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Monoidally Graded Geometry

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Monoidally Graded Geometry

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This note aims to give a generalization of the theory of $\mathbb{Z}_2$-graded geometry [1] [2] to a theory of $\mathcal{I}$-graded geometry, where $\mathcal{I}$ is a commutative semi-ring with some additional properties. We also prove Bachelor-type theorems in this setting. To our knowledge, such proofs are still missing except for some special cases (e.g., $\mathcal{I} = \mathbb{Z}_2$ or $\mathbb{N}$).

1 Commutative Monoids

Let $(\mathcal{I}, 0, +)$ be a commutative monoid. Let $\mathbb{Z}_q$ denote the cyclic group of order q.

Definition 1.1. A parity function is a (non-trivial) monoid homomorphism $p : \mathcal{I} \to \mathbb{Z}_2$.

Not every $\mathcal{I}$ has a non-trivial parity function. For example, there is no non-trivial homomorphism from $\mathbb{Z}_q$ to $\mathbb{Z}_2$ when q is a odd. Let $\mathcal{I}_a$ denote $p^{-1}(a)$ for $a \in \mathbb{Z}_2$.[1] We have $\mathcal{I}_a\mathcal{I}_b \subseteq \mathcal{I}_{a+b}$. Recall that an element x in $\mathcal{I}$ is called cancellative if $x + y = x + z$ implies $y = z$ for all y and z in $\mathcal{I}$. Suppose that there is a cancellative element in $\mathcal{I}_1$. It is easy to see that such an element induces injective maps from $\mathcal{I}_a$ to $\mathcal{I}_{a+1}$. It follows from the Cantor-Bernstein theorem that there exists a bijection between $\mathcal{I}_0$ and $\mathcal{I}_1$. A monoid is called cancellative if every element in it is cancellative. We have shown that

Proposition 1.1. *Let $\mathcal{I}$ be an commutative cancellative monoid. If $\mathcal{I}$ has a non-trivial parity function p, then the submonoid $\mathcal{I}_0$ and its complement $\mathcal{I}_1$ have the same cardinality.*

Remark 1.1. In the finite case, proposition 1.1 is no longer true if we drop the cancellative condition. For example, we can consider the commutative monoid defined by the following table. A non-trivial p is defined by setting $p(0) = p(b) = 0$ and $p(a) = 1$.

	0	a	b
0	0	a	b
a	a	b	a
b	b	a	b

Table 1.1: A commutative non-cancellative monoid of order 3

[1]We sometimes say that $\mathcal{I}_0$ is the even part of $\mathcal{I}$, and that $\mathcal{I}_1$ is the odd part of $\mathcal{I}$. We also say that an element of $\mathcal{I}_a$ has parity a for $a = 0, 1$.

Remark 1.2. In the infinite case, I do not know if there exists any counterexample if we drop the cancellative condition for $\mathcal{I}$.

The question now is, given an appropriate commutative cancellative monoid $\mathcal{I}$, how can one construct a parity function for it? If $\mathcal{I}$ is a finite, it is not hard to show that $\mathcal{I}$ is actually an abelian group. The fundamental theorem of finite abelian groups then tells us that $\mathcal{I}$ is isomorphic to a direct product of cyclic groups of prime-power order. By Proposition 1.1, one of these cyclic groups must be $\mathbb{Z}_{2^k}$, $k \geq 1$. We can write

$$\mathcal{I} = \mathbb{Z}_{2^k} \times \cdots .$$

We then define p by sending $(x, \cdots) \in \mathcal{I}$ to $a - 1 \mod 2$, where a is the order of $x \in \mathbb{Z}_{2^k}$. If $\mathcal{I}$ is infinite, the construction of p is hard, perhaps not possible in general. However, one can easily work out the case when $\mathcal{I}$ is free. ($\mathcal{I}$ is then cancellative, but not a group.) Let $\mathcal{I}_0$ be the submonoid of elements generated by even number of generators. Let $\mathcal{I}_1$ be the subset of elements generated by odd number of generators. Note that $\mathcal{I}_a\mathcal{I}_b \subseteq \mathcal{I}_{a+b}$. We obtain a parity function which sends elements in $\mathcal{I}_a$ to a. As an example, let $\mathcal{I}$ be $\mathbb{N}$, the monoid of natural numbers under addition. p is defined by sending even numbers to 0 and odd numbers to 1.

Let $K(\mathcal{I})$ denote the Grothendieck group of $\mathcal{I}$. Recall that it can be constructed as follows. Let $\sim$ be the equivalence relation on $\mathcal{I} \times \mathcal{I}$ defined by $(a_1, a_2) \sim (b_1, b_2)$ if there exists a $c \in \mathcal{I}$ such that $a_1 + b_2 + c = a_2 + b_1 + c$. The quotient $K(\mathcal{I}) = \mathcal{I} \times \mathcal{I}/\sim$ has a group structure by $[(a_1, a_2)] + [(b_1, b_2)] = [(a_1 + b_1, a_2 + b_2)]$.

Proposition 1.2. *Let p be a parity function for $\mathcal{I}$. The map*

$$\begin{aligned} p' : K(\mathcal{I}) &\to \mathbb{Z}_2 \\ [(a_1, a_2)] &\mapsto p(a_1) + p(a_2) \end{aligned}$$

is well-defined and gives a parity function for $K(\mathcal{I})$.

Proof. Let (a_1, a_2) and (b_1, b_2) represent the same element of $K(\mathcal{I})$, i.e., there exist some c such that $a_1 + b_2 + c = a_2 + b_1 + c$. One then concludes that $a_1 + b_2$ and $a_2 + b_1$ must have the same parity. Note that, for $a, b \in \mathbb{Z}_2$, $a = b$ if and only if $a + b = 0$. But

$$p'([(a_1, a_2)]) + p'(([b_1, b_2)]) = p(a_1 + b_2) + p(a_2 + b_1) = 0.$$

Hence $p'([a_1, a_2]) = p'([b_1, b_2])$. ☐

As an example, consider $K(\mathbb{N}) = \mathbb{Z}$, the monoid of integers under addition. The parity function p' induced from the above p for $\mathbb{N}$ again sends even numbers to 0 and odd numbers to 1.

Remark 1.3. When $\mathcal{I}$ is cancellative, it can be seen as a submonoid of $K(\mathcal{I})$ by the embedding

$$\begin{aligned} \iota : \mathcal{I} &\to K(\mathcal{I}) \\ a &\mapsto [(a, 0)]. \end{aligned}$$

The cancellative property is not necessary for the proof of Proposition 1.2. But it guarantees the non-triviality of p', since p' restricted to $\mathcal{I}$ must coincide with p.

Definition 1.2. $\mathcal{I}$ is said to be Cartesian if it is the Cartesian product $\mathcal{J} \times \mathcal{J}$ with the canonical additive structure. The parity function p of such a monoid should satisfy the following additional conditions

$$\mathcal{J} \times \{0\} \nsubseteq \mathcal{I}_0, \quad \{0\} \times \mathcal{J} \nsubseteq \mathcal{I}_0.$$

In other words, p can not be induced from the parity function on factors of $\mathcal{I}$.

As an example, consider $\mathcal{I} = \mathbb{Z} \times \mathbb{Z}$. Our construction of p for $\mathcal{I}$ is legit. However, the map which sends $(i, j) \in \mathcal{I}$ to $i \mod 2$ is not a parity function for $\mathcal{I}$, as the even part $\mathcal{I}_0$ contains the sub-monoid $\{0\} \times \mathbb{Z}$.

Let $\mathrm{Pow}(\mathcal{I})$ denote the power set of $\mathcal{I}$.

Proposition 1.3. $\mathrm{Pow}(\mathcal{I})$ *is a commutative semi-ring.*

Proof. Let $U, V \in \mathrm{Pow}(\mathcal{I})$, we define the addition of U and V as

$$U + V = U \cup V$$

and the multiplication of U and V as

$$U \cdot V = \bigcup_{x \in U, y \in V} \{x + y\}$$

where $x + y$ is the product of x and y in $\mathcal{I}$. We choose $\emptyset$ as the additive identity and $\{0\}$ as the multiplicative identity. It is easy to verify that $\mathrm{Pow}(\mathcal{I})$ with the above datum forms a commutative semi-ring. For example, one can easily verify the distributive law

$$U \cdot (V_1 + V_2) = \bigcup_{x \in U, y \in V_1 \cup V_2} \{x + y\} = \left(\bigcup_{x \in U, y \in V_1} \{x + y\} \right) \cup \left(\bigcup_{x \in U, y \in V_2} \{x + y\} \right) = U \cdot V_1 + U \cdot V_2,$$

for all $U, V_1, V_2 \in \mathrm{Pow}(\mathcal{I})$. □

Remark 1.4. It is necessary to choose union instead of intersection as the additive operation, otherwise the distributive law will not be satisfied.

Remark 1.5. Sometimes, we will only need the monoid structure of $\mathrm{Pow}(\mathcal{I})$. We use $\mathrm{Pow}(\mathcal{I})^+$ to denote the corresponding commutative monoid under addition and $\mathrm{Pow}(\mathcal{I})^\times$ to denote the corresponding commutative monoid under multiplication.

Lemma 1.1. *Let* $h : \mathcal{I}_1 \to \mathcal{I}_2$ *be a homomorphism between two commutative monoids. The induced map from* $\mathrm{Pow}(\mathcal{I}_1)$ *to* $\mathrm{Pow}(\mathcal{I}_2)$ *by sending* $U \in \mathrm{Pow}(\mathcal{I}_1)$ *to* $h(U) \in \mathrm{Pow}(\mathcal{I}_2)$ *is a semi-ring homomorphism.*

Proof. Trivial. □

Lemma 1.2. *The singleton* $\{x\} \in \mathrm{Pow}(\mathcal{I})^\times$ *is cancellative if and only if* $x \in \mathcal{I}$ *is cancellative.*

Note that $\mathcal{I}$ is a sub-monoid of $\mathrm{Pow}(\mathcal{I})^\times$ by sending its element to the corresponding singleton. So we only need to prove the "$\Leftarrow$" direction.

Proof. Since x is cancellative, multiplying x induces a injective map $m_x : \mathcal{I} \to \mathcal{I}$. Let $U \in \mathrm{Pow}(\mathcal{I})$, note that

$$\{x\} \cdot U = m_x(U).$$

Recall that a map $f : X \to Y$ is injective if and only if the induced map $f : \mathrm{Pow}(X) \to \mathrm{Pow}(Y)$ is injective. We then have

$$\{x\} \cdot U = \{x\} \cdot V \Rightarrow U = V$$

for all $U, V \in \mathrm{Pow}(\mathcal{I})$. □

It will be seen later that the cancellative property is essential for the construction of "monoidally graded categories".

2 Monoidally Graded Categories

Let $\mathcal{I}$ be a commutative cancellative monoid with a non-trivial parity function p. Recall that the hom-set $\mathrm{Hom}(C, C)$ of each object C of a locally small category $\mathcal{C}$ is a monoid. Thus, every monoid can be viewed as a category with a single object.

Definition 2.1. An $\mathcal{I}$-graded category is a category $\mathcal{C}$ with a (full) functor G from $\mathcal{C}$ to $\mathrm{Pow}(\mathcal{I})^\times$. We call G the grading functor of $\mathcal{C}$. An object C of $\mathcal{C}$ is called an $\mathcal{I}$-graded object. In particular, C is called an graded object if $\mathcal{I} = \mathbb{Z}$, and an bi-graded object if $\mathcal{I} = \mathbb{Z} \times \mathbb{Z}$.

In other words, every morphism of $\mathcal{C}$ is labelled by an element in $\mathrm{Pow}(\mathcal{I})^\times$ and the labelling is compatible with compositions of morphisms. Let $\mathcal{J}$ be a sub-monoid of $\mathcal{I}$, its power set $\mathrm{Pow}(\mathcal{J})$ under multiplication is a sub-monoid of $\mathrm{Pow}(\mathcal{I})$. The collection of all $\mathcal{I}$-graded objects C with the collection of morphisms f taking values in $\mathrm{Pow}(\mathcal{J})^\times$ form a sub-category of C, denoted by $\mathcal{C}_\mathcal{J}$. Let 0 denote the trivial sub-monoid of $\mathcal{I}$

Definition 2.2. The sub-category $\mathcal{C}_0$ of $\mathcal{C}$ is called the category of $\mathcal{I}$-graded objects in $\mathcal{C}$.

Example 2.1. [The category $\mathcal{I}$-graded sets] Let's consider the following cateogry $\mathcal{C}$. Objects of $\mathcal{C}$ are $\mathcal{I}$-graded sets. An $\mathcal{I}$-graded set is a set X with a cover $\{X_i\}_{i\in\mathcal{I}}$ indexed by $\mathcal{I}$ such that $X_i \neq X_j$ for $i \neq j$. The cover induces a map $P_X : \mathrm{Pow}(X) \to \mathrm{Pow}(\mathcal{I})$ which sends a subset U of X to a subset of $\mathcal{I}$ in the following form[2]

$$P_X(U) = \bigcup_{i\in\mathcal{I}, U\cap X_i \neq \emptyset} \{i\}.$$

By definition, the image of P_X contains at least one of the singleton of $\mathrm{Pow}(\mathcal{I})$. X is called fully $\mathcal{I}$-graded if $X_i \not\subseteq X_j$ for $i \neq j$, or equivalently, if the image of P_X contains all the singletons of $\mathrm{Pow}(\mathcal{I})$.

Morphisms of $\mathcal{C}$ are $\mathcal{I}$-graded maps. An $\mathcal{I}$-graded map between two $\mathcal{I}$-graded sets X and Y is a map $f : X \to Y$ satisfying

$$P_X(U) \cdot V = P_Y(f(U)) \tag{2.1}$$

[2]Let $\mathrm{Pow}(X)^+$ denote the commutative monoid with union of subsets as addition. P_X is also a monoid homomophism from $\mathrm{Pow}(X)^+$ to $\mathrm{Pow}(\mathcal{I})^+$.

for all $U \in \mathrm{Pow}(X)$ and some $V \in \mathrm{Pow}(\mathcal{I})$. Such a V, if exists, must be unique. To see this, let V' be another subset of $\mathcal{I}$ satisfying (2.1). By definition of a $\mathcal{I}$-graded set, one can find a U with $P_X(U) = \{x\}$ for some $x \in \mathcal{I}$. By Lemma 1.2, $V' = V$. We obtain a map from $\mathrm{Hom}(X, Y)$ to $\mathrm{Pow}(\mathcal{I})$. Denote the above morphism by f_V. Let g_W be another morphism associated to the map $g : Y \to Z$. It is easy to see that $f_V \circ g_W$ is a morphism from X to Z and

$$f_V \circ g_W = (f \circ g)_{V \cdot W}.$$

We obtain a functor from $\mathcal{C}$ to $\mathrm{Pow}(\mathcal{I})^\times$. In other words, the category $\mathcal{C}$ is $\mathcal{I}$-graded. It is then straightforward to define the category of $\mathcal{I}$-graded sets and the category of fully $\mathcal{I}$-graded sets.

Definition 2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{I}$-graded categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ is called a $\mathcal{I}$-graded functor if the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow \quad \swarrow & \\ & \mathrm{Pow}(\mathcal{I})^\times & \end{array}$$

Remark 2.1. By definition, a $\mathcal{I}$-graded functor between two $\mathcal{I}$-graded categories induces a functor between the corresponding categories of $\mathcal{I}$-graded objects.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two commutative cancellative monoids. Recall that a (surjective) homomorphism between two monoids can be seen as a (full) functors between the corresponding categories. By Lemma 1.1, a $\mathcal{I}_1$-graded category is then also a $\mathcal{I}_2$-graded category. Let C be the product category of two $\mathcal{I}$-graded categories. Naturally, it is a $\mathcal{I} \times \mathcal{I}$-graded category. Note that there is a canonical surjective homomorphism

$$\begin{aligned} s : \mathcal{I} \times \mathcal{I} &\to \mathcal{I} \\ (i, j) &\mapsto i + j. \end{aligned}$$

Hence C is also a $\mathcal{I}$-graded category.

Definition 2.4. An $\mathcal{I}$-graded monoidal category $\mathcal{C}$ is a monoidal category which is also $\mathcal{I}$-graded. Moreover, we require the tensor product bi-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to be $\mathcal{I}$-graded.

Example 2.2. Let $\mathcal{C}$ be as in Example 2.1. It is $\mathcal{I}$-graded monoidal by considering the tensor product $X \otimes Y$ which is the Cartesian product space $X \times Y$ equipped with the cover $\{X_i \times Y_j\}_{i,j \in \mathcal{I}}$. Each factor $X_i \times Y_j$ is labeled by $i + j \in \mathcal{I}$.

Recall that a pre-additive category is a category such that every hom-set is an abelian group, and compositions of morphisms are bi-linear.

Definition 2.5. An $\mathcal{I}$-graded pre-additive category $\mathcal{C}$ is a pre-additive category which is also $\mathcal{I}$-graded. Moreover, the grading functor G of C preserves the additive structure, i.e.,

$$G : \mathrm{Hom}(X, Y) \to \mathrm{Pow}(\mathcal{I})$$

is a semi-ring homomorphism.

Let R be a commutative ring. Let $\mathcal{I}$ be a countable commutative cancellative monoid.

Example 2.3 (The category of $\mathcal{I}$-graded R-modules)**.** Let's consider the following cateogry $\mathcal{C}$. Objects of $\mathcal{C}$ are $\mathcal{I}$-graded R-modules. An $\mathcal{I}$-graded R-module is an R-module V with a family of sub-modules $\{V_i\}_{i\in\mathcal{I}}$ indexed by $\mathcal{I}$ such that $V = \bigoplus_{i\in\mathcal{I}} V_i$.

Morphisms of $\mathcal{C}$ are $\mathcal{I}$-graded R-linear maps. An $\mathcal{I}$-graded R-linear map of degree j between two $\mathcal{I}$-graded R-module V and W is a R-linear map $f_j : V \to W$ satisfying

$$f_j(V_i) \subset W_{i+j}$$

for all $i \in \mathcal{I}$. An $\mathcal{I}$-graded R-linear map f is just a linear combination $f = \sum_{j\in\mathcal{I}} c_j f_j$, $c_j \in R$. One can associate a subset U of $\mathcal{I}$ to it. U is defined by

$$U = \bigcup_{j\in\mathcal{I}, c_j\neq 0} \{j\}.$$

In this way, we obtain a map from $\mathrm{Hom}(V, W)$ to $\mathrm{Pow}(\mathcal{I})$. It is easy to check that this map preserves both the additive and the multiplicative structure of $\mathrm{Hom}(V, W)$ and $\mathrm{Pow}(\mathcal{I})$. In other words, the category $\mathcal{C}$ is indeed a $\mathcal{I}$-graded pre-additive category. It is then straightforward to define the category of $\mathcal{I}$-graded R-modules.

Remark 2.2. The category of $\mathcal{I}$-graded R-modules is a sub-category of $\mathcal{I}$-graded sets. In fact, the decomposition $V = \bigoplus_{i\in\mathcal{I}}$ of a $\mathcal{I}$-graded R-module V yields a cover of V by $\{V_i^c\}_{i\in\mathcal{I}}$. V is then a fully $\mathcal{I}$-graded set. It is also easy to see that a $\mathcal{I}$-graded R-linear map is naturally a $\mathcal{I}$-graded map between the underlying $\mathcal{I}$-graded sets.

Recall that a pre-additive category is additive if it has all finite bi-product. It is not hard to see that the category of $\mathcal{I}$-graded R-modules is additive and monoidal. Given two $\mathcal{I}$-graded R-modules V and W, we define the bi-product (or simply the direct sum) $V \oplus W$, the tensor product $V \otimes W$ by setting

$$V \oplus W = \bigoplus_{i\in\mathcal{I}} (V_i \oplus W_i), \quad V \otimes W = \bigoplus_{k\in\mathcal{I}} \left(\bigoplus_{i+j=k} V_i \otimes W_j \right),$$

where $V_i \oplus W_i$ and $V_i \otimes W_j$ are just the ordinary direct sum and tensor product over R, respectively.

One can also make $\mathrm{Hom}(V, W)$ into a $\mathcal{I}$-graded R-module by

$$\mathrm{Hom}(V, W) = \bigoplus_{j\in\mathcal{I}} \mathrm{Hom}(V, W)_j, \quad \mathrm{Hom}(V, W)_j = \{f \in \mathrm{Hom}(V, W) | fV_i \subset W_{i+j}\}.$$

(Note that each $\mathrm{Hom}(V, W)_j$ has a canonical R-module structure.) Morphisms from V to W in the category of $\mathcal{I}$-graded R-modules are then just elements of $\mathrm{Hom}(V, W)_0$. For later use, we give the following definition.

Definition 2.6. Let V be an $\mathcal{I}$-graded R-module. The map

$$d : V \to \mathrm{Pow}(\mathcal{I})$$
$$v = \sum_{i\in\mathcal{I}} v_i \mapsto \bigcup_{v_i\neq 0} \{i\}$$

is called the degree map. $d(v)$ is called the degree of v. v is said to be homogeneous if $d(v)$ is a singleton. With a slight abuse of notation, we also use $d(v)$ to denote the corresponding element of $\mathcal{I}$ when v is homogeneous. We define the parity of a homogeneous element v to be $p(d(v))$, where p is the parity function of $\mathcal{I}$. For simplicity, we often use $p(v)$ to replace $p(d(v))$.

Let $(\mathcal{I}, 0, +)$ be a countable commutative cancellative monoid. Suppose that it has a commutative multiplicative structure which is compatible with the additive structure. That is, it is a commutative cancellative semi-ring. We write ab as the multiplication of a and b in $\mathcal{I}$.

Definition 2.7. An $\mathcal{I}$-graded R-module A is called an $\mathcal{I}$-graded R-algebra if A is a unital associative R-algebra and if the multiplication $\mu : A \otimes A \to A$ is a morphism of $\mathcal{I}$-graded R-modules. We write $xy = \mu(x \otimes y)$ as the shorthand notation for multiplications of A. A is said to be commutative if

$$xy - (-1)^{p(d(x)d(y))} yx = 0, \tag{2.2}$$

for all homogeneous $x, y \in A$.

Remark 2.3. Here we have to be careful about the sign appearing in the right hand side of (2.2). Although both of $\mathcal{I}$ and $\mathbb{Z}_2$ are semi-rings[3], p is not necessarily a semi-ring homomorphism and we do not have $p(d(x)d(y)) = p(x)p(y)$ in general. (They do coincide when $\mathcal{I} = \mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{N}$. But in the case where $\mathcal{I} = \mathbb{Z} \times \mathbb{Z}$, a parity function which preserves the semi-ring structure must be induced from the parity function on factors of $\mathcal{I}$, which contradicts Definition 1.2. Hence $(-1)^{p(d(x)d(y))} \neq (-1)^{p(x)p(y)}$. To choose which as the sign factor in (2.2) is just a matter of convention.) Bearing this in mind, we will use $(-1)^{p(x)p(y)}$ to replace the sign factor $(-1)^{p(d(x)d(y))}$ for simplicity.

To define the category of (commutative) $\mathcal{I}$-graded R-algebras, one simply let the morphisms to be the $\mathcal{I}$-graded R-linear maps of degree 0 which are also algebraic homomorphisms. We use $\mathrm{Alg}_{\mathcal{I}}$ and $\text{Comm-Alg}_{\mathcal{I}}$ to denote the corresponding categories. $\mathrm{Alg}_{\mathcal{I}}$ and $\text{Comm-Alg}_{\mathcal{I}}$ are monoidal by the following definition.

Definition 2.8. The tensor product of two (commutative) $\mathcal{I}$-graded R-algebra A and B is the $\mathcal{I}$-graded R-module $A \otimes B$, together with the multiplication

$$(x \otimes y)(x' \otimes y') = (-1)^{p(x')p(y)} xx' \otimes yy',$$

for all homogeneous $x, x' \in A$ and $y, y' \in B$.

Given an $\mathcal{I}$-graded R-module V, $\text{Comm-Alg}_{\mathcal{I}}$ has a universal object associated to V.

Definition 2.9. The tensor algebra $\mathrm{T}(V)$ is the $\mathcal{I}$-graded R-module $\mathrm{T}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes^n}$, together with the tensor product $\otimes$ as the canonical multiplication. The symmetric algebra $\mathrm{S}(V)$ is the quotient algebra of $\mathrm{T}(V)$ by the $\mathcal{I}$-graded two-sided ideal generated by

$$v \otimes w - (-1)^{p(v)p(w)} w \otimes v,$$

where $v, w \in V \subset \mathrm{T}(V)$ are homogeneous.

[3] The multiplicative structure on $\mathcal{Z}_p$ is given by the one inherited from the canonical multiplicative structure on $\mathbb{Z}$.

Remark 2.4. $\mathrm{S}(V)$ has a canonical $\mathbb{N}$-grading inherited from $\mathrm{T}(V)$ which should not be confused with its $\mathcal{I}$-grading. We write $\mathrm{S}(V) = \bigoplus_{n\in\mathbb{N}} \mathrm{S}^n(V)$ to indicate that fact. Note that $\mathrm{S}^0(V) = R$, but $\mathrm{S}(V)_0$, the sub-space of homogeneous elements of degree 0, is in general larger than R.

Just like $\mathrm{T}(V)$, $\mathrm{S}(V)$ is universal in the sense that, given a commutative $\mathcal{I}$-graded R-algebra A and a $\mathcal{I}$-graded R-linear map $f : V \to A$. There exists a unique algebraic homomorphism $\tilde{f} : \mathrm{S}(V) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \mathrm{S}(V) \\ & \overset{f}{\searrow} & \downarrow \tilde{f} \\ & & A \end{array}$$

where $\iota : V \to \mathrm{S}(V)$ is the canonical embedding. Note that $\tilde{f}$ preserves the $\mathcal{I}$-grading, i.e., it is a morphism in Comm-Alg$_{\mathcal{I}}$. Choosing A to be R (viewed as an $\mathcal{I}$-graded R-algebra whose components of degree other than zero are 0.) and f to be the zero map, we obtain an R-algebra homomorphism from $\mathrm{S}(V)$ to R. We denote this map by ϵ. Note that $\ker\epsilon = I$, where $I = \bigoplus_{n>0} \mathrm{S}^n(V)$.

Let k be a field and R be a commutative k-algebra. Let A be a commutative $\mathcal{I}$-graded k-algebra.

Definition 2.10. A k-algebra epimorphism $\epsilon : A \to R$ is called a body map of A if $\ker\epsilon \supset I$, where I is the ideal in A generated by homogeneous elements of non-zero degree.

By definition, ϵ preserves the $\mathcal{I}$-grading of A.

Definition 2.11. Let ϵ be a body map of A. A is said to be projected if the short exact sequence

$$0 \longrightarrow \ker\epsilon \longrightarrow A \overset{\epsilon}{\longrightarrow} R \longrightarrow 0$$

splits.

The splitting gives A an R-module structure depending on ϵ, with respect to which ϵ becomes an R-algebra homomorphism. Conversely, A is projected if A has an R-module structure and ϵ preserves that structure.

Lemma 2.1. *Let V be an $\mathcal{I}$-graded R-module and $V_0 = 0$. Let ϵ be an R-linear body map of* $\mathrm{S}(V)$*. Then ϵ is unique.*

Proof. In this case, $\mathrm{S}(V) = R \oplus I$ where $I = \bigoplus_{n>0} \mathrm{S}^n(V)$. Since $I \subset \ker\epsilon$ and ϵ is R-linear, the only possible choice of ϵ is the canonical one. □

Remark 2.5. Let V be as in Lemma 2.1. Suppose $A \cong \mathrm{S}(V)$ as $\mathcal{I}$-graded k-algebras. In particular, this implies that A admits a decomposition $A = A' \oplus I$ where $A' \cong R$ and I is the ideal generated by homogeneous elements of non-zero degree. Let ϵ be a body map of A. Since $I \subset \ker\epsilon$, ϵ is determined by $\epsilon|_{A'}$. In other words, ϵ of A is unique up to a k-algebra epimorphism of R.

More can be said if V is free.

Lemma 2.2. *Let V be a free $\mathcal{I}$-graded R-module and $V_0 = 0$. Let ϵ be a R-linear body map of* $\mathrm{S}(V)$*. (By Lemma 2.1, ϵ is the canonical one.) Let I denote the kernel of ϵ. Then there exists an R-algebra isomorphism*

$$\mathrm{S}(V) \cong \mathrm{S}(I/I^2),$$

where I^2 is the square of the ideal I.

Proof. Let $\iota : V \hookrightarrow \mathrm{S}(V)$ be the canonical embedding. Since $I = \bigoplus_{n>0} \mathrm{S}^n(V)$, we have $\iota(V) \subset I$, which yields another embedding $V \hookrightarrow I/I^2 \hookrightarrow \mathrm{S}(I/I^2)$, which induces the desired isomorphic map between $\mathrm{S}(V)$ and $\mathrm{S}(I/I^2)$. □

Definition 2.12. The $\mathcal{I}$-graded algebra of formal power series on V is the R-module

$$\mathrm{S}[V] = \prod_{n\in\mathbb{N}} S^n(V)$$

equipped with the canonical algebraic multiplication.

Let I be the kernel of the canonical body map of $\mathrm{S}(V)$. One can equip $\mathrm{S}(V)$ with the so-called I-adic topology. (To each point x of $\mathrm{S}(V)$ one assigns a collection of subsets $\mathcal{B}(x) = \{x + I^n\}_{x\in A, n>0}$. The I-adic topology is then the unique topology on $\mathrm{S}[V]$ such that $\mathcal{B}(x)$ forms a neighborhood base of x for all x.) Moreover, one can consider the I-adic completion of $\mathrm{S}(V)$ which is defined as the inverse limit

$$\widehat{\mathrm{S}(V)}_I := \varprojlim \mathrm{S}(V)/I^n.$$

By the universal property of the inverse limit, one obtains a natural map

$$\iota_I : \mathrm{S}(V) \to \widehat{\mathrm{S}(V)}_I$$

with kernel equal to $\bigcap_{n>0} I^n$. In our case, there is a canonical isomorphism $\widehat{\mathrm{S}(V)}_I \cong \mathrm{S}[V]$ under which ι_I coincides with the canonical embedding of $\mathrm{S}(V)$ into $\mathrm{S}[V]$. In fact, $\mathrm{S}(V)$ is Hausdorff and can be made into a metric space such that $\mathrm{S}[V]$ is the completion of $S(V)$ with respect to that metric structure.

Lemma 2.3. *Let A be a commutative $\mathcal{I}$-graded R-algebra. Let J be an ideal of A such that A is J-adic complete. $S[V]$ is universal in the sense that, given an $\mathcal{I}$-graded R-linear map $f : V \to A$ such that $f(V) \subset J$, there exists a unique (continuous) algebraic homomorphism $\tilde{f} : \mathrm{S}[V] \to A$ such that the following diagram commutes*

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \mathrm{S}[V] \\ & \searrow^{f} & \downarrow{\tilde{f}} \\ & & A \end{array}$$

Proof. We already know that f induces a unique map $f' : \mathrm{S}(V) \to A$ such that $f' \circ \iota = f$. By assumption, f' extends naturally to a map $\tilde{f} : \mathrm{S}[V] \cong \widehat{\mathrm{S}(V)}_I \to \widehat{A}_J \cong A$.
<u>Claim:</u> $\tilde{f}$ is continuous.
<u>Proof:</u> It suffices to show that $\tilde{f}^{-1}(J^m)$ is a neighborhood of 0 for any $m \in \mathbb{N}$. By assumption, $I \subset \tilde{f}^{-1}(J)$. It follows that $I^m \subset \tilde{f}^{-1}(J)^m \subset \tilde{f}^{-1}(J^m)$. ■

Now since $\mathrm{S}(V)$ is dense in $\mathrm{S}[V]$ and $\tilde{f}|_{\mathrm{S}(V)} = f'$, $\tilde{f}$ is unique. □

Remark 2.6. Likewise, we have a canonical body map of $\mathrm{S}[V]$ induced from the zero map $V \to R$. Similar results like Lemma 2.1 and Lemma 2.2 also hold. For example, we have

$$\mathrm{S}[V] \cong \mathrm{S}[I/I^2],$$

where V and I are as in Lemma 2.2.

Lemma 2.4. *Let ϵ be the canonical body map of* $\mathrm{S}[V]$*. Then for* $f \in \mathrm{S}[V]$*,* f *is invertible if and only if* $\epsilon(f)$ *is invertible.*

Proof. "$\Rightarrow$": Trivial.
"$\Leftarrow$": Suppose $\epsilon(f) = c$ where $c \in R$ is invertible. We can write $f = c + f'$ where $f' \in \prod_{n\geq 1} \mathrm{S}^n(V)$. Note that $(f')^k \in \prod_{n\geq k} \mathrm{S}^n(V)$ for all $k > 0$. We can then set the inverse of f to be the formal sum $f^{-1} := c^{-1} \sum_{k\in\mathbb{N}} (-1)^k (c^{-1}f')^k$. ($f^{-1}$ is well-defined because the formal sum restricted to each $S^n(V)$ is a finite sum.) ☐

Corollary 2.1. $\mathrm{S}[V]$ *is local if* R *is local.*

Proof. Choose a non-unit $f \in \mathrm{S}[V]$. Let $c = \epsilon(f)$. By Lemma 2.4, c is a non-unit. Since R is local, $1 - c$ is invertible. But then $1 - f$ is a unit by Lemma 2.4. ☐

As one will see in the Section 4, it is actually crucial to work with $\mathrm{S}[V]$ instead of $\mathrm{S}(V)$ because the former allows us to have a description of morphisms between $\mathcal{I}$-graded domains in term of coordinates, a notion of partition of unity for "$\mathcal{I}$-graded manifolds", and more.

3 Monoidally Graded Ringed Spaces

Recall that a ringed space $(X, \mathcal{O})$ is a topological space X with a sheaf of rings $\mathcal{O}$ on X.

Definition 3.1. An $\mathcal{I}$-graded ringed space is a ringed space $(X, \mathcal{O})$ such that

1. $\mathcal{O}(U)$ is an $\mathcal{I}$-graded algebra for any open subset U of X;
2. the restriction morphism

$$\rho_{V,U} : \mathcal{O}(U) \to \mathcal{O}(V)$$

   preserves the $\mathcal{I}$-grading.

A morphism between two $\mathcal{I}$-graded ringed spaces $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ is just a morphism $\varphi = (\tilde{\varphi}, \varphi^*)$ between ringed spaces such that $\varphi^*_U : \mathcal{O}_2(U) \to \mathcal{O}_1(\tilde{\varphi}^{-1}(U))$ preserves the $\mathcal{I}$-grading for any open subset U of X_2.

Let (X, C) be a ringed space such that $C(U)$ are commutative rings. $\mathcal{I}$-graded C-modules and commutative $\mathcal{I}$-graded C-algebras and the corresponding morphisms can be defined in a similar way to Definition 3.1. In particular, the structure sheaf $\mathcal{O}$ of an $\mathcal{I}$-graded ringed space can be viewed as an $\mathcal{I}$-graded C-algebra if C is a sub-sheaf of $\mathcal{O}$ and $C(U)$ are homogeneous sub-algebras of degree 0 of $\mathcal{O}(U)$.

Definition 3.2. Let $\mathcal{F}$ be an $\mathcal{I}$-graded C-module. The formal symmetric power $\mathrm{S}[\mathcal{F}]$ of $\mathcal{F}$ is the sheafification of the presheaf

$$U \to \mathrm{S}[\mathcal{F}(U)],$$

where $\mathrm{S}[\mathcal{F}(U)]$ is the $\mathcal{I}$-graded algebra of formal power series on the $C(U)$-module $\mathcal{F}(U)$.

By definition, $\mathrm{S}[\mathcal{F}]$ is a commutative $\mathcal{I}$-graded C-algebra.

Lemma 3.1. *Let $\mathcal{A}$ be a commutative $\mathcal{I}$-graded C-algebra. Let $\mathcal{B}$ be a sub-sheaf of $\mathcal{A}$ such that $\mathcal{A}(U)$ is $\mathcal{B}(U)$-adic complete for all open subsets U.* $\mathrm{S}[\mathcal{F}]$ *is universal in the sense that, given a morphism of $\mathcal{I}$-graded C-modules $F : \mathcal{F} \to \mathcal{A}$ such that $F(\mathcal{F}(U)) \subset \mathcal{B}(U)$ for all open subsets U, there exists a unique morphism of $\mathcal{I}$-graded C-algebras $\tilde{F} : \mathrm{S}[\mathcal{F}] \to \mathcal{A}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F} & \overset{\iota}{\hookrightarrow} & \mathrm{S}[\mathcal{F}] \\ & \searrow^{F} & \downarrow{\scriptstyle \tilde{F}} \\ & & \mathcal{A} \end{array}$$

where $\iota : \mathcal{F} \to \mathrm{S}[\mathcal{F}]$ is the canonical monomorphism.

Proof. This follows directly from the universal property of sheafification[4], and the universal property of $S[\mathcal{F}(U)]$ stated in Lemma 2.3. □

To end this section, we state the following lemma taken from [2].

Lemma 3.2. *Let*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{3.1}$$

be a short exact sequence of $\mathcal{I}$-graded C-modules where $\mathcal{F}$ and $\mathcal{G}$ are locally free $\mathcal{I}$-graded C-modules. Then the obstruction of the existence of a splitting of (3.1) can be represented as an element in the first sheaf cohomology group $H^1(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G})_0)$ of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})_0$.

4 Monoidally Graded Domains

Throughout this section, V is a real $\mathcal{I}$-graded vector space with $V_0 = 0$ and the dimension of the homogeneous sub-space V_i of V is m_i. We also assume that only finitely many of V_i are non-trivial.

Definition 4.1. Let U be a domain of $\mathbb{R}^n$. An $\mathcal{I}$-graded domain $\mathcal{U}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded ringed space $(U, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of $\mathrm{S}[V]$-valued smooth functions.

Remark 4.1. $\mathcal{U}$ is a locally ringed space by Corollary 2.1.

For example, a domain U with the sheaf C^∞ of smooth functions on U is an $\mathcal{I}$-graded domain of dimension $n|0$, which is denoted again by U for simplicity.

Lemma 4.1. *Let $F : C^\infty \to C^\infty$ be a endomorphism of sheaves of commutative rings on U. Then F must be the identity.*

Proof. First, we show that F is actually an endomorphism of sheaves of unital $\mathbb{R}$-algebras on U. It suffices to show that F restricted to any open subset of U sends a constant function to itself. We know this is true for $\mathbb{Q}$-valued constant functions. Now, if F sends a constant function f to a non-constant function g, then one can find two rational number b_1 and b_2 such that $g - b_1$ and $g - b_2$ are non-invertible. But then the pre-images $f - b_1$ and $f - b_2$ are non-invertible, which implies that

[4] That is, given a presheaf $\mathcal{F}$, a sheaf $\mathcal{G}$, and a presheaf morphism $F : \mathcal{F} \to \mathcal{G}$, there exists a unique sheaf morphism $\tilde{F} : \mathcal{F}^\sharp \to \mathcal{G}$ such that $\tilde{F} \circ \iota = F$, where $\mathcal{F}^\sharp$ is the sheafification of $\mathcal{F}$ and $\iota : \mathcal{F} \to \mathcal{F}^\sharp$ is the canonical morphism.

f is non-constant: a contradiction. To show that g actually equals f, use the fact that the only field endomorphism of $\mathbb{R}$ is the identity.

Let $p \in U$. F induces a map F_p on the stalk C_p^∞ which is a unital ring homomorphism. On the other hand, for any open neighborhood $U_p \subset U$ of p, the evaluation map

$$\begin{aligned} \mathrm{ev} : C^\infty(U_p) &\to \mathbb{R} \\ f &\mapsto f(p) \end{aligned}$$

induces a map $\mathrm{ev}_p : C_p^\infty \to \mathbb{R}$. For $f_p \in C_p^\infty$, it is easy to see that f_p is invertible if and only if $\mathrm{ev}_p(f_p) \neq 0$. Let $c = \mathrm{ev}_p(F_p(f_p))$. $f_p - c$ is non-invertible. Hence $\mathrm{ev}_p(f_p) = c$. In other words, for any open subset U' of U, we have $F_{U'}(f)(p) = f(p)$ for all $f \in C^\infty(U')$ and all $p \in U'$. This implies $F = \mathrm{id}$. □

A morphism between $\mathcal{I}$-graded domains is just a morphism of locally ringed spaces which preserves the $\mathcal{I}$-grading. In particular, we have a canonical morphism $U \hookrightarrow \mathcal{U}$ induced by the canonical body map $\epsilon : C^\infty(U) \otimes \mathrm{S}[V] \to C^\infty(U)$.

Proposition 4.1. *There exists a unique monomorphism $\varphi : U \to \mathcal{U}$ with $\tilde{\varphi} = \mathrm{id}$.*

Proof. Existence is guaranteed by ϵ. Uniqueness follows from Remark 2.5 and Lemma 4.1. □

We also have a canonical morphism for the other direction $\mathcal{U} \to U$ induced by the canonical embedding $\iota : C^\infty(U) \to C^\infty(U) \otimes \mathrm{S}[V]$. [5] Note that $\epsilon \circ \iota = \mathrm{id}$ on $C^\infty(U)$.

Proposition 4.2. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be a morphism from $\mathcal{U}_1 = (U_1, \mathcal{O}_1)$ to $\mathcal{U}_2 = (U_2, \mathcal{O}_2)$. The following diagram commutes*

$$\begin{array}{ccc} \mathcal{U}_1 & \xrightarrow{\varphi} & \mathcal{U}_2 \\ \uparrow & & \uparrow \\ U_1 & \xrightarrow{\tilde{\varphi}} & U_2. \end{array}$$

Proof. Let U be an open subset of U_2. Let $f \in \mathcal{O}_2(U)$. We need to show that

$$\epsilon(\varphi^*(f)) = \epsilon(f) \circ \tilde{\varphi}.$$

Suppose this does not hold. One can find a $p \in \tilde{\varphi}^{-1}(U)$ such that $\epsilon(\varphi^*(f))(p) = c \neq \epsilon(f)(\tilde{\varphi}(p))$. Then there exists an open neighborhood $U' \subset U$ of $\tilde{\varphi}(p)$ such that $\epsilon(f) - c$ is invertible. By Lemma 2.4, $f - c$ is also invertible on U', which implies that $\varphi^*(f - c)$ is invertible on $\tilde{\varphi}^{-1}(U') \subset \tilde{\varphi}^{-1}(U)$, which contradicts the fact that $\epsilon(\varphi^*(f - c))$ is non-invertible on $\tilde{\varphi}^{-1}(U')$. □

A coordinate system of $\mathcal{U}$ is a collection of functions $(x^\mu, \theta_{i,a})$ such that

1. x^μ are elements of $\mathcal{O}(U)_0$ such that $\epsilon(x^\mu)$ form a coordinate system of U;

2. $\theta_{i,a}$ are homogeneous elements of $\mathcal{O}(U)$ of degree $d(\theta_{i,a}) = i$, for all non-zero $i \in \mathcal{I}$ and $a = 1, \cdots, m_i$, which generate $\mathcal{O}(U)$ as a $C^\infty(U)$-algebra.

[5] There will be no longer such a canonical choice if we go the category of $\mathcal{I}$-graded manifolds.

Suppose that $\mathcal{I}$ can be given a total order $<$. It follows that any function $f \in \mathcal{O}(U)$ can be written uniquely in the form

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(x^\mu) \prod_{j \in \mathcal{J}} \theta_j^{\beta^j}, \tag{4.1}$$

where

- $\mathcal{J} \in \mathrm{Pow}(\mathcal{I})$, $\beta = (\beta^j)_{j\in\mathcal{J}}$, $\beta^j = (\beta^j_1, \dots, \beta^j_{m_j})$;
- $\beta^j_k \in \{0,1\}$ if $p(j) = 1$, $\beta^j_k \in \mathbb{N}$ if $p(j) = 0$;
- $\theta_j^{\beta^j} = \theta_{j,1}^{\beta^j_1} \cdots \theta_{j,m_j}^{\beta^j_{m_j}}$, the product $\prod_{j\in\mathcal{J}} \theta_j^{\beta^j}$ is arranged properly such that $\theta_j^{\beta^j}$ is on the left of $\theta_{j'}^{\beta^{j'}}$ whenever $j < j'$;
- $f(x^\mu) \in \mathcal{O}(U)_0$ should be understood as

  $$f(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} f(\epsilon(x^\mu))(x^1 - \epsilon(x^1))^{i_1} \cdots (x^n - \epsilon(x^n))^{i_n}, \tag{4.2}$$

  where $f(\epsilon(x^\mu))$ is a smooth function on U.

The sum in (4.1) is well-defined because, by assumption, only finitely many of m_j are non-zero.

Remark 4.2. One may wonder how we get (4.1) and (4.2). In fact, by definition, every function f can be written as

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(\epsilon(x^\mu)) \prod_{j \in \mathcal{J}} \theta_j^{\beta^j}.$$

One can then define a map from $\mathcal{O}(U)$ to itself by sending the coefficients $f(\epsilon(x^\mu))$ to $f(x^\mu)$. Using the binomial theorem, one can easily show that this map actually has an inverse which sends $f(\epsilon(x^\mu))$ to $f^-(x^\mu)$, where

$$f^-(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} f(\epsilon(x^\mu))(\epsilon(x^1) - x^1)^{i_1} \cdots (\epsilon(x^n) - x^n)^{i_n}.$$

Corollary 4.1. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be as in Proposition 4.2. $\tilde{\varphi}$ is uniquely determined by φ^*.*

Proof. Let $(x^\mu, \theta_{i,a})$ be a coordinate system of $\mathcal{U}_2$. By Proposition 4.2, one has $\tilde{\varphi}^\mu = \epsilon(\varphi^* x^\mu)$, where $(\tilde{\varphi}^\mu)$ is $\tilde{\varphi}$ expressed in the coordinate system $(\epsilon(x^\mu))$ of U_2. □

Let ev_p be the evaluation map of $C^\infty(U)$ at $p \in U$. Let I_p denote the kernel of $\mathrm{ev}_p \circ \epsilon$.

Lemma 4.2. *For any functions $f \in \mathcal{O}(U)$ and any integer $k \geq 0$, there is a polynomial P_k in the coordinates $(x^\mu, \theta_{i,a})$ such that $f - P_k \in I_p^{k+1}$.*

Proof. Use the classical Hadamard lemma and the decomposition (4.1). □

Lemma 4.3. *Let f and g be functions of $\mathcal{O}(U)$, then $f = g$ if and only if $f - g \in I_p^k$ for all $k \in \mathbb{N}$ and $p \in U$. In other words, $\bigcap_{p\in U}\bigcap_{k\in\mathbb{N}} I_p^k = \{0\}$.*

Proof. Let $h = f - g$. Apply the decomposition (4.1) to h, then by Lemma 4.2, $h_{\mathcal{J},\beta} = 0$ for all $\mathcal{J}$, β and $p \in U$. Hence $h = 0$. □

Theorem 4.1. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be a morphism from $\mathcal{U}_1 = (U_1, \mathcal{O}_1)$ to $\mathcal{U}_2 = (U_2, \mathcal{O}_2)$. Let $(x^\mu, \theta_{i,a})$ be a coordinate system of $\mathcal{U}_2$. Then φ^* is uniquely determined by the data $(\varphi^* x^\mu, \varphi^*\theta_{i,a})$.*

Proof. Let $f \in \mathcal{O}_2(U_2)$. By (4.1), to construct $\varphi^* f$, we only need to define $\varphi^* f_{\mathcal{J},\beta}$. But this is straightforward: one just replaces x^μ with $\varphi^* x^\mu$ in (4.2). By construction, we have $\varphi^* 1 = 1$, $\varphi^*(f + g) = \varphi^* f + \varphi^* g$, and $\varphi^*(fg) = \varphi^* f \varphi^* g$, hence φ^* is well-defined.

Now suppose there exists another φ'^* which equals φ^* on coordinates. Then they also equals on all polynomials of $(x^\mu, \theta_{i,a})$. By Lemma 4.2 and Lemma 4.3, $\varphi'^* = \varphi^*$. □

Corollary 4.2. *Let $\varphi^* : \mathcal{O}_2(U_2) \to \mathcal{O}_1(U_1)$ be a ring homomorphism which preserves the $\mathcal{I}$-grading. Then there exists a unique morphism $\varphi' : \mathcal{U}_1 \to \mathcal{U}_2$ such that $\varphi'^* = \varphi^*$.*

Proof. First, one can easily show that φ^* is actually an $\mathbb{R}$-algebra homomorphism using similiar arguments to those in Lemma 4.1. Next, observe that a coordinate system of U_2 restricted to any open subset of it gives a coordinate system of that open subset. Now apply Theorem 4.1 and Corollary 4.1. □

5 Monoidally Graded Manifolds

Definition 5.1. Let M be a n-dimensional manifold. An $\mathcal{I}$-graded manifold $\mathcal{M}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded locally ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to an $\mathcal{I}$-graded domain of dimension $n|(m_i)_{i\in\mathcal{I}}$. That is, for each $x \in M$, there exists an open neighborhood U_x of x, an $\mathcal{I}$-graded domain $\mathcal{U}$, and an isomorphism of locally ringed spaces

$$\varphi = (\tilde{\varphi}, \varphi^*) : (U_x, \mathcal{O}_M|_{U_x}) \to \mathcal{U}.$$

φ is called a chart of $\mathcal{M}$ on U_x.[6]

Let $x \in M$. An open neighborhood U of x on which $\mathcal{O}(U) \cong C^\infty(U) \otimes \mathrm{S}[V]$ is called a splitting neighborhood. Clearly, every chart is a splitting neighborhood, but not vice versa. The set of open neighborhoods form a base of the topology of M.

For a splitting U, there exists sub-algebras $C(U)$ and $D(U)$ of $\mathcal{O}(U)$ such that $C(U) \cong C^\infty(U)$, $D(U) \cong \mathrm{S}(V)$ and $\mathcal{O}(U) = C(U) \otimes D(U)$. This induces an epimorphism

$$\epsilon : \mathcal{O}(U) \to C^\infty(U)$$

of graded commutative algebras, which is again referred to as the body map.

Now, let U be an arbitrary open subset of M. We can choose a collection of charts $\{U_\alpha\}$ such that $U = \bigcup_\alpha U_\alpha$. For $f \in \mathcal{O}(U)$, one can apply the restriction morphisms to f to get a sequence of sections $\{f_\alpha\}$ in $\{\mathcal{O}(U_\alpha)\}$. Now, apply ϵ to each of them to get a sequence of smooth functions $\{\tilde{f}_\alpha\}$ in $\{C^\infty(U_\alpha)\}$. By Proposition 4.1, $\tilde{f}_\alpha$ are compatible with each other and can be glued together to get a smooth function $\tilde{f}$ over U. In this way, we construct a body map for every open subset of M, which are compatible with restrictions.

[6] Though we often refer to U_x as a chart too.

Proposition 5.1. *There exists a unique monomorphism* $\varphi : M \to \mathcal{M}$ *with* $\tilde{\varphi} = \mathrm{id}$.

M with the sheaf of smooth functions is an $\mathcal{I}$-graded manifold of dimension $n|0$. By Proposition 5.1, it can canonically embedded into $\mathcal{M}$. We call (M, ϵ) the underlying manifold of $\mathcal{M}$.

Proposition 5.2. *Let* $\varphi = (\tilde{\varphi}, \varphi^*)$ *be a morphism from* $\mathcal{M} = (M, \mathcal{O}_M)$ *to* $\mathcal{N} = (N, \mathcal{O}_N)$*. For any* U *open in* N*, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{O}_N(U) & \xrightarrow{\varphi_U^*} & \mathcal{O}_M(\phi^{-1}(U)) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ C_N^\infty(U) & \xrightarrow{\tilde{\varphi}_U^*} & C_M^\infty(\phi^{-1}(U)) \end{array}$$

where $\tilde{\varphi}^*$ *is the pullback induced by* $\tilde{\varphi}$.

Proof. The proof is essentially the same as the one of Proposition 4.2. □

Lemma 5.1. *Let* $\mathcal{O}^1$ *be the kernel of* ϵ*. Then* $\mathcal{O}$ *is* $\mathcal{O}^1$*-adic complete. That is, for any open subset* U*,* $\mathcal{O}(U)$ *is* $\mathcal{O}^1(U)$*-adic complete.*

Proof. Let $\widehat{\mathcal{O}}$ be the $\mathcal{O}^1$-adic completion of $\mathcal{O}$.[7] There exists a canonical morphism $\iota : \mathcal{O} \to \widehat{\mathcal{O}}$. Since $\mathcal{O}$ is locally $\mathcal{O}^1$-adic complete, the induced morphism $\iota_p : \mathcal{O}_p \to \widehat{\mathcal{O}}_p$ on stalks is an isomorphism for each $p \in M$. It follows that $\mathcal{O}$ is $\mathcal{O}^1$-adic complete. □

Definition 5.2. An $\mathcal{I}$-graded manifold $\mathcal{M}$ is called projected if there exists a splitting of the short exact sequence of sheaves of rings

$$0 \longrightarrow \mathcal{O}^1 \longrightarrow \mathcal{O} \xrightarrow{\epsilon} C^\infty \longrightarrow 0, \tag{5.1}$$

where $\mathcal{O}^1$ is the kernel of ϵ.

The structure sheaf $\mathcal{O}$ of a projected manifold is a $\mathcal{C}^\infty$-module.

Definition 5.3. A projected $\mathcal{I}$-graded manifold $\mathcal{M}$ is called split if there exists a splitting of the short exact sequence of sheaves of $\mathcal{C}^\infty$-modules

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}^1 \xrightarrow{\pi} \mathcal{O}^1/\mathcal{O}^2 \longrightarrow 0, \tag{5.2}$$

where $\mathcal{O}^2$ is the square of $\mathcal{O}^1$.

Let $\mathcal{O}$ be the structure sheaf of a projected $\mathcal{I}$-graded manifold. Let $\mathcal{F}$ denote the sheaf $\mathcal{O}^1/\mathcal{O}^2$. $\mathcal{F}$ is an $\mathcal{I}$-graded C^∞-module and we can define its formal symmetric power $\mathrm{S}[\mathcal{F}]$. By construction, the ringed space $\mathcal{M}_{Sym} = (M, \mathrm{S}[\mathcal{F}])$ is also a projected $\mathcal{I}$-graded manifold.

Lemma 5.2. *Let* $\mathcal{M} = (M, \mathcal{O})$ *be a projected* $\mathcal{I}$*-graded manifold. Then* $\mathcal{M}$ *is split if and only if* $\mathcal{M} \cong \mathcal{M}_{Sym}$.

[7] For each open subset U, one has $\widehat{\mathcal{O}}(U) = \varprojlim \mathcal{O}(U)/\mathcal{O}^n(U)$, where $\mathcal{O}^n(U)$ is the n-th power of $\mathcal{O}^1(U)$.

Proof. Let $\iota : \mathcal{F} \to \mathrm{S}[\mathcal{F}]$ be the canonical monomorphism. ι splits the short exact sequence (5.2). Now if $\mathcal{M}$ is split, one can find a monomorphism $F : \mathcal{F} \to \mathcal{O}$ of C^∞-modules such that $F(\mathcal{F}(U)) \subset \mathcal{O}^1(U)$ for any open subset U. By Lemma 3.1 and Lemma 5.1, there exists a unique C^∞-algebra morphism $\tilde{F} : \mathrm{S}[\mathcal{F}] \to \mathcal{O}$ such that $\tilde{F} \circ \iota = F$. By Remark 2.6, $\tilde{F}$ induces an isomorphism for each stalk. Hence $\mathcal{M} \cong \mathcal{M}_{Sym}$. ☐

Theorem 5.1. *Every projected $\mathcal{I}$-graded manifold is split.*

Proof. Due to the existence of a smooth partition of unity, $H^q(M, \mathrm{Hom}(\mathcal{O}^1/\mathcal{O}^2), \mathcal{O}^2)_0)$ vanishes for $q \geq 1$. By Lemma 3.2, there is no obstruction of the existence of a splitting of (5.2). ☐

Theorem 5.2. *Every $\mathcal{I}$-graded manifold is projected.*

Proof. Let $\mathcal{O}_{(i)} = \mathcal{O}/\mathcal{O}^{i+1}$. Let $\phi_{(0)} : C^\infty \to \mathcal{O}_{(0)}$ be the identity. ($\mathcal{O}_{(0)} \cong C^\infty$ and there is no non-trivial automorphism of C^∞.) One can construct by induction on i mappings $\phi_{(i+1)} : C^\infty \to \mathcal{O}_{(i+1)}$ such that $\pi_{i+1,i} \circ \phi_{(i+1)} = \phi_{(i)}$, where $\pi_{i+1,i} : \mathcal{O}_{i+1} \to \mathcal{O}_i$ is the canonical epimorphism. As is shown in [2], one can construct an obstruction

$$\omega(\phi_{(i)}) \in H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0)$$

of the existence of $\phi_{(i+1)}$, where $\mathcal{T}$ is the tangent sheaf of M. In the smooth case, $H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0) = 0$ and $\omega(\phi_{(i)}) = 0$. It follows that there exists a unique morphism $\phi : C^\infty \to \varprojlim \mathcal{O}_{(i)}$ such that $\pi_i \circ \phi = \phi_{(i)}$, where $\pi_i : \varprojlim \mathcal{O}_{(i)} \to \mathcal{O}_i$ is the canonical epimorphism. By Lemma 5.1, ϕ is actually a morphism from C^∞ to $\mathcal{O}$. Note that $\pi_0 = \epsilon$ and $\pi_0 \circ \phi = \phi_{(0)} = \mathrm{id}$. ϕ splits (5.1). ☐

Corollary 5.1. *Every $\mathcal{I}$-graded manifold is split.*

One can also prove the existence of partition of unity exists for an $\mathcal{I}$-graded manifold $\mathcal{M}$.

Lemma 5.3. *Let $f \in \mathcal{O}(M)$ such that $\epsilon(f)(x) \neq 0$ for all $x \in M$. f is invertible.*

Proof. Choose an open cover $\{U_\alpha\}$ of $\mathcal{I}$-graded charts of M. Let f_α denote $\rho_{U_\alpha,M}(f)$. Each f_α is invertible by Lemma 2.4. Let f_α^{-1} denote the inverse of f_α. By uniqueness, f_α^{-1} are compatible with each other, hence can be glued to give a section $f^{-1} \in \mathcal{O}(M)$, which is the inverse of f. ☐

Lemma 5.4. *Let $\{U_\alpha\}$ be an open cover of M. There exists a locally finite refinement $\{V_\beta\}$ of $\{U_\alpha\}$ and a family of functions $\{l_\beta \in \mathcal{O}(M)_0\}$ such that*

1. $\mathrm{supp}\, l_\beta \subset V_\beta$ *is compact and* $\epsilon(l_\beta) \geq 0$ *for all* β*;*
2. $\sum_\beta l_\beta = 1$.

Proof. First, find a partition of unity $\{\tilde{l}_\beta\}$ of M subordinate to $\{V_\beta\}$. Choose $l'_\beta \in \mathcal{O}(V_\beta)$ such that $\epsilon(l'_\beta) = \tilde{l}_\beta$. Then set l_β to be $(\sum_\beta l'_\beta)^{-1} l'_\beta$. ☐

We convince the reader that one can use partitions of unity to prove the following result as a analogue of Corollary 4.2.

Theorem 5.3. *Let $\mathcal{M}_1 = (M_1, \mathcal{O}_1)$ and $\mathcal{M}_2 = (M_2, \mathcal{O}_2)$ be two $\mathcal{I}$-graded manifolds. Let $\varphi^* : \mathcal{O}_2(M_2) \to \mathcal{O}_1(M_1)$ be a ring homomorphism which preserves the $\mathcal{I}$-grading. Then there exists a unique morphism $\varphi' : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\varphi'^* = \varphi^*$.*

The proof can be easily obtained by generalizing the one in the ungraded case [3].

References

[1] Leites, D.A., 1980. Introduction to the theory of supermanifolds. *Russian Mathematical Surveys*, 35(1), p.1.

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