers to the dual action of $\text{Der}_0(\mathcal{J})$, $\mathcal{J} \subset \hat{\mathfrak{g}}(\mathcal{J})$ on \mathcal{J}^* .

If this is the case, then the maximal integral leaf through ξ is the connected component of ξ in $\mathcal{O}_{\mathfrak{m}, \mathcal{J}}^{\text{reg}} \subset \mathcal{O} = \text{G}(\mathcal{J}) \cdot \xi$.

Proof. The map ϕ from equation (42) defines an action of $\hat{\text{G}}(\mathcal{J})$ on \mathcal{J}^* such that, by equation (37), it is $\mathcal{H}_\xi^{\mathcal{J}} = \mathcal{D}_\xi^{\mathcal{J}}$. Evidently, $\hat{\text{G}}(\mathcal{J}) \cdot \xi = \text{G}(\mathcal{J}) \cdot \xi$. Since $(\hat{\mathfrak{g}}(\mathcal{J}), \text{Der}_0(\mathcal{J}))$ is a transvective symmetric pair, equation (21) is satisfied for $\mathfrak{m} := \mathcal{J} \subset \hat{\mathfrak{g}}(\mathcal{J})$, and the assertion now follows from Corollary 1, as by equation (19) it is

$$\mathcal{D}_\xi^{[\mathcal{J}, \mathcal{J}]} = \mathcal{D}_\xi^{\text{Der}_0(\mathcal{J})} = \text{Der}_0(\mathcal{J}) \cdot \xi, \quad \mathcal{D}_\xi^{\mathcal{J}} = \mathcal{J} \cdot \xi. \quad \square$$

4.1 Jordan frames and the Peirce decomposition

Let \mathcal{J} be a real finite-dimensional Jordan algebra. We define the symmetric bilinear form τ on \mathcal{J} by

$$\tau(x, y) := \text{tr } l_{\{x, y\}}. \quad (43)$$

Observe that for $x, y \in \mathcal{J}$

$$\tau(gx, gy) = \tau(x, y), \quad g \in \text{Aut}(\mathcal{J}), \quad \tau(dx, y) + \tau(x, dy) = 0, \quad d \in \text{Der}(\mathcal{J}).$$

Namely, if $g \in \text{Aut}(\mathcal{J})$, then $\tau(gx, gy) = \text{tr } l_{g\{x, y\}} = \text{tr } g l_{\{x, y\}} g^{-1} = \tau(x, y)$, and the second identity follows as $\text{Der}(\mathcal{J})$ is the Lie algebra of $\text{Aut}(\mathcal{J})$.

A symmetric bilinear form β on \mathcal{J} is called *associative*, if for all $x, y, z \in \mathcal{J}$

$$\beta(\{x, y\}, z) = \beta(x, \{y, z\}), \quad (44)$$

i.e., if all l_x are self-adjoint w.r.t. β . Then the following is known.

Proposition 3. [27, p. 59] *The bilinear form τ from equation (43) is associative.*

An element $c \in \mathcal{J}$ is called an *idempotent* if $c^2 := \{c, c\} = c$. Such an idempotent is called *primitive*, if there is no decomposition $c = c_1 + c_2$ with idempotents $c_1, c_2 \neq 0$. For an idempotent $c \in \mathcal{J}$, l_c is diagonalizable with eigenvalues in $\{0, \frac{1}{2}, 1\}$ [16, Proposition III.1.2]. Therefore, it follows that

$$\tau(c, c) = \text{tr } l_{\{c, c\}} = \text{tr } l_c \geq 1, \quad (45)$$

as the trace is the sum of the eigenvalues, and c is in the 1-eigenspace of l_c .

If \mathcal{J} has an identity element $\mathbf{1}_{\mathcal{J}}$, then a *Jordan frame* of \mathcal{J} is a set $(c_i)_{i=1}^r \subset \mathcal{J}$ of primitive idempotents such that

$$\{c_i, c_j\} = \delta_{ij} c_i \quad \text{and} \quad c_1 + \cdots + c_r = \mathbf{1}_{\mathcal{J}}. \quad (46)$$

Note that

$$\tau(c_i, c_j) = \tau(\{c_i, c_i\}, c_j) \stackrel{(44)}{=} \tau(c_i, \{c_i, c_j\}) = 0, \quad \text{for } i \neq j,$$

so that $(c_i)_{i=1}^r$ is an τ -orthogonal system. From this, one can show that the maps l_{c_i} commute pairwise [16, Lemma IV.1.3].

Let $\mathfrak{c} := \text{span}\{c_i\}$. Then, as all l_{c_i} are diagonalizable, there is a τ -orthogonal decomposition of \mathcal{J} into the common eigenspace of l_{c_i} , i.e., into spaces of the form

$$\mathcal{J}_\rho := \{x \in \mathcal{J} \mid l_c(x) = \rho(c)x, c \in \mathfrak{c}\}$$

for some $\rho \in \mathfrak{c}^*$. Since c_i has only eigenvalues $\{0, \frac{1}{2}, 1\}$ and $\rho(\mathbf{1}_\mathcal{J}) = 1$, it follows that $\rho = \frac{1}{2}(\theta_i + \theta_j)$, $i \leq j$, where $(\theta_i)_{i=1}^r \in \mathfrak{c}^*$ is the dual basis to $(c_i)_{i=1}^r$. That is, we have the τ -orthogonal eigenspace decomposition

$$\mathcal{J} = \bigoplus_{i \leq j} \mathcal{J}_{ij}, \quad (47)$$

where $\mathcal{J}_{ij} := \mathcal{J}_{\frac{1}{2}(\theta_i + \theta_j)}$. This is called the *Peirce decomposition of \mathcal{J} with respect to the Jordan frame $(c_i)_{i=1}^r$* . For convenience, we let $\mathcal{J}_{ji} := \mathcal{J}_{ij}$ for $i < j$.

4.2 Semi-simple and positive Jordan algebras

For a Jordan algebra \mathcal{J} , we define the *radical of \mathcal{J}* as the null space of τ , i.e.,

$$\mathfrak{r}(\mathcal{J}) := \{a \in \mathcal{J} \mid \tau(a, x) = 0 \text{ for all } x \in \mathcal{J}\}. \quad (48)$$

Evidently, $\mathfrak{r}(\mathcal{J}) \subset \mathcal{J}$ is an ideal by Proposition 3.

We call \mathcal{J} *semi-simple* if $\mathfrak{r}(\mathcal{J}) = 0$, i.e., if τ is non-degenerate. Moreover, we call \mathcal{J} *positive* or *formally real*, if τ is positive definite.

We shall now collect some known results on semi-simple and positive Jordan algebras.

Proposition 4. *Let \mathcal{J} be a semi-simple real Jordan algebra. Then the following hold.*

1. \mathcal{J} has a decomposition $\mathcal{J} = \mathcal{J}_1 \oplus \dots \oplus \mathcal{J}_k$ into simple Jordan algebras \mathcal{J}_i , i.e., such that \mathcal{J}_i does not contain a non-trivial ideal [27, Theorem III.11].
2. \mathcal{J} has an identity element $\mathbf{1}_\mathcal{J}$ [27, Theorem III.9].
3. \mathcal{J} is positive iff it admits a positive definite associative bilinear form β [16, p.61].
4. If \mathcal{J} is positive, then for every $x \in \mathcal{J}$ there is a Jordan frame $\{c_i\}_{i=1}^r$ with $x \in \text{span}(\{c_i\}_{i=1}^r)$ [16, Theorem III. 1.2]. In particular, \mathcal{J} has Jordan frames.
5. If \mathcal{J} is simple and positive and $(c_i)_{i=1}^r$ and $(c'_i)_{i=1}^r$ are Jordan frames, then there is an automorphism $h \in \text{Aut}_0(\mathcal{J})$ with $h(c_i) = c'_i$ for all i [16, Theorem IV.2.5]², where $\text{Aut}_0(\mathcal{J}) \subset \text{Aut}(\mathcal{J})$ is the identity component. In particular, all Jordan frames have the same number r of elements, and r is called the rank of \mathcal{J} .
6. If \mathcal{J} is positive, then for the Peirce spaces in (47) we have [16, Theorem IV.2.1]

$$\{\mathcal{J}_{ij}, \mathcal{J}_{kl}\} \subset \begin{cases} 0 & \text{if } \{i, j\} \cap \{j, k\} = \emptyset \\ \mathcal{J}_{jl} & \text{if } i = k, j \neq l, \\ \mathcal{J}_{ii} + \mathcal{J}_{jj} & \text{if } \{i, j\} = \{k, l\}. \end{cases}$$

²In [16] it is only stated that there exists an element $h \in \text{Aut}(\mathcal{J})$ with the asserted property; however, $h \in \text{Aut}_0(\mathcal{J})$ follows from the proof.

7. If \mathcal{J} is positive, then $\mathcal{J}_{ii} = \text{span}(c_i)$ is one-dimensional for all i .

8. If \mathcal{J} is positive, simple and of rank r , then $\mathcal{J}_{jk} \neq 0$ for all $1 \leq j \leq k \leq r$ [16, Theorem IV.2.3].

Proof. We only need to show point 7, as it appears not to be explicitly stated in the literature. Note that \mathcal{J}_{ii} is a subalgebra, as $\{\mathcal{J}_{ii}, \mathcal{J}_{ii}\} \subset \mathcal{J}_{ii}$ by the product relations in point 6, and by definition $c_i = \mathbf{1}_{\mathcal{J}_{ii}}$. Since $\tau|_{\mathcal{J}_{ii}}$ is a positive definite associative bilinear form, it follows from point 3 that \mathcal{J}_{ii} is a positive Jordan algebra as well. However, since $c_i = \mathbf{1}_{\mathcal{J}_{ii}}$ is primitive, it follows that each Jordan frame of \mathcal{J}_{ii} consists of c_i only, so that by 4 each $x \in \mathcal{J}_{ii}$ must be a multiple of $c_i = \mathbf{1}_{\mathcal{J}_{ii}}$. \square

The fourth of these results is called the *spectral theorem of positive Jordan algebras*. It shows that each $x \in \mathcal{J}$ admits a decomposition

$$x = \sum_{i=1}^r \lambda_i c_i \quad (49)$$

for a Jordan frame $(c_i)_{i=1}^r$, and the decomposition in equation (49) is referred to as the *spectral decomposition of x* . The λ_i 's are called the *spectral coefficients of x* . Evidently, the tuple $(\lambda_i)_{i=1}^r$ is defined only up to permutation of the entries. Furthermore, we call the pair (n_+, n_-) , where n_+ and n_- are the number of positive and negative spectral coefficients of x the *spectral signature of x* .

Lemma 3. *Let \mathcal{J} be a semi-simple, positive Jordan algebra, $(c_i)_{i=1}^r$ a Jordan frame of \mathcal{J} and $x = \sum_i \lambda_i c_i$. Then, it holds*

$$\begin{aligned} l_x(\mathcal{J}) &= \left(\bigoplus_{\lambda_a + \lambda_b \neq 0} \mathcal{J}_{ab} \right) \oplus \left(\bigoplus_{a,\mu} \mathcal{J}_{a\mu} \right) \\ \text{Der}_0(\mathcal{J}) \cdot x &= \left(\bigoplus_{\lambda_a - \lambda_b \neq 0} \mathcal{J}_{ab} \right) \oplus \left(\bigoplus_{a,\mu} \mathcal{J}_{a\mu} \right), \\ \mathfrak{g}(\mathcal{J}) \cdot x &= \bigoplus_{a,i} \mathcal{J}_{ai}, \end{aligned} \quad (50)$$

where we use the index convention that i, j run over $1, \dots, r$, while a, b run over those indices with $\lambda_a \neq 0$, and ν, μ over those indices with $\lambda_\nu = 0$.

Proof. By point 7 in Proposition 4, the Peirce decomposition in equation (47) reads

$$\mathcal{J} = \mathfrak{c} \oplus \bigoplus_{a < b} \mathcal{J}_{ab} \oplus \bigoplus_{a,\mu} \mathcal{J}_{a\mu} \oplus \bigoplus_{\mu < \nu} \mathcal{J}_{\mu\nu}, \quad \mathfrak{c} := \text{span}\{c_i\}. \quad (51)$$

As $l_x(x_{ij}) = \frac{1}{2}(\lambda_i + \lambda_j)x_{ij}$ for $x_{ij} \in \mathcal{J}_{ij}$, the first equality in equation (50) is immediate.

For the second equality, recall that $\text{Der}_0(\mathcal{J})$ is spanned by $[l_{x_{ij}}, l_{y_{kl}}]$ for $x_{ij}, y_{ij} \in \mathcal{J}_{ij}$, and we compute

$$\begin{aligned} [l_{x_{ij}}, l_{y_{kl}}](x) &= \left\{ x_{ij}, \frac{1}{2}(\lambda_k + \lambda_l)y_{kl} \right\} - \left\{ y_{kl}, \frac{1}{2}(\lambda_i + \lambda_j)x_{ij} \right\} \\ &= \frac{1}{2}(\lambda_k + \lambda_l - \lambda_i - \lambda_j)\{x_{ij}, y_{kl}\}. \end{aligned}$$

Therefore, the relation “ \subset ” in the second equality in equation (50) follows easily from the bracket relation of the Peirce spaces in point 6 of Proposition 4. For the converse inclusion, we compute

$$\begin{aligned} [l_x, l_{x_{ij}}](x) &= \{x, \{x_{ij}, x\}\} - \{x_{ij}, x^2\} \\ &= \frac{1}{4}(\lambda_i + \lambda_j)^2 x_{ij} - \frac{1}{2}(\lambda_i^2 + \lambda_j^2)x_{ij} \\ &= -\frac{1}{4}(\lambda_i - \lambda_j)^2 x_{ij}. \end{aligned}$$

The third equation then follows as $\mathfrak{g}(\mathcal{J}) \cdot x = l_x(\mathcal{J}) + \mathcal{D}er_0(\mathcal{J}) \cdot x$, and $\lambda_a + \lambda_b = \lambda_a - \lambda_b = 0$ cannot both hold for $\lambda_a, \lambda_b \neq 0$. \square

Theorem 1. *For a positive simple Jordan algebra \mathcal{J} , the following hold:*

1. *The orbits of $\mathcal{A}ut_0(\mathcal{J})$ are the sets of elements with equal spectral coefficients.*
2. *The orbits of the structure group $G(\mathcal{J})$ consist of all elements with equal spectral signature.*

Proof. Any automorphism maps (primitive) idempotents to (primitive) idempotents and fixes $\mathbf{1}_{\mathcal{J}}$, whence it maps Jordan frames to Jordan frames. Thus, if $x = \sum_{i=1}^r \lambda_i c_i$ for a Jordan frame $(c_i)_{i=1}^r$, it follows that for $h \in \mathcal{A}ut_0(\mathcal{J})$

$$h(x) = \sum_{i=1}^r \lambda_i h(c_i),$$

and $(h(c_i))_{i=1}^r$ is again a Jordan frame, so that $x, h(x)$ have the same spectral coefficients $(\lambda_i)_{i=1}^r$. Conversely, if x, y have the same spectral coefficients $(\lambda_i)_{i=1}^r$, then

$$x = \sum_{i=1}^r \lambda_i c_i, \quad y = \sum_{i=1}^r \lambda_i c'_i$$

for Jordan frames $(c_i)_{i=1}^r$ and $(c'_i)_{i=1}^r$. Thus, by point 5 of Proposition 4, there is a $h \in \mathcal{A}ut_0(\mathcal{J})$ with $h(c_i) = c'_i$ and hence, $h(x) = y$. This shows the first statement.

Concerning the second statement, we define the following subsets of \mathcal{J} :

$$\begin{aligned} \Sigma^m &:= \{x \in \mathcal{J} \mid m \text{ spectral coefficients of } x \text{ are } \neq 0\}, \\ \Sigma^{\leq m} &:= \{x \in \mathcal{J} \mid \text{at most } m \text{ spectral coefficients of } x \text{ are } \neq 0\}, \\ \Sigma_{n_+, n_-} &:= \{x \in \mathcal{J} \mid x \text{ has spectral signature } (n_+, n_-)\}. \end{aligned}$$

Evidently,

$$\Sigma^m = \bigcup_{n_+ + n_- = m} \Sigma_{n_+, n_-}. \quad (52)$$

Moreover, by the first assertion, all these sets are $\mathcal{A}ut_0(\mathcal{J})$ -invariant.

There are continuous (in fact, polynomial) functions $a_k : \mathcal{J} \rightarrow \mathbb{R}$ such that

$$f(x, \lambda) = \lambda^r + \lambda^{r-1} a_{r-1}(x) + \cdots + a_0(x) \quad (53)$$

is the minimal polynomial of all *generic* $x \in \mathcal{J}$, i.e., elements with pairwise distinct spectral coefficients $\lambda_i(x)$ [16, Proposition II.2.1]. If $x = \sum_i \lambda_i(x) c_i$ is the spectral decomposition of x ,

then $\prod_i(x - \lambda_i(x)x) = 0$ by equation (46), and as the roots $\lambda_i(x)$ are pairwise distinct, it follows that

$$f(x, \lambda) = \prod_{i=1}^r (\lambda - \lambda_i(x)), \quad (54)$$

and as generic x 's are dense in \mathcal{J} [16, Proposition II.2.1], it follows that equation (54) holds for *any* $x \in \mathcal{J}$. Then $\Sigma^{\leq m}$ are those elements where $\lambda = 0$ is a root of $f(x, \cdot)$ of multiplicity $\geq r - m$; that is,

$$\Sigma^{\leq m} = \{x \in \mathcal{J} \mid a_0(x) = \cdots = a_{r-m-1}(x) = 0\} \subset \mathcal{J}. \quad (55)$$

As the spectral coefficients of x are unchanged under the automorphism group, equation (53) and equation (54) imply

$$a_k(h \cdot x) = a_k(x), \quad x \in \mathcal{J}, h \in \mathcal{A}ut_0(\mathcal{J}). \quad (56)$$

We assert that $\Sigma^{\leq m}$ is invariant under $G(\mathcal{J})$. For this, fix a Jordan frame $(c_i)_{i=1}^r$ and let $x = \lambda_a c_a \in \mathfrak{c} \cap \Sigma^m$, using the index summation convention from Lemma 3. Define the map

$$\Phi_x : \mathcal{A}ut_0(\mathcal{J}) \times \mathbb{R}^m \longrightarrow G(\mathcal{J}) \cdot x, \quad (h, (t_i)) \mapsto h \cdot \exp(l_{t_i c_i}) \cdot x.$$

Since $l_{t_i c_i}^k x = t_i^k \lambda_a c_a$ by equation (46), it follows that

$$\Phi_x(h, (t_i)) = h \cdot \exp(l_{t_i c_i}) \cdot x = h \cdot (e^{t_i} \lambda_a c_a), \quad (57)$$

and as $e^{t_i} \lambda_a c_a \in \Sigma^m$, the $\mathcal{A}ut_0(\mathcal{J})$ -invariance of Σ^m implies that $\text{Im}(\Phi_x) \subset \Sigma^m$. Moreover, it follows that the image of the differential $d_{(e,0)}\Phi_x$ is

$$\text{Im } d_{(e,0)}\Phi_x = \text{span}\{c_a\} \oplus \mathcal{D}er_0(\mathcal{J}) \cdot x \stackrel{(50)}{\subset} \mathfrak{g}(\mathcal{J}) \cdot x = T_x(G(\mathcal{J}) \cdot x). \quad (58)$$

In fact, equation (50) implies that $\text{Im } d_{(e,0)}\Phi_x = T_x(G(\mathcal{J}) \cdot x)$ if $x = \lambda_a c_a \in \mathfrak{c}$ is *generic* in Σ^m , that is, if $\lambda_a \neq \lambda_b$ for all $a \neq b$.

This implies that for $x \in \mathfrak{c} \cap \Sigma^m$ generic, there is an open neighborhood $U \subset G(\mathcal{J})$ of the identity such that

$$U \cdot x \subset \text{Im}(\Phi_x) \subset \Sigma^m. \quad (59)$$

Thus, for $X \in \mathfrak{g}(\mathcal{J})$ and $x \in \mathfrak{c} \cap \Sigma^m$ generic, equation (55) implies

$$a_k(\exp(tX) \cdot x) = 0, \quad k = 0, \dots, r - m - 1 \quad (60)$$

for $|t|$ small enough such that $\exp(tX) \in U$. As all a_k are polynomials, the expressions in equation (60) are real analytic in t , whence their vanishing for $|t|$ small implies that they vanish for *all* $t \in \mathbb{R}$, in particular for $t = 1$. That is, we conclude that

$$a_k(\exp(X) \cdot x) = 0, \quad k = 0, \dots, r - m - 1, \quad X \in \mathfrak{g}(\mathcal{J}) \quad (61)$$

for $x \in \mathfrak{c} \cap \Sigma^m$ generic, and taking the closure, it follows that equation (61) holds for *all* $x \in \mathfrak{c} \cap \Sigma^{\leq m}$. Moreover, by the first part, each $x \in \Sigma^{\leq m}$ can be written as $x = h \cdot \tilde{x}$ for $\tilde{x} \in \mathfrak{c} \cap \Sigma^{\leq m}$ and $h \in \mathcal{A}ut_0(\mathcal{J})$. Thus, it holds

$$a_k(\exp(X) \cdot x) = a_k(h \cdot \text{Ad}_{h^{-1}}(X)\tilde{x}) \stackrel{(56)}{=} a_k(\text{Ad}_{h^{-1}}(X) \cdot \tilde{x}) \stackrel{(61)}{=} 0,$$

so that equation (61) holds for all $x \in \Sigma^{\leq m}$ and $X \in \mathfrak{g}(\mathcal{J})$. Thus, by equation (55) it follows that

$$\exp(\mathfrak{g}(\mathcal{J})) \cdot \Sigma^{\leq m} \subset \Sigma^{\leq m},$$

and as the connected group $G(\mathcal{J})$ is generated by $\exp(\mathfrak{g}(\mathcal{J}))$, the asserted $G(\mathcal{J})$ -invariance of $\Sigma^{\leq m}$ follows.

Since $\Sigma^m = \Sigma^{\leq m} \setminus \Sigma^{\leq m-1}$ is the difference of two $G(\mathcal{J})$ -invariant sets, it follows that Σ^m is $G(\mathcal{J})$ -invariant as well.

Next, we assert that $\Sigma_{n_+, n_-} \subset \Sigma^m$ is relatively closed. For if $(x_k)_{k \in \mathbb{N}} \in \Sigma_{n_+, n_-}$ converges to $x_0 \in \Sigma^m$, then, fixing a Jordan frame $(c_i)_{i=1}^r$, we find $h_k \in \mathcal{A}ut_0(\mathcal{J})$ such that

$$y_k := h_k \cdot x_k = \sum_a \lambda_{a,k} c_a, \quad \lambda_{1,k} \geq \dots \geq \lambda_{m,k}.$$

Since $y_k \in \Sigma_{n_+, n_-}$ as well, it follows that the signs of $0 \neq \lambda_{a,k}$ are equal for all k . As $\mathcal{A}ut_0(\mathcal{J})$ is compact, we may pass to a subsequence to assume that $h_k \rightarrow h_0$, whence $y_k \rightarrow h_0 x_0$, i.e.

$$h_0 x_0 = \sum_a \lambda_{a,0} c_a, \quad \lambda_{a,0} = \lim_{k \rightarrow \infty} \lambda_{a,k}.$$

Since x_0 and hence $h_0 x_0 \in \Sigma^m$, it follows that $\lambda_{a,0} \neq 0$ for all a , whence $\lambda_{a,0}$ has the same sign as all $\lambda_{a,k}$, so that $h_0 x_0 \in \Sigma_{n_+, n_-}$, i.e., $x_0 \in \Sigma_{n_+, n_-}$.

Thus, equation (52) is the disjoint decomposition of Σ^m into finitely many relatively closed subsets, and since $G(\mathcal{J})$ and hence all orbits are connected, it follows that each $G(\mathcal{J})$ -orbit must be contained in some Σ_{n_+, n_-} .

On the other hand, as elements with equal spectral coefficients lie in the same $\mathcal{A}ut_0(\mathcal{J})$ -orbit, equation (57) immediately implies that $G(\mathcal{J})$ acts transitively on Σ_{n_+, n_-} , which completes the proof. \square

For a positive Jordan algebra \mathcal{J} we identify \mathcal{J} and \mathcal{J}^* by the isomorphism

$$\begin{aligned} \flat : \mathcal{J} &\longrightarrow \mathcal{J}^*, & x &\longmapsto x^\flat := \tau(x, \cdot), \\ \sharp : \mathcal{J}^* &\longrightarrow \mathcal{J}, & \xi &:= \flat^{-1} \end{aligned}$$

By the spectral theorem (cf. point 4 of Proposition 4), for each $\xi \in \mathcal{J}^*$ there is a Jordan frame $(c_i)_{i=1}^r$ on \mathcal{J} such that

$$\xi^\sharp = \lambda_i c_i, \quad \text{and} \quad \xi = \lambda_i c_i^\flat, \quad (62)$$

and we define the spectral coefficients $(\lambda_i)_i$ and the spectral signature (n_+, n_-) of ξ to be the spectral coefficients and signature of ξ^\sharp . We let

$$\mathcal{O}_{n_+, n_-} \subset \mathcal{J}^*$$

be the set of elements of spectral signature (n_+, n_-) . Furthermore, we define the *dual* of τ to be the scalar product on \mathcal{J}^* given by

$$\tau^\flat(\eta_1, \eta_2) := \tau(\eta_1^\sharp, \eta_2^\sharp) \quad \text{or} \quad \tau^\flat(x_1^\flat, x_2^\flat) := \tau(x_1, x_2). \quad (63)$$

For $x, y \in \mathcal{J}$ and $\xi \in \mathcal{J}^*$, we have $(l_x^* \xi)(y) = \xi(l_x y) = \tau(\xi^\sharp, l_x y) = \tau(l_x \xi^\sharp, y) = (l_x \xi^\sharp)^\flat(y)$, so that

$$l_x^* \xi = (l_x \xi^\sharp)^\flat. \quad (64)$$

Therefore, with the help of the definition of the dual action in equation (36), it follows that

$$\mathcal{A}ut_0(\mathcal{J}) \cdot \xi = (\mathcal{A}ut_0(\mathcal{J}) \cdot \xi^\#)^\flat, \quad \mathsf{G}(\mathcal{J}) \cdot \xi = (\mathsf{G}(\mathcal{J}) \cdot \xi^\#)^\flat, \quad (65)$$

so that, by Theorem 1, we obtain that the orbits of the action of $\mathsf{G}(\mathcal{J})$ on \mathcal{J}^* are the sets \mathcal{O}_{n_+, n_-} .

The open cone of squares in \mathcal{J} is

$$\Omega_{\mathcal{J}} := \text{Int}\{x^2 \mid x \in \mathcal{J}\}$$

Looking at the spectral decomposition in equation (49), it follows that $x \in \Omega_{\mathcal{J}}$ iff all its spectral coefficients are positive iff l_x is positive definite, and the latter description shows that $\Omega_{\mathcal{J}}$ is indeed a convex cone; in fact, it easily follows from this characterization that

$$\begin{aligned} \Omega_{\mathcal{J}} &= \mathcal{O}_{r,0} = \mathfrak{g}(\mathcal{J}) \cdot \mathbf{1}_{\mathcal{J}}, \\ \bar{\Omega}_{\mathcal{J}} &= \dot{\bigcup}_{n_+ \geq 0} \mathcal{O}_{n_+,0}. \end{aligned} \quad (66)$$

Theorem 2. *Let \mathcal{J} be a positive, simple Jordan algebra with structure group $G(\mathcal{J}) \subset \text{Gl}(\mathcal{J})$. Then $\xi \in \mathcal{J}^*$ is $\mathfrak{m}_{\mathcal{J}}$ -regular iff the spectral coefficients (λ_i) of ξ satisfy:*

$$\lambda_a + \lambda_b \neq 0 \quad \text{whenever } \lambda_a, \lambda_b \neq 0. \quad (67)$$

In particular, the $\mathsf{G}(\mathcal{J})$ -orbit \mathcal{O}_{n_+, n_-} is $\mathfrak{m}_{\mathcal{J}}$ -regular iff $n_+ = 0$ or $n_- = 0$, i.e., iff it is contained in $\bar{\Omega}_{\mathcal{J}}$ or $-\bar{\Omega}_{\mathcal{J}}$.

Proof. Let $x := \xi^\# \in \mathcal{J}$. By equation (65), $\mathcal{A}ut_0(\mathcal{J}) \cdot \xi \subset \mathcal{J} \cdot \xi$ iff $\mathcal{A}ut_0(\mathcal{J}) \cdot x \subset \mathcal{J} \cdot x = l_x(\mathcal{J})$, and, recalling point 8 in Proposition 4, by equation (50) this condition is satisfied iff equation (67) holds. Recalling Proposition 2, the first statement follows.

If $n_+, n_- > 0$, then evidently, \mathcal{O}_{n_+, n_-} contains elements two of whose spectral coefficients satisfy $\lambda_a = -\lambda_b \neq 0$, so that \mathcal{O}_{n_+, n_-} is not $\mathfrak{m}_{\mathcal{J}}$ -regular.

On the other hand, on $\mathcal{O}_{n_+,0}$ (\mathcal{O}_{0,n_-} , respectively) $\lambda_a, \lambda_b > 0$ (< 0 , respectively) so that equation (67) holds; whence $\mathcal{O}_{n_+,0}$ and \mathcal{O}_{0,n_-} are the only $\mathfrak{m}_{\mathcal{J}}$ -regular orbits, and by equation (66) these are the orbits contained in $\bar{\Omega}_{\mathcal{J}}$ or $-\bar{\Omega}_{\mathcal{J}}$, respectively. \square

Let us now describe the pseudo-Riemannian metric \mathcal{G} on $\mathcal{O}_{\mathfrak{m}_{\mathcal{J}}}^{\text{reg}}$. Take $\xi \in \mathcal{J}^*$ with spectral decomposition

$$\xi = \lambda_a c_a^\flat \in \mathcal{J}^* \quad \Rightarrow \quad x := \xi^\# = \lambda_a c_a \in \mathcal{J} \quad (68)$$

for some Jordan frame $(c_i)_{i=1}^r$, and assume it satisfies equation (67). Then, it holds

$$T_\xi \mathcal{O} = \mathfrak{g}(\mathcal{J}) \cdot \xi = (\mathfrak{g}(\mathcal{J}) \cdot x)^\flat \stackrel{(50)}{=} \bigoplus_{a,i} \mathcal{J}_{ai}^\flat, \quad (69)$$

and we have the following Proposition.

Proposition 5. *Let $\xi = \lambda_a c_a^\flat \in \mathcal{J}^*$ be as above. Then, it holds*

$$\mathcal{G}_\xi = \sum_{a,i} \frac{2}{\lambda_a + \lambda_i} \tau^\flat|_{\mathcal{J}_{ai}^\flat}, \quad (70)$$

which is equivalent to

$$\mathcal{G}_\xi(x_{ai}^\flat, y_{bj}^\flat) = \begin{cases} \frac{2}{\lambda_a + \lambda_i} \tau^\flat(x_{ai}^\flat, y_{ai}^\flat) & \text{if } (a, i) = (b, j) \\ 0 & \text{else} \end{cases}.$$

Proof. For $x_{ij} \in \mathcal{J}_{ij}$ we have

$$l_{x_{ij}}^*(\xi) \stackrel{(64)}{=} (l_{x_{ij}} \xi^\#)^{\flat} = \frac{1}{2}(\lambda_i + \lambda_j)x_{ij}^{\flat}. \quad (71)$$

Therefore, evaluating both sides of equation (38) gives us

$$\begin{aligned} \mathcal{G}_\xi(l_{x_{ai}}^*(\xi), l_{y_{bj}}^*(\xi)) &= \frac{1}{4}(\lambda_a + \lambda_i)(\lambda_b + \lambda_j)\mathcal{G}_\xi(x_{ai}^{\flat}, y_{bj}^{\flat}) \\ \mathcal{G}_\xi(l_{x_{ai}}^*(\xi), l_{y_{bj}}^*(\xi)) &= \xi(\{x_{ai}, y_{bj}\}) = \tau(\xi^\#, \{x_{ai}, y_{bj}\}) \\ &= \tau(\{\xi^\#, x_{ai}\}, y_{bj}) \stackrel{(68)}{=} \frac{1}{2}(\lambda_a + \lambda_i)\tau(x_{ai}, y_{bj}). \end{aligned}$$

Since both equations must be equal, equation (70) follows as the Peirce decomposition $\mathcal{J} = \bigoplus_{ij} \mathcal{J}_{ij}$ is τ -orthogonal. \square

Remark 2. 1. Comparing the the description of the regular points in \mathcal{O} in Theorem 2 and equation (70), it follows that \mathcal{G}_ξ has a pole of order 1 on $\mathcal{O} \setminus \mathcal{O}_{\mathfrak{m}_{\mathcal{J}}}^{\text{reg}}$.

2. As \mathcal{G}_ξ is positive or negative definite on \mathcal{J}_{aa}^{\flat} , depending on the sign of $\lambda_a \neq 0$, it follows that \mathcal{G}_ξ is indefinite at any regular point of spectral signature (n_+, n_-) with $n_+, n_- > 0$.

That is, \mathcal{G} on \mathcal{O} is definite (and hence defines a Riemannian metric) iff $\mathcal{O} = \mathbb{G}(\mathcal{J}) \cdot \xi$ is a regular orbit, iff $\mathcal{O} \subset \bar{\Omega}_{\mathcal{J}}$ is contained in the closure of the cone of squares ($\mathcal{G} > 0$) or $\mathcal{O} \subset -\bar{\Omega}_{\mathcal{J}}$ ($\mathcal{G} < 0$).

3. It is also evident from the description of regular points in Theorem 2 that for a non-regular orbit \mathcal{O}_{n_+, n_-} with $n_+, n_- > 0$ the regular part $(\mathcal{O}_{n_+, n_-})_{\mathfrak{m}_{\mathcal{J}}}^{\text{reg}}$ is not path connected.

4. Note that the Riemannian metric \mathcal{G} on $\mathcal{O}_{n_+, 0}$ (and similarly, $-\mathcal{G}$ on \mathcal{O}_{0, n_-}) is not complete. Namely, for $t > 0$, the curve

$$\alpha(t) := t^2(c_1 + \dots + c_{n_+})^{\flat} \in \mathcal{O}_{n_+, 0}$$

for a Jordan frame $(c_i)_{i=1}^{n_+}$ has constant speed with respect to \mathcal{G} because

$$\mathcal{G}_{\alpha(t)}(\dot{\alpha}, \dot{\alpha}) = \sum_{a=1}^{n_+} \frac{1}{t^2} \tau(2tc_a, 2tc_a) = 4 \sum_{a=1}^{n_+} \tau(c_a, c_a).$$

However, α cannot be extended in $\mathcal{O}_{n_+, 0}$ at $t = 0$.

4.3 Examples

We shall now describe the metric \mathcal{G} for the standard examples of positive Jordan algebras.

1. The Fisher-Rao metric for finite sample spaces

We regard $\mathcal{J} := \mathbb{R}^n$ as a positive Jordan algebra whose algebraic operations are defined in a component-wise way. Then, it is not difficult to see that $\Omega_{\mathcal{J}}$ can be identified with the first orthant $\mathbb{R}_+^n \subset \mathbb{R}^n \cong \mathcal{J}^*$. The metric \mathcal{G}_ξ at $\xi = (\xi_1, \dots, \xi_n) \in \Omega_{\mathcal{J}}$ is given by

$$\mathcal{G}_\xi(u, v) = \sum_i \frac{1}{\xi_i} u_i v_i, \quad u = (u_i)_{i=1}^n, v = (v_i)_{i=1}^n \in \mathbb{R}^n.$$

When interpreting $\Omega_{\mathcal{G}}$ as the set of positive finite measures on $\mathcal{X}_n = \{1, \dots, n\}$, it is clear that \mathcal{G} is such its pullback to the submanifold of strictly positive probability distributions on \mathcal{X}_n (i.e., open interior of the unit simplex inside \mathbb{R}_+^n) coincides with the *Fisher-Rao-metric tensor* which naturally occurs in Classical Information Geometry [2, 3]. As partially noted in [8, 9], this instance shows that we may look at the non-normalized Fisher-Rao metric tensor on $\Omega_{\mathcal{G}} = \mathbb{R}_+^n$ as the analogue of the homogeneous symplectic form on co-adjoint orbits in the case of Lie algebras.

2. The Jordan algebras $M_n^{sa}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Let \mathbb{K} denote either the real, complex or quaternionic numbers, and we define the Jordan algebra of self-adjoint matrices

$$M_n^{sa}(\mathbb{K}) := \{A \in \mathbb{K}^{n \times n} \mid A = A^*\}, \quad \{A, B\} := \frac{1}{2}(AB + BA).$$

For convenience, we replace τ from equation (43) by the associative inner product

$$\hat{\tau}(A, B) := \text{Tr}(AB),$$

so that τ and $\hat{\tau}$ only differ by the multiplicative constant $\frac{1}{n} \dim_{\mathbb{R}} M_n^{sa}(\mathbb{K})$.

Let $E_{ij} \in \mathbb{K}^{n \times n}$ denote the matrix with a 1 in the (i, j) -entry. Then $\{E_{11}, \dots, E_{nn}\}$ is a Jordan frame of $M_n^{sa}(\mathbb{K})$, and the remaining Pierce spaces with respect to this frame are given as

$$(M_n^{sa}(\mathbb{K}))_{ij} = \{zE_{ij} + \bar{z}E_{ji} \mid z \in \mathbb{K}\}, \quad i < j.$$

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and \mathbb{H} the automorphism group of $M_n^{sa}(\mathbb{K})$ is $SO(n), U(n)$ and $Sp(n)$, respectively, acting on $M_n^{sa}(\mathbb{K})$ by conjugation. Thus, in particular, each $A \in M_n^{sa}(\mathbb{K})$ is diagonalizable by an element in the automorphism group, so that the spectral coefficients are the eigenvalues of A . Thus, by Proposition 5, for $\xi = \lambda_a E_{aa} \in (M_n^{sa}(\mathbb{K}))^*$ the metric \mathcal{G}_{ξ} reads

$$\mathcal{G}_{\xi}(zE_{ai}^b, wE_{bj}^b) = \begin{cases} \frac{2}{\lambda_a + \lambda_i}(z\bar{w} + w\bar{z}) & \text{if } (a, i) = (b, j) \\ 0 & \text{else} \end{cases}.$$

Moreover, because of Proposition 1, it follows that \mathcal{G} is preserved by the automorphism group of $M_n^{sa}(\mathbb{K})$.

As already mentioned in the introduction, and in accordance with the results put forward in [8, 9], an interesting link between Jordan algebras and Quantum Information Geometry appears when $\mathbb{K} = \mathbb{C}$. In this case, we may identify $M_n^{sa}(\mathbb{C})$ with the Jordan algebra of self-adjoint observables of a finite-level quantum system with Hilbert space $\mathcal{H} \cong \mathbb{C}^n$. Then, if we focus on the $\mathfrak{m}_{\mathcal{G}}$ -regular orbit $\Omega_{\mathcal{G}}$ of faithful, non-normalized quantum states, which can be identified with the dual of the orbit of invertible positive matrices in $M_n^{sa}(\mathbb{C})$, the metric tensor \mathcal{G} is such that its pullback to the submanifold of faithful quantum states, determined by the condition $\text{Tr}(A) = 1$, coincides with the so-called Bures-Helstrom metric tensor [5, 10, 11, 18, 19, 20, 21, 34, 40, 42, 43]. Analogously, if we focus on the $\mathfrak{m}_{\mathcal{G}}$ -regular orbit through non-normalized pure states, which are identified with rank one matrices in $M_n^{sa}(\mathbb{C})$, the metric tensor \mathcal{G} is such that its pullback to the submanifold of pure states, determined by the condition $\text{Tr}(A) = 1$, is a multiple of the Fubini-Study metric tensor, essentially because of its unitary invariance. Accordingly, and in analogy

with the Fisher-Rao metric tensor seen before, we may think of the non-normalized version of the Bures-Helstrom metric tensor and of the Fubini-Study metric tensor as the analogue of the Kostant-Kirillov-Souriau symplectic form in the case of the Jordan algebra $M_n^{sa}(\mathbb{C})$.

3. The spin-factor Jordan algebra $\mathcal{JSpin}(n)$.

Denoting the standard inner product of \mathbb{R}^n by $\langle \cdot, \cdot \rangle$, we let

$$\mathcal{JSpin}(n) := \mathbb{R}^{n+1} = \mathbb{R}\mathbf{1} \oplus \mathbb{R}^n, \quad \{x, y\} := \langle x, y \rangle \mathbf{1}, \quad x, y \in \mathbb{R}^n,$$

and where $\mathbf{1}$ is the identity element of $\mathcal{JSpin}(n)$. An associative inner product is given by

$$\hat{\tau}|_{\mathbb{R}^n} = \langle \cdot, \cdot \rangle, \quad \hat{\tau}(\mathbf{1}, \mathbf{1}) := 1, \quad \hat{\tau}(\mathbf{1}, \mathbb{R}^n) = 0.$$

The automorphism group is $SO(n)$, acting on \mathbb{R}^n and fixing $\mathbf{1}$. Every Jordan frame is given by

$$\left\{ \frac{1}{2}(\mathbf{1} + e_0), \frac{1}{2}(\mathbf{1} - e_0) \right\}$$

for a fixed unit vector $e_0 \in \mathbb{R}^n$, and the Peirce space complementary to the Jordan frame is

$$\mathcal{JSpin}(n)_{12} := e_0^\perp.$$

The two spectral coefficients of an element $X = t\mathbf{1} + x$ are

$$\lambda_1 = \frac{1}{2}(t + \|x\|), \quad \lambda_2 = \frac{1}{2}(t - \|x\|),$$

where $\|\cdot\|$ denotes the norm on \mathbb{R}^n induced by $\langle \cdot, \cdot \rangle$. Therefore, $\xi \in \mathcal{JSpin}(n)^*$ is regular iff $0 \neq \lambda_1 + \lambda_2 = \hat{\tau}^b(\xi, \mathbf{1}^b)$. If

$$\xi = t_0 \mathbf{1}^b + s_0 e_0^b \in \mathcal{JSpin}(n)^*, \quad t_0 \neq 0$$

is regular, where $e_0 \in \mathbb{R}^n$ is a unit vector, then the tangent vectors $X_1, X_2 \in T_\xi \mathcal{O}$ are of the form

$$X_i = t_i \mathbf{1}^b + s_i e_0^b + x_i^b,$$

where $x_i \in e_0^\perp$, and where $t_0 = \pm s_0 \Rightarrow t_i = \pm s_i$. The spectral coefficients of ξ are $\lambda_1 = \frac{1}{2}(t_0 + s_0)$ and $\lambda_2 = \frac{1}{2}(t_0 - s_0)$, and

$$X_i = (t_i + s_i) \frac{1}{2}(\mathbf{1} + e_0)^b + (t_i - s_i) \frac{1}{2}(\mathbf{1} - e_0)^b + x_i^b.$$

Therefore

$$\begin{aligned} \mathcal{G}_\xi(X_1, X_2) &= \frac{2}{t_0 + s_0} (t_1 + s_1)(t_2 + s_2) + \frac{2}{t_0 - s_0} (t_1 - s_1)(t_2 - s_2) \\ &\quad + \frac{2}{t_0} \langle x_1, x_2 \rangle \end{aligned}$$

This metric is positive definite if $t_0 \geq |s_0|$ and negative definite if $t_0 \leq -|s_0|$, as predicted by Proposition 5.

Remark 3. According to the classification given in [16, Theorem V.3.7], the first two classes of examples discussed above give a complete list of simple, positive Jordan algebras up to the Albert algebra, a 27-dimensional simple Jordan algebra of rank 3. Its automorphism group is F_4 and the structure algebra is of type E_6 .

While it would be possible but elaborate to calculate the regular points and the inner product \mathcal{G} on the tangent to the orbit at a regular point, our results in Theorem 2 and Proposition 5, allow to understand the structure without these explicit calculations.

4. Non-simple, semi-simple positive Jordan algebras

By point 1 of Proposition 4, each positive, semi-simple Jordan algebra admits a decomposition $\mathcal{J} = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_k$ into positive *simple* Jordan algebras, so that both the automorphism and the structure group of \mathcal{J} is the direct sum of the automorphism and structure group of the simple factors \mathcal{J}_i , respectively. Then, applying Theorem 1, Theorem 2, and Proposition 5, it follows that the $G(\mathcal{J})$ -orbits are of the form

$$\mathcal{O}_{n_+, n_-}^1 \times \cdots \times \mathcal{O}_{n_+, n_-}^k \subset \mathcal{J}^* = \mathcal{J}_1^* \oplus \cdots \oplus \mathcal{J}_k^*,$$

where $\mathcal{O}_{n_+, n_-}^i \subset \mathcal{J}_i^*$ are $G(\mathcal{J}_i)$ -orbits. In particular, such an orbit is regular iff $n_+^i n_-^i = 0$ for all i , and the metric \mathcal{G} on the regular part of this orbit is given by (70).

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