ADDENDUM 11

MATHAI - QUILLEN EULER FORMS FOR TS^2

HERE WE CONTINUE WITH THE EXAMPLE IN APPENDIX 5 TO PRODUCE SOME REPRESENTATIVES OF THE EULER CLASS OF S^2. THUS, WE BEGIN WITH

$$U = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (\Omega' + du_1 du_2 + \omega' \wedge (u_1 du_1 + u_2 du_2))$$

AND PULLBACK BY

$$(\Delta, S\circ\Delta)^* = ((1, S)\circ\Delta)^* = \Delta^* \circ (1, S)^*$$

WHERE Δ IS A SECTION OF $F_{SO}(TS^2)$ AND $S : F_{SO}(TS^2) \to \mathbb{R}^2$ IS EQUIVARIANT FOR THE $SO(2)$-ACTIONS (WE MUST, OF COURSE, CHOOSE ω AND THESE IN ORDER TO GET CONCRETE REPRESENTATIVES OF THE EULER CLASS $e(TS^2)$).

WE PULL THE $\mathbb{R}^2$-PARTS OF U BACK BY S TO GET A FORM ON $F_{SO}(TS^2)$ WHICH IS THEN PULLED BACK TO S^2 BY Δ.

AN EQUIVARIANT MAP $S : F_{SO}(TS^2) \to \mathbb{R}^2$ IS NOTHING OTHER THAN A SECTION OF TS^2, I.E., A VECTOR FIELD V ON S^2 (MORE PRECISEY, $x \to [p, S(p)]$, WHERE $p \in \pi_{SO}^{-1}(x)$, IS A SECTION OF $F_{SO}(TS^2) \times_{id} \mathbb{R}^2 = TS^2$).

A SECTION Δ OF $F_{SO}(TS^2)$ IS NOTHING OTHER THAN AN ORIENTED, ORTHONORMAL FRAME FIELD ON S^2

AS OUR Δ WE CHOOSE THE ORIENTED, ORTHONORMAL FRAME FIELD CORRESPONDING TO THE SPHERICAL COORDINATE CHART :

$$s(\phi,\theta) = (\phi, \theta, \frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta})$$

FOR S WE MAY CHOOSE ANY VECTOR FIELD ON S^2. WE HAVE SELECTED A CONSTANT (γ) MULTIPLE OF THE INFINITESIMAL GENERATOR FOR ROTATIONS ABOUT THE x-AXIS :

$$\gamma\left(\sin\theta\frac{\partial}{\partial\phi} + \cos\theta\cot\phi\frac{\partial}{\partial\theta}\right)$$

$$= \gamma\sin\theta\frac{\partial}{\partial\phi} + \gamma\cos\theta\cos\phi\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)$$

AS AN EQUIVARIANT MAP $S : F_{SO}(TS^2) \longrightarrow \mathbb{R}^2$ THIS IS DEFINED ON THE IMAGE OF s BY

$$(S\circ s)(\phi,\theta) = S(s(\phi,\theta)) = S(\phi, \theta, \frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta})$$

$$= (\gamma\sin\theta, \gamma\cos\theta\cos\phi)$$

AND ELSEWHERE BY EQUIVARIANCE ($S(p\cdot g) = g^{-1}\cdot S(p)$).

FOR THE CONNECTION ω ON $F_{SO}(TS^2)$ WE CHOOSE, AS IN APPENDIX 1, THE LEVI-CIVITA CONNECTION

$$\omega = -\cos\phi\, d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

WHOSE CURVATURE IS

$$\Omega = \sin\phi\, d\phi\wedge d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

NOW WE COMPUTE THE PULLBACK OF THE $\mathbb{R}^2$- PARTS OF $\mho$ BY S, I.E., SUBSTITUTE $\mu_1 = \gamma\sin\theta$ AND $\mu_2 = \gamma\cos\theta\cos\phi$:

• $(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} = (2\pi)^{-1} e^{-\frac{1}{2}r^2(\sin^2\theta + \cos^2\theta\cos^2\phi)}$

$du_1 = r\cos\theta\, d\theta$

$du_2 = -r\sin\theta\cos\phi\, d\theta - r\cos\theta\sin\phi\, d\phi$

$$u_1 du_1 + u_2 du_2 = r^2\sin\theta\cos\theta\, d\theta - r^2\sin\theta\cos\theta\cos^2\phi\, d\theta - r^2\cos^2\theta\sin\phi\cos\phi\, d\phi$$

$$= r^2\sin\theta\cos\theta(1-\cos^2\phi)\, d\theta - r^2\cos^2\theta\sin\phi\cos\phi\, d\phi$$

• $u_1 du_1 + u_2 du_2 = r^2\sin\theta\cos\theta\sin^2\phi\, d\theta - r^2\cos^2\theta\sin\phi\cos\phi\, d\phi$

$$\omega' \wedge (u_1 du_1 + u_2 du_2) = -\cos\phi\, d\theta \wedge (r^2\sin\theta\cos\theta\sin^2\phi\, d\theta - r^2\cos^2\theta\sin\phi\cos\phi\, d\phi)$$

$$= r^2\cos^2\theta\sin\phi\cos^2\phi\, d\theta \wedge d\phi$$

• $\omega' \wedge (u_1 du_1 + u_2 du_2) = -r^2\cos^2\theta\sin\phi\cos^2\phi\, d\phi \wedge d\theta$

$$du_1 du_2 = (r\cos\theta\, d\theta) \wedge (-r\sin\theta\cos\phi\, d\theta - r\cos\theta\sin\phi\, d\phi)$$

$$= -r^2\cos^2\theta\sin\phi\, d\theta \wedge d\phi$$

• $du_1 du_2 = r^2\cos^2\theta\sin\phi\, d\phi \wedge d\theta$

SUBSTITUTING THESE AND $\Omega' = \sin\phi\, d\phi \wedge d\theta$ INTO THE EXPRESSION FOR $\mho$ GIVES

$$(2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta\cos^2\phi)} \left(\sin\phi \, d\phi \wedge d\theta + \gamma^2\cos^2\theta \sin\phi \, d\phi \wedge d\theta \right.$$

$$\left. - \gamma^2\cos^2\theta \sin\phi \cos^2\phi \, d\phi \wedge d\theta \right) =$$

$$(2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta\cos^2\phi)} \sin\phi \, (1 + \underbrace{\gamma^2\cos^2\theta - \gamma^2\cos^2\theta\cos^2\phi}_{\gamma^2\cos^2\theta\sin^2\phi}) \, d\phi \wedge d\theta$$

$$(2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta\cos^2\phi)} \sin\phi \, (1 + \gamma^2\cos^2\theta\sin^2\phi) \, d\phi \wedge d\theta$$