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Gauge theory on graphs

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Gauge Theory on Graphs

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Abstract

In this paper, we provide the notions of connections, exterior covariant derivatives, and connection Laplacians on a graph. We prove a Weitzenböck formula in this setting. We also define a discrete Yang-Mills functional and study the solutions to its Euler-Lagrange equations.

1 Introduction

Gauge theory is of fundamental importance in both modern physics and mathematics. From a physical perspective, it provides an elegant geometric framework to describe electromagnetism, weak interactions, and strong interactions uniformly. From a mathematical perspective, the study of gauge invariant functionals reveals astounding information about smooth structures of low dimensional manifolds [1, 2].

To understand the quantum aspects of gauge theory better, the physicist Kenneth Wilson formulated it on a lattice in 1974 [3]. His methods were later developed into a mature area of research nowadays known as lattice gauge theory [4]. Lattice gauge theory has many successful applications in quantum chromodynamics calculations [5]. However, the geometry flavor of gauge theory is lost after this discretization. On the other hand, there exist other discretizations of physical theories where the geometry flavors of the original theories are preserved, for example, Robin Forman's discretization [6] of Witten-Morse theory [7]. Inspired by Forman's work, here we present a formulation of gauge theory on a general graph. The paper is organized as follows:

In Section 2, we review the standard definitions of a differential form and an exterior derivative on a graph. We also introduce the notion of a (Hodge) Laplacian and derive an explicit formula for it. In Section 3, we extend the notion of an exterior derivative d to an exterior covariant derivative d_A . We define the curvature 2-form F through the standard formula $F = d_A \circ d_A$ and show that F satisfies the second Bianchi identity $d_A F = 0$. In Section 4, we extend the notion of a Laplacian Δ to a connection Laplacian Δ_A and prove an analogue of the Weitzenböck formula in this discrete setting. In Section 5, we provide the definition of a Yang-Mills functional and derive its Euler-Lagrange equations. Unlike the continuous case, this functional is bounded from above. We prove that the maximum of the functional can always be achieved if the graph satisfies some topological conditions. We also compute a few examples for an abelian gauge group. In Section 6, we generalize the discrete Yang-Mills functional by introducing a Higgs scalar field. Section 7 discusses a few possible future research directions of the present work.

The research for this paper was performed in 2020. Later, the paper by [8] by Ginestra Bianconi appeared where she also presented an approach to gauge theory on graphs. More recently, she has

further developed her approach. It turns out, however, that her and my approach are different from each other.

2 Differential forms

Definition 2.1. A (finite simple) graph is a pair $\Gamma = (V, E)$, where $V = \{1, \dots, n\}$ is a finite set, $E \subseteq \binom{V}{2}$ is a set of 2-subsets of V . The elements of V are called vertices, and the elements of E are called edges.

Once the set E is specified, we automatically get sets of cliques of higher order.

Definition 2.2. The set of k -cliques $K_k(\Gamma) \subseteq \binom{V}{k}$ is defined by

$$\{i_1, \dots, i_k\} \in K_k(\Gamma) \iff \{i_p, i_q\} \in E, \quad \text{for all } 1 \leq p < q \leq k.$$

In other words, a k -clique is just a complete k -subgraph of Γ .

A maximum clique is a clique of maximum possible size in Γ . Its size is known as the clique number of Γ , denoted by $\omega(\Gamma)$. Let $K(\Gamma)$ be the set of all cliques of a graph Γ , i.e.,

$$K(\Gamma) = \cup_{k=1}^{\omega(\Gamma)} K_k(\Gamma).$$

By construction, $K(\Gamma)$ is an abstract simplicial complex and is called as the clique complex of Γ .

Definition 2.3. A k -form is a map $f : V \times \dots \times V \rightarrow \mathbb{R}$ such that

$$f(i_{\sigma(0)}, \dots, i_{\sigma(k)}) = (-1)^{|\sigma|} f(i_0, \dots, i_k)$$

for $\{i_0, \dots, i_k\} \in K_{k+1}(\Gamma)$, where $\sigma \in S_{k+1}$ and $|\sigma|$ is the sign of σ , and

$$f(i_0, \dots, i_k) = 0$$

for $\{i_0, \dots, i_k\} \notin K_{k+1}(\Gamma)$.

We denote the space of k -forms by $\Omega^k(\Gamma)$, and put

$$\Omega(\Gamma) = \oplus_{k=0}^{\omega(\Gamma)-1} \Omega^k(\Gamma).$$

Definition 2.4. The k -th exterior derivative $d_k : \Omega^k(\Gamma) \rightarrow \Omega^{k+1}(\Gamma)$ is defined by

$$(d_k f)(i_0, \dots, i_{k+1}) = \sum_{j=0}^{k+1} (-1)^j f(i_0, \dots, \hat{i}_j, \dots, i_{k+1}).$$

By definition, $d_{k+1} \circ d_k = 0$ for all k . The k -th cohomology group of Γ can be then defined as the quotient $H^k(\Gamma) = \ker(d_k) / \text{im}(d_{k-1})$. We use d to denote the exterior derivative on $\Omega(\Gamma)$.

We can put an inner product $\langle \cdot, \cdot \rangle$ on $\Omega(\Gamma)$ by setting

$$\langle f, g \rangle = \sum_{i_0 < \dots < i_k} f(i_0, \dots, i_k) g(i_0, \dots, i_k),$$

where f and g are two k -forms, $k = 0, 1, \dots, \omega(\Gamma) - 1$. $\Omega(\Gamma)$ becomes a finite dimensional Hilbert space. We denote the adjoint of the exterior derivative d by d^* .

Remark 2.1. It is not hard to work out an explicit formula for d^* . For a k -form f , we have

$$d^*f(i_0, \dots, i_{k-1}) = \sum_l f(l, i_0, \dots, i_{k-1}).$$

Definition 2.5. The (Hodge) Laplacian on $\Omega(\Gamma)$ is defined as

$$\Delta = dd^* + d^*d.$$

A k -form $\varphi \in \Omega^k(\Gamma)$ is said to be harmonic if

$$\Delta\varphi = 0.$$

Since Δ preserve the degree of a differential form, the space of harmonic forms $\ker \Delta$ is canonically a graded vector space.

Theorem 2.1. $H(\Gamma) \cong \ker(\Delta)$ as graded vector spaces.

Proof. The proof can be found in, for example, [9, 10]. □

One can work out an explicit expression for Δ . On one hand,

$$\begin{aligned} d^*df(i_0, \dots, i_k) &= \sum_l df(l, i_0, \dots, i_k) \\ &= \sum_l \left(f(i_0, \dots, i_k) - \sum_{j=0}^k (-1)^j f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right), \end{aligned}$$

where the summation outside the bracket is taken over vertices l which together with i_0, \dots, i_k form a $(k+2)$ -cliques. On the other hand,

$$\begin{aligned} dd^*f(i_0, \dots, i_k) &= d^*f(i_1, \dots, i_k) + \sum_{j=1}^k (-1)^j d^*f(i_0, \dots, \widehat{i}_j, \dots, i_k) \\ &= \sum_l f(l, i_1, \dots, i_k) + \sum_{j=1}^k (-1)^j \sum_l f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \\ &= \sum_{j=0}^k (-1)^j \sum_l f(l, i_0, \dots, \widehat{i}_j, \dots, i_k), \end{aligned}$$

where the second summation in the third line is taken over vertices l which together with $i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k$ form a $(k+1)$ -cliques. It follows that

$$\Delta f(i_0, \dots, i_k) = \deg(i_0, \dots, i_k) f(i_0, \dots, i_k) + \sum_{j=0}^k (-1)^j \sum_l f(l, i_0, \dots, \widehat{i}_j, \dots, i_k),$$

where the second summation is taken over vertices l which are non-adjacent to i_j , $\deg(i_0, \dots, i_k)$ is the number of $(k+2)$ -cliques containing i_0, \dots, i_k , $k > 0$. For $k = 0$, we have

$$\Delta f(i) = d^*df(i) = \deg(i)f(i) - \sum_l f(l),$$

where the summation is taken over vertices l which are adjacent to i .

3 Exterior covariant derivatives

Definition 3.1. Let G be a Lie group. Let $\Gamma = (V, E)$ be a graph. A principal G -bundle over Γ is a pair $P = (V \times G, E)$. Let $(V \times V)_E = \{(i, j) \in V \times V \mid \{i, j\} \in E\}$. A connection on P is a map $A : (V \times V)_E \rightarrow G$ such that $A(j, i) = A(i, j)^{-1}$. A gauge transformation of P is a map $g : V \rightarrow G$.

Let \mathcal{A} be the space of connections on P . Let \mathcal{G} be the set of gauge transformations on P . Note that \mathcal{G} is a group and has a canonical (left) action on \mathcal{A} defined by

$$(gA)(i, j) = g(i)A(i, j)g(j)^{-1}$$

for $g \in \mathcal{G}$ and $A \in \mathcal{A}$.

Definition 3.2. Let W be a vector space. We call $W_\Gamma = (V \times W, E)$ an associated bundle to P if W carries a representation of G through ρ . A section of W_Γ is a map $s : V \rightarrow W$.

Let s be a section of W_Γ , a gauge transformation g acts on s via

$$(gs)(i) = \rho(g(i))s(i).$$

Definition 3.3. A W -valued k -form on Γ is an element in $\Omega^k(\Gamma) \otimes W$.

We denote the space of W -valued k -forms by $\Omega^k(\Gamma, W)$, and put $\Omega(\Gamma, W) = \bigoplus_{k=0}^{\omega(\Gamma)-1} \Omega^k(\Gamma, W)$. A gauge transformation g acts on a W -valued k -form f via

$$(gf)(i_0, \dots, i_k) = \rho(g(i_0))f(i_0, \dots, i_k),$$

where $i_0 < \dots < i_k$.

Definition 3.4. An $\text{End}(W)$ -valued k -form on Γ is a map $\varphi : V \times \dots \times V \rightarrow \text{End}(W)$ such that $\varphi(i_0, \dots, i_k) = 0$ for all $\{i_0, \dots, i_k\} \notin K_{k+1}(\Gamma)$.

Remark 3.1. An $\text{End}(W)$ -valued k -form φ is not necessarily alternating. Therefore, $d(d\varphi) \neq 0$ in general.

We denote the space of $\text{End}(W)$ -valued k -forms by $\Omega^k(\Gamma, \text{End}(W))$, and put $\Omega(\Gamma, \text{End}(W)) = \bigoplus_{k=0}^{\omega(\Gamma)-1} \Omega^k(\Gamma, \text{End}(W))$. Let φ_1 be an $\text{End}(W)$ -valued p -form and φ_2 be an $\text{End}(W)$ -valued q -form, their product $\varphi_1\varphi_2$ is a $\text{End}(W)$ -valued $(p+q)$ -form defined by

$$\varphi_1\varphi_2(i_0, \dots, i_{p+q}) = \varphi_1(i_0, \dots, i_p)\varphi_2(i_{p+1}, \dots, i_{p+q}).$$

Definition 3.5. Let φ be an $\text{End}(W)$ -valued p -form and f be a W -valued q -form. The action of φ on f is defined by setting

$$(\varphi f)(i_0, \dots, i_{p+q}) = \varphi(i_0, \dots, i_p)f(i_{p+1}, \dots, i_{p+q})$$

and

$$(\varphi f)(i_{\sigma(0)}, \dots, i_{\sigma(p+q)}) = (-1)^{|\sigma|}(\varphi f)(i_0, \dots, i_{p+q})$$

for $i_0 < \dots < i_p < \dots < i_{p+q}$, $\sigma \in S_{p+q+1}$.

A gauge transformation g should act on φ via

$$(g\varphi)(i_0, \dots, i_k) = \rho(g(i_0))\varphi(i_0, \dots, i_k)\rho(g(i_k))^{-1}.$$

From this point of view, we can interpret the connection A as an $\text{End}(W)$ -valued 1-form on Γ .

Definition 3.6. Let s be a W -valued 0-form, i.e., a section of W_Γ . The covariant derivative ∇ of W_Γ is defined by setting

$$(\nabla s)(i, j) = \rho(A(i, j))s(j) - s(i),$$

and

$$(\nabla s)(j, i) = -(\nabla s)(i, j)$$

for $i < j$, $\{i, j\} \in E$, where s is a section of W_Γ .

Remark 3.2. Since $s(j)$ and $s(i)$ live on different vertices, to compare them, A is used to “transport” the vector $s(j)$ from j to i , hence the name “connection”.

We have

$$\begin{aligned} \rho((gA)(i, j))(gs)(j) - (gs)(i) &= \rho(g(i))\rho(A(i, j))\rho(g(j))^{-1}\rho(g(j))s(j) - \rho(g(i))s(i) \\ &= \rho(g(i))(\rho(A(i, j))s(j) - s(i)) \\ &= (g\nabla s)(i, j). \end{aligned}$$

In other words, ∇ is compatible with gauge transformations.

Definition 3.7. The (exterior) covariant derivative $d_A : \Omega^k(\Gamma, W) \rightarrow \Omega^{k+1}(\Gamma, W)$ is defined by

$$(d_A f)(i_0, \dots, i_{k+1}) = \rho(A(i_0, i_1))f(i_1, \dots, i_{k+1}) + \sum_{j=1}^{k+1} (-1)^j f(i_0, \dots, \widehat{i_j}, \dots, i_{k+1})$$

and

$$(d_A f)(i_{\sigma(0)}, \dots, i_{\sigma(k+1)}) = (-1)^{|\sigma|} (d_A f)(i_0, \dots, i_{k+1})$$

for $i_0 < \dots < i_{k+1}$, where $\sigma \in S_{k+2}$.

Proposition 3.1. $d_A \circ d_A = F$, where F is an $\text{End}(W)$ -valued 2-form defined by

$$F(i, j, k) = \rho(A(i, j)A(j, k)) - \rho(A(i, k))$$

We call F as the curvature 2-form of A .

Proof.

$$\begin{aligned}
d_A(d_A f)(i_0, \dots, i_{k+2}) &= \rho(A(i_0, i_1))(d_A f)(i_1, \dots, i_{k+2}) + \sum_{j_1=1}^{k+2} (-1)^{j_1} (d_A f)(i_0, \dots, \widehat{i_{j_1}}, \dots, i_{k+2}) \\
&= \rho(A(i_0, i_1)) \left(\rho(A(i_1, i_2)) f(i_2, \dots, i_{k+2}) + \sum_{j_2=2}^{k+2} (-1)^{j_2-1} f(i_1, \dots, \widehat{i_{j_2}}, \dots, i_{k+2}) \right) - \\
&\quad (d_A f)(i_0, i_2, \dots, i_{k+2}) + \sum_{j_1=2}^{k+2} (-1)^{j_1} (d_A f)(i_0, \dots, \widehat{i_{j_1}}, \dots, i_{k+2}) \\
&= \rho(A(i_0, i_1)) \left(\rho(A(i_1, i_2)) f(i_2, \dots, i_{k+2}) + \sum_{j_2=2}^{k+2} (-1)^{j_2-1} f(i_1, \dots, \widehat{i_{j_2}}, \dots, i_{k+2}) \right) - \\
&\quad \left(\rho(A(i_0, i_2)) f(i_2, \dots, i_{k+2}) + \sum_{j_3=2}^{k+2} (-1)^{j_3-1} f(i_0, \widehat{i_1}, \dots, \widehat{i_{j_3}}, \dots, i_{k+2}) \right) + \\
&\quad \sum_{j_1=2}^{k+2} (-1)^{j_1} \left(\rho(A(i_0, i_1)) f(i_1, \dots, \widehat{i_{j_1}}, \dots, i_{k+2}) + \sum_{j_4=1}^{k+2} (-1)^{j_4} f(i_0, \dots, \widehat{i_{j_4}}, \dots, \widehat{i_{j_1}}, \dots, i_{k+2}) \right) \\
&= (\rho(A(i_0, i_1)A(i_1, i_2)) - \rho(A(i_0, i_2))) f(i_2, \dots, i_{k+2}) - \\
&\quad \sum_{j_3=2}^{k+2} (-1)^{j_3-1} f(i_0, \widehat{i_1}, \dots, \widehat{i_{j_3}}, \dots, i_{k+2}) + \sum_{j_4=1, j_1=2}^{k+2} (-1)^{j_1+j_4} f(i_0, \dots, \widehat{i_{j_4}}, \dots, \widehat{i_{j_1}}, \dots, i_{k+2}) \\
&= F(i_0, i_1, i_2) f(i_2, \dots, i_{k+2}) - \\
&\quad \sum_{j_3=2}^{k+2} (-1)^{j_3-1} f(i_0, \widehat{i_1}, \dots, \widehat{i_{j_3}}, \dots, i_{k+2}) + \sum_{j_1=2}^{k+2} (-1)^{j_1+1} f(i_0, \widehat{i_1}, \dots, \widehat{i_{j_1}}, \dots, i_{k+2}) \\
&= (Ff)(i_0, \dots, i_{k+2}).
\end{aligned}$$

To pass to the second last equality, we use the alternating property of f . \square

Remark 3.3. Let $\bar{A} = A - 1$. We can write $d_A = d + \bar{A}$. For a trivial connection, $\bar{A} = 0$ and d_A reduces to the exterior derivative d . Moreover, it is not hard to show that $F = d\bar{A} + \bar{A}\bar{A}$.

Remark 3.4. Let G be an abelian Lie group, for example, $U(1)$. Let P be a trivial principal G -bundle over a n -manifold M , $n \geq 2$. Let A be a connection 1-form and F be its corresponding curvature 2-form, $F = dA$, applying Stokes' theorem, we have

$$\int_{\gamma} A = \int_{\sigma} F,$$

where σ is a homology cycle of dimension 2 on M and $\gamma = \partial\sigma$. From this point of view, another reasonable definition of the curvature 2-form on a graph is

$$\tilde{F}(i, j, k) = A(i, j)A(j, k)A(k, i).$$

For later use, we denote $\rho(\tilde{F})$ by \tilde{F}_ρ .

Let s be a section of W_Γ . Let φ be an $\text{End}(W)$ -valued 0-form. We have

$$\begin{aligned} (\nabla(\varphi s) - \varphi(\nabla s))(i, j) &= \rho(A(i, j))\varphi(j)s(j) - \varphi(i)s(i) - \varphi(i)(\rho(A(i, j))s(j) - s(i)) \\ &= (\rho(A(i, j))\varphi(j) - \varphi(i)\rho(A(i, j)))s(j). \end{aligned}$$

Definition 3.8. The covariant derivative $\nabla : \text{End}(W_\Gamma) \rightarrow \Omega^1(\Gamma, \text{End}(W))$ is defined by

$$(\nabla\varphi)(i, j) = \rho(A(i, j))\varphi(j) - \varphi(i)\rho(A(i, j))$$

for $\{i, j\} \in E$.

By definition, ∇ satisfies Leibniz's rule. It is also easy to check that it is compatible with gauge transformations.

Definition 3.9. The (exterior) covariant derivative $d_A : \Omega^k(\Gamma, \text{End}(W)) \rightarrow \Omega^{k+1}(\Gamma, \text{End}(W))$ is defined by

$$\begin{aligned} (d_A\varphi)(i_0, \dots, i_{k+1}) &= \rho(A(i_0, i_1))\varphi(i_1, \dots, i_{k+1}) + \sum_{j=1}^k (-1)^j \varphi(i_0, \dots, \widehat{i_j}, \dots, i_{k+1}) \\ &\quad + (-1)^{k+1} \varphi(i_0, \dots, i_k) \rho(A(i_k, i_{k+1})). \end{aligned}$$

Proposition 3.2. Let f be a W -valued q -form. Let φ be a $\text{End}(W)$ -valued p -form. We have

$$d_A(\varphi f) = (d_A\varphi)f + (-1)^p \varphi(d_A f).$$

Proof. This follows from direct computations.

$$\begin{aligned} (d_A\varphi)f(i_0, \dots, i_{p+q+1}) + (-1)^p \varphi(d_A f)(i_0, \dots, i_{p+q+1}) &= \left(\rho(A(i_0, i_1))\varphi(i_1, \dots, i_{p+1}) + \right. \\ &\quad \left. \sum_{j=1}^{p+1} (-1)^j \varphi(i_1, \dots, \widehat{i_j}, \dots, i_{p+1}) + (-1)^{p+1} \varphi(i_0, \dots, i_p) \rho(A(i_p, i_{p+1})) \right) f(i_{p+1}, \dots, i_{p+q+1}) + \\ &\quad (-1)^p \varphi(i_0, \dots, i_p) \left(\rho(A(i_p, i_{p+1}))f(i_{p+1}, \dots, i_{p+q+1}) + \sum_{j=p+1}^{p+q+1} (-1)^{j-p} f(i_{p+1}, \dots, \widehat{i_j}, \dots, i_{p+q+1}) \right) \\ &= \rho(A(i_0, i_1))(\varphi f)(i_1, \dots, i_{p+q+1}) + \sum_{j=1}^{p+q+1} (-1)^j (\varphi f)(i_1, \dots, \widehat{i_j}, \dots, i_{p+q+1}) \\ &= d_A(\varphi f)(i_0, \dots, i_{p+q+1}) \end{aligned}$$

for $i_0 < \dots < i_{p+q+1}$. □

Proposition 3.3 (Second Bianchi identity). $d_A F = 0$.

Proof. This follows from the observation

$$\begin{aligned} 0 &= (d_A \circ d_A \circ d_A)(\cdot) - (d_A \circ d_A \circ d_A)(\cdot) \\ &= d_A(F(\cdot)) - F(d_A(\cdot)) \\ &= (d_A F)(\cdot), \end{aligned}$$

where we use Propositions 3.1 and 3.2. □

4 Weitzenböck Formula

Let W be a vector space endowed with an (hermitian) inner product (\cdot, \cdot) . Let ρ be a representation of G on W that preserves (\cdot, \cdot) . We put a gauge invariant inner product $\langle \cdot, \cdot \rangle$ on $\Omega(\Gamma, W_\Gamma)$ by setting

$$\langle f_1, f_2 \rangle = \sum_{i_0 < \dots < i_k} (f_1(i_0, \dots, i_k), f_2(i_0, \dots, i_k)),$$

where f_1, f_2 are two W -valued k -forms. It is not hard to show that the adjoint d_A^* of d_A with respect to $\langle \cdot, \cdot \rangle$ takes the form

$$d_A^* f(i_0, \dots, i_{k-1}) = \sum_{l < i_0} \rho(A(i_0, l)) f(l, i_0, \dots, i_{k-1}) + \sum_{l > i_0} f(l, i_0, \dots, i_{k-1}).$$

Definition 4.1. The connection Laplacian on $\Omega(\Gamma, W_\Gamma)$ is defined as

$$\Delta_A = d_A d_A^* + d_A^* d_A.$$

Let $\{i_0, \dots, i_k\}$ be a $(k+1)$ -clique, $i_0 < \dots < i_k$. We have

$$\begin{aligned} d_A d_A^* f(i_0, \dots, i_k) &= \rho(A(i_0, i_1)) d_A^* f(i_1, \dots, i_k) + \sum_{j=1}^k (-1)^j d_A^* f(i_0, \dots, \widehat{i}_j, \dots, i_k) \\ &= \rho(A(i_0, i_1)) \left(\sum_{l < i_1} \rho(A(i_1, l)) f(l, i_1, \dots, i_k) + \sum_{l > i_1} f(l, i_1, \dots, i_k) \right) + \\ &\quad \sum_{j=1}^k (-1)^j \left(\sum_{l < i_0} \rho(A(i_0, l)) f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) + \sum_{l > i_0} f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right) \\ &= \sum_{l < i_1} \rho(A(i_0, i_1) A(i_1, l)) f(l, i_1, \dots, i_k) + \sum_{l > i_1} \rho(A(i_0, i_1)) f(l, i_1, \dots, i_k) + \\ &\quad \sum_{j=1}^k (-1)^j \left(\sum_{l < i_0} \rho(A(i_0, l)) f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) + \sum_{l > i_0} f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right), \end{aligned}$$

where the summations inside the bracket of the last line are taken over vertices l which together with $i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k$ form a $(k+1)$ -cliques, $j = 1, \dots, k$, and the summations in the second last line are taken over vertices l which together with i_1, \dots, i_k form a $(k+1)$ -cliques.

We also have

$$\begin{aligned}
d_A^* d_A f(i_0, \dots, i_k) &= \sum_{l < i_0} \rho(A(i_0, l)) d_A f(l, i_0, \dots, i_k) + \sum_{l > i_0} d_A f(l, i_0, \dots, i_k) \\
&= \sum_{l < i_0} \rho(A(i_0, l)) \left(\rho(A(l, i_0)) f(i_0, \dots, i_k) - \sum_{j=0}^k (-1)^j f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right) + \\
&\quad \sum_{i_0 < l < i_1} \left(f(i_0, \dots, i_k) - \rho(A(i_0, l)) f(l, i_1, \dots, i_k) - \sum_{j=1}^k (-1)^j f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right) + \\
&\quad \sum_{l > i_1} \left(f(i_0, \dots, i_k) - \rho(A(i_0, i_1)) f(l, i_1, \dots, i_k) - \sum_{j=1}^k (-1)^j f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right) \\
&= \deg(i_0, \dots, i_k) f(i_0, \dots, i_k) - \sum_{l < i_1} \rho(A(i_0, l)) f(l, i_1, \dots, i_k) - \sum_{l > i_1} \rho(A(i_0, i_1)) f(l, i_1, \dots, i_k) \\
&\quad - \sum_{j=1}^k (-1)^j \left(\sum_{l < i_0} \rho(A(i_0, l)) f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) + \sum_{l > i_0} f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right),
\end{aligned}$$

where the summations in the last two lines are taken over vertices l which together with i_0, \dots, i_k form a $(k+2)$ -cliques.

Proposition 4.1 (Weitzenböck formula). *For $k > 0$ and $i_0 < \dots < i_k$, we have*

$$\Delta_A f(i_0, \dots, i_k) = \Delta' f(i_0, \dots, i_k) + \sum_{l < i_1} F(i_0, i_1, l) f(l, i_1, \dots, i_k), \quad (4.1)$$

where the summation is taken over vertices l which together with i_0, \dots, i_k form a $(k+2)$ -cliques. Δ' can be viewed as a gauged version of the Hodge Laplace operator Δ , it takes the form

$$\begin{aligned}
\Delta' f(i_0, \dots, i_k) &= \deg(i_0, \dots, i_k) f(i_0, \dots, i_k) \\
&\quad + \sum_{l < i_1} \rho(A(i_0, i_1) A(i_1, l)) f(l, i_1, \dots, i_k) + \sum_{l > i_1} \rho(A(i_0, i_1)) f(l, i_1, \dots, i_k) \\
&\quad + \sum_{j=1}^k (-1)^j \left(\sum_{l < i_0} \rho(A(i_0, l)) f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) + \sum_{l > i_0} f(l, i_0, \dots, \widehat{i}_j, \dots, i_k) \right),
\end{aligned} \quad (4.2)$$

$$(4.3)$$

where the summations in the second line are taken over vertices l which are non-adjacent to i_0 and the summations inside the bracket of the third line are taken over vertices l which are non-adjacent to i_j .

Remark 4.1. In (4.2), fix i_0 and i_1 , we can apply a gauge transformation g such that $A(i_0, l) = 1$ when l is adjacent to i_0 and $A(i_0, i_1) = A(i_1, l) = 1$ when l is non-adjacent to i_0 . In such a gauge, we have $\Delta' = \Delta$ at the k -cliques containing the edge $\{i_0, i_1\}$. Note that, however, the curvature term in (4.1) can not be eliminated by a gauge transformation.

Remark 4.2. For $k = 0$, we have

$$\Delta_A f(i) = d_A^* d_A f(i) = \deg(i) f(i) - \sum_l \rho(A(i, l)) f(l), \quad (4.4)$$

where the summation is taken over vertices l which are adjacent to i . We can apply a gauge transformation such that $\Delta_A = \Delta$ at $i \in V$. A similar formula like (4.4) is given in [11] and is used there to prove a generalized version of the matrix-tree theorem in graph theory. A natural question is then if the Weitzenböck formula (4.1) can be applied to study problems in graph theory.

5 Yang-Mills functional

Let W be a vector space endowed with a (hermitian) inner product (\cdot, \cdot) . Let ρ be a representation of G on W that preserves (\cdot, \cdot) . For an endomorphism φ , we denote its adjoint w.r.t (\cdot, \cdot) as φ^\dagger . The gauge invariant inner product $\langle \cdot, \cdot \rangle$ on $\Omega(\Gamma, \text{End}(W))$ is defined by

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{(k+1)!} \sum_{i_0, \dots, i_k} \text{Tr}(\varphi_1(i_0, \dots, i_k) \varphi_2(i_0, \dots, i_k)^\dagger),$$

where φ_1, φ_2 are two $\text{End}(W)$ -valued k -forms. It is not hard to check that $\langle \cdot, \cdot \rangle$ is (conjugate) symmetric and non-degenerate, hence well-defined.

Definition 5.1. Let Γ be a graph with $\omega(\Gamma) \geq 3$. The Yang-Mills functional on \mathcal{A} is defined as

$$YM_\rho(A) = \frac{1}{2} \langle F, F \rangle. \quad (5.1)$$

Remark 5.1. Note that

$$F(i_0, i_1, i_2) F(i_1, i_1, i_2)^\dagger = 2 - \tilde{F}_\rho(i_0, i_1, i_2) - \tilde{F}_\rho(i_0, i_1, i_2)^\dagger$$

and

$$\text{Tr}(\tilde{F}_\rho(i_0, i_1, i_2)) = \text{Tr}(\tilde{F}_\rho(i_{\sigma(0)}, i_{\sigma(1)}, i_{\sigma(2)})^{\text{sgn}(\sigma)}),$$

where $\sigma \in S_3$, and

$$\tilde{F}_\rho(i_0, i_1, i_2)^{-1} = \tilde{F}_\rho(i_0, i_1, i_2)^\dagger.$$

We can also write the Yang-Mills functional as

$$YM_\rho(A) = \frac{1}{2} \sum_{i < j < k} \text{Tr}(2 - \tilde{F}_\rho(i, j, k) - \tilde{F}_\rho(i, j, k)^\dagger). \quad (5.2)$$

This form of the functional can be found in a lot of literature on the lattice gauge theory, it is referred to as the ‘‘Wilson action’’. Here we managed to define it on a general graph. It corresponds to the leading order term in the lattice approximation to the continuous Yang-Mills functional. See, for example, [12] for a more detailed explanation.

Proposition 5.1. YM_ρ does not depend on the order of the vertices.

Proof. This follows directly from the cyclic property of the trace and formula (5.2). \square

Since we have obtained the functional, it is very natural to ask what its Euler-Lagrange equations are. Note that

$$\delta F(i, j, k) = \delta A(i, j)A(j, k) - \delta A(j, k) + A(i, j)\delta A(j, k) = (d_A \delta A)(i, j, k),$$

where δA is the variation of A , i.e., an $\text{End}(W)$ -valued 1-form, and δF is the variation of F , i.e., an $\text{End}(W)$ -valued 2-form. We then have

$$\delta Y M_\rho(A) = \langle d_A(\delta A), F \rangle = \langle \delta A, d_A^* F \rangle,$$

where d_A^* is the adjoint of d_A with respect to $\langle \cdot, \cdot \rangle$. Note that δA cannot be an arbitrary $\text{End}(W)$ -valued 1-form. Since $A(j, i) = A(i, j)^{-1}$, we must have

$$\delta A(j, i) = -A(j, i)\delta A(i, j)A(j, i).$$

It follows that

$$\begin{aligned} \langle \delta A, d_A^* F \rangle &= \sum_{i, j} \text{Tr}(\delta A(i, j)(d_A^* F(i, j))^\dagger) \\ &= \sum_{i < j} (\text{Tr}(\delta A(i, j)(d_A^* F(i, j))^\dagger) + \text{Tr}(\delta A(j, i)(d_A^* F(j, i))^\dagger)) \\ &= \sum_{i < j} \text{Tr}(\delta A(i, j)((d_A^* F(i, j))^\dagger - A(j, i)(d_A^* F(j, i))^\dagger A(j, i))). \end{aligned}$$

Proposition 5.2. *The Euler-Lagrange equations of the Yang-Mills functional is*

$$d_A^* F(i, j) = A(i, j)d_A^* F(j, i)A(i, j) \tag{5.3}$$

for all $\{i, j\} \in E$. We also refer to (5.3) as the Yang-Mills equations.

It is not hard to work out an explicit formula for d_A^* , which is

$$\begin{aligned} d_A^* \varphi(i_0, \dots, i_{k-1}) &= \frac{1}{k+1} \sum_l \left(\rho(A(i_0, l))\varphi(l, i_0, \dots, i_{k-1}) \right. \\ &\quad \left. + \sum_{j=1}^{k-1} (-1)^j \varphi(i_0, \dots, i_{j-1}, l, i_j, \dots, i_{k-1}) + (-1)^k \varphi(i_0, \dots, i_{k-1}, l)\rho(A(l, i_{k-1})) \right). \end{aligned}$$

With this formula in hands, it is not hard to show that

$$d_A^* F(i_0, i_1) = - \sum_l F(i_0, l, i_1).$$

Thus, the Yang-Mills equations (5.3) become

$$\sum_l F(i, l, j) = A(i, j) \sum_l F(j, l, i)A(i, j).$$

Or equivalently,

$$\sum_l \tilde{F}_\rho(i, l, j) = \sum_l \tilde{F}_\rho(i, j, l).$$

Since the functional (5.1) is gauge invariant, we can choose a proper gauge to simplify the calculations. Let Γ be a connected graph. Let $T = (V_t, E_t)$ be a spanning tree of Γ , i.e., a subgraph that is a tree which includes all of the vertices of Γ . One can then consider the following procedure.

1. Pick a vertex $a \in V_t$, let $g(a) = 1$.
2. For any vertex b adjacent to a , choose $g(b) = A(a, b)$.
3. For any vertex c adjacent to b , choose $g(c) = A(c, b)$. Since T is a tree, c is distinct from a and its adjacent vertices b , $g(c)$ is well defined.
4. Repeat the Step 3 until all vertices of T are exhausted.

We then have $A(i, j) = 1$ for all $\{i, j\} \in E_t$. In particular, we have $\mathcal{A}/\mathcal{G} = \text{pt}$ when Γ is a tree.

Definition 5.2. The above procedure is called the spanning tree gauge fixing.

Example 5.1. Let $G = \text{U}(1)$. We choose the spanning tree T as in Figure 5.1. There is only one

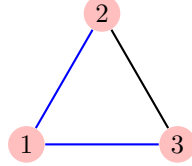


Figure 5.1: The complete graph K_3 .

degree of freedom, namely, $A(2, 3)$. The solutions to (5.3) are

$$A(2, 3) = \pm 1.$$

The solution $A(2, 3) = 1$ is the (global) minimizer of the Yang-Mills functional, and the solution $A(2, 3) = -1$ is the (global) maximizer of the Yang-Mills functional.

Note that the Yang-Mills is bounded from both below and above. More precisely, we have $0 \leq YM_\rho(A) \leq 2\omega_3$, where ω_3 is the number of 3-cliques in Γ . The global minimum can be easily achieved by setting the connection A to be the identity.

Proposition 5.3. For a (connected) graph Γ with $\omega(\Gamma) \geq 3$ and trivial $\pi_1(K(\Gamma))$, the global maximum of the Yang-Mills functional on Γ can always be achieved.

Proof. Let T be a spanning tree of Γ . For any two vertices i and j of T , we set $A(i, j) = (-1)^{d_{ij}-1}$ if $\{i, j\}$ is an edge of Γ , where d_{ij} is the length of the path connecting i and j in T . In particular, $A(i, j) = 1$ if $\{i, j\}$ is an edge of T , i.e., we are in the spanning tree gauge. Now fix an edge $\{i, j\}$ of Γ which is not in T . $\{i, j\}$ together with the path connecting i and j in T forms a cycle C in Γ .

Since $\pi_1(K(\Gamma))$ is trivial, C can be partitioned into triangles. For any such triangle $\{i', j', k'\}$, we assume that, without loss of generality, $d_{i'k'} = d_{i'j'} + d_{j'k'}$. It follows that

$$F_\rho(i', j', k') = A(i', j')A(j', k')A(k', i') = (-1)^{d_{i'j'}-1}(-1)^{d_{j'k'}-1}(-1)^{-d_{i'k'}+1} = -1.$$

Therefore, A is a global maximizer of the Yang-Mills functional on Γ . \square

Definition 5.3. We call a solution to the Yang-Mills equations trivial if it is a global minimizer/maximizer of the Yang-Mills functional.

Example 5.2. Let $G = U(1)$. We choose the spanning tree T as in Figure 5.2. There are three

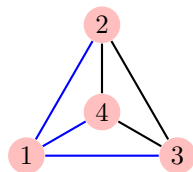


Figure 5.2: The complete graph K_4 .

degrees of freedom, namely, $A(2, 3)$, $A(2, 4)$, $A(3, 4)$. Let $G = U(1)$. The solutions to (5.3) are

$$\begin{aligned} A(2, 3) &= A(2, 4) = A(3, 4) = \pm 1, \\ A(2, 3) &= A(2, 4)^{-1} = -A(3, 4)^{-1} = \exp(i\alpha), \\ A(2, 3) &= -A(2, 4) = A(3, 4) = \exp(i\alpha), \\ A(2, 3) &= -A(2, 4) = -A(3, 4)^{-1} = \exp(i\alpha). \end{aligned}$$

The solution $A(2, 3) = A(2, 4) = A(3, 4) = 1$ is the minimizer of the Yang-Mills functional, and

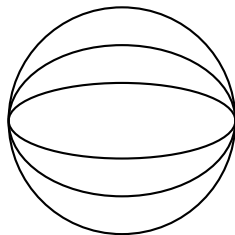


Figure 5.3: The space of nontrivial solutions to the $U(1)$ Yang-Mills equations on K_4 .

the solution $A(2, 3) = A(2, 4) = A(3, 4) = -1$ is the maximizer of the Yang-Mills functional. The space of nontrivial solutions is depicted in Figure 5.3. The two intersection points correspond to the solutions

$$A(2, 3) = -A(2, 4) = A(3, 4) = \pm i.$$

This space has a S_3 -symmetry, which is inherited from the S_3 -symmetry of the pair (K_4, T) .

Remark 5.2. The computation of solutions to the U(1) Yang-Mills equations on the complete graph K_n becomes much more harder for $n \geq 5$. However, we know the space of nontrivial solutions (if nonempty) should have a S_{n-1} -symmetry.

To simplify the computation, a natural idea is to decompose the graph into easily computable pieces. For example, if a (connected) graph Γ can be obtained by connecting two graphs Γ_1 and Γ_2 connected by a path (see Figure 5.4), then it is easy to show that the space of solutions to the Yang-Mills equations on Γ is the Cartesian product of the spaces of solutions to the Yang-Mills equations on Γ_1 and Γ_2 .

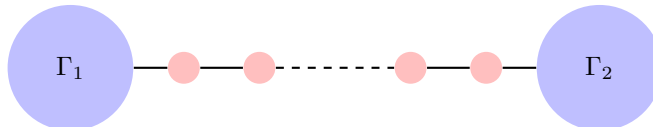


Figure 5.4: A graph Γ obtained by two graphs Γ_1 and Γ_2 connected by a path.

6 Yang-Mills-Higgs functional

Definition 6.1. Let Γ be a graph with $\omega(\Gamma) \geq 3$. The Yang-Mills-Higgs functional is defined as

$$YMH_\rho(A, \phi) = \frac{1}{2} \langle F, F \rangle + \frac{1}{2} \langle d_A \phi, d_A \phi \rangle + V(\phi), \quad (6.1)$$

where ϕ is a section of W_Γ and V is a non-negative function on the space of sections of W_Γ .

Let's derive the Euler-Lagrange equations to (6.1). For simplicity, we set $V(\phi) = 0$. The variation of the second term of (6.1) with respect to A is

$$\begin{aligned} \langle \delta A \phi, d_A \phi \rangle &= \sum_{i < j} \text{Tr}(\delta A \phi(i, j) (d_A \phi(i, j))^\dagger) \\ &= \sum_{i < j} \text{Tr}(\delta A(i, j) \phi(j) (A(i, j) \phi(j) - \phi(i))^\dagger) \\ &= \sum_{i < j} \text{Tr}(\delta A(i, j) (\phi(j) \phi(j)^\dagger A(j, i) - \phi(j) \phi(i)^\dagger)). \end{aligned}$$

Recall that

$$\langle \delta A, d_A^* F \rangle = \sum_{i < j} \text{Tr}(\delta A(i, j) ((d_A^* F(i, j))^\dagger - A(j, i) (d_A^* F(j, i))^\dagger A(j, i))).$$

We arrive at the Euler-Lagrange equations for the connection A , which are

$$d_A^* F(i, j) - A(i, j) (d_A^* F(j, i)) A(i, j) = \phi(i) \phi(j)^\dagger - A(i, j) \phi(j) \phi(j)^\dagger, \quad (6.2)$$

or equivalently,

$$\sum_l \tilde{F}_\rho(i, l, j) - \sum_l \tilde{F}_\rho(i, j, l) = A(i, j) \phi(j) \phi(j)^\dagger A(j, i) - \phi(i) \phi(j)^\dagger A(j, i). \quad (6.3)$$

The variation of the second term of (6.1) with respect to ϕ is

$$\langle d_A \delta \phi, d_A \phi \rangle = \langle \delta \phi, \Delta_A \phi(i) \rangle = 0,$$

where Δ_A is the connection Laplacian. By (4.4), the Euler-Lagrange equations for the scalar field ϕ are

$$\deg(i)\phi(i) - \sum_l A(i, l)\phi(l) = 0. \tag{6.4}$$

7 Conclusions and future directions

In this paper, we have developed a discrete setting for gauge theories which preserves most of the flavor of the original continuous setting. Possible future research directions can be a more detailed study of the solutions to the Yang-Mills(-Higgs) equations, and incorporations of further concepts and theorems (e.g., Wilson’s area law [3]) from lattice gauge theory into this framework.

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