ADDENDUM 1 : CHERN-WEIL CHARACTERISTIC CLASSES

HERE WE WILL SIMPLY RECORD THE PERTINENT DEFINITIONS AND RESULTS AND, AS AN ILLUSTRATION, SKETCH ONE SIMPLE CALCULATION.

A SMOOTH PRINCIPAL BUNDLE CONSISTS OF A SMOOTH (C^∞) MANIFOLD P (BUNDLE SPACE), A SMOOTH MANIFOLD M (BASE SPACE), A SMOOTH MAP π (PROJECTION) OF P ONTO M, A LIE GROUP G (STRUCTURE GROUP) AND A SMOOTH RIGHT ACTION

$$\sigma : P \times G \longrightarrow P$$

$$\sigma(p,g) = p \cdot g = \sigma_p(g) = \sigma_g(p)$$

OF G ON P SUCH THAT THE FOLLOWING ARE SATISFIED :

1. σ PRESERVES FIBERS OF π : $\pi(p \cdot g) = \pi(p)$ $\forall p \in P \; \forall g \in G$

2. (LOCAL TRIVIALITY) FOR EACH $x \in M$ $\exists$ OPEN NEIGHBORHOOD U OF x IN M (TRIVIALIZING NBD) AND A DIFFEOMORPHISM $\Psi : \pi^{-1}(U) \longrightarrow U \times G$ (TRIVIALIZATION) OF THE FORM $\Psi(p) = (\pi(p), \psi(p))$, WHERE $\psi : \pi^{-1}(U) \longrightarrow G$ SATISFIES

$$\psi(p \cdot g) = \psi(p) \cdot g$$

$\forall p \in \pi^{-1}(U) \; \forall g \in G$ (ψ IS EQUIVARIANT WITH RESPECT TO σ AND THE NATURAL ACTION OF G ON $U \times G$).

$$G \hookrightarrow P \xrightarrow{\pi} M$$

(LOCAL) SECTION / GAUGE : $s : U \to \pi^{-1}(U)$, $\pi \circ s = id_U$

$$\text{SECTIONS} \longleftrightarrow \text{TRIVIALIZATIONS}$$

$$s(x) \cdot g \longleftrightarrow (x, g)$$

TRANSITION FUNCTIONS :

$$s_i : U_i \to \pi^{-1}(U_i)$$

$$s_j : U_j \to \pi^{-1}(U_j)$$

$$s_j(x) = s_i(x) \cdot g_{ij}(x)$$

ASSOCIATED FIBER BUNDLES : F SMOOTH MANIFOLD WITH LEFT ACTION OF G

$$(g, \xi) \to g \cdot \xi$$

RIGHT ACTION OF G ON $P \times F$: $(p, \xi) \cdot g = (p \cdot g, g^{-1} \cdot \xi)$

ORBIT SPACE :

$$\begin{array}{c} P \times_G F \\ \downarrow \\ M \end{array}$$

IF $F = V$ IS A VECTOR SPACE AND THE ACTION OF G COMES FROM A REPRESENTATION $\rho : G \to GL(V)$, THEN THE ASSOCIATED BUNDLE $P \times_\rho V$ IS A VECTOR BUNDLE, E.G., ADJOINT BUNDLE

$$\mathrm{ad}\, P = P \times_{\mathrm{ad}} \mathfrak{g}$$

THREE (EQUIVALENT) DEFINITIONS OF A CONNECTION ON $G \hookrightarrow P \xrightarrow{\pi} M$:

ASSUME G IS A MATRIX LIE GROUP WITH LIE ALGEBRA $\mathfrak{g}$ AND $\dim M = n$

1. n - DIMENSIONAL DISTRIBUTION $\quad p \in P \longrightarrow HOR_p(P) \subseteq T_p(P)$ S.T.

   (a) $T_p(P) = HOR_p(P) \oplus VERT_p(P)$, WHERE

   $$VERT_p(P) = T_p(\pi^{-1}(\pi(p)))$$

   (b) $HOR_{p \cdot g}(P) = (\sigma_g)_{*p}(HOR_p(P))$

2. $\mathfrak{g}$ - VALUED 1 - FORM ω ON P S.T.

   (a) $\omega_{p \cdot g}((\sigma_g)_{*p}(v_p)) = g^{-1} \omega_p(v_p) g$

   (b) $\omega_p(\xi^{\#}(p)) = \xi \quad \forall \xi \in \mathfrak{g}$, WHERE

   $$\xi^{\#}(p) = \frac{d}{dt}(p \cdot \exp(t\xi))\Big|_{t=0}$$

3. A TRIVIALIZING COVER $\{(U_j, \Psi_j)\}$ OF M FOR $G \hookrightarrow P \xrightarrow{\pi} M$ AND, FOR EACH j, A $\mathfrak{g}$ - VALUED 1 - FORM $\mathcal{A}_j$ ON U_j S.T. WHENEVER $U_j \cap U_i \neq \phi$

   $$\mathcal{A}_j = g_{ij}^{-1} \mathcal{A}_i \, g_{ij} + g_{ij}^{-1} dg_{ij}$$

RELATIONS : $\quad HOR_p(P) = \ker \omega_p \quad$ AND $\quad \mathcal{A}_j = s_j^{*} \omega$

<u>CURVATURE</u> OF A CONNECTION 1-FORM ω IS THE $\mathfrak{g}$-VALUED 2-FORM Ω ON P DEFINED BY

$$\Omega_p(v_p, w_p) = d\omega_p(v_p^{HOR}, w_p^{HOR})$$

WHERE v_p^{HOR} AND w_p^{HOR} ARE THE PROJECTIONS OF v_p AND w_p ONTO $HOR_p(P)$ IN $T_p(P) = HOR_p(P) \oplus VERT_p(P)$.

<u>CARTAN STRUCTURE EQUATION</u>: $\Omega = d\omega + \omega \wedge \omega$

<u>LOCAL FIELD STRENGTHS</u>: $\mathfrak{F}_j = s_j^* \Omega = d\mathfrak{A}_j + \mathfrak{A}_j \wedge \mathfrak{A}_j$

$$\mathfrak{F}_j = g_{ij}^{-1} \mathfrak{F}_i \, g_{ij}$$

$\Rightarrow$ THE $\mathfrak{F}_j$ PIECE TOGETHER INTO A 2-FORM

$$F_\omega \in \Omega^2(M, ad\,P)$$

ON M WITH VALUES IN THE ADJOINT BUNDLE $ad\,P$. THIS IS ALSO CALLED THE <u>CURVATURE</u> OF ω.

<u>CHERN-WEIL PROCEDURE</u>:

ASSUME NOW THAT G IS A COMPACT MATRIX LIE GROUP WITH LIE ALGEBRA $\mathfrak{g}$ AND CONSIDER THE ALGEBRA

$$\mathbb{C}[\mathfrak{g}]^G$$

OF $\mathrm{ad}\,G$ - INVARIANT, COMPLEX-VALUED POLYNOMIAL FUNCTIONS ON $\mathfrak{g}$.

<u>MORE DETAIL</u>: LET $\{\xi_1, \dots, \xi_n\}$ BE A BASIS FOR $\mathfrak{g}$ AND $\{x^1, \dots, x^n\}$ THE DUAL BASIS FOR $\mathfrak{g}^*$ $(x^a(\xi_b) = \delta^a_b)$. THE SYMMETRIC ALGEBRA $S(\mathfrak{g}^*) = S[x^1, \dots, x^n]$ IS THE ALGEBRA OF POLYNOMIALS WITH REAL COEFFICIENTS IN $x^1, \dots, x^n$ (REAL-VALUED POLYNOMIAL FUNCTIONS ON $\mathfrak{g}$). THERE IS A LEFT ACTION OF G ON $S(\mathfrak{g}^*)$: $\rho \in S(\mathfrak{g}^*)$ AND $g \in G$ GIVE $g \cdot \rho \in S(\mathfrak{g}^*)$ DEFINED BY

$$(g \cdot \rho)(\xi) = \rho(\mathrm{ad}_{g^{-1}} \xi) = \rho(g^{-1} \xi g).$$

$S(\mathfrak{g}^*)^G = \{\rho \in S(\mathfrak{g}^*) : g \cdot \rho = \rho \ \forall g \in G\}$, I.E.,

$$\rho(g^{-1} \xi g) = \rho(\xi).$$

$\mathbb{C}[\mathfrak{g}] = S(\mathfrak{g}^*) \otimes \mathbb{C}$ AND $\mathbb{C}[\mathfrak{g}]^G = S(\mathfrak{g}^*)^G \otimes \mathbb{C}$. EVERY $\rho \in \mathbb{C}^k[\mathfrak{g}]^G$ GIVES RISE, VIA POLARIZATION, TO A G-INVARIANT k-MULTILINEAR FUNCTION, ALSO DENOTED

$$\rho : \mathfrak{g} \times \overset{k}{\cdots} \times \mathfrak{g} \longrightarrow \mathbb{C}$$

CHERN-WEIL IS BASICALLY A MAP

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \Omega^*(P)_{\text{BASIC}}$$

FROM G-INVARIANT POLYNOMIALS ON $\mathfrak{g}$ TO DIFFERENTIAL FORMS ON P THAT ARE <u>BASIC</u>, I.E., <u>G-INVARIANT</u> $(\sigma_g^* \varphi = \varphi \ \forall g \in G)$ AND <u>HORIZONTAL</u> $(\iota_V \varphi = 0 \ \forall$ VERTICAL VECTOR FIELD V ON $P)$

<u>NOTE</u>: BASIC FORMS ON THE PRINCIPAL BUNDLE SPACE P ARE PRECISELY THOSE $\varphi \in \Omega^*(P)$ WHICH DESCEND TO THE BASE MANIFOLD M, I.E., FOR WHICH $\exists\ \bar{\varphi} \in \Omega^*(M)$ WITH $\pi^*\bar{\varphi} = \varphi$.

GIVEN $\rho \in \mathbb{C}[\mathfrak{g}]^G$ (THOUGHT OF AS A k-MULTILINEAR FUNCTION) CHOOSE A CONNECTION ω ON $G \hookrightarrow P \xrightarrow{\pi} M$ AND LET Ω BE ITS CURVATURE. WRITE $\omega = \omega^a \xi_a$ AND $\Omega = \Omega^a \xi_a$. DEFINE $\rho(\Omega) \in \Omega^{2k}(P)$ BY

$$\begin{aligned}\rho(\Omega) &= \rho(\Omega^{a_1}\xi_{a_1}, \dots, \Omega^{a_k}\xi_{a_k}) \\ &= \rho(\xi_{a_1}, \dots, \xi_{a_k})\,\Omega^{a_1}\wedge \dots \wedge \Omega^{a_k}\end{aligned}$$

THEN $\rho(\Omega)$ IS A CLOSED, BASIC $2k$-FORM ON P SO IT DESCENDS TO A CLOSED $2k$-FORM $\bar{\rho}(\Omega)$ ON M:

$$\pi^*(\bar{\rho}(\Omega)) = \rho(\Omega)$$

ESSENTIALLY, $\bar{\rho}(\Omega)$ IS JUST THE RESTRICTION OF $\rho(\Omega)$ TO THE ω-HORIZONTAL SPACES IN TP.

THE COHOMOLOGY CLASS $[\bar{\rho}(\Omega)] \in H^{2k}(M; \mathbb{R})$ DOES NOT DEPEND ON THE CHOICE OF ω OR THE BASIS $\{\xi_1, \dots, \xi_n\}$ FOR $\mathfrak{g}$ AND IS CALLED A CHARACTERISTIC CLASS OF $G \hookrightarrow P \xrightarrow{\pi} M$.

AS AN EXAMPLE WE WILL CONSIDER THE 2^{ND} CHERN CLASS OF A PRINCIPAL $SU(2)$ - BUNDLE

$$SU(2) \hookrightarrow P \xrightarrow{\pi} M$$

OVER A COMPACT, CONNECTED, ORIENTED 4 - MANIFOLD M

<u>NOTE</u> : ANOTHER EXAMPLE (THE EULER CLASS) WILL BE DISCUSSED IN THE NEXT SECTION ON " EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN " .

NEED TO CHOOSE

1. A BASIS $\{\xi_1, \xi_2, \xi_3\}$ FOR $su(2)$ (2×2 COMPLEX MATRICES THAT ARE SKEW-HERMITIAN AND TRACEFREE), E.G.,

$$\xi_1 = -\frac{i}{2}\sigma_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\xi_2 = -\frac{i}{2}\sigma_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\xi_3 = -\frac{i}{2}\sigma_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

2. A CONNECTION ω WITH CURVATURE Ω WHICH WE WRITE AS

$$\omega = \omega^a \xi_a = \frac{1}{2}\begin{pmatrix} -\omega^3 i & -\omega^2 - i\omega^1 \\ \omega^2 - i\omega^1 & \omega^3 i \end{pmatrix}$$

$$\Omega = \Omega^a \xi_a = \frac{1}{2}\begin{pmatrix} -\Omega^3 i & -\Omega^2 - i\Omega^1 \\ \Omega^2 - i\Omega^1 & \Omega^3 i \end{pmatrix}$$

3. AN INVARIANT POLYNOMIAL ρ ON $su(2)$. FOR THIS WE TAKE

$$\rho : su(2) \to \mathbb{C}$$

$$\rho(A) = \frac{1}{8\pi^2} \, Tr(A^2)$$

THE CORRESPONDING BILINEAR FORM ON $su(2)$ IS CLEARLY

$$\rho : su(2) \times su(2) \to \mathbb{C}$$

$$\rho(A,B) = \frac{1}{8\pi^2} \, Tr(AB)$$

THUS,

$$\rho(\Omega) = \rho(\Omega^{a_1}\xi_{a_1}, \Omega^{a_2}\xi_{a_2})$$

$$= \rho(\xi_{a_1}, \xi_{a_2})\,\Omega^{a_1} \wedge \Omega^{a_2}$$

$$= \frac{1}{8\pi^2} \, Tr(\xi_{a_1}\xi_{a_2})\,\Omega^{a_1} \wedge \Omega^{a_2}$$

FOR THE BASIS CHOSEN ABOVE

$$Tr(\xi_{a_1}\xi_{a_2}) = \begin{cases} 0 & , \; a_1 \neq a_2 \\ -\frac{1}{2} & , \; a_1 = a_2 \end{cases}$$

SO

$$\rho(\Omega) = \frac{1}{8\pi^2} \left(-\tfrac{1}{2}\right)\left(\Omega^1 \wedge \Omega^1 + \Omega^2 \wedge \Omega^2 + \Omega^3 \wedge \Omega^3\right)$$

$$= \frac{1}{8\pi^2} \, Tr(\Omega \wedge \Omega)$$

WHERE $\Omega \wedge \Omega$ IS THE MATRIX PRODUCT WITH ENTRIES MULTIPLIED BY THE ORDINARY WEDGE PRODUCT.

WE KNOW THAT

$$\frac{1}{8\pi^2} \, Tr \, (\Omega \wedge \Omega)$$

IS A CLOSED, BASIC 4-FORM ON P AND SO DESCENDS TO A CLOSED 4-FORM ON M (I.E., IS π^* OF SOME CLOSED 4-FORM ON M).

THIS CLOSED 4-FORM ON M CAN BE DESCRIBED LOCALLY IN TERMS OF LOCAL FIELD STRENGTHS

$$\mathfrak{F}_j = s_j^* \Omega$$

SINCE $\pi^* \mathfrak{F}_j = \pi^*(s_j^* \Omega) = (s_j \circ \pi)^* \Omega = id^* \Omega = \Omega$,

$$\pi^* \left(\frac{1}{8\pi^2} Tr (\mathfrak{F}_j \wedge \mathfrak{F}_j) \right) = \frac{1}{8\pi^2} Tr (\Omega \wedge \Omega)$$

ON ITS DOMAIN. SINCE THE LOCAL FIELD STRENGTHS TRANSFORM UNDER THE ADJOINT REPRESENTATION ($\mathfrak{F}_j = g_{ij}^{-1} \mathfrak{F}_i \, g_{ij}$) THE SAME IS TRUE OF $\mathfrak{F}_j \wedge \mathfrak{F}_j$ AND Tr IS ad-INVARIANT SO THE

$$\frac{1}{8\pi^2} Tr (\mathfrak{F}_j \wedge \mathfrak{F}_j)$$

PIECE TOGETHER INTO THE GLOBALLY DEFINED 4-FORM ON M TO WHICH $\frac{1}{8\pi^2} Tr (\Omega \wedge \Omega)$ DESCENDS. WE WRITE THIS AS

$$\frac{1}{8\pi^2} Tr (F_\omega \wedge F_\omega)$$

AND CALL ITS COHOMOLOGY CLASS

$$c_2(P) \in H^4(M;\mathbb{R})$$

THE $2^{\underline{ND}}$ <u>CHERN CLASS</u> OF $SU(2) \hookrightarrow P \xrightarrow{\pi} M$. ITS INTEGRAL OVER M

$$c_2(P)[M] = \int_M c_2(P) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(F_\omega \wedge F_\omega),$$

WHICH IS ACTUALLY AN INTEGER, IS CALLED THE $2^{\underline{ND}}$ <u>CHERN NUMBER</u> OF $SU(2) \hookrightarrow P \xrightarrow{\pi} M$

<u>NOTE</u> : FOR THE QUATERNIONIC HOPF BUNDLE

$$SU(2) \hookrightarrow S^7 \xrightarrow{\pi} S^4$$

ONE CAN USE ANY OF THE BPST INSTANTON CONNECTIONS TO COMPUTE REPRESENTATIVES OF THE CHERN CLASS. MOREOVER, THE BUNDLE IS TRIVIAL OVER $S^4 - \{\text{POINT}\}$ WHICH STEREOGRAPHICALLY PROJECTS TO $\mathbb{R}^4$ SO THE CHERN NUMBER CAN BE COMPUTED AS AN INTEGRAL OVER $\mathbb{R}^4$. THE CALCULATION WAS SKETCHED IN THE LECTURE. THE RESULT IS 1.

FOR THE COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLDS M OF INTEREST TO US HERE, PRINCIPAL $SU(2)$-BUNDLES OVER M ARE CLASSIFIED UP TO EQUIVALENCE BY THEIR 2^{ND} CHERN CLASS / NUMBER.