Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Shrinking rates of horizontal gaps for generic translation surfaces

by

Jon Chaika and Samantha Fairchild

Preprint no.: 22 2022

SHRINKING RATES OF HORIZONTAL GAPS FOR GENERIC TRANSLATION SURFACES

JON CHAIKA AND SAMANTHA FAIRCHILD

ABSTRACT. A translation surface is given by polygons in the plane, with sides identified by translations to create a closed Riemann surface with a flat structure away from finitely many singular points. Understanding geodesic flow on a surface involves understanding saddle connections. Saddle connections are the geodesics starting and ending at these singular points and are associated to a discrete subset of the plane. To measure the behavior of saddle connections of length at most R, we obtain precise decay rates as $R \to \infty$ for the difference in angle between two almost horizontal saddle connections.

1. INTRODUCTION

Consider a finite collection of polygons in the plane, where all sides come in pairs of equal length with opposite orientations on the boundary of the polygons. Identifying these sides gives a compact finite type Riemann surface. The form dz on the plane endows this Riemann surface with a holomorphic 1-form ω. This structure is called a *translation surface*. Translation surfaces are stratified by their zeros and each connected component of each stratum supports a natural measure class called *Masur–Veech* measure. More background on translation surfaces can be found, for example in [Mas22, MS91]. A *saddle connection* γ is a geodesic starting and ending at the zeroes of ω with no zeroes in between. Associated to γ, we define the *holonomy vector* $v_\gamma = \int_\gamma d\omega \in \mathbb{C}$. To understand the geometry of a typical surface, much effort has gone into understanding the asymptotic behavior of holonomy vectors of saddle connections of length at most R [Doz19, EM01, EMM15, EMZ03, NRW20, Mas88, Mas90, Vee98, Vor05], and more recently the asymptotic behavior of pairs of saddle connections [ACM19, AFM22]. We will consider

$$\Lambda_\omega(R) = \{v_\gamma \in \mathbb{C} \cap B(0,R) : \gamma \text{ is a saddle connection}\} \text{ and } \Theta_\omega(R) = \{\arg(v_\gamma) : v_\gamma \in \Lambda_\omega(R)\},$$

where $\arg(v) \in [-\pi, \pi)$ is the angle v makes with the horizontal. The sets $\Lambda_\omega(R)$ and $\Theta_\omega(R)$ are discrete subsets of $\mathbb{C}$ and $[-\pi,\pi)$, respectively, so we can define

$$\zeta_\omega(R) = \min\{\phi \in \Theta_\omega(R) : \phi \geq 0\} - \max\{\phi \in \Theta_\omega(R) : \phi < 0\}.$$

The main result of this paper is:

Theorem 1.1. *Let $\psi : [1,\infty) \to [1,\infty)$ be a nondecreasing function. In any connected component of a stratum of translation surfaces of genus at least 2,*

(1) If $\int_1^\infty \frac{1}{t\psi(t)^2}\, dt < \infty$, then for Masur–Veech almost every ω,

$$\liminf_{R\to\infty} \psi(R) R^2 \zeta_\omega(R) = \infty.$$

(2) If $\int_1^\infty \frac{1}{t\psi(t)^2}\, dt = \infty$, then for Masur–Veech almost every ω,

$$\liminf_{R\to\infty} \psi(R) R^2 \zeta_\omega(R) = 0.$$

Remark 1.2. Note that the choice of a horizontal gap is a convenience. Apply a rotation to the full measure set in Theorem 1.1, and we obtain the same result in a different direction. Consider a countable subset $D_n = \{\theta_n\}_{n\in\mathbb{N}} \subseteq [0, 2\pi)$. Since a countable union of measure 0 subsets is still measure zero, we obtain a natural corollary that the smallest gap of any of the directions in D_n has the same decay rate as given in Theorem 1.1.

We first provide an explanation for the scaling factor of R^2. Masur ([Mas88, Mas90]) showed $|\Lambda_\omega(R)|$ has *quadratic growth* in the sense that for each ω there exist constants c_1, c_2 so that

$$c_1R^2 \leq |\Lambda_\omega(R)| \leq c_2R^2$$

for all large enough R. There are at most $4g - 4$ saddle connections in the same direction, so $|\Theta_\omega(R)|$ also has quadratic growth. This result explains the scaling factor of R^2 in Theorem 1.1. The quadratic growth of saddle connections was subsequently built on in [Vee98, EM01, Vor05, EMM15, NRW20].

For almost every translation surface, Theorem 1.1 gives partial information on how $\Theta_\omega(R)$ is distributed in $[-\pi, \pi)$. For every translation surface, [Mas86] shows that $\Theta_\omega = \bigcup_R \Theta_\omega(R)$ is dense in $[-\pi, \pi)$. If we order the points $\theta_1 \leq \cdots \leq \theta_{\ell(R)}$ in $\Theta_\omega(R)$, then by density the adjacent differences $\theta_{j+1} - \theta_j \to 0$ as $R \to \infty$ for every ω. Thus $\zeta_\omega(R) \to 0$ as $R \to \infty$ for every ω. However, we cannot expect Theorem 1.1 to hold for every translation surface. Indeed when ω is a *lattice surface* (see [Mas22, Sections 5 and 7] for a definition), [AC12] showed for every unbounded ψ we have $\liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) = \infty$.

The results of Theorem 1.1 considers the behavior of a single gap. There is also substantial work done on studying the behavior of the family of gaps. Namely one can study the entire set $\Theta_\omega(R)$. The distribution of normalized gaps exists for almost every ω by [AC12]. In many cases, the distribution has also been computed [ACL15, BMMM+21, KSW21, UW16, San21]. Considering the behavior of a single gap, which is the focus of the current paper, is orthogonal because the behavior of a single gap does not affect the distribution of gaps.

1.1. **Outline of proof.** The proof follows the now standard strategy of relating a problem about the geometry of a translation surface ω to the orbit of ω under Teichmüller geodesic flow, $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. In Section 2 we reduce our problem about gaps to a *shrinking target problem* for g_t. The shrinking targets are sets A_t obtained by relating Theorem 1.1 to whether or not $g_t\omega \in A_t$ for arbitrarily large t. To prove Theorem 1.1 (2) we use independence results for the sets $g_{-t}A_t$. A key tool to do this is the fact that the g_t action is exponentially mixing (Section 3). However, because our targets are not $SO(2)$-invariant, the estimates from exponential mixing are not sufficient to treat all non-increasing sequences. See Remark 2.11 and Assumptions (3) and (4) in Proposition 2.10. (c.f. [KL21, Proposition 11.1]). Section 4 establishes that Assumption (4) of Proposition 2.10 is satisfied and provides a different argument to overcome the limitations from exponential mixing. In fact, in Section 4 even the targets we consider are different.

Acknowledgements. We would like to thank Jayadev Athreya, Osama Khalil, and Howard Masur for useful discussions. The first author is supported by NSF grants DMS-2055354 and DMS-452762, the Sloan foundation, Poincaré chair, and Warnock chair. The second author is supported by the Deutsche Forschungsgemeinschaft (DFG) – Projektnummer 445466444.

2. Reductions

In this section we present a series of reductions:

(1) We use renormalization to relate gaps to a shrinking target problem in Section 2.1.
(2) In Section 2.3, we consider the convergence case Theorem 1.1(1) and prove it suffices to show that for almost every ω we have $\liminf_{R\to\infty}\psi(R)R^2\zeta_\omega(R)>0$.
(3) In Section 2.4, we consider the divergence case Theorem 1.1(2) and prove it suffices to show that for a positive measure set of ω we have $\liminf_{R\to\infty}\psi(R)R^2\zeta_\omega(R)<\infty$.
(4) In Section 2.5 we state and prove a partial converse to the Borel–Cantelli lemma that we use as the framework for proving Theorem 1.1.
(5) We prove Theorem 1.1 assuming Proposition 2.14.

2.1. **Definition and measure of the shrinking target sets.**

Fix a stratum $\mathcal{H}$, with complex dimension $2g+s-1$ with s the number of distinct singularities. We will sample ψ along a discrete set $\psi(b^k)$ for $k\in\mathbb{N}$ where $b=e^{\ell_0}$, and $\ell_0\geq 1$ is as in Corollary 4.6, for $0<\delta<1$, and $I=(-\frac{\pi}{12},\frac{\pi}{12})$.

Definition 1 (Definition of the A's)**.** *Fix $0<\sigma<1$, and let $0\leq c<1$. Define $T^{\pm}_{c,\sigma,j}$ to be the trapezoids with corners $(c,0),(1,0),(1,\frac{\pm\sigma}{\psi(b^j)}),(c,c\frac{\pm\sigma}{\psi(b^j)})$. Set*

$$H_{c,\sigma,j}=\{\omega\in\mathcal{H}:\omega \text{ has a holonomy vector in } T^+_{c,\sigma,j} \text{ and a holonomy vector in } T^-_{c,\sigma,j}\}.$$

Finally for $k\in\mathbb{N}$, define $A_k(c,\sigma)=g_{\log(b^k)}H_{c,\sigma,k}$, where $g_t=\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}$.

Remark 2.1. When the choice of c or σ is clear, or arbitrary, then for clarity we will suppress the dependence and simply write A_k instead of $A_k(c,\sigma)$.

We next obtain measure estimate for A_k. Let μ denote the Masur–Veech measure on $\mathcal{H}$. Since μ is $SL(2,R)$ invariant, it suffices to understand $\mu(H_{c,\sigma,j})$. Before obtaining the desired estimates, we quote the following result of Masur–Smillie (as quoted from [AC12])

Lemma 2.2. *There is a constant M so that for all $\epsilon,\kappa>0$, the subset of $\mathcal{H}$ consisting of flat surfaces which have a saddle connection of length at most ϵ has measure at most $M\epsilon^2$. The subset of flat surfaces which have a saddle connection of length at most ϵ and another nonhomologous saddle connection of length at most κ has measure at most $M\epsilon^2\kappa^2$.*

Following [MS91, p.464-5] μ is defined on the flat structures of area 1 via a cone measure $\widetilde{\mu}$ over flat structures with area at most 1. The cone measure $\widetilde{\mu}$ is inherited from a measure on cohomology, defined via charts coming from the developing map. For $\omega\in\mathcal{H}$ there exists $r_\omega>0$ so that the ball of radius r_ω about the preimage of ω in relative cohomology is sent injectively to the coordinate chart about ω. In a fixed coordinate chart we can write $\omega=(x_1,x_2,x_3)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}^{2g+s-3}$ and $\widetilde{\mu}$ as Lebesgue measure on $\mathbb{C}^{2g+s-1}$. Let $\omega_0=(x_1^0,x_2^0,x_3^0)$ have two horizontal holonomy vectors of length 1, and let $r>0$ be the radius for the chart as above. One can find such a surface in every connected component of every stratum. Indeed, [Zor08] explicitly constructs surfaces, which are a single horizontal cylinder, in each connected component of every stratum of genus at least two. From such a surface one can vary the length of a pair of boundary horizontal saddle connections to obtain ω.

Lemma 2.3. *Given a stratum $\mathcal{H}$, there exists positive finite constants $m = m(\mathcal{H})$, $M = M(\mathcal{H})$, $n_{\mathcal{H}} \in \mathbb{N}$, and $\sigma_{\mathcal{H}} < 1$ chosen so that for all $0 < \sigma < \sigma_{\mathcal{H}}$ and for $c_{\mathcal{H}} = 1 - 2^{-n_{\mathcal{H}}}$*

$$\frac{m\sigma^2}{\psi(b^j)^2} \leq \mu(H_{c_{\mathcal{H}},\sigma,j}) \leq \mu(H_{0,\sigma,j}) \leq \frac{M\sigma^2}{\psi(b^j)^2}.$$

Proof. Since μ is defined via $\widetilde{\mu}$, it suffices to prove the statement for $\widetilde{\mu}$. Fix $r > 0$ as above for the surface $\omega_0 \in \mathcal{H}$. If necessary, shrink r so that $r < 1$ and the image of the chart is contained in a compact subset of $\mathcal{H}$. Notice that for any c, since $\psi(t) \geq 1$, $T^{\pm}_{c,\sigma,j}$ is always contained in the trapezoid $T_{c,\sigma,*}$ given by $(c, c\sigma), (c, 0), (1, 0), (1, \sigma)$. So we can guarantee $T_{c,\sigma,j}$ is always contained in the chart whenever $T_{c,\sigma,*} \subset B_{\mathbb{C}}(1, r)$. This geometric condition can be satisfied for some $0 < \sigma_{\mathcal{H}} < r$ and $n_{\mathcal{H}} \in \mathbb{N}$, where $c_{\mathcal{H}} = 1 - 2^{-n_{\mathcal{H}}}$ and $\sigma < \sigma_{\mathcal{H}}$. Define the set in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{2g+s-3}$ by

$$\widetilde{H}_{c_{\mathcal{H}},\sigma,j} = T^{+}_{c_{\mathcal{H}},\sigma,j} \times T^{-}_{c_{\mathcal{H}},\sigma,j} \times B,$$

where $B = B_{\mathbb{C}^{2g+s-3}}(x_3^0, r)$. By our choice of r the measure $\widetilde{\mu}$ on $\mathcal{H}$ is locally Lebesgue. Hence if m_j is Lebesgue measure on $\mathbb{C}^j$, by symmetry of $T^{\pm}_{c_{\mathcal{H}},\sigma,j}$,

$$\widetilde{\mu}(\widetilde{H}_{c_{\mathcal{H}},\sigma,j}) = m_1(T^{+}_{c_{\mathcal{H}},\sigma,j})^2 m_{2g+s-3}(B).$$

Note that

$$m_1(T^{+}_{c_{\mathcal{H}},\sigma,j}) = \frac{1}{2}\left(\frac{\sigma}{\psi(b^j)} + c_{\mathcal{H}}\frac{\sigma}{\psi(b^j)}\right)(1 - c_{\mathcal{H}}) = \frac{\sigma}{\psi(b^j)}\frac{(1 - c_{\mathcal{H}}^2)}{2}.$$

Hence $m = \frac{1-c_{\mathcal{H}}^2}{2} m_{2g+s-3}(B)$ and $\widetilde{\mu}(H_{0,\sigma,j}) \geq \widetilde{\mu}(H_{c_{\mathcal{H}},\sigma,j})$ gives the desired bounds.

For the upper bound, flowing by geodesic flow which preserves measure, $g_{\log\left(\sqrt{\sigma/\psi(b^k)}\right)} H_{0,\sigma,k}$ has two non-homologous vectors of length at most $\sqrt{\frac{2\sigma}{\psi(b^k)}}$, so by Lemma 2.2 we obtain the desired upper bound. □

2.2. **A lemma on targets.** For the following lemma and corollary, for generality we allow any $b > 1$. Note that $b = e^{\ell_0}$ as defined in Definition 1 satisfies $b > 1$ since we can choose $\ell_0 > 0$.

Lemma 2.4. *Let $\phi : [1, \infty) \to [1, \infty)$ be non-decreasing. We have $\int_1^\infty (t\phi(t))^{-1} dt = \infty$ if and only if $\sum_{j=1}^\infty \phi(b^j)^{-1} = \infty$ for any $b > 1$.*

Proof. For ease of exposition we assume $b = 2$, the general case is similar, but requires using floor and ceiling functions. For each $k \in \mathbb{N}$ we have

$$2^k \frac{1}{2^k \phi(2^k)} \geq \int_{2^k}^{2^{k+1}} (t\phi(t))^{-1} dt \geq 2^k \frac{1}{2^{k+1}\phi(2^{k+1})}.$$

It follows that $\sum_{j=1}^\infty \phi(2^j)^{-1} \geq \int_2^\infty (t\phi(t))^{-1} dt \geq \frac{1}{2}\sum_{j=2}^\infty \phi(2^j)^{-1}$. □

Corollary 2.5. *Let $\psi : [1, \infty) \to [1, \infty)$ be non-decreasing. We have $\int_1^\infty (t\psi(t)^2)^{-1} dt = \infty$ if and only if $\sum_{j=1}^\infty \psi(b^j)^{-2} = \infty$ for any $b > 1$.*

2.3. **Convergence reduction.**

Lemma 2.6. *If whenever $\int_1^\infty \frac{1}{t\psi(t)^2}\, dt < \infty$, $\mu\left(\{\omega : \liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) > 0\}\right) = 1$, then whenever $\int_1^\infty \frac{1}{t\psi(t)^2}\, dt < \infty$ we have*

$$\mu\left(\left\{\omega : \liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) = \infty\right\}\right) = 1.$$

Proof. We first construct a slightly smaller function $\psi_0(t)$ so that

$$\int_1^\infty \frac{1}{t\psi_0(t)^2}\,dt < \infty \text{ and } \lim_{t\to\infty}\frac{\psi(t)}{\psi_0(t)} = \infty.$$

To do this let $n_1 = 1$ and set

$$n_j = \sup\left\{T > n_{j-1} : \int_{n_{j-1}}^{T} \frac{1}{t\psi(t)^2}\,dt \le 2^{-j}\int_1^\infty \frac{1}{t\psi(t)^2}\,dt.\right\}$$

For $j \in \mathbb{N}$ we piecewise define

$$\psi_0(t) = j^{-1}\psi(t) \text{ whenever } n_j \le t < n_{j+1}.$$

Then since $j \to \infty$ as $t \to \infty$, $\lim_{t\to\infty}\psi(t)/\psi_0(t) = \infty$. By our choice of n_j, we also have $\int_1^\infty (t\psi_0(t)^2)^{-1}\,dt < \infty$. By the assumption of Lemma 2.6 there is a full measure set of ω so that $\liminf_{t\to\infty}\psi_0(t)t^2\zeta_\omega(t) > 0$. From this we have the desired result that for a full measure set of ω,

$$\liminf_{t\to\infty}\psi(t)t^2\zeta_\omega(t) \ge \left(\liminf_{t\to\infty}\frac{\psi(t)}{\psi_0(t)}\right)\left(\liminf_{t\to\infty}\psi_0(t)t^2\zeta_\omega(t)\right) = \infty.$$

□

2.4. **Divergence Case.**

Proposition 2.7. *If for all $\sigma > 0$, $\mu(\limsup A_k(0,\sigma)) > 0$ then $\liminf_{R\to\infty}\psi(R)R^2\zeta_\omega(R) = 0$ for μ-a.e. ω.*

Remark 2.8. Note that $A_k(0,\sigma) \subseteq A_k(0,\sigma')$ whenever $\sigma < \sigma'$. Thus the assumption of Proposition 2.7 is satisfied as long as $\mu(\limsup A_k(0,\sigma)) > 0$ for all σ small enough.

Proof. We first claim $\mu(\limsup A_k) = 1$. We will show $\limsup A_k$ is invariant under forward geodesic flow $g_{\log(b^t)}$ for $t > 0$. Let $\omega \in g_{\log(b^{-t})}\limsup_{k\to\infty} g_{\log(b^k)}H_{0,\sigma,k}$. For $n \in \mathbb{N}$, there exists $k \ge n + t$ so that the monotonicity of ψ implies

$$\omega \in g_{\log(b^{-t+k})}H_{0,\sigma,k} \subseteq g_{\log(b^{-t+k})}H_{0,\sigma,-t+k}.$$

Hence $\omega \in \limsup A_k$. By ergodicity of the geodesic flow and the assumption that $\mu(\limsup A_k) > 0$, we conclude that $\mu(\limsup A_k) = 1$.

We now translate $\limsup A_k$ by a small backwards geodesic flow to a set where we have the desired result. Namely we claim that for $s_0 = \left\lceil \frac{\log(2)}{2\log(b)} \right\rceil > 0$, any $\widetilde{\omega} \in g_{-s_0\log(b)}\limsup A_k$ satisfies

$$\liminf_{R\to\infty}\psi(R)R^2\zeta_{\widetilde{\omega}}(R) = 0.$$

First note that we are still working with a full measure set since μ is invariant under geodesic flow

$$\mu(g_{-s_0\log(b)}\limsup A_k) = \mu(\limsup A_k) = 1.$$

Let $\widetilde{\omega} = g_{-s_0\log(b)}\omega$ for some $\omega \in \limsup A_k$. Since $\omega \in \limsup A_k$, for any $m \in \mathbb{N}$ we can find $\rho_m \ge m$ so that $\omega \in g_{\rho_m\log(b)}H_{0,\sigma,\rho_m}$. The choice of s_0 guarantees that the longest possible holonomy vectors in $g_{-s_0\log(b)}T^{\pm}_{0,\sigma,b^{\rho_m}}$ are at most b^{ρ_m} since the choice of s_0 is sufficient so that

$$b^{2(\rho_m-s_0)} + \frac{\sigma^2}{b^{2(\rho_m-s_0)}\psi(b^{\rho_m})^2} \le b^{2\rho_m}.$$

Thus the holonomy vectors detected by $g_{-s_0 \log(b)} T^{\pm}_{0,\sigma,b^{\rho_m}}$ have length at most b^{ρ_m} and an upper bound on the angle around zero, giving the following upper bound

$$\zeta_{g_{-\log(b^s)}\omega}(b^{\rho_m}) \leq \arctan \frac{\sigma}{\psi(b^{\rho_m})b^{2\rho_m - 2s_0}} < \frac{\sigma b^{2s_0}}{\psi(b^{\rho_m})b^{2\rho_m}}.$$

Since s_0 is fixed, we can take $\sigma \to 0$ to obtain $\liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) = 0$. □

2.5. **Axiomatic framework.** We will first recall the Borel–Cantelli lemma, and then spend the remainder of the section stating and proving a partial converse.

Lemma 2.9 (Borel–Cantelli lemma). *Suppose $(A_k)_{k=1}^\infty$ are measurable sets with $\sum_{k=1}^\infty \mu(A_k) < \infty$. Then $\mu\left(\cap_{N=1}^\infty \cup_{i=N}^\infty A_i\right) = \mu(\limsup A_i) = 0$.*

Proposition 2.10 (Exponential Decay Borel–Cantelli). *Let $C \geq 1$, $0 < \delta < 1$, and $(A_k)_{k=1}^\infty$ be measurable sets. Suppose the following hold.*

(1) $\sum_{k=1}^\infty \mu(A_k) = \infty$.
(2) For all $i \leq j$, $\mu(A_i) \geq \mu(A_j)$.
(3) For all i, for all j so that $j > i + C\log\left(\frac{1}{\mu(A_i)}\right)$ we have

$$\mu(A_i \cap A_j) \leq C\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right].$$

(4) For all i, for all j so that $i < j \leq i + C\log\left(\frac{1}{\mu(A_i)}\right)$ we have that there exists measurable sets B_i, C_j so that
   (a) $B_i \subset A_i$ and $A_j \subset C_j$,
   (b) $\mu(B_i) > \frac{1}{C}\mu(A_i)$,
   (c) $\mu(C_j) < C\mu(A_j)^{\frac{1}{2}}$,
   (d) $\mu(B_i \cap C_j) < C\mu(B_i)\left(2^{-(j-i)(1-\delta)} + \mu(C_j)^{\frac{1+\delta}{2}}\right)$.

Then $\cap_{N=1}^\infty \cup_{i=N}^\infty A_i = \limsup A_i$ has positive measure.

Remark 2.11. One can observe that the decay of correlations in Assumption (3) is insufficient to handle the generality of Theorem 1.1. Indeed consider $\psi(t) = \sqrt{\log(t+4)\log(\log(t+4))}$. Since $\int_1^\infty \frac{1}{t\psi(t)^2}\,dt = \infty$, ψ satisfies the second assumption of Theorem 1.1. Motivated by Lemma 2.3 we assume $\mu(A_k)$ is proportional to $\psi(b^k)^{-2} = \frac{1}{\log(b^k+4)\log(\log(b^k+4))} < \frac{1}{k\log(k)\log(b)}$. Notice Assumption (3) gives no bounds on $\mu(A_i \cap A_j)$ when $i < j < C'\log(i) + i$. Define n_k recursively by $n_1 = 2$ and $n_{i+1} = \lceil n_i + C'\log(n_i)\rceil$. Observe $\sum_{i=1}^\infty \mu(A_{n_i}) < \infty$, so $\mu(\limsup A_{n_k}) = 0$ by the Borel–Cantelli lemma. Thus we cannot draw any conclusions on $\mu(\limsup A_k)$ without Assumption (4).

We now prove Proposition 2.10. This proof is inspired by [Pet02, Theorem 2.1], which invokes the Chung–Erdős inequality.

Lemma 2.12 (Chung–Erdős inequality). *Suppose$(A_k)_{k=1}^\infty$ is a sequence of measurable sets with $\mu\left(\bigcup_{k=1}^N A_k\right) > 0$, then*

$$\mu\left(\bigcup_{k=1}^N A_k\right) \geq \frac{\left(\sum_{k=1}^N \mu(A_k)\right)^2}{\sum_{j,k=1}^n \mu(A_j \cap A_k)}.$$

Proof of Lemma 2.12. Beginning with the numerator on the right hand side,

$$\left(\sum_{k=1}^{N}\mu(A_k)\right)^2 = \left[\int \mathbb{1}_{\{\sum_{k=1}^{N}\mathbb{1}_{A_k}>0\}}\left(\sum_{k=1}^{N}\mathbb{1}_{A_k}\right)d\mu\right]^2.$$

By the Cauchy–Schwarz inequality

$$\left(\sum_{k=1}^{N}\mu(A_k)\right)^2 \leq \mu\left(\bigcup_{k=1}^{N}A_k\right)\int\left(\sum_{k=1}^{N}\mathbb{1}_{A_k}\right)^2 d\mu = \mu\left(\bigcup_{k=1}^{N}A_k\right)\left[\sum_{j,k=1}^{N}\mu(A_j\cap A_k)\right].$$

Rearranging we obtain the desired inequality. □

The next lemma will also be used to prove Proposition 2.10, which states the conditions on the sets $B_k \subseteq A_k$ that we use to show $\limsup(B_k) > 0$, and thus $\limsup(A_k) > 0$.

Lemma 2.13. *Under the assumptions of Proposition 2.10, there exists some $\widetilde{C} \geq 1$ depending only on C large enough so that if $\widetilde{m}_i = i + \widetilde{C} + \widetilde{C}\log\left(\frac{1}{\mu(B_i)}\right)$, the following hold.*

(a) $\sum_{k=1}^{\infty}\mu(B_k) = \infty$.
(b) For all $i \leq j$, $\mu(B_i) \geq \frac{1}{C}\mu(B_j)$.
(c) For all i and for all j so that $j > \widetilde{m}_i$,

$$\mu(B_i\cap B_j) \leq \widetilde{C}\mu(B_i)\left[\mu(B_j) + e^{-\frac{\delta}{4}|i-j|}\right].$$

(d) For all i and for all j with $i < j < \widetilde{m}_i$,

$$\mu(B_i\cap B_j) < \widetilde{C}\mu(B_i)\left[2^{-|i-j|(1-\delta)} + \mu(B_j)^{\frac{1+\delta}{4}}\right].$$

Moreover there exists constants $D, D' > 0$ so that for any $n > 0$ and $N \geq n$,

$$\sum_{i,j=n}^{N}\mu(B_i\cap B_j) \leq \widetilde{C}\left[D\sum_{k=n}^{N}\mu(B_k) + D' + \left(\sum_{k=n}^{N}\mu(B_k)\right)^2\right]. \tag{2.1}$$

Proof of Lemma 2.13. The first 4 parts use assumptions on A_k in Proposition 2.10.

(a) Follows from Assumptions (1) and (4b).
(b) Follows from Assumptions (2) and (4b).
(c) Combine Assumption (3) with Assumptions (4a) and (4b).
(d) Combine Assumptions (4b) and (4c) with that fact that $\frac{1+\delta}{4} \in (\frac{1}{4}, \frac{1}{2})$ and $C \geq 1$ to obtain

$$\mu(C_j)^{\frac{1+\delta}{2}} \leq C\mu(A_j)^{\frac{1+\delta}{4}} \leq C(C\mu(B_j))^{\frac{1+\delta}{4}} \leq \widetilde{C}^2\mu(B_j)^{\frac{1+\delta}{4}}.$$

Now we move to Eq. (2.1) where we want to find an upper bound for the denominator of Lemma 2.12 applied to the sets B_k. First since $\widetilde{C} \geq 1$,

$$\sum_{i,j=n}^{N}\mu(B_i\cap B_j) \leq 2\sum_{i=n}^{N}\sum_{j>i}\mu(B_i\cap B_j) + \widetilde{C}\sum_{k=n}^{N}\mu(B_k). \tag{2.2}$$

We split the double sum on the right hand side of Eq. (2.2) into cases when $j > \widetilde{m}_i$ and $j \leq \widetilde{m}_i$.

In the first case when $j > \widetilde{m}_i$, part (c) implies

$$2\sum_{i=n}^{N}\sum_{j>\widetilde{m}_i}\mu(B_i\cap B_j) \le 2\widetilde{C}\sum_{i=n}^{N}\sum_{j>\widetilde{m}_i}\left(\mu(B_i)\mu(B_j)+\mu(B_i)e^{-\frac{\delta}{4}|i-j|}\right). \tag{2.3}$$

In the first sum of Eq. (2.3), we re-expand the square so that

$$2\sum_{i=n}^{N}\sum_{j>\widetilde{m}_i}^{N}\mu(B_i)\mu(B_j) \le \left(\sum_{i=n}^{N}\mu(B_i)\right)^2 - \sum_{i=n}^{N}\mu(B_i)^2 \le \left(\sum_{i=n}^{N}\mu(B_i)\right)^2. \tag{2.4}$$

In the second sum of Eq. (2.3) we making the change of variables $k = j - i$, the geometric series formula gives

$$2\sum_{i=n}^{N}\sum_{j>\widetilde{m}_i}^{N}\mu(B_i)e^{-\frac{\delta}{4}|i-j|} \le 2\sum_{i=n}^{N}\mu(B_i)\frac{1}{1-e^{-\frac{\delta}{4}}}. \tag{2.5}$$

Combining Eq. (2.3), Eq. (2.4), and Eq. (2.5), we obtain

$$2\sum_{i=n}^{N}\sum_{j>\widetilde{m}_i}\mu(B_i\cap B_j) \le \widetilde{C}\left[\left(\sum_{k=n}^{N}\mu(B_k)\right)^2 + \frac{2}{1-e^{-\frac{\delta}{4}}}\sum_{k=1}^{N}\mu(B_k)\right]. \tag{2.6}$$

We now consider the case for $i < j \le \widetilde{m}_i$. By (d)

$$2\sum_{i=n}^{N}\sum_{i<j\le\widetilde{m}_i}\mu(B_i\cap B_j) \le 2\widetilde{C}\sum_{i=n}^{N}\sum_{i<j\le\widetilde{m}_i}\left(\mu(B_i)2^{-|i-j|(1-\delta)}+\mu(B_i)\mu(B_j)^{\frac{1+\delta}{4}}\right). \tag{2.7}$$

We again use a geometric series formula so that the first sum is bounded by

$$2\sum_{i=n}^{N}\sum_{i<j\le\widetilde{m}_i}\mu(B_i)2^{-|i-j|(1-\delta)} \le \frac{2}{1-2^{-(1-\delta)}}\sum_{i=n}^{N}\mu(B_i). \tag{2.8}$$

Note $\frac{5+\delta}{4} \in \left(\frac{5}{4},\frac{6}{4}\right)$ so $\frac{5+\delta}{4}$ is a power bigger than 1 with $\mu(B_i) \le 1$. Therefore $\mu(B_i)^{\frac{5+\delta}{4}} \le \mu(B_i)$. Combining this fact with (b),

$$\begin{aligned}2\sum_{i=n}^{N}\sum_{i<j\le\widetilde{m}_i}\mu(B_i)\mu(B_j)^{\frac{1+\delta}{4}} &\le 2C\sum_{i=n}^{N}\mu(B_i)^{\frac{5+\delta}{4}}(\widetilde{m}_i - i)\\ &\le 2C\widetilde{C}\sum_{i=n}^{N}\mu(B_i) + 2C\widetilde{C}\sum_{i=n}^{N}\mu(B_i)^{\frac{5+\delta}{4}}\log\left(\frac{1}{\mu(B_i)}\right).\end{aligned} \tag{2.9}$$

Now choose n_0 large enough so that for all $i \ge n_0$,

$$\log\left(\frac{1}{\mu(B_i)}\right) \le \mu(B_i)^{\frac{-1-\delta}{4}}.$$

Then if $n \ge n_0$,

$$\sum_{i=n}^{N}\mu(B_i)^{\frac{5+\delta}{4}}\log\left(\frac{1}{\mu(B_i)}\right) \le \sum_{i=n}^{N}\mu(B_i).$$

Otherwise if $n \le n_0$

$$\sum_{i=n}^{N}\mu(B_i)^{\frac{5+\delta}{4}}\log\left(\frac{1}{\mu(B_i)}\right) \le \sum_{i=n}^{n_0-1}\mu(B_i)\log\left(\frac{1}{\mu(B_i)}\right) + \sum_{i=n_0}^{N}\mu(B_i) \le C' + \sum_{i=n}^{N}\mu(B_i).$$

where $C' > 0$ is the bound for the finite sum. Therefore there is a constant C' so that

$$\sum_{i=n}^{N} \mu(B_i)^{\frac{5+\delta}{4}} \log\left(\frac{1}{\mu(B_i)}\right) \leq C' + \sum_{i=n}^{N} \mu(B_i). \tag{2.10}$$

Thus we conclude by combining Eq. (2.7), Eq. (2.8), Eq. (2.9), and Eq. (2.10) so that

$$2\sum_{i=n}^{N} \sum_{i<j\leq \widetilde{m}_i} \mu(B_i \cap B_j) \leq \widetilde{C}\left[2CC'\widetilde{C} + \left(\frac{2}{1-2^{-(1-\delta)}} + 4C\widetilde{C}\right)\sum_{i=n}^{N} \mu(B_i)\right]. \tag{2.11}$$

The proof of Eq. (2.1) is completed by combining Eq. (2.2), Eq. (2.6), and Eq. (2.11) where $D' = 2C\widetilde{C}C'$ and $D = 1 + \frac{2}{1-e^{-\frac{\delta}{4}}} + \frac{2}{1-2^{-(1-\delta)}} + 4C\widetilde{C}$. □

Proof of Proposition 2.10. It suffices show that that the measure of $\limsup B_k$ has positive measure. By the Chung–Erdős inequality (Lemma 2.12), Lemma 2.13 (a), and Eq. (2.1),

$$\begin{aligned}\liminf_{N\to\infty} \mu\left(\bigcup_{k=n}^{N} B_k\right) &\geq \liminf_{N\to\infty} \frac{\left(\sum_{k=n}^{N} \mu(B_k)\right)^2}{\widetilde{C}\left[D\sum_{k=n}^{N} \mu(B_k) + D' + \left(\sum_{k=n}^{N} \mu(B_k)\right)^2\right]} \\ &= \liminf_{N\to\infty} \frac{1}{\widetilde{C}\left[\frac{D}{\sum_{k=n}^{N} \mu(B_k)} + \frac{D'}{\left(\sum_{k=n}^{N} \mu(B_k)\right)^2} + 1\right]} = \frac{1}{\widetilde{C}}\end{aligned}$$

Hence $\mu\left(\bigcup_{k=n}^{\infty} B_k\right) \geq \frac{1}{\widetilde{C}}$. Notice $\bigcup_{k=n}^{\infty} B_k$ is a nested decreasing sequence of sets, so we conclude

$$\mu(\limsup B_n) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} B_k\right) \geq \frac{1}{\widetilde{C}} > 0.$$

□

2.6. **Proof of Theorem 1.1.**

We now prove Theorem 1.1 conditional on the next proposition, whose proof is completed in Section 3 and Section 4.

Proposition 2.14. *For some $c > 0$, there exist sets B_k, C_k so that along with the sets A_k defined in Definition 1, the assumptions of Proposition 2.10 are satisfied.*

Before outlining how to prove Proposition 2.14, we first show that it is sufficient to obtain Theorem 1.1.

Proof of Theorem 1.1. We first prove the convergence case. By Corollary 2.5, $\sum_{j=1}^{\infty} \psi(e^j)^{-2} < \infty$, and thus $\lim_{j\to\infty} \psi(e^j) = \infty$. Set

$$L_k = \{\omega : \exists\, R \in [e^k, e^{k+1}] \text{ so that } R^2\zeta_\omega(R) < \psi(e^k)^{-1}\}.$$

For k large enough, if $\zeta_\omega(R) < \frac{\psi(e^k)^{-1}}{R^2}$, then there are two saddle connections on ω with horizontal holonomy at most R and vertical holonomy of magnitude at most $\frac{2\psi(e^k)^{-1}}{R}$. It follows that for $\omega \in L_k$, $g_{-\log\left(e^k\sqrt{\psi(e^k)}\right)}\omega$ has two saddle connections of length at most $\psi(e^k)^{-1/2}e\sqrt{2}$. By Lemma 2.2, there is some constant C' so that $\mu(L_k) \leq C'\psi(e^k)^{-2}$ so that $\sum_{k=1}^{\infty} \mu(L_k) < \infty$.

By the Borel–Cantelli lemma (Lemma 2.9), $\mu(\limsup L_k) = 0$. Taking the complement, we have a full measure set of ω so that for all $k \geq k_0$, and for all $R \in [e^k, e^{k+1}]$,

$$\psi(R)R^2\zeta_\omega(R) \geq \psi(e^k)R^2\zeta_\omega(R) \geq 1.$$

Therefore

$$\mu\left(\{\omega : \liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) > 0\}\right) \geq \mu\left(\{\omega : \liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) \geq 1\}\right) = 1.$$

By Lemma 2.6, the convergence case of Theorem 1.1 is verified.

We now prove the divergence case. By Corollary 2.5, $\sum_{k=1}^{\infty} \psi(b^k)^{-2} = \infty$. By Lemma 2.3 for any $\sigma < \sigma_{\mathcal{H}}$ our set A_k has measure proportional to $\psi(b^k)^{-2}$. By Proposition 2.10 and Proposition 2.14, for a positive measure set of ω we have $g_t\omega \in A_k$ for infinitely many k. Following Remark 2.8 we satisfy the assumptions of Proposition 2.7, which implies the divergence case of Theorem 1.1. □

Proof outline of Proposition 2.14. We verify or state where each of the assumptions of Proposition 2.10 are verified.

(1) Assumption (1) follows by the measure bounds of Lemma 2.3, Corollary 2.5 and the fact that g_t preserves measure:

$$\sum_{k=1}^{\infty} \mu(A_k) \geq \sum_{k=1}^{\infty} m\frac{\sigma^2}{\psi(b^k)^2} = \infty.$$

(2) Assumption (2) follows by Lemma 2.3: $i \leq j$ implies $\psi(b^i) \leq \psi(b^j)$, so

$$\mu(A_i) = \frac{m\sigma^2}{\psi(b^i)^2} \geq \frac{m\sigma^2}{\psi(b^j)^2} = \mu(A_j).$$

(3) Assumption (3) is proved in Section 3.1 using Corollary 3.4, Lemma 3.5, and Lemma 3.7.
(4) Assumption (4) is proved as follows. The construction of the B_k and C_k sets along with proofs of Assumptions (4a), (4b), and (4c) are given in Section 4.1. The proof is completed by verifying Assumption (4d) in Section 4.3.

□

3. Verifying Proposition 2.10 Assumption (3): exponential decay of correlations for far away pairs

We begin by stating our key exponential mixing result:

Theorem 3.1 (Stated from [AEZ16] Theorem C.4, see [AGY06])**.** *Fix $\mathcal{H}$ a connected component of the stratum and μ be Masur-Veech measure as above. There exists $C > 0$ and $\delta > 0$ so that for all h_1, h_2 Lipschitz and compactly supported, there exists a C_K depending only on the shortest sytole of a surface in the compact set so that for all $t \geq 0$*

$$\left|\int h_1(h_2 \circ g_t)\, d\mu - \int h_1\, d\mu \int h_2\, d\mu\right| \leq C(C_K + \|h_1\|_\infty + \|h_2\|_{Lip})(C_K + \|h_2\|_\infty + \|h_2\|_{Lip})e^{-\delta t}.$$

Remark 3.2. The metric used to define the Lipschitz norm in Theorem 3.1 is given in [AGY06, Section 2.2.2]. The metric is uniformly comparable to the metric we use below, which uses period coordinates. Indeed, the denominator in [AGY06, Equation (2.6)] is bounded away from zero on compact sets.

In order to apply Theorem 3.1, we need to use bump functions to approximate the sets A_i and A_j. By g_t-invariance of μ, $\mu(A_i \cap A_j) = \mu(H_{c_{\mathcal{H}},\sigma,i} \cap g_{\log(b^{j-i})} H_{c_{\mathcal{H}},\sigma,j})$. So it suffices to define our bump function to approximate $H_{c_{\mathcal{H}},\sigma,i}$ and $H_{c_{\mathcal{H}},\sigma,j}$.

Definition 2. *For each i and each $j > i + \frac{4}{\delta}\log\left(\frac{1}{\mu(A_i)}\right)$, define*

$$\epsilon_{i,j} = e^{-\frac{\delta}{4}|i-j|}.$$

Then for $\ell \in \{i, j\}$, define

$$\rho^{\ell}_{i,j}(x_1, x_2, x_3) = f^{\ell}_1(x_1) f^{\ell}_2(x_2) f_3(x_3)$$

where

$$f^{\ell}_1(x_1) = \min\left\{1, \frac{1}{\epsilon_{i,j}} \operatorname{dist}\left(x_1, \partial T_{c_{\mathcal{H}},\sigma,\psi(b^{\ell})}\right)\right\} \cdot \mathbb{1}_{T_{c_{\mathcal{H}},\sigma,\psi(b^{\ell})}},$$

$$f^{\ell}_2(x_2) = \min\left\{1, \frac{1}{\epsilon_{i,j}} \operatorname{dist}\left(x_2, \partial T_{c_{\mathcal{H}},\sigma,\psi(b^{\ell})}\right)\right\} \mathbb{1}_{T_{c_{\mathcal{H}},\sigma,\psi(b^{\ell})}},$$

and

$$f^{\ell}_3(x_3) = \min\left\{1, \frac{1}{\epsilon_{i,j}} \operatorname{dist}\left(x_3, \partial B(0,1)\right)\right\} \mathbb{1}_{B(0,1)}.$$

Lemma 3.3. *The functions $\rho^{\ell}_{i,j}$ are $\frac{1}{\epsilon_{i,j}}$-Lipschitz.*

Proof. Note that f^{ℓ}_k for $k = 1, 2, 3$ are all $\frac{1}{\epsilon_{i,j}}$-Lipschitz as the distance function being 1-Lipschitz implies

$$\left|f^{\ell}_k(x_k) - f^{\ell}_k(y_k)\right| \le \frac{1}{\epsilon_{i,j}} \left|\operatorname{dist}\left(x_k, \partial T_{c_{\mathcal{H}},\sigma,\psi(b^{\ell})}\right) - \operatorname{dist}\left(y_k, \partial T_{c_{\mathcal{H}},\sigma,\psi(b^{\ell})}\right)\right| \le \frac{1}{\epsilon_{i,j}} |x_k - y_k|.$$

Now we claim that the function $\rho^{\ell}_{i,j}$ is $\frac{1}{\epsilon_{i,j}}$-Lipschitz with respect to the distance on $\mathcal{H}$ given by

$$d_{\mathcal{H}}((x_1, x_2, x_3) - (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|.$$

To see this, let (x_1, x_2, x_3) and (y_1, y_2, y_3) be fixed. We compute

$$\begin{aligned}
&|\rho(x_1, x_2, x_3) - \rho(y_1, y_2, y_3)| \\
&\le |f_1(x_1)f_2(x_2)f_3(x_3) - f_1(y_1)f_2(x_2)f_3(x_3)| \\
&\quad + |f_1(y_1)f_2(x_2)f_3(x_3) - f_1(y_1)f_2(y_2)f_3(x_3)| \\
&\quad + |f_1(y_1)f_2(y_2)f_3(x_3) - f_1(y_1)f_2(y_2)f_3(y_3)| \\
&\le \frac{1}{\epsilon_{i,j}} \left(f_2(x_2)f_3(x_3)|x_1 - y_1| + f_1(y_1)f_3(x_3)|x_2 - y_2| + f_1(y_1)f_2(y_2)|x_3 - y_3|\right) \\
\text{(Since } f_i \le 1\text{)} \quad &\le \frac{1}{\epsilon_{i,j}} d_{\mathcal{H}}((x_1, x_2, x_3) - (y_1, y_2, y_3)).
\end{aligned}$$

□

We can now use the definition of the $\rho^{\ell}_{i,j}$ to state a corollary of Theorem 3.1.

Corollary 3.4. *Fix $0 < \delta < 1$. Then there exists a constant C so that for all $j > m_i$,*

$$\int \rho^{i}_{i,j}(\rho^{j}_{i,j} \circ g_{\ell_0(j-i)}) \le C\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right].$$

Proof. By definition $\left\|\rho_{i,j}^{\ell}\right\|_{\infty} = 1$, and by Lemma 3.3 $\left\|\rho_{i,j}^{\ell}\right\| = \frac{1}{\epsilon_{i,j}} = e^{\frac{\delta}{4}|i-j|}$. By Theorem 3.1, the fact that $b \geq e$, and writing $C_K + 1 = C_K^{+}$,

$$\left|\int \rho_{i,j}^{i}(\rho_{i,j}^{j} \circ g_{\ell_0(j-i)})) - \int \rho_{i,j}^{i} \int \rho_{i,j}^{j}\right| \leq C(C_K^{+} + e^{\frac{\delta}{4}|i-j|})^2 e^{-\delta|i-j|}$$

(Since $e^{-\frac{\delta}{2}|i-j|} \geq e^{-\frac{3}{4}\delta|i-j|} \geq e^{-\delta|i-j|}$ as $e^{-\delta|i-j|} < 1$)

$$= C(C_K^{+})^2 e^{-\delta|i-j|} + 2CC_K^{+} e^{-\frac{3\delta}{4}|i-j|} + Ce^{-\frac{\delta}{2}|i-j|}$$
$$\leq \widetilde{C} e^{-\frac{\delta}{2}|i-j|}.$$

By construction of $\rho_{i,j}^{\ell}$, $\int \rho_{i,j}^{\ell} = \mu(H_{c_{\mathcal{H}},\sigma,\ell}) = \mu(A_\ell)$, so

$$\int \rho_{i,j}^{i}(\rho_{i,j}^{j} \circ g_{\ell_0(j-i)})) \leq \int \rho_{i,j}^{i} \int \rho_{i,j}^{j} + \widetilde{C} e^{-\frac{\delta}{2}|i-j|} \leq \mu(A_i)\mu(A_j) + \widetilde{C} e^{-\frac{\delta}{2}|i-j|}.$$

Since $j > m_i$, we have $e^{-\frac{\delta}{4}|i-j|} < \mu(A_i)$. Thus, using $\widetilde{C} > 1$,

$$\int \rho_{i,j}^{i}(\rho_{i,j}^{j} \circ g_{\ell_0(j-i)})) \leq \widetilde{C}\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right].$$

□

Next our goal is to get a relationship between $\mu(A_i \cap A_j)$ and $\int \rho_{i,j}^{i}(\rho_{i,j}^{j} \circ g_{\ell_0(j-i)}))$.

Lemma 3.5. *For all $j > m_i$,*

$$\mu(A_i \cap A_j) \leq \int \rho_{i,j}^{i}(\rho_{i,j}^{j} \circ g_{\ell_0(j-i)})) + \mu(E_j) + \mu(E_i)$$

where $E_\ell = \{\omega : \rho_{i,j}^{\ell} \in (0,1)\}$.

Proof. We first make a general claim.

Claim If $0 \leq g \leq f \leq 1$ then

$$\int f \leq \int g + \mu\{f \neq g\}.$$

To see this is true, we write

$$\int f = \int g + f - g = \int g + \int_{\{f \neq g\}} f - g \leq \int g + \mu\{f \neq g\}$$

where the last inequality follows since $f - g \leq 1$.

The proof now follows from the claim where $f = \mathbb{1}_{H_{c_{\mathcal{H}},\sigma,i} \cap g_{\ell_0(j-i)}) H_{c_{\mathcal{H}},\sigma,j}}$ and $g = \rho_{i,j}^{i}(\rho_{i,j}^{j} \circ g_{\ell_0(j-i)}))$, combined with the fact that $f \neq g$ occurs when $\rho_{i,j}^{i} \in (0,1)$ or $\rho_{i,j}^{j} \in (0,1)$. So $\mu\{f \neq g\} \leq \mu(E_i) + \mu(E_j)$. □

3.1. **Proof that Proposition 2.10 Assumption (3) holds.** To finish the proof of Proposition 2.10 Assumption (3) we will relate $\int \rho_{i,j}^{i} \int \rho_{i,j}^{j}$ to $\mu(E_i) + \mu(E_j)$ from Lemma 3.5. In order to do so, we will show it suffices to add a technical assumption about the behavior of ψ.

Lemma 3.6. *We may assume that*
(3.1)
$$\forall i\, \forall j \text{ so that } j > i + \frac{4}{\delta}\log\left(\frac{1}{\mu(A_i)}\right), \text{ we in fact have } e^{-\frac{\delta}{4}|i-j|} \le \min\left\{\frac{\sigma^4}{7^4}\frac{1}{\psi(b^j)^4}, \frac{r}{2}, 2^{-(n_{\mathcal{H}}+2)}\right\},$$
where r is a fixed injectivity radius for a stratum $\mathcal{H}$, and $n_{\mathcal{H}}$ is chosen as in Lemma 2.3.

Lemma 3.7. *Under the assumptions of Lemma 3.6 there exists a constant $C > 1$ so that*
$$C\int \rho^i_{i,j}\int \rho^j_{i,j} > \mu(E_i) + \mu(E_j).$$

Indeed Lemma 3.7 is sufficient to conclude the proof of (3) as follows:

Proof of Proposition 2.14, part (3). Fix i and let $j > m_i$. Then

(By Lemma 3.5) $$\mu(A_i \cap A_j) \le \int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{\ell_0(j-i)})) + \mu(A_i) + \mu(A_j)$$

(By Lemma 3.7) $$\le \int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{\ell_0(j-i)})) + C\int \rho^i_{i,j}\int \rho^j_{i,j}$$

(by Corollary 3.4, and since $\int \rho^\ell_{i,j} \le \mu(H_{c_{\mathcal{H}},\sigma,\ell}) = \mu(A_\ell)$)
$$\le C\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right] + C\mu(A_i)\mu(A_j)$$

(Setting $\widetilde{C} = 2C$.) $$\le \widetilde{C}\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right]$$

□

Since Lemma 3.7 depends on the additional assumptions in Lemma 3.6, we will first prove Lemma 3.6 by replacing the function ψ with a function $\widetilde{\psi}$ which satisfies Eq. (3.1) and is still sufficient to prove the desired conclusion for ψ. The proof of Lemma 3.6 uses Lemma 3.8. After stating Lemma 3.8, we will then give a proof of Lemma 3.6, then Lemma 3.7, and the section concludes with the proof of Lemma 3.8.

Lemma 3.8. *Let $\{a_j\}_{j\in\mathbb{N}}$ be a non-increasing sequence of positive numbers with $\sum_{j=1}^\infty a_j = \infty$. For any $\rho, k, \tau > 0$ with $\tau < 1$, there exists a constant $C > 0$ and a non-increasing sequence $\{c_j\}_{j\in\mathbb{N}}$ with $c_j = \min\left\{\frac{1}{j}, \max\{a_j, \frac{1}{j^2}\}\right\}$, so that $\sum_{j=1}^\infty c_j = \infty$, $\sum_{j:c_j > a_j} a_j < \infty$, and*

$$\text{whenever } i \ge 3, \text{ for all } j > \max\{i - C\log(c_i), 9\}, \text{ we have } e^{-\rho(j-i)} < \tau c_j^k. \tag{3.2}$$

Proof of Lemma 3.6. We first note that from the proof of Lemma 2.3 that we can choose $n_{\mathcal{H}}$ large enough so that we always have $2^{-n_{\mathcal{H}}} < 2r$, and thus
$$\min\left\{\frac{\sigma^4}{7^4}\frac{1}{\psi(b^j)^4}, \frac{r}{2}, 2^{-(n_{\mathcal{H}}+2)}\right\} = \min\left\{\frac{\sigma^4}{7^4}\frac{1}{\psi(b^j)^4}, 2^{-(n_{\mathcal{H}}+2)}\right\}$$

We next construct a modification to remove the constant upper bound. That is, we replace ψ by $\widehat{\psi}$ so that $e^{-\frac{\delta}{4}|i-j|} \le \frac{\sigma^4}{7^4}\frac{1}{\widehat{\psi}}(b^j)^4$ implies $e^{-\frac{\delta}{4}|i-j|} \le \min\left\{\frac{\sigma^4}{7^4}\frac{1}{\psi(b^j)^4}, \frac{r}{2}, 2^{-(n_{\mathcal{H}}+2)}\right\}$. Namely using the constants from Lemma 2.3, define

$$\widehat{\psi}(b^i) = \begin{cases} \psi(b^i) & \frac{M\sigma^2}{\psi(b^i)} < 2^{-(n_{\mathcal{H}}+2)} \\ M\sigma^2 2^{(n_{\mathcal{H}}+2)} & \text{otherwise.} \end{cases}$$

We now have the constant upper bounds are trivially satisfied for $\widehat{\psi}$ for all $j > i + \frac{4}{\delta}\log\left(\frac{1}{\mu(A_i)}\right)$. Moreover for sets $\widehat{A}_j$ corresponding to $\widehat{\psi}$, $\widehat{A}_j \subseteq A_j$, and so $\mu(\limsup A_j) \geq \mu(\limsup \widehat{A}_j)$. Since our goal is to prove positive measure, we may now always assume the constant upper bounds hold.

We now construct $\widetilde{\psi}$ from ψ which satisfies Eq. (3.1) by defining $\widetilde{\psi}(b^j) = c_j^{-1/2}$ where $c_j = \min\{1/j, \max\{a_j, 1/j^2\}\}$ is the sequence from Lemma 3.8 for $a_j = \frac{1}{\psi(b^j)^2}$, $\rho = \frac{\delta}{4}$, $k = 2$, and $\tau = \frac{\sigma^4}{7^4} < 1$. Let $\widetilde{A}_i$ be the set corresponding to $\widetilde{\psi}$.

Notice that $c_i = \frac{1}{\widetilde{\psi}(2^i)^2}$, so by Lemma 3.8 whenever $i \geq 3$ and $j > \max\{i - C\log(\widetilde{\psi}(2^i)^2), 9\}$, we indeed have for i large enough that making C larger if necessary

$$e^{-\frac{\delta}{4}|i-j|} < (\widetilde{\psi}(2^i))^{-2\cdot C\frac{\delta}{4}} < m\sigma^2(\widetilde{\psi}(2^i))^{-2} \leq \mu(\widetilde{A}_i)$$

and moreover we have the desired inequality that

$$e^{-\frac{\delta}{4}|i-j|} < \frac{\sigma^4}{7^4}\frac{1}{\widetilde{\psi}(2^j)^4}.$$

The restrictions on $i \geq 3$ and $j > 9$ do not play a role in the conclusion for the limsup sets. For $j \geq n > 9$, notice $c_j \leq a_j$ implies $\widetilde{A}_j \subseteq A_j$, so

$$\bigcup_{j=n}^{\infty} A_j \supseteq \bigcup_{\substack{j=n \\ c_j \leq a_j}} \widetilde{A}_j \cup \bigcup_{\substack{j=n \\ c_j > a_j}} A_j \supseteq \bigcup_{\substack{j=n \\ c_j \leq a_j}} \widetilde{A}_j.$$

On the other hand since $c_j > a_j$ exactly when $c_j = \frac{1}{j^2}$,

$$\mu\left(\bigcup_{j=n}^{\infty} \widetilde{A}_j\right) \leq \mu\left(\bigcup_{\substack{j=n \\ c_j \leq a_j}}^{\infty} \widetilde{A}_j\right) + \mu\left(\bigcup_{\substack{j=n \\ c_j > a_j}}^{\infty} \widetilde{A}_j\right) \leq \mu\left(\bigcup_{\substack{j=n \\ c_j \leq a_j}}^{\infty} \widetilde{A}_j\right) + \sum_{\substack{j=n \\ c_j > a_j}}^{\infty} \frac{m\sigma^2}{j^2}.$$

Since the tail sum of convergent series goes to zero, for any ϵ we can choose n large enough so that $\sum_{j>n, c_j > a_j} \frac{m\sigma^2}{j^2} < \epsilon$, Hence,

$$\mu\left(\bigcup_{j=n}^{\infty} A_j\right) \geq \mu\left(\bigcup_{\substack{j=n \\ c_j \leq a_j}} \widetilde{A}_j\right) \geq \mu\left(\bigcup_{j=n}^{\infty} \widetilde{A}_j\right) - \epsilon.$$

Thus in the limit we conclude $\mu(\limsup A_j) \geq \mu(\limsup \widetilde{A}_j)$. $\square$

With our preliminary work, Lemma 3.7 follows from geometric estimates on measures of balls and trapezoids.

Proof of Lemma 3.7. Since $\rho^j_{i,j} \le \rho^j_{i,j}$, we have

$$\int \rho^i_{i,j} \int \rho^j_{i,j} \ge \left(\int \rho^j_{i,j} \right)^2 \ge \mu\{\rho^j_{i,j} = 1\}^2. \tag{3.3}$$

Now we want to get a lower bound for the measure of the set where $\rho^j_{i,j} = 1$. To do this, we need to find the area of the subset of the trapezoid $T^+_{c_{\mathcal{H}},\sigma,j}$ where the $\rho^j_{i,j} = 1$. To do this, note the horizontal line $y = \epsilon_{i,j}$ from $c_{\mathcal{H}} + \epsilon$ to $1 - \epsilon$ give the height of the inner trapezoid. The line which is length $\epsilon_{i,j}$ away from the line $y = \frac{\sigma}{\psi(b^j)} x$ is given by

$$y = \frac{\sigma}{\psi(b^j)} x - \epsilon \sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}}.$$

Thus the four corners of the trapezoid where $\rho^j_{i,j} = 1$ are given by $(c_{\mathcal{H}} + \epsilon, \epsilon), (1 - \epsilon, \epsilon)$,

$$\left(1 - \epsilon, \frac{\sigma}{\psi(b^j)}(1 - \epsilon) - \epsilon\sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}}\right), \left(c_{\mathcal{H}} + \epsilon, \frac{\sigma}{\psi(b^j)}(c_{\mathcal{H}} + \epsilon) - \epsilon\sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}}\right).$$

Hence the area of subset of $T^+_{c_{\mathcal{H}},\sigma,j}$ where $\rho^j_{i,j} = 1$ is given by

$$\frac{(1 - c_{\mathcal{H}} - 2\epsilon_{i,j})}{2} \left(\frac{\sigma}{\psi(b^j)}(1 + c_{\mathcal{H}}) - 2\epsilon_{i,j}\sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}} - 2\epsilon_{i,j} \right).$$

By symmetry, the area of $T^-_{c_{\mathcal{H}},\sigma,j}$ where $\rho^j_{i,j} = 1$ is the same as the area for $T^+_{c_{\mathcal{H}},\sigma,j}$. Thus the total area is the product of the two trapezoids with the product of the ball where $n + 4 = 2g + s - 1$ and $\sigma(n)$ is gives the volume of the n-ball. That is

$$\mu(\{\rho^j_{i,j} = 1\}) = \frac{(1 - c_{\mathcal{H}} - 2\epsilon_{i,j})^2}{4} \left(\frac{\sigma}{\psi(b^j)}(1 + c_{\mathcal{H}}) - 2\epsilon_{i,j}\sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}} - 2\epsilon_{i,j} \right)^2 \cdot \sigma(n)(r - \epsilon_{i,j})^n$$

(by (3.1), $\epsilon_{i,j} \le \frac{r}{2}$ so d_n is some constant depending only on n, and since $1 + c_{\mathcal{H}} \ge 1$)

$$\ge d_n(1 - c_{\mathcal{H}} - 2\epsilon_{i,j})^2 \left(\frac{\sigma}{\psi(b^j)} - 2\epsilon_{i,j}\sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}} - 2\epsilon_{i,j} \right)^2$$

(substituting $1 - c_{\mathcal{H}} = 2^{-n_{\mathcal{H}}}$ from Lemma 2.3 and and the trivial bounds $\sigma^2/\psi(b^j)^2 < 1 < 3$)

$$= d_n \left(2^{-n_{\mathcal{H}}} - 2\epsilon_{i,j}\right)^2 \left[\frac{\sigma}{\psi(b^j)} - 6\epsilon_{i,j} \right]^2$$

(assuming by (3.1) $\epsilon_{i,j} < 2^{-(n_{\mathcal{H}}+2)}$, which implies $2^{-n_{\mathcal{H}}} - 2\epsilon_{i,j} > 2^{-(n_{\mathcal{H}}+1)}$)

$$\ge \frac{d_n}{2^{2(n_{\mathcal{H}}+1)}} \left[\frac{\sigma}{\psi(b^j)} - 6\epsilon_{i,j} \right]^2$$

(assuming by (3.1) that $\epsilon_{i,j} \le \frac{\sigma^4}{7^4 \psi(b^j)^4}$, which implies $6\epsilon_{i,j} + \epsilon_{i,j}^{\frac{1}{4}} \le 7\epsilon_{i,j}^{\frac{1}{4}} \le \frac{\sigma}{\psi(b^j)}$)

$$\ge \frac{d_n}{2^{2(n_{\mathcal{H}}+1)}} \epsilon_{i,j}^{\frac{1}{2}}.$$

Combining this fact with (3.3), we obtain

$$\int \rho^i_{i,j} \int \rho^j_{i,j} \ge \frac{d_n^2}{2^{4(n_{\mathcal{H}}+1)}} \epsilon_{i,j}. \tag{3.4}$$

Now from the other end we want an upper bound for $\mu\{\rho^i \in (0,1)\} + \mu\{\rho^j \in (0,1)\} \le 2C\epsilon_{i,j}$.

Given $\ell = i$ or $\ell = j$, we have the area of the $\epsilon_{i,j}$-boundary of one of the trapezoids $T^{\pm}_{c_{\mathcal{H}},\sigma,2^\ell}$ is given by

$$\mu(\partial_{\epsilon_{i,j}} T^{\pm}_{c_{\mathcal{H}},\sigma,2^\ell})$$

(taking area of $T^{\pm}_{c_{\mathcal{H}},\sigma,\ell}$ less the area where $\rho^{\ell}_{i,j} = 1$)

$$= \frac{\sigma}{2\psi(b^\ell)}(1 - c_{\mathcal{H}}^2) - \left(\frac{1 - c_{\mathcal{H}}}{2} - \epsilon_{i,j}\right)\left(\frac{\sigma}{\psi(b^\ell)}(1 + c_{\mathcal{H}}) - 2\epsilon_{i,j}\sqrt{1 + \frac{\sigma^2}{\psi(b^\ell)^2}}\right)$$

$$= \epsilon_{i,j}\left[\frac{\sigma}{\psi(b^\ell)}(1 + c_{\mathcal{H}}) + (1 - c_{\mathcal{H}} - 2\epsilon_{i,j})\left[1 + \sqrt{1 + \frac{\sigma^2}{\psi(b^\ell)^2}}\right]\right]$$

(assuming $\sigma < \psi(b^\ell)$ which is easy since $\sigma < 1$ is fixed and $\psi \geq 1$ is non-decreasing)

$$\leq \epsilon_{i,j}\left[1 + c_{\mathcal{H}} + (1 - c_{\mathcal{H}})(1 + \sqrt{2})\right]$$

(where C depends on $c_{\mathcal{H}}$)

$$\leq \epsilon_{i,j} C.$$

Thus

$$\mu(\{\rho^i_{i,j} \in (0,1)\}) + \mu(\{\rho^j_{i,j} \in (0,1)\}) \leq 2C\epsilon_{i,j}. \tag{3.5}$$

Combining Equation 3.5 with Equation 3.4,

$$\int \rho^i_{i,j} \int \rho^j_{i,j} \geq \frac{d_n^2}{2^{4(n_{\mathcal{H}}+1)}}\epsilon_{i,j} \geq \frac{d_n^2}{2^{4(n_{\mathcal{H}}+1)}(2C)}\left[\mu\{\rho^i_{i,j} \in (0,1)\} + \mu\{\rho^j_{i,j} \in (0,1)\}\right]. \tag{3.6}$$

Setting $\widetilde{C} = \frac{2^{4(n_{\mathcal{H}}+1)}(2C)}{d_n^2}$, we can assume $\widetilde{C} > 1$ since d_n is bounded above by a fixed constant, and we can make $\widetilde{C}$ larger if necessary. □

Proof of Lemma 3.8. We first claim the following:

Claim: The sequence c_j is non-increasing, has divergent sum and $\sum_{j:c_j > a_j} a_j < \infty$ and $\sum_{j:c_j \neq a_j} c_j < \infty$.

Proof of Claim. The maximum of two non-increasing sequences is non-increasing and so $\max\{a_j, \frac{1}{j^2}\}$ is a non-increasing sequence (in j). Similarly the minimum of two non-increasing sequences is non-increasing and so c_j is non-increasing.

If $\max\{a_j, \frac{1}{j^2}\} = \frac{1}{j^2}$, then $c_j = j^{-2} > a_j$. Otherwise $c_j = \min\{1/j, a_j\} \leq a_j$.

If $c_j > a_j$ then $a_j < \frac{1}{j^2}$ and so clearly $\sum_{c_j > a_j} a_j < \sum \frac{1}{j^2} < \infty$. On the other hand, since $c_j > a_j$ is only possible when $c_j = j^{-2}$,

$$\sum_{c_j > a_j} c_j = \sum_{c_j > a_j} j^{-2} < \infty.$$

Now observe that

$$\sum_{j=2^k}^{2^{k+1}-1} c_j \geq \min\{\frac{1}{2}, 2^k a_{2^{k+1}}\}. \tag{3.7}$$

Indeed we are estimating the sum from below by $2^k c_{2^{k+1}}$ and considering the different possibilities of $c_{2^{k+1}}$. Notice that $2\sum_k 2^k a_{2^{k+1}} \geq \sum_j a_j$ and so $\sum 2^k a_{2^{k+1}}$ diverges and thus $\sum_k \min\{\frac{1}{2}, 2^k a_{2^{k+1}}\}$ diverges. So $\sum_j c_j = \sum_k \sum_{j=2^k}^{2^{k+1}-1} c_j$ diverges. □

We have proved the Claim and now proceed with the remainder of the proof of Lemma 3.8. We now show if $C > 4\frac{\widetilde{k}}{\rho}(\frac{1}{\widetilde{\tau}} + 1)$ then Eq. (3.2) holds where $\widetilde{\tau} = \min\{\tau, 1\}$ and $\widetilde{k} = \max\{\frac{1}{2}, k\}$. Indeed it suffices to show that for all $j > i + C\log(i)$ we have that

$$e^{-\rho(j-i)} < \frac{\tau}{j^{2k}}. \tag{3.8}$$

Clearly smaller τ and larger $2k$ make (3.8) harder to satisfy. So from here we assume $\tau \leq 1$ and $2k \geq 1$, which motivated our choice of $\widetilde{\tau}$ and $\widetilde{k}$ Under these assumptions, Equation (3.8) is implied by $j - i > \frac{2k}{\rho}\log(\frac{j}{\tau})$.

If $j \leq 2i$ this follows from our condition on C. Indeed

$$j - i > 4\frac{k}{\rho}\left(\frac{1}{\tau} + 1\right)\log(i) > 4\frac{k}{\rho}\left(\frac{1}{\tau} + 1\right)(\log(j) - \log(2)).$$

And so

$$j - i > 4\frac{k}{\rho}(\log\left(j^{\frac{1}{\tau}+1}\right) - \log(2^{\frac{1}{\tau}+1})) > 2\frac{k}{\rho}\log(j^{\frac{1}{\tau}+1}) > \frac{2k}{\rho}\log\left(\frac{j}{\tau}\right).$$

Note that the second inequality uses that $\log(j^{\frac{1}{\tau}+1}) > 2\log(2^{\frac{1}{\tau}+1})$ because $j \geq 9 > 2^2$ and the third inequality uses that $j^{\frac{1}{\tau}+1} > \frac{1}{\tau}j$ for all $j \geq 9$ and $\tau > 0$.

For the case when $j > 2i$, set $f(x) = x - i$ and $g(x) = \frac{2k}{\rho}\log\left(\frac{x}{\tau}\right)$. Note that $f(2i) > g(2i)$ from the case where $j \leq 2i$. Moreover, $f'(x) = 1 > g'(x) = \frac{2k}{x\rho\tau}$ for all $x > \frac{2k}{\rho\tau}$. Since $i \geq 3$ and $\log(3) > 1$,

$$j > i + C\log(i) > 3 + \frac{4k}{\rho}(\frac{1}{\tau} + 1)\log(3) > \frac{2k}{\rho\tau}.$$

So for all $j > 2i$ we have $f'(j) > g'(j)$ and $f(2i) > g(2i)$. Hence $f(j) > g(j)$ for all $j \geq 2i$ as desired. □

4. Verifying Proposition 2.10 Assumption (4)

We begin this section by defining the sets B_i and C_i required for Proposition 2.14, and then verify these sets satisfy Assumption (4) of Proposition 2.10.

Definition 3 (Definition of the B's). *Set* $I \overset{def}{=} (-\frac{\pi}{12}, \frac{\pi}{12})$. *For* $k \in \mathbb{N}$ *define*

$$B_k = g_{\log(b^k)} g_{\log\left(\sqrt{\frac{\psi(b^k)}{\sigma}}\right)} \bigcup_{\theta \in I} r_\theta W_k$$

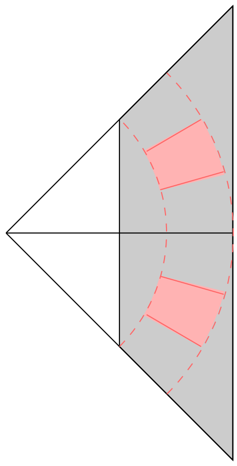
where we have the following definitions. We first pull back the set A_k *so that trapezoids in* $H_{c,\sigma,k}$ *makes a 45 degree angle so*

$$\widetilde{W_k} = g_{log\left(\sqrt{\frac{\sigma}{\psi(b^k)}}\right)} g_{-log(b^k)} A_k.$$

Then we restrict to a smaller subset of $\widetilde{W_k}$ denoted W_k so that $r_\theta W_k \subset \widetilde{W_k}$ for $\theta \in I$. That is W_k is the set of ω with two holonomy vectors v_1 and v_2 satisfying

$$c_{\mathcal{H}}\sqrt{\frac{2\sigma}{\psi(b^k)}} \leq |v_1|, |v_2| \leq \sqrt{\frac{\sigma}{\psi(b^k)}} \quad \arg(v_1) \in \left(\frac{\pi}{12}, \frac{\pi}{6}\right) \text{ and } \arg(v_2) \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right).$$

In the graphic below, the gray shaded region corresponds to the region $\widetilde{W_k}$ and the pink region corresponds to W_k.



Definition 4 (Definition of the C's)**.** *Define*

$$C_k = g_{\log(b^k)} g_{\log\left(\sqrt{\frac{\psi(b^k)}{\sigma}}\right)} S(c_{\mathcal{H}}, \sigma, b^k)$$

where

$$S(c_{\mathcal{H}}, \sigma, t) = \left\{\omega : \omega \text{ has a holonomy vector } v \text{ with } |v| \in \left(c_{\mathcal{H}}\sqrt{\frac{\sigma}{\psi(t)}}, \sqrt{\frac{2\sigma}{\psi(t)}}\right)\right\}.$$

4.1. **Proof that (4) (a)-(c) hold.** We now verify assumptions (4) (a), (b) and (c), where we set $c = c_{\mathcal{H}}$. Fix i and take j so that $i < j \leq i + C\log\left(\frac{1}{\mu(A_i)}\right)$.

(a) As constructed $B_i \subseteq A_i$ and $A_j \subseteq C_j$.

(b) Following the strategy of Lemma 2.3 where we compute the area of the sectors instead of trapezoids,

$$\mu(W_i) \geq \frac{\sigma^2}{\psi(b^i)^2}\left(\frac{\pi}{24}(1 - 2c_{\mathcal{H}})\right)^2 m_{2g+s-3}(B).$$

Thus by Lemma 2.3,

$$\mu(B_i) \geq \mu(W_i) \geq \mu(A_i)\frac{1}{m}\left(\frac{\pi}{24}(1 - 2c_{\mathcal{H}})\right)^2 m_{2g+s-3}(B).$$

(c) Since the measure is invariant under geodesic flow and by Lemma 2.3,

$$\mu(A_j) = \mu(H_{c_{\mathcal{H}},\sigma,j}) \geq m\frac{\sigma^2}{\psi(b^j)^2}.$$

By Masur–Smillie Lemma 2.2,

$$\mu(C_j) = \mu(S(c_{\mathcal{H}}, \sigma, 2^j)) \leq M\frac{\sigma}{\psi(b^j)}.$$

Thus

$$\mu(C_j) \le \frac{M}{\sqrt{m}}\sqrt{\frac{m\sigma^2}{\psi(b^j)^2}} = \frac{M}{\sqrt{m}}\mu(A_j)^{\frac{1}{2}}.$$

4.2. **Construction and circle averages of logsmooth functions.** The main goal of this section is to prove Corollary 4.6, which extends the statements of [Doz19] (giving averages over intervals) to include so-called logsmooth functions from [Ath06] (which gives averages over the full circle)

Definition 5. *A complex K in ω is a closed subset of X whose boundary ∂K consists of a union of disjoint (in the interior) saddle connections such that if ∂K contains three saddle connections bounding a triangle, then the interior of that triangle is in K. Given a complex K the complexity of K is the number of saddle connections needed to triangulate K. For any $\delta > 0$ and $k \in \mathbb{N}$, if M is the complexity of ω,*

$$\alpha_k(\omega) = \max_{\substack{K \text{ complexity } k \\ \text{area}(K) < 2^{k-M-1}}} \frac{1}{|\partial K|^{1+\delta}}.$$

If the set over which we take the maximum is empty, then we set $\alpha_k(\omega) = 0$.

Definition 6. *Given a function f on $\mathcal{H}$ and a point $\omega \in \mathcal{H}$, we let*

$$Ave_t(f)(\omega) = \frac{1}{2\pi}\int_0^{2\pi} f(g_t r_\theta \omega)\, d\theta.$$

Note that $\alpha_1(\omega) = \frac{1}{\ell(\omega)^{1+\delta}}$ where $\ell(\omega)$ is the length of the shortest saddle connection. Since M is finite for all k large enough, $\alpha_k(\omega) = 0$. For more information and intuition for the α_k, see Section 5.3 of [Doz19]. From Proposition 5.3 of [Doz19], we have

Proposition 4.1. *Fix a stratum $\mathcal{H}$, and $0 < \delta < \frac{1}{2}$. We can find a constant b such that for any interval $I \subseteq S^1$, there exists a constant c_I such that for all $\omega \in \mathcal{H}$ and $T \ge 0$,*

$$\int_I \alpha_k(g_T r_\theta \omega)\, d\theta < c_I e^{-(1-2\delta)T} \sum_{j \ge k} \alpha_j(\omega) + b|I|.$$

The strategy we will take is to extend this theorem to a function V_δ which is a weighted average of the α_k functions. We want to weight the average to have nice properties, so our goal is to recreate the following theorem for integrating over an interval I instead of $[0, 2\pi)$.

Lemma 4.2 (Lemma 6.2 [AG13], Proof in [Ath06])**.** *Let $\mathcal{V}$ be a neighborhood of the identity in $SL(2,\mathbb{R})$. Fix $\mathcal{H}$ a conncted stratum of $\mathcal{H}(\alpha)$. For every $0 < \delta < 1$ there exists $c_1 > 0$ so that for all $t > 0$ there exists a function $V_\delta^{(t)} : \mathcal{H} \to [1, \infty)$ and a scalar b_t satisfying the following properties. For all $\omega \in \mathcal{H}$,*

$$(Ave)_t(V_\delta^{(t)})(\omega) = \int_0^{2\pi} V_\delta^{(t)}(g_t r_\theta \omega)\, d\theta \le c_1 e^{-(1-\delta)t} V_\delta^{(t)}(\omega) + b_t.$$

Moreover, V_δ is logsmooth. That is

$$V_\delta^{(t)}(g\omega) \le c_3 V_\delta^{(t)}(\omega) \tag{4.1}$$

for all $\omega \in \mathcal{H}$ and $g \in \mathcal{V}$.

Finally, there exists a constant $C_{\delta,t}$ so that

$$\frac{V_\delta^{(t)}(\omega)}{V_\delta(\omega)} \in [C_{\delta,t}^{-1}, C_{\delta,t}] \tag{4.2}$$

where $V_\delta = \max\{1, \alpha_1(\omega)\} = \max\{1, \frac{1}{\ell(\omega)^{1+\delta}}\}$.

We want to change Lemma 4.2 to restricting over an interval I. We now explicitly construct V_δ using the following result.

Proposition 4.3 (Proposition 5.4 [Doz19]). *Fix $\mathcal{H}$ and $0 < \delta < 1$. There exists $C > 0$ so that for any $t > 0$, there exists constants b_t and w_t so that for any k and any $\omega \in \mathcal{H}$,*

$$Ave_t(\alpha_k)(\omega) \leq Ce^{-t(1-\delta)}\alpha_k(\omega) + w_t \sum_{j>k} \alpha_j(\omega) + b_t. \tag{4.3}$$

Definition 7. *Fix δ and $t > 0$. Define*

$$\lambda_k^{(t)} = \left(\frac{w_t}{C} + 1\right)^k$$

where w_t and C are the constants of 4.3. Define

$$V_\delta^{(t)}(\omega) = \sum_{k=0}^{M} \lambda_k^{(t)} \alpha_k(\omega)$$

where M is the maximum complexity of ω.

Proof of Lemma 4.2. We first claim

$$\lambda_k Ce^{-(1-\delta)t} + w_t \sum_{j=0}^{k-1} \lambda_j \leq 2C\lambda_k e^{-(1-\delta)t}. \tag{4.4}$$

To see this holds, note that $e^{-(1-\delta)t} \geq 1$. Thus since $\lambda_k \geq 1$, we have

$$1 - \frac{1}{\lambda_k} \leq 1 \leq \left(\frac{w_t}{C} + 1 - 1\right) \frac{C}{w_t} e^{-(1-\delta)t}.$$

Simplifying and using the finite geometric series formula, this implies

$$\sum_{j=0}^{k-1} \lambda_j = \frac{\lambda_k - 1}{\lambda_1 - 1} \leq \lambda_k \frac{C}{w_t} e^{-(1-\delta)t}.$$

Multiplying by w_t and adding $\lambda_k Ce^{-(1-\delta)t}$ to each side yields Equation 4.4.

We now want to prove that on average V_δ shrinks over circles of radius t. To see this, we compute

$$\begin{aligned}
\mathrm{Ave}_t(V_\delta)(\omega) &= \sum_{k=0}^{n} \lambda_k \mathrm{Ave}_t(\alpha_k)(\omega) \\
\text{(by 4.3)} \qquad &\leq \sum_{k=0}^{n} \lambda_k \left(Ce^{-t(1-\delta)}\alpha_k(\omega) + w_t \sum_{j>k} \alpha_j(\omega) + b_t \right)
\end{aligned}$$

(replacing $b_t = b_t \left(\sum_{j=1}^{n} \lambda_j\right)$)

$$= \sum_{k=0}^{n} \lambda_k Ce^{-t(1-\delta)}\alpha_k(\omega) + w_t \sum_{k=1}^{n} \alpha_k(\omega) \left(\sum j = 0^{k-1}\lambda_j\right) + b_t$$

(by Equation 4.4 and replacing C with $2C$)

$$\begin{aligned}
&\leq 2Ce^{-t(1-\delta)}\lambda_0\alpha_0(\omega) + \sum_{k=1}^{n} \alpha_k(\omega) \left[\lambda_k Ce^{-t(1-\delta)} + w_t \sum_{j=0}^{k-1} \lambda_j \right] + b_t \\
&\leq Ce^{-t(1-\delta)} \sum_{k=0}^{n} \lambda_k \alpha_k(\omega) + b_t \\
&= Ce^{-t(1-\delta)} V_\delta(\omega) + b_t.
\end{aligned}$$

The logsmoothness of the V_δ follows from [EM01]. □

Now that we have defined V_δ with the logsmooth property, we now proceed to extending the results of [Doz19] to include the V_δ function.

Lemma 4.4. *There exists a constant $c_2 > 0$ so that for any $\tau \geq 0$ and $I \subseteq S^1$ an interval, there exists $t_0(\tau, |I|) \geq 0$ so that for any $\omega \in \mathcal{H}$ and $t > t_0$, we have*

$$\int_I V_\delta^{(\tau)}(g_{t+\tau} r_\theta \omega)\, d\theta \leq c_2 \int_J Ave_\tau(V_\delta^{(\tau)})(g_t r_\theta \omega)\, d\theta$$

where $J \subseteq S^1$ is an interval (that could depend on all other parameters) with $|J| = |I|$.

Proof. Note that this result would follow directly from linearity combined with Lemma 5.2 of [Doz19], except as stated in Lemma 5.2 the interval J could depend on α_i. However following the proof exactly using linearity to replace each α_i with $V_\delta^{(\tau)}$, we take the interval $2I$ with the same center as I and twice the length. Then in the last 5 lines of the proof, we write $2I = J_1 \cup J_2$ as a union of two intervals with $|J_1| = |J_2| = |I|$. Then

$$\max_{j=1,2} \int_{J_j} \mathrm{Ave}_\tau(V_\delta^{(\tau)}(g_t r_\theta \omega) \geq \frac{1}{2} \int_{2I} \mathrm{Ave}_\tau V_\delta^{(\tau)}(g_t r_\theta \omega).$$

Now define J (which now depends on V_δ^τ instead of individual α_k to be the interval on which the maximum is achieved, and the proof follows by linearity as desired. □

Now we state Proposition 5.3 of [Doz19] for the V_δ functions.

Proposition 4.5. *Fix a stratum $\mathcal{H}$ and $0 < \delta < 1$. Let c_1 and c_2 be the constants of Lemma 4.2 and Lemma 4.4, respectively. Choose $\tau \geq 0$ large enough so that*

$$c_1 c_2 e^{-(1-\delta)\tau} < \frac{1}{2}.$$

Let $I \subseteq S^1$ be an interval and by Lemma 4.4 let m be the smallest possible integer so that $(m-1)\tau > t_0(\tau, |I|)$. That is $m = 1 + \left\lceil \frac{t_0(\tau,|I|)}{\tau} \right\rceil$. There are constants $c = c(\tau, \delta, |I|) > 0$ and $b_\tau = b(\tau, \delta)$ so that for all $n \geq m$ and for any $\omega \in \mathcal{H}$,

$$\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega)\, d\theta < ce^{-(1-\delta)n\tau} V_\delta^{(\tau)}(\omega) + b_\tau |I|. \tag{4.5}$$

Proof. Let $n \geq m$ and $\omega \in \mathcal{H}$. Our goal is to construct the constants c and b_τ to that Equation 4.5 holds. Indeed applying Lemma 4.4 followed by Lemma 4.2, we have

$$\begin{aligned}
\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega)\, d\theta &\leq c_2 \int_{J_{n-1}} \mathrm{Ave}_\tau(V_\delta^{(\tau)})(g_{(n-1)\tau} r_\theta \omega)\, d\theta \\
&\leq c_2 \left(\int_{J_{n-1}} c_1 e^{-(1-\delta)\tau} V_\delta^{(\tau)}(g_{(n-1)\tau} r_\theta \omega)\, d\theta + b_\tau \right) \\
&= c_2 c_1 e^{-(1-\delta)\tau} \int_{J_{n-1}} V_\delta^{(\tau)}(g_{(n-1)\tau} r_\theta \omega)\, d\theta + c_2 b_\tau |I|
\end{aligned}$$

where the last equality follows from the fact that $|J_{n-1}| = |I|$.

Now repeatedly applying this inequality for $n-1, n-2, \ldots, m$ with I replaced by J_{n-1}, J_{n-2} through J_m which all have length $|I|$, we obtain

$$\begin{aligned}
\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega) \leq \left(c_1 c_2 e^{-(1-\delta)\tau}\right)^{n-m+1} &\int_{J_{m-1}} V_\delta^{(\tau)}(g_{(m-1)\tau} r_\theta \omega)\, d\theta \\
+ |I| b_\tau c_2 &\sum_{j=0}^{n-m+1} \left(c_1 c_2 e^{-(1-\delta)\tau}\right)^j
\end{aligned}$$

(by our choice of τ the geometric sum is at most 2, so replacing b_τ with $2c_2 b_\tau$,)

$$\leq \left(c_1 c_2 e^{-(1-\delta)\tau}\right)^{n-m+1} \int_{J_{m-1}} V_\delta^{(\tau)}(g_{(m-1)\tau} r_\theta \omega)\, d\theta + |I| b_\tau.$$

By the logsmooth property of V_δ from Lemma 4.2, splitting into small steps, there exists some $k(m, \tau)$ so that

$$V_\delta^{(\tau)}(g_{(m-1)\tau} r_\theta \omega) \leq c_3^{k(m,\tau)} V_\delta^{(\tau)}(\omega).$$

Thus we can write our constant c as

$$c = (c_1 c_2)^{n-m+1} \left(e^{-(-1\delta)\tau}\right)^{-m+1} c_3^{k(m,\tau)} |I|$$

where we note m depends on τ and $|I|$, so c depends only on δ, τ and $|I|$.

Thus we obtain

$$\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega)\, d\theta \leq ce^{-\tau n(1-\delta)} V_\delta^{(\tau)}(\omega) + b_\tau |I|.$$

□

Corollary 4.6. *Fix a stratum $\mathcal{H}$ and $0 < \delta < 1$. There exists $\tau \geq 0$ so that for any interval $I \subseteq S^1$, there exists constants $c = c(\tau, \delta, |I|) > 0$ and $b_\tau = b(\tau, \delta)$ so that there exists an ℓ_0 so that for all $\ell \geq \ell_0$ and for any $\omega \in \mathcal{H}$,*

$$\int_I V_\delta^{(\tau)}(g_\ell r_\theta \omega)\, d\theta \leq ce^{-(1-\delta)\ell} V_\delta^{(\tau)}(\omega) + b_\tau |I|.$$

Proof. We choose τ to satisfy the assumption of Proposition 4.5. Choose $\ell \geq m\tau$ where m is defined in Proposition 4.5. Pick $n_0 = \min\{n \in \mathbb{N} : n\tau > \ell\}$ and note $n_0 - 1 \geq m$. Let $r = n_0\tau - \ell$. Choose step sizes of r_0 so that $r = kr_0$ for some $k \in \mathbb{N}$ and $g_{-r_0} \in \mathcal{V}$ so we can apply (4.1).

Then from Proposition 4.5,

$$\int_I V_\delta^{(\tau)}(g_\ell r_\theta \omega)\, d\theta = \int_I V_\delta^{(\tau)}(g_{-kr_0} g_{n_0\tau} r_\theta \omega)\, d\theta$$

$$\text{(by (4.1))} \qquad \leq c_3^k \int_I V_\delta^{(\tau)}(g_{n_0\tau} r_\theta\omega)\, d\theta$$

$$\text{(by (4.5))} \qquad \leq c_3^k \left[ce^{-(1-\delta)n_0\tau} V_\delta^{(\tau)}(\omega) + b_\tau |I|\right]$$

$$\text{(since } n_0\tau \geq \ell \text{ and } r \leq \tau) \qquad \leq c_3^{\frac{\tau}{r_0}} \left[ce^{-(1-\delta)\ell} V_\delta^{(\tau)}(\omega) + b_\tau |I|\right].$$

Thus the final constants only depend on τ and not ℓ and we obtain the desired result. □

4.3. **Completion of the verification of (4) (d).** To obtain an upper bound for $\mu(B_i \cap C_j)$, by g_t-invariance of μ, it suffices to find an upper bound for $\mu(\widetilde{B}_i \cap \widetilde{C}_j)$ where $\widetilde{B}_i = \bigcup_{\theta\in I} r_\theta W_i = I\cdot W_i$ and $\widetilde{C}_j = g_{f(i,j)} S(c_{\mathcal{H}}, \sigma, 2^j)$ for $f(i,j) = \log\left(b^{j-i}\sqrt{\frac{\psi(b^j)}{\psi(b^i)}}\right)$.

We first use the fact that in $S(c_{\mathcal{H}}, \sigma, 2^j)$, the shortest possible saddle connection has length $\ell(\omega) \in \left(c_{\mathcal{H}}\sqrt{\frac{\sigma}{\psi(b^j)}}, \sqrt{\frac{2\sigma}{\psi(b^j)}}\right)$. Choose τ large enough to satisfy the assumption of Corollary 4.6. By (4.2),

$$C_{\delta,\tau}^{-1}\left(\frac{\psi(b^j)}{2\sigma}\right)^{\frac{1+\delta}{2}} \leq V_\delta^{(\tau)}(\omega) \leq \frac{C_{\delta,\tau}}{c_{\mathcal{H}}}\left(\frac{\psi(b^j)}{\sigma}\right)^{\frac{1+\delta}{2}}.$$

Thus by Markov's inequality,

$$\mu(\widetilde{B}_i \cap \widetilde{C}_j) \leq \mu\left(\left\{\omega \in \widetilde{B}_i : V_\delta^{(\tau)}(g_{-f(i,j)}\omega) \geq C_{\delta,\tau}^{-1}\left(\frac{\psi(b^j)}{2\sigma}\right)^{\frac{1+\delta}{2}}\right\}\right) \tag{4.6}$$

$$\leq C_{\delta,\tau}\left(\frac{\psi(b^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \int_{I\cdot W_i} V_\delta^{(\tau)}(g_{-f(i,j)}\omega)\, d\mu(\omega).$$

Disintegrating the measure $\mu = d\theta\, d\widetilde{\mu}$ on $SO(2) \times (\mathcal{H}/SO(2))$ and increasing to a full $SO(2)$ orbit

$$(4.6) \leq C_{\delta,\tau}\left(\frac{\psi(b^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \int_{SO(2)\cdot W_i/SO(2)} \int_I V_\delta^{(\tau)}(g_{-f(i,j)} r_\theta \widetilde{\omega})\, d\theta\, d\widetilde{\mu}(\widetilde{\omega})$$

(by Corollary 4.6 and monotonicity of ψ, $f(i,j) \geq \log(b^{j-i}) = \ell_0(j-i) > \ell_0$)

$$\leq C_{\delta,\tau}\left(\frac{\psi(b^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \int_{SO(2)\cdot W_i/SO(2)} ce^{-(1-\delta)(f(i,j))} V_\delta^{(\tau)}(\widetilde{\omega}) + b_\tau|I|\, d\widetilde{\mu}(\widetilde{\omega})$$

(since $SO(2)W_i$ is $SO(2)$-invariant)

$$= C_{\delta,\tau}\left(\frac{\psi(b^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \left(ce^{-(1-\delta)(f(i,j))} \int_{W_i} V_\delta^{(\tau)}(\omega)\, d\mu(\omega) + b_\tau|I|\mu(W_i)\right).$$

Since all holonomy vectors in W_i are contained in the circle we apply the upper bound for $V_\delta^{(\tau)}$

$$(4.6) \leq C_{\delta,\tau}\left(\frac{\psi(b^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}}\mu(W_i)\left(ce^{-(1-\delta)(f(i,j))}\frac{C_{\delta,\tau}}{c_{\mathcal{H}}}\left(\frac{\psi(b^j)}{\sigma}\right)^{\frac{1+\delta}{2}}+b_\tau|I|\right)$$

$$\text{(since } W_i \subseteq \widetilde{B}_i.) \qquad \leq \mu(\widetilde{B}_i)\left(\frac{c}{c_{\mathcal{H}}}e^{-(1-\delta)f(i,j)}C_{\delta,\tau}^2 2^{\frac{1+\delta}{2}}+b_\tau|I|C_{\delta,\tau}\left(\frac{2\sigma}{\psi(b^j)}\right)^{\frac{1+\delta}{2}}\right).$$

Our goal is to compare the equation on the right hand side to the volume of C_j. Note by the construction of the Masur–Veech measure, there is a constant m so that

$$\mu(C_j) = \mu(S(c_{\mathcal{H}},\sigma,2^j)) \geq m\left(\sqrt{\frac{2\sigma}{\psi(b^j)}}\right)^2 = m\frac{2\sigma}{\psi(b^j)}.$$

Thus we have

$$\begin{aligned}
\mu(B_j\cap C_j) &= \mu(\widetilde{B}_j\cap\widetilde{C}_j)\\
&\leq \mu(B_i)\left(\frac{c}{c_{\mathcal{H}}}C_{\delta,\tau}^2 2^{\frac{1+\delta}{2}}e^{-(1-\delta)f(i,j)}+b_\tau|I|C_{\delta,\tau}\left(\frac{\mu(C_j)}{m}\right)^{\frac{1+\delta}{2}}\right)
\end{aligned}$$

(by the definition of $f(i,j)$)

$$= \mu(B_i)\left(\frac{c}{c_{\mathcal{H}}}C_{\delta,\tau}^2 2^{\frac{1+\delta}{2}}2^{-(j-i)(1-\delta)}\left(\frac{\psi(b^i)}{\psi(b^j)}\right)^{\frac{1-\delta}{2}}+b_\tau|I|C_{\delta,\tau}\left(\frac{\mu(C_j)}{m}\right)^{\frac{1+\delta}{2}}\right)$$

(since $\psi(R)$ is a non-decreasing sequence and $i<j$, $\psi(b^i)\leq\psi(b^j)$)

$$\leq \mu(B_i)\left(\frac{c}{c_{\mathcal{H}}}C_{\delta,\tau}^2 2^{\frac{1+\delta}{2}}2^{-(j-i)(1-\delta)}+\frac{b_\tau|I|C_{\delta,\tau}}{m^{\frac{1+\delta}{2}}}\mu(C_j)^{\frac{1+\delta}{2}}\right)$$

(defining $c_1 = \frac{c}{c_{\mathcal{H}}}C_{\delta,\tau}^2 2^{\frac{1+\delta}{2}}$ and $c_2 = \frac{b_\tau|I|C_{\delta,\tau}}{m^{\frac{1+\delta}{2}}}$)

$$\leq \mu(B_i)\left(c_1 2^{-(j-i)(1-\delta)}+c_2\mu(C_j)^{\frac{1+\delta}{2}}\right).$$

Picking $C > \max\{c_1,c_2\}$ we obtain the desired inequality.

References

[AC12] Jayadev S. Athreya and Jon Chaika. The distribution of gaps for saddle connection directions. *Geometric and Functional Analysis*, 22(6):1491–1516, 2012.

[ACL15] Jayadev S. Athreya, Jon Chaika, and Samuel Lelièvre. The gap distribution of slopes on the golden L. In *Recent trends in ergodic theory and dynamical systems*, volume 631 of *Contemp. Math.*, pages 47–62. Amer. Math. Soc., Providence, RI, 2015.

[ACM19] Jayadev S. Athreya, Yitwah Cheung, and Howard Masur. Siegel-Veech transforms are in L^2. *J. Mod. Dyn.*, 14:1–19, 2019. With an appendix by Athreya and Rene Rühr.

[AEZ16] Jayadev S Athreya, Alex Eskin, and Anton Zorich. Right-angled billiards and volumes of moduli spaces of quadratic differentials on $\mathbf{C}P^1$. *Ann. scient. Éc. Norm Sup.* , 49(6):1311–1386, 2016.

[AFM22] Jayadev S. Athreya, Samantha Fairchild, and Howard Masur. Counting pairs of saddle connections. *arXiv: 2201.08628*, 2022.

[AG13] Artur Avila and Sébastien Gouëzel. Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow. *Ann. of Math. (2)*, 178(2):385–442, 2013.

[AGY06] Artur Avila, Sébastien Gouëzel, and Jean-Christophe Yoccoz. Exponential mixing for the Teichmüller flow. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 104(1):143–211, 2006.

[Ath06] Jayadev S. Athreya. Quantitative recurrence and large deviations for Teichmueller geodesic flow. *Geom. Dedicata*, 119:121–140, 2006.

[BMMM+21] Jonah Berman, Taylor McAdam, Ananth Miller-Murthy, Caglar Uyanik, and Hamilton Wan. Slope gap distribution of saddle connections on the $2n$-gon. *arXiv: 2109.04495*, 2021.

[Doz19] Benjamin Dozier. Equidistribution of saddle connections on translation surfaces. *J. Mod. Dyn.*, 14:87–120, 2019.

[EM01] Alex Eskin and Howard Masur. Asymptotic formulas on flat surfaces. *Ergodic Theory Dynam. Systems*, 21(2):443–478, 2001.

[EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space. *Annals of Mathematics*, pages 673–721, 2015.

[EMZ03] Alex Eskin, Howard Masur, and Anton Zorich. Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel–Veech constants. *Publ. Math. Inst. Hautes Études Sci.*, (97):61–179, 2003.

[KL21] Osama Khalil and Manuel Luethi. Random walks, spectral gaps, and Khintchine's theorem on fractals. *arXiv: 2101.05797*, 2021.

[KSW21] Luis Kumanduri, Anthony Sanchez, and Jane Wang. Slope gap distributions of Veech surfaces. *arXiv: 2102.10069*, 2021.

[Mas86] Howard Masur. Closed trajectories for quadratic differentials with an application to billiards. *Duke Math. J.*, 53(2):307–314, 1986.

[Mas88] Howard Masur. Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential. In *Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)*, volume 10 of *Math. Sci. Res. Inst. Publ.*, pages 215–228. Springer, New York, 1988.

[Mas90] Howard Masur. The growth rate of trajectories of a quadratic differential. *Ergodic Theory Dynam. Systems*, 10(1):151–176, 1990.

[Mas22] Daniel Massart. A short introduction to translation surfaces, Veech surfaces, and Teichmueller dynamics. In *Surveys in Geometry I*, pages 343–388. Springer, 2022.

[MS91] Howard Masur and John Smillie. Hausdorff dimension of sets of nonergodic measured foliations. *Ann. of Math. (2)*, 134(3):455–543, 1991.

[NRW20] Amos Nevo, Rene Rühr, and Barak Weiss. Effective counting on translation surfaces. *Adv. Math.*, 360:106890, 29, 2020.

[Pet02] Valentin V. Petrov. A note on the Borel-Cantelli lemma. *Statist. Probab. Lett.*, 58(3):283–286, 2002.

[San21] Anthony Sanchez. Gaps of saddle connection directions for some branched covers of tori. *Ergodic Theory and Dynamical Systems*, page 1–55, 2021.

[UW16] Caglar Uyanik and Grace Work. The distribution of gaps for saddle connections on the octagon. *Int. Math. Res. Not. IMRN*, (18):5569–5602, 2016.

[Vee98] William A. Veech. Siegel measures. *Ann. of Math. (2)*, 148(3):895–944, 1998.

[Vor05] Yaroslav Vorobets. Periodic geodesics on generic translation surfaces. In *Algebraic and topological dynamics*, volume 385 of *Contemp. Math.*, pages 205–258. Amer. Math. Soc., Providence, RI, 2005.

[Zor08] Anton Zorich. Explicit Jenkins-Strebel representatives of all strata of abelian and quadratic differentials. *Journal of Modern Dynamics*, 2(1):139–185, 2008.