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**Mathematical structures of
cohomological field theories**

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Mathematical Structures of Cohomological Field Theories

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Abstract

A mathematical framework of cohomological field theories (CohFTs) is formulated in the language of bigraded manifolds. Algebraic properties of operators in CohFTs are studied. Methods of constructing CohFTs, with or without gauge symmetries, are discussed. In particular, a generalization of the Mathai-Quillen formalism is given. Examples such as topological quantum mechanics, topological sigma model, topological M-theory, and topological Yang-Mills theory can be obtained uniformly using this new formalism.

1 Introduction

A cohomological field theory (CohFT) is a Lagrangian field theory which possesses a scalar supersymmetry Q with $Q^2 = 0$ [1, 2]. The physical operators of interest in such a theory are solutions to the following equations

$$Q\mathcal{O}^{(p)} = d\mathcal{O}^{(p-1)} \quad (1.1)$$

for $1 \leq p \leq n$ with $Q\mathcal{O}^{(0)} = 0$, where d is the de Rham differential of an n -dimensional Riemannian manifold M . The expectation value of $\int_{\gamma_p} \mathcal{O}^{(p)}$, where γ_p is a p -cycle in M , does not depend the Riemannian metric. That is to say, it can be seen as a smooth invariant of M . Many famous invariants in mathematics, e.g., the Donaldson invariants, Gromov-Witten invariants, and Seiberg-Witten invariants, can be obtained in this way.

There exist various mathematical approaches to CohFTs based on the BRST cohomology [3, 4], the equivariant cohomology [5], and the master equation in the BV formalism [6]. However, most of these approaches focus only on the construction of Lagrangians, lacking a careful treatment of the algebraic structures of operators in CohFTs. It is the goal of this paper to build a new mathematical framework for CohFTs systematizing the previous ones, within which a complete classification of the solutions to (1.1) is available.

The first step toward such a framework is to generalize the theory of supermanifolds (or \mathbb{Z}_2 -graded manifolds) to a theory with richer grading structures. Such a theory was developed by the author in [7], where \mathbb{Z}_2 is replaced by a cancellative commutative semiring \mathcal{I} to yield the notion of a monoidally graded manifold. Here we are mainly interested in the case of $\mathcal{I} = \mathbb{Z} \times \mathbb{Z}$, i.e., bigraded manifolds. The operator $\mathcal{O}^{(p)}$ can be viewed (classically) as a function of degree $(p, n-p)$ over a bigraded manifold. It follows that Q and d should be viewed as vector fields of degrees $(0, 1)$ and $(1, 0)$, respectively. Moreover, the supersymmetry algebra can be extended to include a vector

supersymmetry K , a vector field of degree $(1, -1)$. K together with Q and d satisfies the following relations

$$Q^2 = 0, \quad QK + KQ = d, \quad Kd + dK = 0.$$

We call the corresponding extended supersymmetry algebra as the QK -algebra. K can be used to produce particular solutions to (1.1) called K -sequences by setting $\mathcal{O}^{(p)} = \sum_{q=0}^p \frac{1}{(p-q)!} K^{p-q} \mathcal{W}^{(q)}$, where $\mathcal{W}^{(0)} = \mathcal{O}^{(0)}$ and $\mathcal{W}^{(q)}$ is any (non-exact) Q -closed function of degree $(q, n-q)$ for $1 \leq q \leq n$. The main result of this paper is

Theorem. *Every solution to (1.1) is cohomologically a K -sequence.*

The paper is organized as follows: In Section 2, knowledge about equivariant cohomology is reviewed to motivate mathematical constructions later. In Sections 3 and 4, we build the geometric settings for CohFTs and prove the main theorem. In Section 5, we investigate Witten's idea of topological twistings [1] and show that the twisting of a super Poincaré algebra gives naturally rise to a QK -algebra. In Section 6, we provide a systematic formalism of constructing CohFTs as a generalization of the methods in [1–5, 8–11]. In this new formalism, one can create CohFTs by specifying just a small collection of data, as shown in Section 7.

2 Equivariant Cohomology

In this section, we briefly describe three different approaches to equivariant cohomology, namely, the Weil model, the Cartan model, and the Kalkman model. We use the last one to reformulate the Mathai-Quillen construction of the Thom class.

Naturally, one should expect that the equivariant cohomology of a G -space X tells us both the topological information of X and the information about the G -action on X . A naive choice is the cohomology of the quotient space X/G . This is not right, since X/G remembers nothing about the stabilizer of the group action at each point $x \in X$ unless the group action is free. The key idea here is to consider the Cartesian product of a “universal G -space” EG and X where

1. EG is contractible, hence it does not provide any new topological information;
2. EG has a free G -action. (It follows that G acts also freely on $EG \times X$.)

Definition 2.1. A contractible space EG with a free G -action is called a universal G -space. Let X be a G -space. The quotient space $X_G = (EG \times X)/G$ is called a homotopy orbit space of X . In particular, the homotopy orbit space of a one-point space is called a classifying space for G , denoted by BG .

A standard result is that EG always exists for a topological Lie group G . Moreover, all homotopy orbit spaces of a G -space X are homotopic equivalent to each other.

Definition 2.2. The equivariant cohomology of a G -space X , denoted by $H_G(X)$, is defined as the singular cohomology of the homotopy orbit space X_G of X .

Let's switch to the smooth category and assume G to be compact. By the quotient manifold theorem, the homotopy orbit space X_G is also smooth and we can replace the singular cohomology of it in Def 2.2 by its de Rham cohomology.

Remark 2.1. The universal G -space EG is often obtained as a direct limit of some direct system of finite dimensional manifolds. The de Rham complex of EG can be then defined as the inverse limit of the induced inverse system of de Rham complexes over such finite dimensional manifolds.

Let G be a Lie group with Lie algebra \mathfrak{g} . To each $\xi \in \mathfrak{g}$ we can associate a vector field v_ξ on a G -manifold, which again induces a contraction ι_ξ and a Lie derivative Lie_ξ on the de Rham complex of the G -manifold. Let d denote the de Rham differential. Fix a basis $\{\xi_a\}$ of \mathfrak{g} . Let ι_a and Lie_a denote the contraction and Lie derivative associated to ξ_a . d, ι_a, Lie_a satisfy the following relations

$$[\text{Lie}_a, \text{Lie}_b] = f_{ab}^c \text{Lie}_c, \quad [\text{Lie}_a, \iota_b] = f_{ab}^c \iota_c, \quad [\text{Lie}_a, d] = 0, \quad (2.1)$$

$$\{d, d\} = 0, \quad \{\iota_a, \iota_b\} = 0, \quad \{d, \iota_a\} = \text{Lie}_a. \quad (2.2)$$

Remark 2.2. These relations are not independent from each other. It is an easy exercise to show that $[\text{Lie}_a, \text{Lie}_b] = f_{ab}^c \text{Lie}_c$ and $[\text{Lie}_a, d] = 0$ can be derived from the others.

Recall that a Lie superalgebra $L = L_{\text{even}} \oplus L_{\text{odd}}$ is specified by [12]

1. a Lie algebra L_{even} ;
2. an L_{even} -module L_{odd} and a bilinear pairing $[\cdot, \cdot]$ such that $[x, \cdot]$ is the action of $x \in L_{\text{even}}$ on L_{odd} and $[\cdot, x] := -[x, \cdot]$;
3. a symmetric bilinear pairing $\{\cdot, \cdot\} : L_{\text{odd}} \times L_{\text{odd}} \rightarrow L_{\text{even}}$ that is a homomorphism of L_{even} -modules and satisfies the Jacobi identity

$$\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0$$

for $x, y, z \in L_{\text{odd}}$.

In this case, L_{even} is spanned by Lie_a . L_{odd} is spanned by d and ι_a . In other words, we have $L_{\text{even}} = \mathfrak{g}$ and $L_{\text{odd}} = \mathbb{R} \oplus \mathfrak{g}$. Elements of L_{even} act on $\mathbb{R} \subset L_{\text{odd}}$ trivially and act on $\mathfrak{g} \subset L_{\text{odd}}$ via the adjoint representation. The symmetric pairing is given by (2.2). Moreover, L is a Lie graded algebra by assigning degrees 0, 1 and -1 to Lie_a, d and ι_a , respectively.

Definition 2.3. Let W be an L -module. Let Lie_a, d, ι_a be a basis of L . An element $\alpha \in W$ is called horizontal if $\iota_a \alpha = 0$. A horizontal element α is called basic if in addition $\text{Lie}_a \alpha = 0$.

Let W_{bas} denote the submodule of basic elements in W . Note that for $\alpha \in W_{\text{bas}}$, $d\alpha$ is also in W_{bas} . As an alternative approach to equivariant cohomology, one can define the equivariant cohomology of a G -manifold X as the cohomology of the basic submodule $(\Omega(EG) \otimes \Omega(X))_{\text{bas}}$ of the L -module $\Omega(EG) \otimes \Omega(X)$. In practice, $\Omega(EG)$ is too large to work with. It is shown in [13] that one can replace it with the following algebra.

Definition 2.4. The Weil Algebra is the commutative graded algebra

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$$

where elements of $\mathfrak{g}^* \subset \Lambda(\mathfrak{g}^*)$ have degree 1 and elements of $\mathfrak{g}^* \subset S(\mathfrak{g}^*)$ have degree 2.

Let's show that $W(\mathfrak{g})$ has an L -module structure. Let ξ^a be a basis of \mathfrak{g}^* , which yields generators $\{\theta^a\}$ and $\{\phi^a\}$ for $\Lambda(\mathfrak{g}^*)$ and $S(\mathfrak{g}^*)$, respectively. We can now define ι_a and d by setting

$$\iota_a \theta^b = \delta_a^b, \quad \iota_a \phi^b = 0, \quad (2.3)$$

$$d\theta^a = \phi^a - \frac{1}{2} f_{bc}^a \theta^b \theta^c, \quad d\phi^a = f_{bc}^a \phi^b \theta^c. \quad (2.4)$$

Consequently, we have

$$\text{Lie}_a \theta^b = -f_{ac}^b \theta^c, \quad \text{Lie}_a \phi^b = -f_{ac}^b \phi^c. \quad (2.5)$$

It is an easy exercise to verify that Lie_a , d , ι_a satisfy (2.1) and (2.2). Moreover, one can show that $W(\mathfrak{g})$ is acyclic, which is not surprising because $\Omega(EG)$ is also acyclic.

Definition 2.5. The universal connection and curvature of the Weil algebra are

$$\theta = \theta^a \otimes \xi_a \in W^1(\mathfrak{g}) \otimes \mathfrak{g}, \quad \phi = \phi^a \otimes \xi_a \in W^2(\mathfrak{g}) \otimes \mathfrak{g}.$$

Let P be a principal G -bundle. $\Omega(P)$ is an L -module. The connection and curvature forms on P determine maps

$$\mathfrak{g}^* \rightarrow \Omega^1(P), \quad \mathfrak{g}^* \rightarrow \Omega^2(P)$$

which induce a homomorphism of commutative graded algebras

$$\phi_W : W(\mathfrak{g}) \rightarrow \Omega(P)$$

known as the Weil homomorphism.

Theorem 2.1 ([14]). ϕ_W is a morphism of L -modules. Moreover, it sends the universal connection and curvature of $W(\mathfrak{g})$ to the connection and curvature forms on P .

Now, let's consider the tensor product $W(\mathfrak{g}) \otimes \Omega(X)$. It has an L -module structure by defining the contractions, the Lie derivatives and the differential to be

$$\iota_a \otimes 1 + 1 \otimes \iota_a, \quad \text{Lie}_a \otimes 1 + 1 \otimes \text{Lie}_a, \quad d \otimes 1 + 1 \otimes d.$$

Let $\Omega_G(X)$ denote the basic part of $W(\mathfrak{g}) \otimes \Omega(X)$. Let d_W denote the differential on $\Omega_G(X)$ induced from the differential on $W(\mathfrak{g}) \otimes \Omega(X)$.

Definition 2.6. $(\Omega_G(X), d_W)$ is called the Weil Model for the equivariant cohomology of a G -manifold X . d_W is called the Weil differential.

Recall that differential forms on $P \times_G X$ can be identified with basic forms on $P \times X$. The Weil homomorphism induces a new homomorphism ϕ_{CW} of commutative graded algebras through the commutative diagram

$$\begin{array}{ccc} W(\mathfrak{g}) \otimes \Omega(X) & \xrightarrow{\phi_W} & \Omega(P \times X) \\ \uparrow & & \uparrow \\ \Omega_G(X) & \xrightarrow{\phi_{CW}} & \Omega(P \times_G X) \end{array}$$

ϕ_{CW} is known as the Chern-Weil homomorphism. It induces a homomorphism of cohomologies

$$H_G^*(X) \rightarrow H^*(P \times_G X)$$

which does not depend on the choice of connection on P . In particular, the Chern-Weil homomorphism induces a homomorphism $S(\mathfrak{g}^*)^G \rightarrow H(P/G)$ because $H_G(\text{pt}) \cong S(\mathfrak{g}^*)^G$.

Definition 2.7. The Mathai-Quillen map is the automorphism map $j = \exp(-\theta^a \otimes \iota_a)$ of $W(\mathfrak{g}) \otimes \Omega(X)$. The differential $d_K = j \circ d_W \circ j^{-1}$ is called the Kalkman differential. $(W(\mathfrak{g}) \otimes \Omega(X), d_K)$ is called the Kalkman model of the equivariant cohomology of a G -manifold X .

One can show that

$$d_K = d \otimes 1 + 1 \otimes d + \theta^a \otimes \text{Lie}_a - \phi^a \otimes \iota_a,$$

and that $\iota_a \otimes 1 = j \circ (\iota_a \otimes 1 + 1 \otimes \iota_a) \circ j^{-1}$, $(\text{Lie}_a \otimes 1 + 1 \otimes \text{Lie}_a) = j \circ (\text{Lie}_a \otimes 1 + 1 \otimes \text{Lie}_a) \circ j^{-1}$. Consequently, the basic part of $W(\mathfrak{g}) \otimes \Omega(X)$ in the Kalkman model of X is just $(S(\mathfrak{g}^*) \otimes \Omega(X))^G$, and the restriction of the Kalkman differential to $(S(\mathfrak{g}^*) \otimes \Omega(X))^G$, denoted by d_C , takes the form $d_C = d \otimes 1 - \phi^a \otimes \iota_a$.

Definition 2.8. $((S(\mathfrak{g}^*) \otimes \Omega(X))^G, d_C)$ is called the Cartan Model of a G -manifold X . d_C is called the Cartan differential.

Let $G = SO(n)$, $n = 2m$. Let ρ be the standard representation of G on $V = \mathbb{R}^n$. We also use V to denote the n -dimensional translation group and its Lie algebra. Let θ and ϕ be the universal connection and curvature of the Weil algebra of G . Let w^i and b_i denote the coordinate functions of V and V^* respectively. Let χ_i denote the odd coordinate functions of ΠV^* . In [14], Mathai and Quillen defined the following element¹

$$U = (2\pi)^{-n} \int d\chi db \exp \left(-b^t(b/2 + iw) + \frac{1}{2} \chi^t \phi \chi + i(dw + \theta w)^t \chi \right)$$

of degree n in $\Omega_G(V)$, where $\phi \chi = \phi^a \otimes \rho(\xi_a) \chi$, $\theta w = \theta^a \otimes \rho(\xi_a) w$, $\int d\chi db$ is the Berezin integral over the even variables b_i and the odd variables χ_i .

Proposition 2.1. U is closed in $\Omega_G(V)$.

To prove this proposition, let

$$L = b^t(b/2 + iw) - \frac{1}{2} \chi^t \phi \chi - i \chi^t (dw + \theta w). \quad (2.6)$$

L can be seen as an element in $W(\mathfrak{g}) \otimes \Omega(V^*) \otimes \Omega(V)$. Let $G \ltimes V$ be the semi-direct product of G and V induced by ρ . Let $G \ltimes V$ act on V through ρ only, i.e., we require that the translation part of the group acts trivially. Let $\mathfrak{g} \ltimes V$ denote its Lie algebra. With a slight abuse of notation, we use $\Omega(V^*)$ to denote $S(V^*) \otimes \Lambda(V^*)$. Note that $W(\mathfrak{g} \ltimes V)$ is isomorphic to $W(\mathfrak{g}) \otimes \Omega(V^*)$ as commutative graded algebras. Moreover,

¹The imaginary unit i is introduced so that we will get an integrable Gaussian function of x after integrating out the ‘‘auxiliary’’ field b .

Lemma 2.1. *The Weil model $(W(\mathfrak{g} \times V) \otimes \Omega(V), d_W)$ is isomorphic to the differential graded algebra $(W(\mathfrak{g}) \otimes \Omega(V^*) \otimes \Omega(V), s)$. The differential s takes the form*

$$s = d_K \otimes 1 + 1 \otimes d,$$

where $d_K = d \otimes 1 + 1 \otimes \delta_K + \theta^a \otimes \text{Lie}_a - \phi^a \otimes \iota_a$ is the Kalkman differential of $W(\mathfrak{g}) \otimes \Omega(V^*)$, with δ_K denoting the Koszul differential on $\Omega(V^*)$.

Proof. We need to show that the differential d on $W(\mathfrak{g} \times V)$ is equivalent to the Kalkman differential d_K on $W(\mathfrak{g}) \otimes \Omega(V^*)$. Let ξ_a and t^i be bases of \mathfrak{g} and V , respectively. We can write $\rho(\xi_a)t^i = \rho_{aj}^i t^j$. The Lie bracket of $\mathfrak{g} \times V$ is given by

$$[(\xi_a, t^i), (\xi_b, t^j)] = ([\xi_a, \xi_b], \rho(\xi_a)t^j - \rho(\xi_b)t^i) = (f_{ab}^c \xi_c, \rho_{ak}^j t^k - \rho_{bk}^i t^k).$$

It then follows from (2.4) that

$$\begin{aligned} d\theta^a &= \phi^a - \frac{1}{2} f_{bc}^a \theta^b \theta^c, & d\phi^a &= -f_{bc}^a \theta^b \phi^c, \\ d\chi^i &= b^i - \rho_{aj}^i \theta^a \chi^j, & db^i &= \rho_{aj}^i \phi^a \chi^j - \rho_{aj}^i \theta^a b^j. \end{aligned}$$

On the other hand, one easily can check that

$$\begin{aligned} \text{Lie}_a b^i &= -\rho_{aj}^i b^j, & \text{Lie}_a \chi^i &= -\rho_{aj}^i \chi^j, \\ \iota_a b^i &= -\rho_{aj}^i \chi^j, & \iota_a \chi^i &= 0. \end{aligned}$$

The rest of the proof is straightforward. \square

$W(\mathfrak{g} \times V) \otimes \Omega(V)$, or equivalently, $W(\mathfrak{g}) \otimes \Omega(V^*) \otimes \Omega(V)$, is also an L -module by setting

$$\text{Lie}_a = (\text{Lie}_a \otimes 1 + 1 \otimes \text{Lie}_a) \otimes 1 + 1 \otimes 1 \otimes \text{Lie}_a, \quad \iota_a = (\iota_a \otimes 1) \otimes 1 + 1 \otimes 1 \otimes \iota_a.$$

For simplicity, we omit the indices of the coordinate functions. The action of s on coordinate functions can then be written as

$$s\theta = \phi - \theta\theta, \quad s\phi = [\phi, \theta], \tag{2.7}$$

$$s w = d w, \quad s b = -\theta b + \phi \chi, \tag{2.8}$$

$$s d w = 0, \quad s \chi = b - \theta \chi. \tag{2.9}$$

Remark 2.3. It is always fun to check $s^2 = 0$ by direct computations. The non-trivial ones are $s^2 b = -(s\theta)b + \theta(sb) + (s\phi)\chi + \phi(s\chi) = -(\phi - \theta\theta)b + \theta(-\theta b + \phi\chi) + ([\phi, \theta])\chi + \phi(b - \theta\chi) = 0$, and $s^2 \chi = sb - s(\theta)\chi + \theta(s\chi) = (-\theta b + \phi\chi) - (\phi - \theta\theta)\chi + \theta(b - \theta\chi) = 0$.

Lemma 2.2. *L is exact in $(W(\mathfrak{g} \times V) \otimes \Omega(V))_{bas}$.*

Proof. The exactness of L follows from direct computations.

$$\begin{aligned} s(\chi^t(iw + b/2)) &= (b - \theta\chi)^t(iw + b/2) - \chi^t(i dw + (-\theta b + \phi\chi)) \\ &= b^t(iw + b/2) - (-\chi^t \theta^t)(iw + b/2) - \chi^t(i dw - \theta b/2) - \frac{1}{2} \chi^t \phi \chi \\ &= b^t(iw + b/2) - \frac{1}{2} \chi^t \phi \chi - \chi^t((i\theta w + \theta b/2) + (i dw - \theta b/2)) \\ &= b^t(iw + b/2) - \frac{1}{2} \chi^t \phi \chi - \chi^t i(dw + \theta w). \end{aligned}$$

We use the skew-symmetric property of θ in the third step. L is basic because $\chi^t(iw + b/2)$ is \mathfrak{g} -invariant and does not contain θ and dw . \square

Let $\mathcal{S}(V^*)$ denote the space of Schwartz functions over V^* . For every $\alpha \in \mathcal{S}(V^*) \otimes \Lambda(V^*)$, the super Fourier transform \mathcal{F} of α is defined by

$$\mathcal{F}(\alpha) = \int_{V^*} \alpha \exp(-i(b^t w - \chi^t dw)) \in \Omega(V).$$

It is not hard to show that \mathcal{F} commutes with both d and ι_a [10], hence also Lie_a .

Now, apply the Mathai-Quillen map $j = \exp(-\theta^a \otimes \iota_a)$ to $W(\mathfrak{g}) \otimes \Omega(V^*)$. This is equivalent to a change of coordinates which sends b to $b - \theta\chi$. In the new coordinates, we have $s = d \otimes 1 \otimes 1 + 1 \otimes \delta_K \otimes 1 + 1 \otimes 1 \otimes d$, and $L = s\alpha + i(b^t w - \chi^t dw)$, where $\alpha = \chi^t(b - \theta\chi)/2$. It is easy to see that α is a basic element in $W(\mathfrak{g}) \otimes \Omega(V^*)$. Moreover, the element $\exp(-s\alpha)$ is a closed basic element in $W(\mathfrak{g}) \otimes \Omega(V^*)_S$ due to the Gaussian factor $\exp(-b^t b/2)$. Proposition 2.1 is then proved by observing that

$$\int d\chi db \exp(-L) = \mathcal{F}(\exp(-s\alpha)).$$

Note that integrating out b and χ will give us a factor $(2\pi)^m$. The component of U with top de Rham degree is $(2\pi)^{-m} \exp(-\frac{1}{2}w^2) dw^1 \dots dw^n$. It follows that $\int_V U = 1$. Let P be a principal G -bundle over a manifold Σ . Let A be a connection 1-form on P . Let E be an associated vector bundle to P of rank $2m$ equipped with a metric (\cdot, \cdot) and a metric connection ∇ induced by A .

Theorem 2.2 ([14]). *U is a universal Thom form in the sense that for any such E , the Chern-Weil homomorphism ϕ_{CW} sends U to a form representing the Thom class of E .*

Remark 2.4. We can also consider the Kalkman model of $W(\mathfrak{g} \ltimes V) \otimes \Omega(V)$. The differential s is locally given by

$$sw = dw - \theta w, \quad sb = -\theta b + \phi\chi, \tag{2.10}$$

$$sdw = -\theta dw + \phi w, \quad s\chi = b - \theta\chi. \tag{2.11}$$

This is not the right choice for constructing a universal Thom class, as s restricted to $\Omega(V)$ is not the de Rham differential. However, we will need this kind of differentials later to construct cohomological field theories with gauge symmetries.

Let v be a section of E , we can also obtain a representative for the Euler class $e_{\nabla}(E)$ of E by setting

$$e_{\nabla}(E) = v^* \phi_{CW}(U) = (2\pi)^{-n} \int d\chi db \exp(-L),$$

where $L = (b, b/2 + iv) - \frac{1}{2}(\chi, R\chi) - i\chi(\nabla v)$. Let x^μ denote the coordinate functions of Σ . Identifying dx^μ with the odd variables η^μ , we have

$$L = (b, b/2 + iv) - i\chi(\nabla_\eta v) - \frac{1}{4}(\chi, R(\eta, \eta)\chi). \tag{2.12}$$

Let's take E to be $T\Sigma$, (\cdot, \cdot) to be a Riemannian metric g , and ∇ to be the Levi-Civita connection determined by g . We have

$$sx^\mu = \eta^\mu, \quad sb_\mu = \Gamma_{\rho\mu}^\nu \eta^\rho b_\nu - \frac{1}{2} R_{\mu\rho\sigma}^\nu \eta^\rho \eta^\sigma \chi_\nu \quad (2.13)$$

$$s\eta^\mu = 0, \quad s\chi_\mu = b_\mu + \Gamma_{\rho\mu}^\nu \eta^\rho \chi_\nu, \quad (2.14)$$

where $\Gamma_{\rho\mu}^\nu$ is the Christoffel symbol and $R_{\mu\rho\sigma}^\nu$ is the Riemann curvature tensor. (2.12) together with the supersymmetry transformations (2.13) and (2.14) give us a 0-dimensional supersymmetric theory [11].

Remark 2.5. The Mathai-Quillen map j of $W(\mathfrak{g}) \otimes \Omega(V^*)$ induces a change of coordinates, namely,

$$b_\mu \rightarrow b_\mu + \Gamma_{\rho\mu}^\nu \eta^\rho \chi_\nu.$$

The differential s in the new coordinates takes the form

$$\begin{aligned} sx^\mu &= \eta^\mu, & sb_\mu &= 0, \\ s\eta^\mu &= 0, & s\chi_\mu &= b_\mu, \end{aligned}$$

which is exactly the BRST differential s appearing in [4]. The price one pays for this simplification is that L will no longer be covariant unless b is integrated out.

3 Graded and Bigraded Manifolds

Let \mathcal{I} be a cancellative commutative semiring. Let $V = \bigoplus_{i \in \mathcal{I}} V_i$ be an \mathcal{I} -graded vector space with $\dim V_i = m_i$. The symmetric algebra $S(V)$ is the quotient algebra of the tensor algebra $T(V)$ by the two-sided ideal generated by

$$v \otimes w - (-1)^{p(d(v)d(w))} w \otimes v, \quad (3.1)$$

where we use $d(u)$ to denote the degree of $u \in V$, $p : \mathcal{I} \rightarrow \mathbb{Z}_2$ is the parity function, $(-1)^{(\cdot)}$ is the sign function of \mathbb{Z}_2 which sends $0 \in \mathbb{Z}_2$ to 1 and $1 \in \mathbb{Z}_2$ to -1 .

Remark 3.1. Here we have to be careful about the sign factor appearing on the right hand side of (3.1). Although both \mathcal{I} and \mathbb{Z}_2 are semi-rings, p is not necessarily a semi-ring homomorphism and we do not have $p(d(v)d(w)) = p(d(v))p(d(w))$ in general. Bearing this in mind, we will use the more familiar notation $p(v)p(w)$ to replace $p(d(v)d(w))$ from now on.

Remark 3.2. From now on, we will often use

$$[v, w] = vw - (-1)^{p(d(v)d(w))} wv$$

to denote the commutator bracket of v and w in an \mathcal{I} -graded algebra.

Definition 3.1. An \mathcal{I} -graded manifold \mathcal{M} of dimension $n|(m_i)_{i \in \mathcal{I}}$ is a ringed space (M, \mathcal{O}) where M is an n -dimensional manifold and \mathcal{O} is a sheaf of commutative \mathcal{I} -graded algebras over M which is locally isomorphic to $C^\infty(\mathbb{R}^n) \otimes S(V)$.

Remark 3.3. In fact, one should consider the algebra of formal power series on V instead of $S(V)$ in Definition 3.1 [7]. Since this subtlety will not infect our discussion later, we stick with the $S(V)$ version for simplicity.

An \mathcal{I} -graded vector bundle $\pi : E \rightarrow M$ is a vector bundle such that the local trivialization map $\varphi_U : \pi^{-1}(U) \rightarrow U \times V$ is a morphism of \mathcal{I} -graded vector spaces when restricted to $\pi^{-1}(x)$, $x \in U \subset M$. In other words, $E = \bigoplus_{k \in \mathcal{I}} E_k$ where E_k are vector bundles whose fibers consist of elements of degree k . To any \mathcal{I} -graded vector bundle E we can associate an \mathcal{I} -graded manifold $E[l]$ whose structure sheaf is defined as the sheaf of sections of $S(\bigoplus_{k \in \mathcal{I}} (E_{k+l})^*)$, $l \in \mathcal{I}$. It is shown in [7] that every \mathcal{I} -graded manifold can be obtained (non-canonically) in this way.

Remark 3.4. It is also possible to generalize other geometric/algebraic objects to the \mathcal{I} -graded case. We refer the reader to [7] for more detail. In this paper, we often use such objects directly without giving their definitions.

3.1 L-manifolds

Definition 3.2. A Q -manifold is a graded manifold \mathcal{M} equipped with a vector field Q of degree 1 such that $Q^2 = \frac{1}{2}[Q, Q] = 0$. Q is called the cohomological vector field of \mathcal{M} .

Example 3.1. Let M be a manifold. Let $\mathcal{M} = T[1]M$. Let (x^μ, η^μ) be a local coordinate system of \mathcal{M} . We define

$$Q = \eta^\mu \frac{\partial}{\partial x^\mu}.$$

The Q -cohomology of \mathcal{M} is the de Rham cohomology of M .

Example 3.2. Let E be a vector bundle over M . Let $\mathcal{M} = E[1]$. Let (x^μ, θ^a) be a local coordinate system of \mathcal{M} . We define

$$Q = -\frac{1}{2} f_{bc}^a(x) \theta^b \theta^c \frac{\partial}{\partial \theta^a} + \rho_a^\mu(x) \theta^a \frac{\partial}{\partial x^\mu}.$$

The condition $Q^2 = 0$ is equivalent to requiring that E equipped with the Lie bracket $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ and the anchor map $\rho : E \rightarrow TM$ induced by $f_{bc}^a(x)$ and $\rho_a^\mu(x)$ is a Lie algebroid [15].

In the case where $E = TM$, $f_{bc}^a(x) = 0$ and $\rho_a^\mu(x) = \delta_a^\mu$, we recover Example 3.1.

As another special case, we can take M to be a G -manifold, E to be the trivial bundle $M \times \mathfrak{g}$ over M , $f_{bc}^a(x)$ to be the structure constants of \mathfrak{g} , and $\rho_a^\mu(x)$ to be induced from the infinitesimal action of \mathfrak{g} on M . The Q -cohomology of \mathcal{M} is nothing but the Lie algebra cohomology of \mathfrak{g} with coefficients in the \mathfrak{g} -module $C^\infty(M)$. The Q -closed condition of a function S of degree 0 is equivalent to the G -invariance condition of S .

Q -manifolds provide the most general setting for studying cohomological field theories. However, it remains to find a Q -closed function S of even degree. S will be considered as the action functional of the corresponding physical theory.

Definition 3.3. A P -manifold is a graded manifold \mathcal{M} equipped with a non-degenerate closed 2-form ω of odd degree n .

Since ω is non-degenerate, to any $f \in \mathcal{O}(M)$ of degree p one can associate a Hamiltonian vector field v_f of degree $p - n$ by $\iota_{v_f}\omega + df = 0$. On the other hand, ω induces a (graded) Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{O}(M)$ of degree $-n$ by setting $\{f, g\} = (-1)^{d(f)}\iota_{v_f}\iota_{v_g}\omega$. Note that $\{f, g\} = -(-1)^{d(f)}v_f(g)$. It is straightforward to verify that $v_{\{f, g\}} = [v_f, v_g]$.

Definition 3.4. A QP -manifold is a P -manifold equipped with a Hamiltonian cohomological vector field Q .

QP -manifolds were first defined in [6]. They provide a powerful mechanism to construct the action functional S of a CohFT by setting S to be the Hamiltonian function associated to Q .

Remark 3.5. The cohomological vector field Q can be seen as the generator of a $(0|1)$ -dimensional Lie superalgebra \mathfrak{q} . From this point of view, a Q -manifold is just a graded manifold equipped with an action of \mathfrak{q} . As is shown in Example 3.2, the bosonic symmetries of an action functional S can sometimes be fully captured by the \mathfrak{q} -invariance condition of S . If S has also fermionic symmetries, \mathfrak{q} should be replaced by the Lie graded algebra L defined in Section 2.1. In fact, the fermionic symmetries of S can be captured by the condition $\iota_a S = 0$. The Q -closed condition of S then implies directly that $\text{Lie}_a S = 0$ via Cartan's magic formula $Q\iota_a + \iota_a Q = \text{Lie}_a$. (With a slight abuse of notation, we use Q to denote the differential of L .)

Definition 3.5. An L -manifold is a graded manifold equipped with an L -action.

By definition, every L -manifold is a Q -manifold. The commutative graded algebra $\mathcal{O}(M)$ of functions over \mathcal{M} is an L -module. In particular, the L -module $W(\mathfrak{g}) \otimes \Omega(X)$ associated to a G -manifold X can now be viewed as the commutative graded algebra of functions over the L -manifold $T[1]X \oplus \mathfrak{g}[1] \oplus \mathfrak{g}[2]$, where \mathfrak{g} is the trivial bundle $X \times \mathfrak{g}$. For a general L -manifold \mathcal{M} , the cohomology of $\mathcal{O}(M)_{bas}$ can be then viewed as a generalization of the equivariant cohomology.

We want to construct a basic Q -closed action functional S .

Definition 3.6. An LP -manifold is a P -manifold equipped with a Hamiltonian L -action. Namely, the fundamental vector fields generated by Lie_a , Q , ι_a in L are Hamiltonian vector fields.

We denote the Hamiltonian functions associated to Lie_a and ι_a by L_a and I_a , respectively.

Example 3.3. Let $\mathcal{M} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[2]$, where \mathfrak{g} is viewed as a bundle over a point. Let (θ^a, ϕ_b) be a (local) coordinate system. The canonical symplectic form on \mathcal{M} takes the form $\omega = d\theta^a \wedge d\phi_a$. In this case, the Hamiltonian functions I_a , L_a and S are of degrees 2, 3 and 4, respectively. We can write

$$\begin{aligned} I_a &= f_{1a}^b \phi_b + f_{2abc} \theta^b \theta^c, \\ L_a &= h_{2ac}^b \phi_b \theta^c + h_{3abcd} \theta^b \theta^c \theta^d \\ S &= g_2^{ab} \phi_a \phi_b + g_{3bc}^a \phi_a \theta^b \theta^c + g_{4abcd} \theta^a \theta^b \theta^c \theta^d, \end{aligned}$$

generally. They need to satisfy

$$\{L_a, I_b\} = f_{ab}^c I_c, \quad \{S, S\} = 0, \quad \{I_a, I_b\} = 0, \quad \{S, I_a\} = L_a.$$

When \mathfrak{g} is compact and semi-simple,² one can find a set of solutions

$$I_a = \phi_a, \quad L_a = -f_{ac}^b \phi_b \theta^c, \quad S = \frac{1}{2} g^{ab} \phi_a \phi_b - \frac{1}{2} f_{bc}^a \phi_a \theta^b \theta^c,$$

²This property guarantees that $f_{ab}^c = f_{bc}^a = f_{ca}^b$ in orthonormal coordinates.

where g^{ab} is the Killing metric on \mathfrak{g} . Using g^{ab} to identify \mathfrak{g}^* with \mathfrak{g} , these solutions recover the L -module structure of the Weil algebra $W(\mathfrak{g})$.

The Hamiltonian function S associated to Q can not be basic in general, because

$$\iota_a(S) = -\{I_a, S\} = -L_a.$$

and L_a cannot be 0 when the even part of the L -action is non-trivial. In other words, LP -manifolds cannot help us to find basic Q -closed S like the way QP -manifolds help one to find Q -closed S .

3.2 QK-manifolds

We are particularly interested in the class of L -manifolds where \mathfrak{g} is abelian. The L -structure of such graded manifold is given by $2n + 1$ vector fields satisfying

$$[\text{Lie}_\mu, \text{Lie}_\nu] = 0, \quad [\text{Lie}_\mu, Q] = 0, \quad [\text{Lie}_\mu, \iota_\nu] = 0, \quad (3.2)$$

$$[Q, Q] = 0, \quad [Q, \iota_\mu] = \text{Lie}_\mu, \quad [\iota_\mu, \iota_\nu] = 0, \quad (3.3)$$

for $\mu, \nu = 1, \dots, n$, where $n = \dim \mathfrak{g}$.

Example 3.4. Let $\mathcal{M} = T[1]\mathbb{R}^n$. Let (x^μ, θ^ν) be a (local) coordinate system. The L -structure is given by

$$Q = \theta^\mu \frac{\partial}{\partial x^\mu}, \quad \iota_\mu = \frac{\partial}{\partial \theta^\mu}, \quad \text{Lie}_\mu = \frac{\partial}{\partial x^\mu}. \quad (3.4)$$

It is not hard to check that (3.4) satisfy (3.2) and (3.3). There is no interesting basic Q -closed function S on \mathcal{M} . This is because $I_\mu(S) = 0$ and $L_\mu(S) = 0$ force S to be independent of x^μ and θ^μ , hence the only possible candidates for such S are constant functions.

Example 3.4 can be easily generalized to a non-flat case. To achieve that, we need to work in a bigraded setting instead of the graded setting. The parity function is defined by

$$p : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_2 \\ (i, j) \mapsto i + j \pmod{2}$$

Note that

$$p(d(x)d(y)), \quad p(d(x))p(d(y))$$

are not the same in this case. As mentioned before, we use the first one as our sign convention for the bigraded setting. We say a commutative bigraded algebra A is of the first (or the second) kind if $p(a)p(b)$ is set to be $p(d(a)d(b))$ (or $p(d(x))p(d(y))$), for $a, b \in A$. These two conventions can be connected by the following lemma.

Lemma 3.1. *Let A be a commutative bigraded algebra of the first kind. Let A' be a bigraded algebra with the same underlying bigraded vector space as A and a new algebraic product $\cdot_{A'}$ defined by*

$$a \cdot_{A'} b = (-1)^{j_a i_b} ab$$

where a is of degree (i_a, j_a) and b is of degree (i_b, j_b) . A' is then a commutative bigraded algebra of the second kind. If D is a derivation of A of degree (i, j) , then D' defined by

$$D'(a) = (-1)^{j i_a} D(a)$$

is a derivation of A' .

Proof. By definition, $a \cdot_{A'} b = (-1)^{j_a i_b} ab = (-1)^{j_a i_b + i_a i_b + j_a j_b} ba = (-1)^{(i_a + j_a)(i_b + j_b)} b \cdot_{A'} a$. We need to verify that D' satisfies Leibniz's rule.

$$\begin{aligned} D'(a \cdot_{A'} b) &= (-1)^{j(i_a + i_b)} (-1)^{j_a i_b} D(ab) \\ &= (-1)^{j(i_a + i_b) + j_a i_b} (D(a)b + (-1)^{i_a + j j_a} aD(b)) \\ &= (-1)^{j(i_a + i_b) + j_a i_b} ((-1)^{(j + j_a) i_b} D(a) \cdot_{A'} b + (-1)^{i_a + j j_a + j_a(i + i_b)} a \cdot_{A'} D(b)) \\ &= D'(a) \cdot_{A'} b + (-1)^{(i + j)(i_a + j_a)} a \cdot_{A'} D'(b). \end{aligned}$$

□

Example 3.5. Consider the bigraded manifold $\mathcal{M} = (T \oplus T)[(1, 1)]M$ where M is an n -dimensional manifold. Let $(x^\mu, \eta^\nu, \theta^\sigma)$ be a local coordinate system where η^ν are of degree $(1, 0)$ while θ^σ are of degree $(0, 1)$. Vector fields analogous to those in Example 3.4 are given by

$$Q = \theta^\mu \frac{\partial}{\partial x^\mu}, \quad K = \eta^\mu \frac{\partial}{\partial \theta^\mu}, \quad L = \eta^\mu \frac{\partial}{\partial x^\mu}. \quad (3.5)$$

It is easy to check that instead of (3.2) and (3.3), they satisfy

$$Q^2 = 0, \quad L^2 = 0, \quad (3.6)$$

$$QL - LQ = 0, \quad KL + LK = 0, \quad QK + KQ = L. \quad (3.7)$$

Note that K^2 does not vanish and that the relations $QL - LQ = 0$ and L^2 are not independent of the rest of (3.6) and (3.7). In fact, we have

$$QL - LQ = Q(QK + KQ) - (QK + KQ)Q = Q^2K - KQ^2 = 0$$

by using $Q^2 = 0$ and $L = QK + KQ$, and

$$L^2 = \frac{1}{2}(L(QK + KQ) + (QK + KQ)L) = \frac{1}{2}(QLK - KLQ - QLK + K LQ) = 0$$

by using $KL + LK = 0$ and $L = QK + KQ$.

Remark 3.6. If we use the second convention for $p(x)p(y)$, we will have

$$QL + LQ = 0, \quad KL - LK = 0, \quad QK - KQ = L. \quad (3.8)$$

instead of (3.7).

Definition 3.7. The QK Lie bigraded algebra is the Lie bigraded algebra \mathcal{K} spanned by Q of degree $(0, 1)$, K of degree $(1, -1)$ and L of $(1, 0)$ with brackets

$$[Q, Q] = [K, K] = [L, L] = [Q, L] = [K, L] = 0, \quad [Q, K] = L.$$

Definition 3.8. A QK -manifold is a bigraded manifold $\mathcal{M} = (M, \mathcal{O})$ equipped with a \mathcal{K} -action. Equivalently, \mathcal{M} is a QK -manifold if it has three vector fields Q of degree $(0, 1)$, K of degree $(1, -1)$ and L of degree $(1, 0)$ satisfying

$$Q^2 = 0, \quad QK + KQ = L, \quad LK + KL = 0.$$

Definition 3.9. A QK -algebra of length n is the unital associative bigraded algebra \mathcal{K}_n generated by the generators Q of degree $(0, 1)$, K of degree $(1, -1)$ and L of degree $(1, 0)$ subject to the relations

$$Q^2 = 0, \quad QK + KQ = L, \quad LK + KL = 0. \quad (3.9)$$

and

$$K^{n+1} = 0 \quad (3.10)$$

for some $n \in \mathbb{N}$.

There is a canonical inclusion $i_{n,m} : \mathcal{K}_n \hookrightarrow \mathcal{K}_m$ for $m \geq n$. We refer to the direct limit of the system $((\mathcal{K}_n)_{n \in \mathbb{N}}, (i_{n,m})_{n \leq m \in \mathbb{N}})$ as the QK -algebra of length ∞ and denote it by \mathcal{K}_∞ . Clearly, \mathcal{K}_∞ is the unital associative bigraded algebra generated by the generators Q , K , and L subject only to the relations (3.9).

Remark 3.7. Note that the commutative bigraded algebra of functions over a QK -manifold is canonically a \mathcal{K}_∞ -module (and a \mathcal{K}_n -module if $K^{n+1} = 0$). In particular, it can be viewed as a bicomplex with horizontal differential L and vertical differential Q .

Let A be a unital associative bigraded algebra. It can be viewed as a Lie bigraded algebra in a natural way. Let $\phi : \mathcal{K} \rightarrow A$ be a morphism of Lie bigraded algebras.

Proposition 3.1. \mathcal{K}_∞ is the universal enveloping algebra of \mathcal{K} . That is, there exists a unique morphism $\tilde{\phi} : \mathcal{K}_\infty \rightarrow A$ of unital associative bigraded algebras such that the following diagram commutes,

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\iota} & \mathcal{K}_\infty \\ & \searrow \phi & \downarrow \tilde{\phi} \\ & & A \end{array}$$

where $\iota : \mathcal{K} \hookrightarrow \mathcal{K}_\infty$ is the canonical inclusion.

Proof. This follows directly from the construction of \mathcal{K}_∞ . □

Lemma 3.2. Every element α in \mathcal{K}_n (or \mathcal{K}_∞) can be uniquely written in the form

$$\alpha = p_0(K) + p_1(K)Q + p_2(K)L + p_3(K)QL, \quad (3.11)$$

where $p_i(K)$ are polynomials in K .

Proof. Consider a word w where K appears to the right of Q , i.e.,

$$w = \cdots QK \cdots .$$

One can replace QK by $L - QK$ and move L from the middle to the rightmost using the properties $LQ = QL$ and $LK = -KL$ to get

$$w = \cdots KQ \cdots + \cdots L.$$

Repeating this procedure until there is no K on the right of Q , one has

$$w = \sum w_K w_{QL}$$

where w_K is expressed purely in K , w_{QL} is expressed purely in Q and L . The proof is completed by using $Q^2 = L^2 = 0$ and $QL = LQ$. \square

Corollary 3.1. $\mathcal{K}_\infty \cong S[\mathcal{K}]$ as bigraded vector spaces.

Remark 3.8. Corollary 3.1 can be viewed as a special case of the Poincaré–Birkhoff–Witt theorem in the bigraded setting.

We prove the following lemmas for later use.

Lemma 3.3. Let $p(K)$ be a polynomial in K , Then

$$[Q, p(K)] = Lp'(K),$$

where $[\cdot, \cdot]$ is the canonical Lie bigraded bracket on \mathcal{K}_n (or \mathcal{K}_∞), p' is the derivative of p .

Proof. Note that $[Q, \cdot]$ is a derivation of degree $(0, 1)$, we have

$$\begin{aligned} [Q, K^p] &= [Q, K]K^{p-1} - K[Q, K]K^{p-2} + \cdots \\ &= LK^{p-1} - K L K^{p-2} + \cdots \\ &= p L K^{p-1}, \end{aligned}$$

where we use $KL = -LK$. \square

Note that $\exp(K) := \sum_{p=0}^{\infty} \frac{1}{p!} K^p$ is a well-defined element in \mathcal{K}_n since K is nilpotent.

Lemma 3.4. Let Ω be a \mathcal{K}_n -module. We have

$$\exp(K)(\alpha\beta) = (\exp(K)\alpha)(\exp(K)\beta)$$

for $\alpha, \beta \in \Omega$ with $p(d(\alpha)) = 0$.

Proof. Since $p(d(\alpha)) = 0$, we have $K(\alpha\beta) = (K\alpha)\beta + \alpha(K\beta)$. More generally, we have

$$\begin{aligned} \frac{1}{p!} K^p(\alpha\beta) &= \frac{1}{p!} \sum_{j=0}^p \binom{p}{j} (K^j \alpha)(K^{p-j} \beta) \\ &= \sum_{j=0}^p \frac{1}{j!(p-j)!} (K^j \alpha)(K^{p-j} \beta). \end{aligned}$$

It follows that

$$\begin{aligned}
\exp(K)(\alpha\beta) &= \sum_{p=0}^{\infty} \sum_{j=0}^p \frac{1}{j!(p-j)!} (K^j\alpha)(K^{p-j}\beta) \\
&= \sum_{j=0}^{\infty} \sum_{p=j}^{\infty} \frac{1}{j!(p-j)!} (K^j\alpha)(K^{p-j}\beta) \\
&= \left(\sum_{j=0}^{\infty} \frac{1}{j!} K^j\alpha \right) \left(\sum_{l=0}^{\infty} \frac{1}{l!} K^l\beta \right) \\
&= (\exp(K)\alpha)(\exp(K)\beta).
\end{aligned}$$

□

4 Cohomological Lagrangian Field Theories

4.1 The General Setting of CohFTs

Let $\pi : Y \rightarrow M$ be a fiber bundle over an n -dimensional compact manifold M with fiber Z being an m -dimensional manifold. Let $\Gamma(Y)$ be the space of sections of Y . Let's consider the de Rham complex $\Omega(M \times \Gamma(Y))$ of differential forms on $M \times \Gamma(Y)$. It is a bicomplex bigraded according to the product structure of $M \times \Gamma(Y)$, i.e.,

$$\Omega(M \times \Gamma(Y)) = \bigoplus_{p,q} \Omega^{p,q}(M \times \Gamma(Y)).$$

Correspondingly, the de Rham differential d_{tot} on $M \times \Gamma(Y)$ breaks into two parts $d_{tot} = d + \delta$, where d is the de Rham differential on M and δ is the de Rham differential on $\Gamma(Y)$.

There is a canonical sub-bicomplex $\Omega_{loc}(M \times \Gamma(Y))$ [16]. Let $J^\infty Y$ be the infinite jet bundle of Y over M . Let ev be the evaluation map from $M \times \Gamma(Y)$ to $J^\infty Y$, i.e.,

$$\begin{aligned}
ev : M \times \Gamma(Y) &\rightarrow J^\infty(Y) \\
(x, \psi) &\mapsto j^\infty(\psi)(x),
\end{aligned}$$

where $j^\infty(\psi)(x)$ is the infinite jet prolongation of ψ at x . The pull-back $ev^*\Omega(J^\infty Y)$ is stable under both d and δ , hence a sub-bicomplex, which is called the variational bicomplex of Y and denoted by $\Omega_{loc}(M \times \Gamma(Y))$. Elements in $\Omega_{loc}(M \times \Gamma(Y))$ are called local forms. d and δ restricted to $\Omega_{loc}(M \times \Gamma(Y))$ are called the horizontal differential, denoted by d_h , and the vertical differential, denoted by d_v , respectively.

Let U be an open neighborhood of M such that $\pi^{-1}U \cong U \times Z$. Let V be a coordinate chart of Z . Y can then be covered by coordinate charts of the form $U \times V$ with coordinate functions $x^1, \dots, x^n, u^1, \dots, u^m$. Let $\mathcal{W}(U, V)$ be the set of pairs (x, ψ) such that $\psi(x)$ is in V , it is then an open neighborhood of $M \times \Gamma(Y)$. We can explicitly define functions x^μ and u_I^j on $\mathcal{W}(U, V)$ by setting

$$x^\mu(x, \psi) = x^\mu(x), \quad u_I^j(x, \psi) = \partial_I(u^j(\psi(x))),$$

where ∂_I is the partial derivative in x^μ with respect to the multi-index $I = (\mu_1, \dots, \mu_n)$. By definition, a local function on $\mathcal{W}(U, V)$ depends only on finitely many of x^μ and u_I^j . In particular, x^μ and u_I^j themselves are local functions. Their derivatives dx^μ and δu_I^j can be viewed as local forms of degree $(1, 0)$ and $(0, 1)$ respectively. We can write any local (k, l) -form ω as a finite sum

$$\omega = f_{\mu_1, \dots, \mu_k, j_1, \dots, j_l}^{I_1, \dots, I_l} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \wedge \delta u_{I_1}^{j_1} \wedge \dots \wedge \delta u_{I_l}^{j_l},$$

where each $f_{(\dots)}^{(\dots)}$ is a local function.

In fact, the above discussion can be easily generalized to the case of a ‘‘graded fiber bundle’’ Y . That is, we require Y to be a graded vector bundle over Y_0 with fiber consisting of elements of nonzero degree, where Y_0 is a fiber bundle over M with fiber consisting of elements of degree 0. The only subtleties of this generalization are

1. we should assign degree $(0, d(u^j))$ to the local function u_I^j and degree $(0, d(u^j) + 1)$ to the local form δu_I^j where $d(u^j)$ is the degree of u^j induced from the grading of Y ;
2. a local function should be polynomial in u_I^j when $d(u^j) \neq 0$.

Local forms obtained in this way can be seen as a sheaf of commutative bigraded algebras of the second kind over $M \times \Gamma(Y_0)$. We can turn this sheaf into a sheaf of commutative bigraded algebras of the first kind by applying Lemma 3.1. In other words, we obtain an infinite dimensional bigraded manifold \mathcal{M}_Y from Y . We call \mathcal{M}_Y the variational bigraded manifold of Y . Local forms can be viewed simply as functions over \mathcal{M}_Y .

Remark 4.1. In the physics literature, the vertical degree of a local form ω viewed as a function over \mathcal{M}_Y is called as the ghost number of ω . One should not confuse it with the vertical degree of ω viewed as a differential form.

The differentials d_h and d_v can be viewed as vector fields over \mathcal{M}_Y of degree $(1, 0)$ and $(0, 1)$, respectively. (Note that by Lemma 3.1, $d_h d_v - d_v d_h = 0$.) They act on x^μ , u_I^j , dx^μ and δu_I^j as

$$\begin{aligned} d_h(x^\mu) &= dx^\mu, & d_h(u_I^j) &= u_{I \cup \{\mu\}}^j dx^\mu, & d_h(dx^\mu) &= 0, & d_h(\delta u_I^j) &= \delta u_{I \cup \{\mu\}}^j dx^\mu, \\ d_v(x^\mu) &= 0, & d_v(u_I^j) &= \delta u_I^j, & d_v(dx^\mu) &= 0, & d_v(\delta u_I^j) &= 0. \end{aligned}$$

We write $d_h = d_{h1} + d_{h2}$ where d_{h1} and d_{h2} are defined by

$$\begin{aligned} d_{h1}(x^\mu) &= dx^\mu, & d_{h1}(u_I^j) &= 0, & d_{h1}(dx^\mu) &= 0, & d_{h1}(\delta u_I^j) &= 0, \\ d_{h2}(x^\mu) &= 0, & d_{h2}(u_I^j) &= u_{I \cup \{\mu\}}^j dx^\mu, & d_{h2}(dx^\mu) &= 0, & d_{h2}(\delta u_I^j) &= \delta u_{I \cup \{\mu\}}^j dx^\mu. \end{aligned}$$

Proposition 4.1. *There is a canonical QK-structure on a variational bigraded manifold where $Q = d_v$, $L = d_{h2}$ and K is defined as follows*

$$K(x^\mu) = 0, \quad K(u_I^j) = 0, \quad K(dx^\mu) = 0, \quad K(\delta u_I^j) = u_{I \cup \{\mu\}}^j dx^\mu.$$

Proof. K is a globally well-defined vector field of degree $(1, -1)$. One can easily check that $QK + KQ = L$ and $KL + LK = 0$. \square

Definition 4.1. A QK_v -manifold is a variational bigraded manifold equipped with a \mathcal{K} -action such that the fundamental vector field generated by $L \in \mathcal{K}$ coincides with d_{h2} .

A QK_v -structure on a variational bigraded manifold \mathcal{M}_Y is hence a generalization of the canonical QK -structure on \mathcal{M}_Y .

Definition 4.2. A cohomological (Lagrangian) field theory is a pair $(\mathcal{M}_Y, \mathcal{L})$ where \mathcal{M}_Y is a QK_v -manifold and \mathcal{L} is a Q -closed function on \mathcal{M}_Y of degree $(n, 0)$.

Let Γ_Y denote the graded manifold $(\Gamma(Y), \Omega)$ where Ω is the sheaf of differential forms on $\Gamma(Y)$. (We call Γ the (graded) configuration space of the corresponding theory.) The action functional of \mathcal{L} is a function S of degree 0 on Γ_Y defined by

$$S = \int_M \mathcal{L}.$$

In most cases, the cohomological vector field Q does not depend on coordinates x^μ and dx^μ . Hence it can be viewed as a cohomological vector field on Γ_Y . We have $QS = Q(\int_M \mathcal{L}) = \int_M Q\mathcal{L} = 0$. In other words, Γ_Y is a Q -manifold equipped with a Q -closed function S .

Remark 4.2. From now on, every function over a variational bigraded manifold considered by us will be assumed implicitly to be independent of x^μ . For this reason, we will often not distinguish $L = d_{h_2}$ and the de Rham differential d .

Definition 4.3. A pre-observable \mathcal{O} is a function on \mathcal{M}_Y such that $Q\mathcal{O}$ is d -exact and $d\mathcal{O}$ is Q -exact. An observable is a Q -closed function on Γ_Y .

Let γ be a submanifold representative of a p -cycle in M . Let $\mathcal{O}[\gamma]$ be a function on Γ_Y defined by $\mathcal{O}[\gamma] = \int_\gamma \mathcal{O}$. $\mathcal{O}[\gamma]$ is Q -closed, hence an observable on Γ_Y . Note that the Q -cohomology class of $\mathcal{O}[\gamma]$ is independent of the choice of representatives of γ . In other words, we have a well-defined map $H_\bullet(M) \rightarrow H^\bullet(\Gamma)$ defined by sending γ to $\mathcal{O}[\gamma]$.

Definition 4.4. Let $\mathcal{O}^{(0)}$ be a Q -closed function of degree $(0, n)$. The descendant sequence of $\mathcal{O}^{(0)}$ is a sequence of pre-observables $\{\mathcal{O}^{(p)}\}_{p=0}^n$ of degrees $(0, n), (1, n-1), \dots, (n, 0)$, satisfying

$$Q\mathcal{O}^{(p)} = d\mathcal{O}^{(p-1)} \tag{4.1}$$

for $p = 1, \dots, n$. (4.1) is called the (topological) descent equations.

Definition 4.5. Let $\mathcal{O}^{(0)}$ be a Q -closed function of degree $(0, n)$. The standard K -sequence of $\mathcal{O}^{(0)}$ is a sequence $\{\mathcal{O}^{(p)}\}_{p=0}^n$, where

$$\mathcal{O}^{(p)} := \frac{1}{p!} K^p \mathcal{O}^{(0)}$$

for $p = 1, \dots, n$.

Proposition 4.2. *The standard K -sequence is a descendant sequence.*

Proof. We have $Q\mathcal{O}^{(p)} = \frac{1}{p!} QK^p \mathcal{O}^{(0)} = \frac{1}{p!} [Q, K^p] \mathcal{O}^{(0)} = \frac{1}{(p-1)!} LK^{p-1} \mathcal{O}^{(0)} = d\mathcal{O}^{(p-1)}$ for $p \geq 1$, where we use $Q\mathcal{O}^{(0)} = 0$ and Lemma 3.3. \square

Definition 4.6. Let $\mathcal{W}^{(q)}$ be a Q -closed function of degree $(q, n - q)$, $1 \leq q \leq n$. A (general) K -sequence of $\mathcal{O}^{(0)}$ is a sequence $\{\mathcal{O}^{(p)}\}_{p=0}^n$, where

$$\mathcal{O}^{(p)} := \frac{1}{p!} K^p \mathcal{O}^{(0)} + \sum_{q=1}^p \frac{1}{(p-q)!} K^{p-q} \mathcal{W}^{(q)}$$

for $p = 1, \dots, n$.

Likewise, one can show that

Proposition 4.3. *Every (general) K -sequence is a descendant sequence.*

Remark 4.3. In physics literature [17] [18], the vector field K is known as the vector supersymmetry. A similar result as Proposition 4.2 is also proved in [19]. (See Proposition 5.14 there.) What we will show later is that the converse of Proposition 4.3 is also true in a cohomological sense.

Lemma 4.1. *Let $\{\mathcal{O}^{(i)}\}_{i=0}^n$ be such that $\mathcal{O}^{(i)} = Q\rho^{(i)} + d\rho^{(i-1)}$ for $i > 0$ and $\mathcal{O}^{(0)} = Q\rho^{(0)}$, where $\rho^{(i)}$ is an arbitrary function of degree $(i, n - i - 1)$. Then, $\{\mathcal{O}^i\}_{i=0}^n$ is a solution to (4.1).*

Proof. $Q\mathcal{O}^{(p)} = Q(Q\rho^{(p)} + d\rho^{(p-1)}) = d(Q\rho^{(p-1)}) = d\mathcal{O}^{(p-1)}$. □

Definition 4.7. A sequence of the form in Lemma 4.1 is called an exact sequence.

Recall that the functions over a bigraded manifold form a bicomplex

$$\Omega = \bigoplus_{(p,q) \in [0, \dots, n] \times \mathbb{Z}} \Omega^{p,q}$$

with commuting differentials Q and L . Moreover, L is homotopic to 0 since $L = QK + KQ$, where K is interpreted as a homotopy operator.

$$\begin{array}{ccccccc} \dots & \xrightarrow{Q} & \Omega^{p,q-1} & \xrightarrow{Q} & \Omega^{p,q} & \xrightarrow{Q} & \Omega^{p,q+1} & \xrightarrow{Q} & \dots \\ & \swarrow K & \downarrow L & \swarrow K & \downarrow L & \swarrow K & \downarrow L & \swarrow K & \\ \dots & \xrightarrow{Q} & \Omega^{p+1,q-1} & \xrightarrow{Q} & \Omega^{p+1,q} & \xrightarrow{Q} & \Omega^{p+1,q+1} & \xrightarrow{Q} & \dots \end{array}$$

Let H_L and H_Q denote the horizontal and vertical cohomology of Ω , respectively. Note that both of them are naturally bigraded. We have obtained the following result.

Proposition 4.4. $H_L^{p,q}(H_Q) \cong H_Q^{p,q}$ for all $0 \leq p \leq n$ and $q \in \mathbb{Z}$.

Let Ω_{tot} denote the total complex of Ω . Let H_{tot} denote the cohomology of Ω_{tot} . Let's consider the filtration $\Omega_{tot,0}^r \subset \Omega_{tot,1}^r \subset \Omega_{tot,2}^r \subset \dots \subset \Omega_{tot}^r$ on Ω_{tot}^r , where $\Omega_{tot,i}^r = \bigoplus_{p+q=r} \Omega^{p,q}$. This filtration is preserved by the total differential, hence induces a filtration $H_{tot,0}^r \subset \overset{p \geq n-i}{H_{tot,1}^r} \subset H_{tot,2}^r \subset \dots \subset H_{tot}^r$ on H_{tot}^r . Let $GH_{tot,i}^r$ denote $H_{tot,i}^r / H_{tot,i-1}^r$. We have $H_{tot}^r \cong \bigoplus_{i=0}^n GH_{tot,i}^r$.

Theorem 4.1. *For each $r \in \mathbb{Z}$ and $0 \leq i \leq n$, there exists a surjective map*

$$f : H_{tot,i}^r \rightarrow H_L^{i,r-l}(H_Q), \tag{4.2}$$

where $l = n - i$. Moreover, (4.2) induces an isomorphism

$$GH_{tot,i}^r \cong H_L^{i,r-l}(H_Q). \tag{4.3}$$

Proof. Let $\mathcal{O} = \sum_{l \leq p \leq n} \mathcal{O}^{p,r-p}$ be a closed element in $\Omega_{tot,i}^r$. We have $Q\mathcal{O}^{l,r-l} = 0$ and $d\mathcal{O}^{l,r-l} = Q\mathcal{O}^{l+1,r-l-1}$. Note that $\mathcal{O}^{l,r-l}$ is Q -exact if \mathcal{O} is exact. We define f to be the map induced by

$$\begin{aligned} \tilde{f} : \Omega_{tot}^r &\rightarrow \Omega^{l,r-l} \\ \mathcal{O} &\mapsto \mathcal{O}^{l,r-l} \end{aligned}$$

f is a well-defined map between cohomologies. To prove the surjectivity of f , note that for each $\mathcal{O}^{l,r-l} \in \Omega^{l,r-l}$, the element $\sum_{p=0}^i \frac{1}{p!} K^p \mathcal{O}^{l,r-l} \in \tilde{f}^{-1}(\mathcal{O}^{l,r-l})$ is closed in $\Omega_{tot,i}^r$. The isomorphism (4.3) then follows directly from the observation that $\text{Ker}(\tilde{f}) = \Omega_{tot,i-1}^r$. \square

Corollary 4.1. *There is an isomorphism $H_{tot}^r \cong \bigoplus_{i=0}^n H_Q^{i,r-i}$ of graded modules.*

Proof. This follows directly from Proposition 4.4 and Theorem 4.1. \square

Remark 4.4. There is a simpler proof for Corollary 4.1 if we adopt the second sign convention for the bigraded setting. By Remark 3.6, Q anticommutes with L and commutes with K . The total differential is just $Q - L$. Consider the ‘‘Mathai-Quillen map’’ $j = \exp(K)$ of Ω . Note that the expression $\exp(K)$ is well-defined because K is nilpotent in this setting. By Lemma 3.3, we have

$$j \circ Q \circ j^{-1} = \exp(K)([Q, \exp(-K)] + \exp(-K)Q) = Q - L.$$

In other words, the total cohomology of Ω is equal to the Q -cohomology of Ω .

The reason we adopt the first sign convention is just that we want to make the algebraic meaning of K more transparent.

One can easily see from the proof of Theorem 4.1 that Corollary 4.1 is equivalent to the following statement.

Theorem 4.2. *Every descendant sequence is a K -sequence up to an exact sequence.*

Let’s assume that there is a well-defined notion of integration \int on Γ_Y . The partition function Z of S is defined as

$$Z = \int \exp(-S).$$

The expectation value of an observable \mathcal{O} is the integration

$$\langle \mathcal{O} \rangle = \int \exp(-S) \mathcal{O}.$$

In [1], assuming that \mathcal{O} does not depend on the Riemannian metric g of M , Witten observed that the expectation value $\langle \mathcal{O} \rangle$ is also independent of the choice of g if the energy momentum tensor T of S is Q -exact, i.e., $T = Q(G)$ for some G .³ More precisely, he computed

$$\delta_g \langle \mathcal{O} \rangle = - \int \exp(-S) T \mathcal{O} = - \int \exp(-S) Q(G) \mathcal{O} = - \int Q(\exp(-S) G \mathcal{O}) = 0,$$

³CohFTs are therefore a special class of topological quantum field theories (TQFTs), though the latter do not have a constructive definition in mathematics.

where he used that S and \mathcal{O} are Q -closed, and that

$$\text{the integration of a } Q\text{-exact function vanishes.} \quad (4.4)$$

(4.4) is a bold assumption. It can be seen as an infinite dimensional analogue of Stokes' theorem. In fact, one can argue that this is indeed the case for a QK_v manifold \mathcal{M}_Y equipped with the canonical QK -structure. For such \mathcal{M}_Y , Q is just the vertical differential d_v , i.e., the de Rham differential on Γ_Y . In Section 6, we will show that, in most CohFTs, Q are just d_v expressed in different coordinates.

Remark 4.5. The condition “ T is Q -exact” is quite technical. One can consider a stronger but simpler condition, that “ \mathcal{L} is Q -exact”. There exists CohFTs where T is Q -exact but \mathcal{L} is not. For example, one has the 2-dimensional BF theory in Lorenz gauge [20]. In this paper, we only consider examples with Q -exact Lagrangians for simplicity.

Fix a sequence $\{\gamma_i\}_{i=0}^n$ of cycles of degrees $0, 1, \dots, n$ in M and a descent sequence $\{\mathcal{O}^{(i)}\}_{i=0}^n$. We can get a sequence $\{\mathcal{O}^{(i)}[\gamma_i]\}_{i=0}^n$ of Q -closed observables. Obviously, $\mathcal{O}^{(i)}[\gamma_i]$ is Q -exact if $\{\mathcal{O}^{(i)}\}_{i=0}^n$ is an exact sequence. Using assumption (4.4), it is easy to see that Q -exact observables have vanishing expectation values. In other words, K -sequences are the only physically interesting solutions to the descent equations (4.1).

4.2 CohFTs with Gauge Symmetries

Let P be a principal G -bundle over M , where G is a compact Lie group. The gauge symmetries are described by the automorphism group

$$\mathcal{G} := \{f : P \rightarrow P \mid \pi \circ f = \pi, f(pg) = f(p)g, \forall p \in P, g \in G\}.$$

Let $\text{Ad}P = P \times_G G$, where G acts on itself by conjugation. We have a natural identification $\mathcal{G} \cong \Gamma(\text{Ad}P)$. The Lie algebra $\text{Lie}(\mathcal{G})$ of \mathcal{G} can then be identified as $\Gamma(\text{ad}P)$, i.e., the space of sections of the adjoint bundle of P .

Recall that a connection 1-form on a principal G -bundle P is a G -equivariant 1-form A with values in the Lie algebra \mathfrak{g} such that $A(K_\xi) = \xi$, $\xi \in \mathfrak{g}$, where K_ξ is the fundamental vector field generated by ξ on P . The curvature 2-form of A is defined to be $F = dA + \frac{1}{2}[A, A]$. F is a basic form, and satisfies the second Bianchi identity $d_A F = 0$, where $d_A = d + [A, \cdot]$ is the covariant derivative associated to A .

Proposition 4.5. *For any principal bundle, the space of all connections \mathcal{A} is an affine space modeled on $\Omega^1(\text{ad}P)$. \mathcal{A} has a natural \mathcal{G} -action, with its infinitesimal action given by*

$$\begin{aligned} \mathcal{A} \times \text{Lie}(\mathcal{G}) &\rightarrow T\mathcal{A} \\ A \times \lambda &\mapsto (A, -d_A \lambda) \end{aligned}$$

where we use identifications $\text{Lie}(\mathcal{G}) \cong \Omega^0(\text{ad}P)$ and $T_A(\mathcal{A}) \cong \Omega^1(\text{ad}P)$.

For our purpose, we need to identify \mathcal{A} with the space of sections of some fiber bundle over M . Let P be a fiber bundle over M . Let J^1P be the first jet bundle of P . J^1P is an affine bundle modeled on the vector bundle $T^*M \otimes_M VP$, where VP is the vertical bundle over P and

the tensor product is taken over M . Let $j^1\phi : J^1P \rightarrow J^1P$ denote the jet prolongation of a bundle automorphism $\phi : P \rightarrow P$. Such operations satisfy the chain rules

$$\begin{aligned} j^1(\phi_1 \circ \phi_2) &= j^1(\phi_1) \circ j^1(\phi_2), \\ j^1(\text{id}_P) &= \text{Id}_{J^1P}. \end{aligned}$$

Thus, J^1P also has a principal G -action. The quotient space $C = J^1P/G$ is then an affine bundle modeled on the vector bundle $(T^*M \otimes_M VP)/G \cong T^*M \otimes \text{ad}P$ over M .

Proposition 4.6 ([21]). *There exists a bijection between \mathcal{A} , the affine space of connection 1-forms on P , and the set $\Gamma(C)$, the affine space of global sections of C .*

From now on, we will consider graded fiber bundles Y of the form

$$Y = C \times_M E,$$

where E is an associated bundle to P . (We assign degree 0 to elements of the fiber of C .) Let L_g denote the Lie graded algebra associated to the Lie algebra $\text{Lie}(\mathcal{G})$. L_g is spanned by elements $\delta_\lambda, \iota_\lambda$ and Q_g for each $\lambda \in \text{Lie}(\mathcal{G})$. They are of degrees 0, -1 , 1, respectively, and satisfy

$$\begin{aligned} [\delta_{\lambda_1}, \delta_{\lambda_2}] &= \delta_{[\lambda_1, \lambda_2]}, \quad [\delta_{\lambda_1}, \iota_{\lambda_2}] = \iota_{[\lambda_1, \lambda_2]}, \quad [\delta_\lambda, Q_g] = 0, \\ [Q_g, Q_g] &= 0, \quad [\iota_{\lambda_1}, \iota_{\lambda_2}] = 0, \quad [Q_g, \iota_\lambda] = \delta_\lambda. \end{aligned}$$

Note that we use the new notation δ_λ to denote the Lie derivatives.

Definition 4.8. Let \mathcal{M}_Y be a QK_v -manifold. An L_g -structure on \mathcal{M}_Y is said to be compatible with the QK_v -structure on \mathcal{M}_Y if

1. Q_g, ι_λ and δ_λ are of degrees $(0, 1)$, $(0, -1)$, $(0, 0)$, respectively;
2. Q_g coincides with Q , ι_λ anticommutes with K .

\mathcal{M}_Y together with the compatible QK_v -structure and L_g -structure is called a QK_{vg} -manifold.

Definition 4.9. Let \mathcal{M}_Y be a QK_{vg} -manifold. \mathcal{M}_Y is said to be simple if

$$[\delta_\lambda, K] = 0. \tag{4.5}$$

It is said to be h-simple if (4.5) only hold true for horizontal functions.

Lemma 4.2. $[\iota_\lambda, L] = [\delta_\lambda, K]$.

Proof. This follows from direct computations.

$$\begin{aligned} [\iota_\lambda, L] &= [\iota_\lambda, [Q, K]] \\ &= [[\iota_\lambda, Q], K] - [Q, [\iota_\lambda, K]] \\ &= [\delta_\lambda, K], \end{aligned}$$

where we use $[\iota_\lambda, Q] = \delta_\lambda$ and $[\iota_\lambda, K] = 0$. □

Definition 4.10. A cohomological (Lagrangian) gauge field theory (CohGFT) is a CohFT $(\mathcal{M}_Y, \mathcal{L})$, where \mathcal{M}_Y is a QK_{vg} -manifold and \mathcal{L} is basic with respect to the L_g -structure on \mathcal{M}_Y . The CohGFT is said to be simple (or h-simple) if \mathcal{M}_Y is simple (or h-simple).

\mathcal{L} being basic can be seen as a generalization of the notion of gauge invariance in bosonic theories. In most cases, δ_λ , ι_λ and Q do not depend on coordinates x^μ and dx^μ . They then gives Γ_Y an L_g -structure. The action functional S is a gauge invariant Q -closed function on Γ_Y .

Definition 4.11. A gauge invariant pre-observable \mathcal{O} is a basic pre-observable on \mathcal{M}_Y . A gauge invariant observable is a basic observable on Γ_Y .

By definition, the observable $\mathcal{O}[\gamma]$ associated to a gauge invariant pre-observable \mathcal{M}_Y and a cycle γ in M is a gauge invariant observable. Let $\mathcal{O}^{(0)}$ be a gauge invariant pre-observable of degree $(0, n)$. A natural question to ask is: Can we find a descendant sequence of $\mathcal{O}^{(0)}$ that is also gauge invariant?

Proposition 4.7. *The basic functions over an h-simple QK_{vg} -manifold \mathcal{M}_Y is preserved by the QK_v -structure.*

Proof. Let f be a basic function over \mathcal{M}_Y . We have

$$\iota_\lambda(Qf) = [\iota_\lambda, Q]f = \delta_\lambda f = 0, \quad \iota_\lambda(Kf) = [\iota_\lambda, K]f = 0.$$

We also have

$$\delta_\lambda(Qf) = [\delta_\lambda, Q]f = 0, \quad \delta_\lambda(Kf) = [\delta_\lambda, K]f = 0,$$

where we use the h-simple property, i.e., that $[\delta_\lambda, K]$ vanishes for horizontal functions. Since $L = QK + KQ$, the basic functions are also preserved by L , hence the QK_v -structure. \square

Recall that a K -sequence of $\mathcal{O}^{(0)}$ is specified by Q -closed functions $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(n)}$.

Corollary 4.2. *In an h-simple CohGFT, a K -sequence of a gauge invariant pre-observable $\mathcal{O}^{(0)}$ is gauge invariant if $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(n)}$ are gauge invariant.*

Now, let's turn back to the world of homological algebras. The QK_{vg} manifold \mathcal{M}_Y give us a bicomplex Ω just like before. But this time we have a canonical sub-bicomplex, namely the sub-bicomplex Ω_{bas} which consists of gauge invariant elements. For an h-simple CohGFT, Ω_{bas} is stable under both Q , K and L . Let H_{tot} and H_Q denote the total cohomology and vertical cohomology of Ω_{bas} , respectively. Likewise, we have the following isomorphism

$$H_{tot}^r(\Omega_{bas}) \cong \bigoplus_{i=0}^n H_Q^{i, r-i}(\Omega_{bas}),$$

which particularly implies that

Theorem 4.3. *In an h-simple CohGFT, every gauge invariant descendant sequence is a K -sequence up to an exact sequence.*

In fact, it is necessary to consider Ω_{bas} instead of Ω . This is because that there is a large class of CohGFTs with trivial Q -cohomologies. (For example, Ω equipped with the canonical QK_v -structure is Q -acyclic if the associated bundle E to P in the construction of Y is a vector bundle.) Therefore, one has to restrict to Ω_{bas} to obtain nontrivial observables. Geometrically, Ω_{bas} determines a QK_v -manifold \mathcal{M}_{bas} as a submanifold of the QK_{vg} -manifold \mathcal{M}_Y . The Lagrangian \mathcal{L} restricted to \mathcal{M}_{bas} is also a Q -closed function of degree $(n, 0)$. Thus, $(\mathcal{M}_{bas}, \mathcal{L})$ is a CohFT. It makes more sense to consider the partition function and expectation values of observables in $(\mathcal{M}_{bas}, \mathcal{L})$, since the path integrals in $(\mathcal{M}_Y, \mathcal{L})$ always carry a redundant factor due the presence of gauge symmetries.

5 Supersymmetric Lagrangian Field Theories

One way to construct Coh(G)FTs is to apply “topological twistings” to supersymmetric theories living on flat spacetimes [1]. In this section, we describe this idea in detail.

5.1 Super Poincaré Algebras

The isometry group of the n -dimensional Minkowski space is the Lie group $\text{IO}(1, n-1) \equiv \mathbb{R}^{1, n-1} \rtimes \text{O}(1, n-1)$. The Lie algebra \mathfrak{p} of this group is usually referred to as the Poincaré algebra. One can then define the Poincaré group to be the simply connected group uniquely determined by this Lie algebra. More precisely, it is the semi-direct product

$$\mathbb{R}^{1, n-1} \rtimes \text{Spin}^0(1, n-1)$$

where $\text{Spin}^0(1, n-1)$ is the identity component of the spin group $\text{Spin}(1, n-1)$. Note that $Z(\text{Spin}^0(1, n-1)) \cong \mathbb{Z}_2$. The representations ρ of the group $\text{Spin}^0(1, n-1)$ can then be classified into two categories. We call ρ a vector representation if $\mathbb{Z}_2 \subset \ker \rho$, and a spinor representation if $-1 \notin \ker \rho$. Elements in vector representations and spinor representations are called bosons and fermions, respectively. By the spin-statistics theorem in physics, bosonic fields must be commutative and fermionic fields must be anti-commutative. This is a hint that there exists a class of theories whose symmetry groups are super generalizations of the Poincaré groups. To define such super groups, we need the following propositions.

Proposition 5.1 ([22]). *Let S be an irreducible real spinor representation of $\text{Spin}^0(1, n-1)$.*

1. *The commutant Z of S is \mathbb{R} for $n-1 = 0, 1, 7 \pmod{8}$, \mathbb{C} for $n-1 = 2, 6 \pmod{8}$, \mathbb{H} for $n-1 = 3, 4, 5 \pmod{8}$.*
2. *Up to a real factor, there exists a unique symmetric morphism $\Gamma : S \otimes S \rightarrow \mathbb{R}^{1, n-1}$, and it is Z_1 -invariant, where Z_1 is the group of unit elements in Z .*

Remark 5.1. For a reducible representation S that is a direct sum of irreducible subrepresentations S^α with such pairings Γ^α , one can consider $\Gamma := \sum \Gamma^\alpha$.

Let S be a real spinor representation of $\text{Spin}^0(1, n-1)$. Let S^\vee be its dual representation. A super Poincaré algebra is a Lie superalgebra of the form

$$\mathfrak{p}_s := \mathfrak{p} \oplus S^\vee.$$

The symmetric pairing $[\cdot, \cdot]$ on the odd part of \mathfrak{p}_s is given by the $\text{Spin}^0(1, n-1)$ -equivariant pairing Γ in Proposition 5.1. \mathfrak{p}_s is said to be an $N = i$ super Poincaré algebra if S is the direct sum of i irreducible spinor representations when $n-2 \not\equiv 0 \pmod{4}$. It is said to be an $N = (i, j)$ super Poincaré algebra if S is the direct sum of i and j copies of the two inequivalent spinor representations when $n-2 \equiv 0 \pmod{4}$. The super Poincaré group can be then defined as the super Harish-Chandra Pair of the Poincaré group and the super Poincaré algebra.

Remark 5.2. Note that there is a subalgebra $\mathfrak{l} := \mathbb{R}^{1, n-1} \oplus S^\vee$ of \mathfrak{p}_s consisting of “super translations”. A physical theory is usually manifestly invariant under the spin group action. Therefore, in practice, it suffices to verify that the theory is invariant under the \mathfrak{l} -action to show that it is supersymmetric.

The largest subgroup of the automorphism group of a super Poincaré algebra which fixes its underlying Poincaré algebra is called its R -symmetry group. Using the second part of Proposition 5.1, one can obtain a nice classification result for R -symmetry groups [23]. See Table 5.1.

$n-2 \pmod{8}$	1, 7	0	3, 5	4	2, 6
\mathcal{I}_R	$\text{SO}(N)$	$\text{SO}(N^+) \times \text{SO}(N^-)$	$\text{Sp}(N)$	$\text{Sp}(N^+) \times \text{Sp}(N^-)$	$\text{U}(N)$

Table 5.1: R -symmetry groups \mathcal{I}_R for n -dimensional Minkowski spaces.

In general, one can not construct a super Poincaré algebra for a quadratic space of arbitrary signature, because a symmetric pairing like the one in Proposition 5.1 may not exist. However, this is not a problem for Euclidean spaces by applying a trick called dimensional reduction. More precisely, a super Poincaré algebra \mathfrak{p}'_s of \mathbb{R}^n can always be obtained from a super Poincaré algebra \mathfrak{p}_s of $\mathbb{R}^{1, n+m}$ for $m \geq 0$ by identifying the spinor representation S of $\text{Spin}^0(1, n+m)$ as a spinor representation of $\text{Spin}(0, n)$ via the canonical inclusion $\text{Spin}(0, n) \hookrightarrow \text{Spin}^0(1, n+m)$. Note that an R -symmetry of \mathfrak{p}_s is automatically an R -symmetry of \mathfrak{p}'_s . Therefore, one can easily find R -symmetry groups (or at least their subgroups) for \mathfrak{p}'_s using Table 5.1.

5.2 Twisted Super Poincaré Algebras and QK -algebras

Let's assume that there is a well-defined supersymmetric theory living on the quadratic space $\mathbb{R}^{r, s}$. Suppose that the R -symmetry group \mathcal{I}_R is also a symmetry of this theory. That is, we assume that the configuration space of the theory carries an action of the semi-direct product of the R -symmetry group and the super Poincaré group, under which the action functional is invariant. Since R -symmetry groups fix the underlying Poincaré algebras, we have a symmetry group of the form

$$\text{Spin}^0(r, s) \times \mathcal{I}_R,$$

which is of course also a symmetry of our theory. We also assume that there exists a non-trivial group homomorphism h_R from $\text{Spin}^0(r, s)$ to \mathcal{I}_R . The term topological twisting refers to the change of the ways of embedding $\text{Spin}^0(r, s) \hookrightarrow \text{Spin}^0(r, s) \times \mathcal{I}_R$ through h_R . More precisely, we change the canonical embedding $(\text{id}, 0)$ to the “twisted” embedding (id, h_R) .

Example 5.1 (Twisting of the $N = (1, 1)$ super Poincaré algebra of \mathbb{R}^4). In this case, we have $\text{Spin}(0, 4) \cong \text{Sp}_+(1) \times \text{Sp}_-(1)$, and a R symmetry group \mathcal{I}_R inherited from the the R symmetry

group of $N = 1$ super Poincaré algebra of $\mathbb{R}^{1,5}$. By checking Table 5.1, we find that $\mathcal{I}_R \cong \text{Sp}(1)$. Thus, we can define the twisting homomorphism h_R by setting

$$\begin{aligned} h_R : \text{Sp}_+(1) \times \text{Sp}_-(1) &\rightarrow \mathcal{I}_R \\ g = (g_+, g_-) &\mapsto g_+ \end{aligned}$$

The irreducible spinor representation on $\mathbb{R}^{1,5}$ gives us two inequivalent irreducible spinor representations S^+ and S^- after applying the dimensional reduction. We have $S^+ \cong S^- \cong \mathbb{H}$ as vector spaces. The spin group acts on S^\pm via

$$\begin{aligned} \text{Spin}(0, 4) \times S^\pm &\rightarrow S^\pm \\ g = (g_+, g_-) \times s^\pm &\mapsto g_\pm s^\pm \end{aligned}$$

and the R symmetry group acts on S^\pm via

$$\begin{aligned} \mathcal{I}_R \times S^\pm &\rightarrow S^\pm \\ g \times s^\pm &\mapsto s^\pm g^* \end{aligned}$$

where all the multiplications are given by quaternionic multiplications, g^* denotes the conjugate of $g \in \mathbb{H}$. The two actions commutes because \mathbb{H} is an associative algebra. The action of $\text{Spin}(0, 4) \times \mathcal{I}_R$ on S^\pm is indeed well defined.

The new actions of $\text{Spin}(0, 4)$ on S^\pm after twisting is given by

$$\begin{aligned} \text{Spin}(0, 4) \times S^+ &\rightarrow S^+ \\ g \times s^+ &\mapsto g_+ s^+ g_+^* \end{aligned}$$

and

$$\begin{aligned} \text{Spin}(0, 4) \times S^- &\rightarrow S^- \\ g \times s^- &\mapsto g_- s^- g_-^* \end{aligned}$$

One can show that after twisting, S^+ becomes $\mathbb{R} \oplus \Lambda^2 \mathbb{R}^4$ and S^- becomes \mathbb{R}^4 as representations of $\text{Spin}(0, 4)$, where $\Lambda^2 \mathbb{R}^4$ is the vector space of anti-self-dual 2-forms. In other words, since all bundles over \mathbb{R}^4 are trivial, the twisting procedure does nothing but reorganize the field components of the original theory in a different way by turning spinor fields into differential forms. The twisted supersymmetric theories can be put on a more general spacetime, because the existence of covariant differential forms put far fewer restrictions on the geometry of the spacetime manifold than the existence of covariant spinors.

Now, let's examine the twisted super Poincaré algebra closely. Before the twisting, the pairing $[\cdot, \cdot]$ on S^\pm is given by [24]

$$\begin{aligned} [\cdot, \cdot] : (S^+ \oplus S^-) \times (S^+ \oplus S^-) &\rightarrow \mathbb{R}^4 \\ (s^+, s^-) \times (t^+, t^-) &\mapsto t^-(s^+)^* + s^-(t^+)^* \end{aligned}$$

Identifying $\mathbb{R}, \Lambda^2 \mathbb{R}^4$ with the real part and imaginary part of \mathbb{H} , respectively. We have

$$\begin{aligned} [\cdot, \cdot] : (\mathbb{R} \oplus \Lambda^2 \mathbb{R}^4 \oplus \mathbb{R}^4) \times (\mathbb{R} \oplus \Lambda^2 \mathbb{R}^4 \oplus \mathbb{R}^4) &\rightarrow \mathbb{R}^4 \\ (\eta_1, \chi_1, \nu_1) \times (\eta_2, \chi_2, \nu_2) &\mapsto \nu_2(\eta_1 - \chi_1) + \nu_1(\eta_2 - \chi_2) \end{aligned}$$

There is a sub-pairing

$$\begin{aligned} [\cdot, \cdot] : (\mathbb{R} \oplus \mathbb{R}^4) \times (\mathbb{R} \oplus \mathbb{R}^4) &\rightarrow \mathbb{R}^4 \\ (\eta_1, v_1) \times (\eta_2, v_2) &\mapsto v_2 \eta_1 + v_1 \eta_2 \end{aligned}$$

which gives us a subalgebra $\mathfrak{l}_t = \mathbb{R}^4 \oplus (\mathbb{R} \oplus \mathbb{R}^4)$ of the twisted super Poincaré algebra. Let $w_i, \eta, v_i, i = 1, \dots, 4$, be a basis of \mathfrak{l}_t . We have

$$[w_i, w_j] = 0, \quad [w_i, \eta] = 0, \quad [w_i, v_j] = 0, \quad (5.1)$$

$$[\eta, \eta] = 0, \quad [\eta, v_i] = w_i, \quad [v_i, v_j] = 0. \quad (5.2)$$

One immediately recognizes that (5.1) and (5.2) are just (3.2) and (3.3) in the disguise. In other words, we have reproduced the Lie graded algebra L associated to an abelian Lie algebra by twisting a super Poincaré algebra. The twisted supersymmetric theory can be then given naturally a QK -structure. Since the Lagrangian is invariant under the \mathfrak{l}_t -action, it is also Q -closed. We then obtain a 4-dimensional CohFT.

Remark 5.3. One can also work with the twisted superalgebra defined by (5.1) and (5.2) directly, and then use the standard superfield method in physics literature to construct action functionals on flat spacetimes. This idea was studied in [25].

As another example, one can twist the $N = (2, 2)$ super Poincaré algebra of \mathbb{R}^4 obtained by applying dimensional reduction to the $N = (1, 1)$ super Poincaré algebra of $\mathbb{R}^{1,5}$. The R -symmetry group in this case is $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, which is isomorphic to the spin group. Therefore, there exists three different homomorphisms h_R (up to automorphisms). The twisting associated to the identity one is called the geometric Langlands twisting [26].

Example 5.2 (Geometric Langlands twisting of the $N = (2, 2)$ super Poincaré algebra of \mathbb{R}^4). In this case, the odd part of the super Poincaré algebra is $S_l \oplus S_r$, where $S_l \cong S_r \cong S^+ \oplus S^-$, $S^\pm \cong \mathbb{H}$. The R -symmetry group acts on $S_l \oplus S_r$ via

$$\begin{aligned} (\mathcal{I}_R \cong \mathrm{Sp}_l(1) \times \mathrm{Sp}_r(1)) \times S_l \oplus S_r &\rightarrow S_l \oplus S_r \\ (g_l, g_r) \times (s_l, s_r) &\mapsto (s_l g_l^*, s_r g_r^*). \end{aligned}$$

It is not hard to see that, after the twisting, S_l becomes $\mathbb{R} \oplus \mathbb{R}^4 \oplus \Lambda_-^2 \mathbb{R}^4$ and S_r becomes $\mathbb{R} \oplus \mathbb{R}^4 \oplus \Lambda_+^2 \mathbb{R}^4$. Again, we are only interested in the $\mathbb{R} \oplus \mathbb{R}^4$ parts of S_l and S_r . The pairings $[\cdot, \cdot]$ on each of them are identical to the one defined in Example 5.1. We then obtain an abelian Lie superalgebra $\mathfrak{l}_t = \mathbb{R}^4 \oplus (\mathbb{R} \oplus \mathbb{R}^4)_l \oplus (\mathbb{R} \oplus \mathbb{R}^4)_r$. The Lie bigraded algebra \mathcal{K}_{GL} associated to \mathfrak{l}_t is spanned by Q_l of degree $(0, 1)$, Q_r of degree $(0, 1)$, K_l of degree $(1, -1)$, K_r of degree $(1, -1)$, and L of degree $(1, 0)$. The only non-trivial brackets between these basis elements are

$$[Q_l, K_l] = [Q_r, K_r] = L.$$

The universal enveloping algebra $\mathcal{K}_{GL\infty}$ of \mathcal{K}_{GL} is generated by Q_l, Q_r, K_l, K_r , and L , which are subject to the relations

$$\begin{aligned} Q_l^2 &= 0, \quad Q_l K_l + K_l Q_l = L, \quad K_l L + L K_l = 0, \\ Q_r^2 &= 0, \quad Q_r K_r + K_r Q_r = L, \quad K_r L + L K_r = 0, \\ Q_l Q_r + Q_r Q_l &= 0, \quad K_l K_r - K_r K_l = 0, \quad Q_l K_r + K_r Q_l = 0, \quad Q_r K_l + K_l Q_r = 0. \end{aligned}$$

There exists a $\overline{T}\mathbb{R}P^1$ -family of QK -algebras as subalgebras of $\mathcal{K}_{GL\infty}$, where $\overline{T}\mathbb{R}P^1$ is an affine bundle modelled on the tangent bundle of the 1-dimensional projective space $\mathbb{R}P^1$. Let $(u_1, u_2, v_1, v_2) \in \mathbb{R}^4$ be such that $u_1v_1 + u_2v_2 = 1$. We define

$$Q_{\vec{u}} = u_1Q_l + u_2Q_r, \quad K_{\vec{v}} = v_1K_l + v_2K_r.$$

It is straightforward to verify that

$$Q_{\vec{u}}^2 = 0, \quad Q_{\vec{u}}K_{\vec{v}} + K_{\vec{v}}Q_{\vec{u}} = L, \quad K_{\vec{v}}L + LK_{\vec{v}} = 0,$$

where $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Let (u'_1, u'_2, v'_1, v'_2) be another element in \mathbb{R}^4 such that $u'_1v'_1 + u'_2v'_2 = 1$. Obviously, (u_1, u_2, v_1, v_2) and (u'_1, u'_2, v'_1, v'_2) determine the same QK -algebra (up to a scaling factor) if there exists an $a \in \mathbb{R}/\{0\}$ such that $u'_1 = au_1$, $u'_2 = au_2$, $v'_1 = v_1/a$, $v'_2 = v_2/a$. On the other hand, if we fix (u_1, u_2, v_1, v_2) and let $\Delta K_{\vec{u}} = -u_2K_l + u_1K_r$, then $Q_{\vec{u}}$, $K_{\vec{v}} + s\Delta K_{\vec{u}}$, and L form a QK -algebra for any $s \in \mathbb{R}$.

Note that there exists a natural $\mathrm{SL}(2, \mathbb{R})$ -action on these QK -algebras. More precisely, for $g \in \mathrm{SL}(2, \mathbb{R})$, we set

$$gQ_{\vec{u}} = Q_{g\vec{u}}, \quad gK_{\vec{v}} = K_{(g^{-1})^t\vec{v}}, \quad gL = L,$$

In other words, the Geometric Langlands twisting of the $N = (2, 2)$ supersymmetric theory gives us a $\mathbb{R}P^1$ -family of CohFTs which can be related to each other via a natural $\mathrm{SL}(2, \mathbb{R})$ -action.

Remark 5.4. One can generalize the above examples by considering bigraded algebra generated by $2k + 1$ generators: Q_i of degree $(1, 0)$, K_j of degree $(1, -1)$, L of degree $(1, 0)$, $i, j = 1, \dots, k$, with non-trivial brackets being

$$[Q_i, K_i] = L, \quad i = 1, \dots, k.$$

Such a bigraded algebra should be obtained from the $N = (k, k)$ super Poincaré algebra of \mathbb{R}^4 . It follows that there exists a $\mathbb{R}P^{k-1}$ -family of CohFTs obtained by twisting $N = (k, k)$ supersymmetric theories, which can be related to each other via a natural $\mathrm{SL}(k, \mathbb{R})$ -action. For $k = 1, 2$, we recover Examples 5.1 and 5.2. However, I do not know if it makes sense to talk about $N = (k, k)$ supersymmetric theories in physics for $k \geq 3$.

6 Mathai-Quillen Formalism Revisited: a Generalization

6.1 Mathai-Quillen Formalism with Gauge Symmetries

In this subsection, we generalize the Mathai-Quillen formalism described in Section 2 to incorporate QK -structures and gauge symmetries.

Let P be a principal G -bundle over an n -dimensional manifold M . Let $\mathrm{ad}P$ denote the adjoint bundle of P . Let \mathcal{A} denote the affine space of connection 1-forms on P . Recall that \mathcal{A} can be identified with $\Gamma(C)$ where C is an affine bundle over M . Let V and W be two associated vector bundles to P . We consider the variational bigraded manifold \mathcal{M}_Y associated to the graded fiber bundle

$$Y = \mathrm{ad}P \times_M C \times_M V \times_M W,$$

where the grading is defined by assigning elements of the fibers of $\text{ad}P$, C , V , and W degrees 1, 0, -2 , and -1 , respectively.

A bundle chart of P induces a local coordinate system

$$(x^\mu, \theta_I^a, A_{\mu;I}^a, w_I^i, \chi_I^i, dx^\mu, \delta\theta_I^a, \delta A_{\mu;I}^a, \delta w_I^i, \delta\chi_I^i) \quad (6.1)$$

for \mathcal{M}_Y . The degrees of the above coordinate functions are

$$(0, 0), (0, 1), (0, 0), (0, -2), (0, -1), (1, 0), (0, 2), (0, 1), (0, -1), (0, 0).$$

For simplicity, we omit the indices of the coordinate functions and use ϕ , v , ψ , b to denote the odd coordinates $\delta\theta$, δA , δw , $\delta\chi$, respectively. There exist a family of QK_v -structures parameterized by $t \in \mathbb{R}$ on \mathcal{M}_Y . The cohomological vector field Q is defined by setting

$$Q\theta = \phi, \quad QA = v, \quad Qw = \psi, \quad Q\chi = b,$$

and the action of Q on the other coordinates to be 0. The homotopy operator K is defined by setting

$$K\theta = tA, \quad K\phi = d\theta - tv, \quad Kv = dA, \quad K\psi = dw, \quad Kb = d\chi,$$

and the action of K on the other coordinates to be 0. Q and K are of degrees $(0, 1)$ and $(1, -1)$, respectively. From now on, we set $t = 1$.

Remark 6.1. The notations we adopt here need more explanation. For example, when we write $Qw = \psi$, we actually mean a family of equations $Qw_I^j = \psi_I^j$. Likewise, when we write $K\psi = dw$, we mean $K\psi_I^j = w_{I \cup \{\mu\}}^j dx^\mu$. The reader may question that the equations $K\theta = A$ and $K\phi = d\theta - v$ are illegal because K need to be of degree $(1, -1)$. However, what we really mean by writing A is $A_{\mu;I}^a dx^\mu$ instead of $A_{\mu;I}^a$. Likewise, we write v to mean $v_{\mu;I}^a dx^\mu$.

By construction, \mathcal{M}_Y can be equipped with an $L_{\mathcal{G}}$ -action. Note that the $\text{Lie}(\mathcal{G})$ -action on \mathcal{A} is not linear. This will cause problems when we apply changes of coordinates later. Hence, we require that $\text{Lie}(\mathcal{G})$ acts on \mathcal{A} through the adjoint action instead. The contractions ι_λ are then defined by setting

$$\iota_\lambda\theta = \lambda, \quad \iota_\lambda\phi = -[\lambda, \theta], \quad \iota_\lambda v = -[\lambda, A], \quad \iota_\lambda\psi = -\lambda w, \quad \iota_\lambda b = -\lambda\chi,$$

and its action on the other coordinates to be 0. However, $\iota_\lambda K + K\iota_\lambda \neq 0$,⁴ i.e., the $L_{\mathcal{G}}$ -structure is not compatible with the QK_v -structure. This issue will be solved later.

We apply the Mathai-Quillen map to express the QK -structure in new coordinates. We have

$$\begin{aligned} Q\theta &= \phi - \frac{1}{2}[\theta, \theta], & Q\phi &= -[\theta, \phi], \\ QA &= v - [\theta, A], & Qv &= -[\theta, v] + [\phi, A] \\ Qw &= \psi - \theta w, & Q\psi &= -\theta\psi + \phi w, \\ Q\chi &= b - \theta\chi, & Qb &= -\theta b + \phi\chi, \end{aligned}$$

⁴One can easily check that $(\iota_\lambda K + K\iota_\lambda)\phi = \iota_\lambda(d\theta - v) - K[\lambda, \theta] = d\lambda + [\lambda, A] - [\lambda, A] = d\lambda$.

as a generalization of the Kalkman differential in Remark 2.4. We also have

$$\begin{aligned} K\theta &= A, & K\phi &= d\theta - v, \\ KA &= 0, & Kv &= dA + [A, A], \\ Kw &= 0, & K\psi &= dw + Aw, \\ K\chi &= 0, & Kb &= d\chi + A\chi. \end{aligned}$$

One can verify $Q^2 = 0$ and $QK + KQ = L$ by direct computations. The advantage of the new coordinates is that we have a simpler expression for ι_λ , namely, we have

$$\iota_\lambda\theta = \lambda,$$

and 0 for ι_λ acting on the other coordinates.

In order to fix the incompatibility between K and ι_λ , and to change the $\text{Lie}(\mathcal{G})$ -action on \mathcal{A} back to the correct one, we apply the following change of coordinates

$$v \rightarrow v - d\theta.$$

We have now

$$\begin{aligned} Q\theta &= \phi - \frac{1}{2}[\theta, \theta], & Q\phi &= -[\theta, \phi], \\ QA &= v + d_A\theta, & Qv &= -[\theta, v] - d_A\phi \\ Qw &= \eta - \theta w, & Q\psi &= -\theta\psi + \phi w, \\ Q\chi &= b - \theta\chi, & Qb &= -\theta b + \phi\chi, \end{aligned}$$

and

$$\begin{aligned} K\theta &= A, & K\phi &= -v, \\ KA &= 0, & Kv &= 2F, \\ Kw &= 0, & K\psi &= d_A w, \\ K\chi &= 0, & Kb &= d_A \chi, \end{aligned}$$

where $d_A = d + A$ and $F = dA + \frac{1}{2}[A, A]$ can be interpreted as the covariant derivative and the curvature of A . We set ι_λ to be of the same form as before. It is then not hard to see that $\delta_\lambda A$ give us the correct gauge transformation of a connection 1-form, and that $\iota_\lambda K + K\iota_\lambda = 0$.⁵

Theorem 6.1. *\mathcal{M}_Y is an h-simple QK_{vg} manifold.*

Proof. It remains to check the h-simple property. Apply the vector fields $K\delta_\lambda - \delta_\lambda K$ to coordinate functions. The only non-vanishing one is

$$(K\delta_\lambda - \delta_\lambda K)\theta = K(-[\lambda, \theta]) - \delta_\lambda A = -[\lambda, A] - d_A \lambda = -d\lambda.$$

However, functions dependent on θ are not in the kernel of ι_λ . □

⁵In fact, we have $\iota_\lambda K = K\iota_\lambda = 0$.

Remark 6.2. Cohomological vector fields Q of the above form were first given in [8]. It was noticed there that Q recovers the scalar supersymmetry in [1] by setting θ to be 0, i.e., by restricting to the horizontal functions over \mathcal{M}_Y .

It remains to find appropriate Lagrangians and observables. For the Lagrangian, we set

$$\mathcal{L} = Q(\langle(\theta, v, \psi, \chi), (0, f_1, f_2, f_3 + b)\rangle) d\text{vol}, \quad (6.2)$$

where $(0, f_1, f_2, f_3)$ is a \mathcal{G} -equivariant vector field over $\Gamma(\text{ad}P) \times \mathcal{A} \times \Gamma(V) \times \Gamma(W)$, $\langle \cdot, \cdot \rangle$ is a \mathcal{G} -invariant inner product on the tangent space, and $d\text{vol}$ is the volume form on M . By construction, \mathcal{L} is a Q -exact basic function over \mathcal{M}_Y . It is easy to see that \mathcal{L} is an infinite dimensional generalization of (2.6) in the Mathai-Quillen construction of an Euler class. Since one cannot have gauge symmetries in the finite dimensional case, this generalization is essentially non-trivial.

Remark 6.3. f_1, f_2 and f_3 are referred to as the gauge fixing functions in the physics literature. They should be carefully chosen such that $d(f_1) = -2$, $d(f_2) = 0$ and $d(f_3) = 0$. In this way, \mathcal{L} is homogeneous of degree $(n, 0)$.

Let $\mathcal{O}^{(0)}$ be a gauge invariant pre-observable of degree 0. It can't be Q -exact, otherwise, the expectation values of the corresponding observables will vanish. For simplicity, let's assume that n is even. A reasonable choice is then $\mathcal{O}^{(0)} = \text{Tr}(\phi^m)$, where $m = n/2$. Using Lemma 3.4, the standard K -sequence of $\mathcal{O}^{(0)}$ can be found as

$$\sum_{p=0}^n \mathcal{O}^{(p)} = \exp(K)\mathcal{O}^{(0)} = \text{Tr}(\phi_K^m), \quad (6.3)$$

where

$$\phi_K = \exp(K)\phi = \phi - v - F$$

can be interpreted as the curvature 2-form on the principal G -bundle $\mathcal{P} = (P \times \mathcal{A})/\mathcal{G} \rightarrow M \times \mathcal{A}/\mathcal{G}$ [3]. In this sense, (6.3) is nothing but a Chern class of \mathcal{P} . With a slight abuse of notation, we call \mathcal{P} the universal G -bundle, and ϕ_K the universal curvature 2-form on \mathcal{P} , though both of them depend apparently on the choice of P . Likewise, we set

$$\theta_K = \exp(K)\theta = \theta + A.$$

θ is called the universal connection 1-form on \mathcal{P} . In fact, the de Rham complex of \mathcal{P} is a commutative bigraded algebra of the second kind. By Lemma 3.1, we should reset Q and K to be $(-1)^p Q$ and $(-1)^q K$, where p and q are the horizontal degrees of the functions acted by Q and K , respectively. The universal curvature 2-form ϕ_K takes the form $\phi - v + F$ instead. Let d_{tot} denote the total differential associated to Q and L . It is not hard to show that

$$\phi_K = d_{tot}\theta_K + \frac{1}{2}[\theta_K, \theta_K], \quad d_{tot}\phi_K + [\theta_K, \phi_K] = 0.$$

In the next subsection, we will consider the case where P is the trivial G -bundle over M . We will give a notion of Weil homomorphism in the infinite dimensional setting which sends θ_K and ϕ_K to connection and curvature forms on a mapping space.

6.2 Chern-Weil Homomorphism for Mapping Spaces

Every physical theory with non-trivial dynamics involves the notion of connections. Theories with gauge symmetries are characterized by the interpretation of connections as variables, while theories with no gauge symmetry usually have their connections fixed. One approach to construct Lagrangians for rigid CohFTs, therefore, is to start with a “universal” CohGFT and apply a generalization of the Chern-Weil homomorphism. In this subsection, we explain this idea in detail.

Let the graded fiber bundle Y be as in the previous subsection, i.e., $Y = \text{ad}P \times_M C \times_M V \times_M W$. Let's consider the trivial principal G -bundle P over M with $G = \text{SO}(2m)$. \mathcal{G} , $\text{Lie}(\mathcal{G})$, \mathcal{A} can then be identified with $C^\infty(M, G)$, $C^\infty(M, \mathfrak{g})$ and $\Gamma(T^*M) \otimes \mathfrak{g}$, respectively. Let V be a real rank $2m$ vector bundle associated to P by the fundamental representation of $\text{SO}(2m)$. Let W be the dual bundle of V . \mathcal{G} acts on $\Gamma(V)$ and $\Gamma(W)$ fiber-wisely. By Remark 2.4, we need to change the QK -structure on \mathcal{M}_Y by resetting

$$Qw = \psi, \quad Q\psi = 0, \quad Kw = 0, \quad K\psi = dw,$$

The L_g -structure also changes correspondingly. Namely, we should redefine ι_λ by setting

$$\iota_\lambda \psi = -\lambda w.$$

Moreover, we reassign degree $(0, 0)$ to w and degree $(0, 1)$ to ψ . We then consider \mathcal{L} of the form

$$\mathcal{L} = Q(\chi(w) + \langle \chi, b \rangle) d\text{vol}. \quad (6.4)$$

It is easy to verify that \mathcal{L} is homogeneous of degree $(n, 0)$.

On the other hand, let Σ be a $2m$ -dimensional Riemannian manifold, $2m \geq n$. Let P_Σ be a principal G -bundle over Σ equipped with a connection 1-form A . Let Y'_0 denote the trivial fiber bundle $M \times P_\Sigma$ over M . Note that $\Gamma(Y'_0) = C^\infty(M, P_\Sigma)$, which can be viewed as a principal \mathcal{G} -bundle over the mapping space $C^\infty(M, \Sigma)$. Therefore, the variational bigraded manifold $\mathcal{M}_{Y'_0}$ associated to Y'_0 carries a canonical L_g -action.

Lemma 6.1. *$\mathcal{M}_{Y'_0}$ equipped with the canonical QK_v -structure and L_g -structure is a simple QK_{vg} -manifold, hence particularly an h -simple QK_{vg} -manifold.*

Proof. We need to show that $[K, \iota_\lambda] = 0$ and $[K, \delta_\lambda] = 0$ (or $[L, \iota_\lambda] = 0$). Let $(x^\mu, u_I^j, dx^\mu, \delta u_I^j)$ be a local coordinate system. It is not hard to see that we only need to check both properties for coordinates δu_I^j . For the first one, we have

$$[K, \iota_\lambda] \delta u_I^j = K(\delta_\lambda u_I^j) + \iota_\lambda(u_{I \cup \{\mu\}}^j dx^\mu) = 0,$$

where we use $[Q, \iota_\lambda] = \delta_\lambda$. For the second one, we have

$$[L, \iota_\lambda] \delta u_I^j = L(\delta_\lambda u_I^j) - (\delta_\lambda u_{I \cup \{\mu\}}^j) dx^\mu = \partial_\mu (\delta_\lambda u_I^j) dx^\mu - (\delta_\lambda u_{I \cup \{\mu\}}^j) dx^\mu = 0.$$

where we use $\delta_\lambda u_I^j = \partial_I (\delta_\lambda u^j)$. □

The evaluation map

$$\text{ev} : M \times C^\infty(M, P_\Sigma) \rightarrow P_\Sigma$$

pulls back A to a \mathfrak{g} -valued 1-form on $M \times C^\infty(M, P_\Sigma)$. It decomposes into two parts: the horizontal part A_h along M , and the vertical part A_v along $C^\infty(M, P_\Sigma)$. Let ∇_h and ∇_v denote the covariant derivatives associated to A_h and A_v , respectively. We can write

$$\nabla_h = d + A_h, \quad \nabla_v = \delta + A_v,$$

where d is the horizontal differential and δ is the vertical differential. We have three types of curvatures:

$$R_h = \nabla_h^2, \quad R_v = \nabla_v^2, \quad R_m = \nabla_v \nabla_h + \nabla_h \nabla_v,$$

where the subscript m of R_m stands for the word "mixed". We have

$$R_h = dA_h + \frac{1}{2}[A_h, A_h], \quad R_v = \delta A_v + \frac{1}{2}[A_v, A_v], \quad R_m = dA_v + \delta A_h + [A_h, A_v].$$

By a simple analysis of degrees, we also have the following four types of Bianchi identities.

$$\nabla_h R_h = 0, \quad \nabla_v R_v = 0, \quad \nabla_v R_m + \nabla_h R_v = 0, \quad \nabla_h R_m + \nabla_v R_h = 0.$$

Remark 6.4. Recall that the sign convention we choose for a variational bigraded manifold is of the first kind. In particular, we should have $d\delta = \delta d$. By Lemma 3.1, this can be achieved by redefining δ to be $(-1)^p \delta$, where p is the horizontal degree of the function acted by δ . Correspondingly, the expressions for the curvature R_m and the third one of the above Bianchi identities change. We have

$$R_m = dA_v - \delta A_h + [A_h, A_v], \quad -\nabla_v R_m + \nabla_h R_v = 0.$$

G can be viewed as a subgroup of \mathcal{G} by identifying its element with the corresponding constant functions in $C^\infty(M, G)$. Thus, A_v, R_v can be viewed as Lie(\mathcal{G})-valued 1-form and 2-form of $\Omega_{loc}(M \times C^\infty(M, P_\Sigma))$, respectively. They determine maps

$$\text{Lie}(\mathcal{G})^* \rightarrow \Omega_{loc}^{\bullet,1}(M \times C^\infty(M, P_\Sigma)), \quad \text{Lie}(\mathcal{G})^* \rightarrow \Omega_{loc}^{\bullet,2}(M \times C^\infty(M, P_\Sigma)).$$

which induce a map

$$\phi_1 : \Omega_{loc}(M \times \Gamma(\text{ad}P)) \rightarrow \Omega_{loc}(M \times C^\infty(M, P_\Sigma))$$

sending θ and ϕ to A_v and R_v , respectively. This is the usual Weil homomorphism in the infinite dimensional setting. Let $(x^\mu, dx^\mu, A_{\mu;I}^a, v_{\mu;I}^a)$ be a coordinate system of $\Omega_{loc}(M \times \mathcal{A})$. We can also define a connection fixing map

$$\phi_2 : \Omega_{loc}(M \times \mathcal{A}) \rightarrow \Omega_{loc}(M \times C^\infty(M, P_\Sigma))$$

which sends $A \in \mathcal{A}$ to A_h and the vertical differential v of A to $-R_m$. Combining ϕ_1 and ϕ_2 , we obtain a map

$$\phi_W : \Omega_{loc}(M \times \Gamma(\text{ad}P) \times \mathcal{A}) \rightarrow \Omega_{loc}(M \times C^\infty(M, P_\Sigma)),$$

which we refer to as the Weil homomorphism for mapping spaces. By definition, it sends the universal connection θ_K and curvature ϕ_K defined in the previous subsection to the connection and curvature on $M \times C^\infty(M, P_\Sigma)$, respectively.

Theorem 6.2. ϕ_W preserves the h -simple QK_{vg} -structure.

Proof. This follows from direct computations. Let's check first that $K\phi_W = \phi_W K$. We have

$$K(\phi_W(A)) = KA_h = 0 = \phi_W(KA), \quad K(\phi_W(\theta)) = KA_v = A_h = \phi_W(K\theta).$$

Since Q is just the vertical differential δ for $\mathcal{M}_{Y'_0}$, we have

$$\begin{aligned} KR_v &= K(QA_v + \frac{1}{2}[A_v, A_v]) = LA_v - QKA_v + [A_h, A_v] = dA_v - \delta A_h + [A_h, A_v] = R_m, \\ KR_m &= K(LA_v - QA_h + [A_h, A_v]) = -LKA_v - LA_h - [A_h, KA_v] = -2dA_h - [A_h, A_h] = -2R_h. \end{aligned}$$

Thus,

$$K(\phi_W(v)) = -KR_m = 2R_h = \phi_W(2F) = \phi_W(Kv), \quad K(\phi_W(\phi)) = KR_v = R_m = \phi_W(K\phi).$$

The next step is to check $Q\phi_W = \phi_W Q$. We have

$$\begin{aligned} \phi_W(Q\theta) &= \phi_W(\phi) - \frac{1}{2}\phi_W([\theta, \theta]) = R_v - [A_v, A_v] = \delta A_v = Q(\phi_W(\theta)), \\ \phi_W(Q\phi) &= \phi_W(-[\theta, \phi]) = -[A_v, R_v] = \delta R_v = Q(\phi_W(\phi)), \end{aligned}$$

where we use the Bianchi identity $\nabla_v R_v = 0$. We also have

$$\begin{aligned} \phi_W(QA) &= \phi_W(v) + \phi_W(d_A\theta) = -R_m + dA_v + [A_h, A_v] = -\delta A_h = Q(\phi_W(A)), \\ \phi_W(Qv) &= \phi_W(-[\theta, v]) - \phi_W(d_A\phi) = [A_v, R_m] - dR_v - [A_h, R_v] = -\delta R_m = Q(\phi_W(v)), \end{aligned}$$

where we use the Bianchi identity $-\nabla_v R_m + \nabla_h R_v = 0$. It follows that

$$L\phi_W = \phi_W L.$$

To prove that ϕ_W preserves the L_g -structure, it suffices to check $\phi_W \iota_\lambda = \iota_\lambda \phi_W$. By construction, we have

$$\iota_\lambda \phi_W(\theta) = \iota_\lambda A_v = \lambda, \quad \iota_\lambda \phi_W(\phi) = \iota_\lambda R_v = 0, \quad \iota_\lambda \phi_W(A) = \iota_\lambda A_h = 0.$$

It remains to show that $\iota_\lambda \phi_W(v) = 0$. In fact, we can check that

$$\begin{aligned} \iota_\lambda R_m &= \iota_\lambda(LA_v - QA_h + [A_h, A_v]) \\ &= [\iota_\lambda, L]A_v + L\lambda - [\iota_\lambda, Q]A_h + Q\iota_\lambda A_h + [A_h, \lambda] \\ &= [\delta_\lambda, K]A_v - \delta_\lambda A_h + [A_h, \lambda] \\ &= -K(\delta_\lambda A_v) + [A_h, \lambda] \\ &= K([\lambda, A_v]) + [A_h, \lambda] \\ &= 0, \end{aligned}$$

where we use $[\iota_\lambda, L] = [\delta_\lambda, K]$. □

Let $Y' = Y'_0 \times_M V \times_M W$. We can extend ϕ_W naturally to a map

$$\Omega_{loc}(M \times \Gamma(Y)) \rightarrow \Omega_{loc}(M \times \Gamma(Y')),$$

which is denoted again by ϕ_W with a slight abuse of notation. There exists a natural map

$$\begin{aligned} \Gamma(Y'_0) &\cong C^\infty(M, P_\Sigma) \rightarrow \Gamma(T^*M) \otimes \mathfrak{g} \cong \Gamma(Y_0) \\ f &\mapsto A(Tf), \end{aligned}$$

where we identify the tangent map Tf of f as a section of $T^*M \otimes f^*TP_\Sigma$, and identify A as a map $\Gamma(TP_\Sigma) \rightarrow \mathfrak{g}$. This map together with ϕ_W determines a morphism $\mathcal{M}_{Y'} \rightarrow \mathcal{M}_Y$ of h-simple QK_{vg} -manifolds.

From now on, we choose P_Σ to be the frame bundle of Σ and A to be the Levi-Civita connection. $(\Omega_{loc}(M \times \Gamma(Y'))_{bas})$ can be identified with $\Omega_{loc}(M \times \Gamma(Y_\Sigma))$, where Y_Σ is the trivial graded fiber bundle $M \times (T\Sigma \times_\Sigma T^*\Sigma)$ over M , with $(Y_\Sigma)_0$ being the trivial bundle $M \times \Sigma$ over M . ϕ_W induces a homomorphism

$$\phi_{CW} : \Omega_{loc}(M \times \Gamma(Y))_{bas} \rightarrow \Omega_{loc}(M \times \Gamma(Y_\Sigma)),$$

which we refer to as the Chern-Weil homomorphism for mapping spaces. Let $(x, u, w, \chi, dx, \delta u, \psi, b)$ be a local coordinate system of \mathcal{M}_{Y_Σ} . The QK -structure on \mathcal{M}_{Y_Σ} is given by

$$\begin{aligned} Qu &= \delta u, & Q\delta u &= 0, & Qw &= \psi, & Q\psi &= 0, & Q\chi &= b - A_v\chi, & Qb &= -A_v b + R_v\chi, \\ Ku &= 0, & K\delta u &= du, & Kw &= 0, & K\psi &= dw, & K\chi &= 0, & Kb &= d\chi + A_h\chi. \end{aligned}$$

To define a CohFT, we simply set the Lagrangian to be

$$\mathcal{L} = Q(\chi(f) + \langle \chi, b \rangle) d\text{vol}, \quad (6.5)$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $C^\infty(M, T^*\Sigma)$ induced by the Riemannian metric on Σ , f is a vector field over $C^\infty(M, \Sigma)$, and $d\text{vol}$ is the volume form on M . (6.5) can be seen as the pullback through f of the image of (6.4) under ϕ_{CW} .

Remark 6.5. More generally, one can take W in the construction of \mathcal{M}_Y and $\mathcal{M}_{Y'}$ to be the tensor product of the dual bundle of V and $\Lambda^p T^*M$, and f to be a section of the bundle $\Lambda^p T^*M \times C^\infty(M, T\Sigma)$ over $M \times C^\infty(M, \Sigma)$.

Remark 6.6. Lagrangians of the form (6.5) can be found in [4, 11]

For pre-observables of the CohFT, consider the map

$$\begin{aligned} e : M \times \Gamma(Y_\Sigma) &\cong M \times C^\infty(M, T\Sigma) \times C^\infty(M, T^*\Sigma) \rightarrow \Sigma \\ (x, f_1, f_2) &\mapsto \pi(f_1(x)), \end{aligned}$$

where $\pi : T\Sigma \rightarrow \Sigma$ is the canonical projection. Let α be a closed n -form on Σ . Let $\mathcal{O} = e^*\alpha$. \mathcal{O} can be decomposed as $\mathcal{O} = \sum_{p=0}^n \mathcal{O}^{(p)}$, where $\mathcal{O}^{(p)}$ is of horizontal degree p . Locally, α can be written as $\alpha_{i_1, \dots, i_n} du^{i_1} \wedge \dots \wedge du^{i_n}$. We then have

$$\mathcal{O}^{(p)} = \binom{n}{p} \alpha_{i_1, \dots, i_n} d_h u^{i_1} \wedge \dots \wedge d_h u^{i_p} \wedge \delta u^{i_{p+1}} \wedge \dots \wedge \delta u^{i_n}.$$

One can check that $\mathcal{O}^{(p)} = \frac{1}{p} K \mathcal{O}^{(p-1)}$. In other words, $\{\mathcal{O}^{(p)}\}_{p=0}^n$ is the standard K -sequence of $\mathcal{O}^{(0)}$.

7 Examples

In this section, various Coh(G)FTs living on spacetimes of dimensions $n \leq 4$ proposed by physicists are constructed using the framework developed in Section 6. Further examples like topological F theory and Kapustin-Witten theory can also be incorporated into this framework with ease.

Example 7.1 (Topological quantum mechanics). The geometric setting is specified by the following data.

1. M is the real line \mathbb{R} ;
2. Σ is a Riemannian manifold equipped with a Morse function h .

The Lagrangian is specified by the gauge fixing function

$$f = \frac{du}{dt} + \text{grad}h,$$

where t is the parameter of \mathbb{R} . The pre-observables are specified by the 1-form $\alpha = dh$ on Σ . We have

$$\begin{aligned}\mathcal{O}^{(0)} &= \partial_i h \delta u^i, \\ \mathcal{O}^{(1)} &= \partial_i h u_t^i dt.\end{aligned}$$

Remark 7.1. In dimension 1, there is no spinor or R -symmetry, hence no topological twisting. The topological quantum mechanics is just the $N = 2$ supersymmetric quantum mechanics.

Example 7.2 (Topological sigma model). The geometric setting is specified by the following data.

1. (M, j) is a Riemann surface;
2. (Σ, ω, J) is a Kähler manifold.

The Lagrangian is specified by the gauge fixing function

$$f = \bar{\partial}_J u,$$

where $\bar{\partial}_J u = \frac{1}{2}(Du + J \circ Du \circ j)$, Du is the total differential of u . Note that f is a section of the bundle $T^*M \times C^\infty(M, T\Sigma)$ over $M \times C^\infty(M, \Sigma)$. The pre-observables are specified by $\alpha = \omega$, the symplectic form on Σ . We have

$$\begin{aligned}\mathcal{O}^{(0)} &= \omega_{i_1 i_2} \delta u^{i_1} \delta u^{i_2}, \\ \mathcal{O}^{(1)} &= 2\omega_{i_1 i_2} u_\mu^{i_1} dx^\mu \delta u^{i_2}, \\ \mathcal{O}^{(2)} &= \omega_{i_1 i_2} u_\mu^{i_1} u_\nu^{i_2} dx^\mu dx^\nu.\end{aligned}$$

Remark 7.2. The topological sigma model can also be obtained by twisting the $N = 2$ supersymmetric non-linear sigma model.

Example 7.3 (Topological M theory). The geometric setting is specified by the following data.

1. M is a 3-dimensional manifold;

2. (Σ, Φ) is a G_2 -manifold.

The Lagrangian is specified by the gauge fixing function

$$f = [Du, Du, Du],$$

where $[\cdot, \cdot, \cdot]$ is the associator bracket of Φ . Note that f is a section of the bundle $\Lambda^3 T^*M \times C^\infty(M, T\Sigma)$ over $M \times C^\infty(M, \Sigma)$. The pre-observables are specified by $\alpha = \Phi$, the closed G_2 -structure on Σ . We have

$$\begin{aligned} \mathcal{O}^{(0)} &= \Phi_{i_1 i_2 i_3} \delta u^{i_1} \delta u^{i_2} \delta u^{i_3}, \\ \mathcal{O}^{(1)} &= 3\Phi_{i_1 i_2 i_3} u_\mu^{i_1} dx^\mu \delta u^{i_2} \delta u^{i_3}, \\ \mathcal{O}^{(2)} &= 3\Phi_{i_1 i_2 i_3} u_\mu^{i_1} u_\nu^{i_2} dx^\mu dx^\nu \delta u^{i_3}, \\ \mathcal{O}^{(3)} &= \Phi_{i_1 i_2 i_3} u_\mu^{i_1} u_\nu^{i_2} u_\sigma^{i_3} dx^\mu dx^\nu dx^\sigma. \end{aligned}$$

Example 7.4 (Topological Yang-Mills theory). The geometric setting is specified by the following data.

1. P is a principal $SU(2)$ -bundle;
2. $V = \text{ad}P$, $W = \text{ad}P \otimes \Lambda_-^2(T^*M)$.

The Lagrangian is specified by the gauge fixing functions

$$f_1 = d_A w, \quad f_2 = [\phi, w], \quad f_3 = F_-,$$

where F_- is the anti-self-dual part of the curvature F . Note that f_1 is of degree -2 , f_2 is of degree 0, and f_3 is of degree 0. The pre-observables are determined by $\mathcal{O}^{(0)} = \text{Tr}(\phi^2)$. We have

$$\begin{aligned} \mathcal{O}^{(0)} &= \text{Tr}(\phi^2), \\ \mathcal{O}^{(1)} &= -2\text{Tr}(\phi v), \\ \mathcal{O}^{(2)} &= \text{Tr}(v \wedge v - 2\phi F), \\ \mathcal{O}^{(3)} &= 2\text{Tr}(v \wedge F), \\ \mathcal{O}^{(4)} &= \text{Tr}(F \wedge F). \end{aligned}$$

Remark 7.3. The topological Yang-Mills theory can also be obtained by twisting the $N = (1, 1)$ supersymmetric Yang-Mills theory. The QK -structure obtained from the twisting is more complicated than the QK -structure defined in Section 6 [27]. More precisely, since χ and b are $\mathfrak{su}(2)$ -valued (anti-self-dual) 2-forms, there exists a family of QK -structures parameterized by $(r, s, t) \in \mathbb{R}^3$ by setting

$$\begin{aligned} Q\theta &= \phi, \quad QA = v, \quad Qw = \psi, \quad Q\chi = b + rF_-, \quad Qb = -r(d_A v)_-, \\ KA &= s\chi, \quad K\theta = tA, \quad K\phi = d\theta - tv, \quad Kv = dA - s(b + rF_-), \quad K\psi = dw, \quad Kb = d\chi + rsd_A\chi, \end{aligned}$$

and the action of Q and K on the other coordinates to be 0. Likewise, we set $t = 1$. After applying the change of coordinates induced by the Mathai-Quillen map, we get

$$\begin{aligned} Q\theta &= \phi - \frac{1}{2}[\theta, \theta], & Q\phi &= -[\theta, \phi], \\ QA &= v + d_A\theta, & Qv &= -[\theta, v] - d_A\phi, \\ Qw &= \psi - [\theta, w], & Q\psi &= -[\theta, \psi] + [\phi, w], \\ Q\chi &= rF_- + b - [\theta, \chi], & Qb &= -r(d_Av)_- - [\theta, b] + [\phi, \chi], \end{aligned}$$

and

$$\begin{aligned} K\theta &= A, & K\phi &= -v, \\ KA &= s\chi, & Kv &= 2F - s(b + rF_-), \\ Kw &= 0, & K\psi &= d_Aw, \\ K\chi &= 0, & Kb &= (1 + rs)d_{A\chi}. \end{aligned}$$

We then have

$$\theta_K = \theta + A + \frac{s}{2}\chi, \quad \phi_K = \phi - v - F + \frac{s}{2}(b + rF_-) + \frac{s}{2}d_{A\chi} + \frac{s^2}{8}[\chi, \chi].$$

Let's set $r = 0$ for simplicity. The standard K -sequence of $\mathcal{O}^{(0)} = \text{Tr}(\phi^2)$ takes the form

$$\begin{aligned} \mathcal{O}^{(0)} &= \text{Tr}(\phi^2), \\ \mathcal{O}^{(1)} &= -2\text{Tr}(\phi v), \\ \mathcal{O}^{(2)} &= \text{Tr}(v \wedge v + \phi(sb - 2F)), \\ \mathcal{O}^{(3)} &= \text{Tr}(s\phi d_{A\chi} - v \wedge (sb - 2F)), \\ \mathcal{O}^{(4)} &= \text{Tr}((sb/2 - F) \wedge (sb/2 - F) - sv \wedge d_{A\chi} + s^2\phi[\chi, \chi]/4). \end{aligned}$$

Let K_0 denote the vector symmetry in the special case of $s = 0$, i.e., the vector symmetry defined in Section 6. One can easily check the above standard K -sequence is nothing but the general K_0 -sequence specified by

$$\mathcal{W}^{(1)} = 0, \quad \mathcal{W}^{(2)} = s\text{Tr}(\phi b), \quad \mathcal{W}^{(3)} = 0, \quad \mathcal{W}^{(4)} = -\frac{s^2}{4}\text{Tr}(b \wedge b + \phi[\chi, \chi]).$$

Note that both $\mathcal{W}^{(2)}$ and $\mathcal{W}^{(4)}$ are gauge invariant and Q -exact. In fact, we have

$$\mathcal{W}^{(2)} = sQ(\text{Tr}(\phi\chi)), \quad \mathcal{W}^{(4)} = -\frac{s^2}{4}Q(\text{Tr}(b \wedge \chi)).$$

The standard K -sequence is equivalent to the standard K_0 -sequence up to an exact sequence.

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