ADDENDUM 15 : SEIBERG-WITTEN INVARIANTS (MORE DETAILS)

HERE WE COLLECT A FEW MISC. OBSERVATIONS AND ARGUMENTS FOR WHICH THERE WAS NO TIME IN LECTURE.

1. SPIN AND SPIN^c STRUCTURES VIA TRANSITION MAPS

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M$$

$\{U_\alpha\}_{\alpha \in \mathcal{A}}$ A TRIVIALIZING COVER OF M WITH TRIVIALIZATIONS

$$\varphi_\alpha : \pi_{SO}^{-1}(U_\alpha) \longrightarrow U_\alpha \times SO(4)$$

$$\varphi_\alpha(p) = (\pi_{SO}(p), g_\alpha(p))$$

CORRESPONDING TRANSITION MAPS

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(4)$$

$$g_\alpha(p) = g_{\alpha\beta}(\pi_{SO}(p))\, g_\beta(p)$$

NOW, A SPIN STRUCTURE ON M CONSISTS OF A PRINCIPAL SPIN(4)-BUNDLE

$$SPIN(4) \hookrightarrow S(M) \xrightarrow{\pi_S} M$$

OVER M TOGETHER WITH A SMOOTH MAP

$$\lambda : S(M) \longrightarrow F_{SO}(M)$$

SUCH THAT

$$\begin{tikzcd} S(M) \arrow[rr, "\lambda"] \arrow[dr, "\pi_S"'] & & F_{SO}(M) \arrow[dl, "\pi_{SO}"] \\ & M & \end{tikzcd}$$

COMMUTES AND

$$\lambda(p \cdot g) = \lambda(p) \cdot \mathrm{SPIN}(g)$$

$\forall\ p \in S(M)\ \forall\ g \in \mathrm{SPIN}(4)$, WHERE $\mathrm{SPIN} : \mathrm{SPIN}(4) \longrightarrow SO(4)$ IS THE DOUBLE COVER (ADDENDUM 14).

<u>THEOREM</u>: LET M BE AN ORIENTED, RIEMANNIAN 4-MANIFOLD WITH ORIENTED ORTHONORMAL FRAME BUNDLE

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M.$$

A SPIN STRUCTURE EXISTS IF AND ONLY IF $\exists$ TRIVIALIZING COVER $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ OF M FOR $F_{SO}(M)$ FOR WHICH THE TRANSITION MAPS $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(4)$ LIFT TO A FAMILY OF MAPS TO SPIN(4)

$$\begin{tikzcd} & \mathrm{SPIN}(4) \arrow[d, "\mathrm{SPIN}"] \\ U_\alpha \cap U_\beta \arrow[r, "g_{\alpha\beta}"'] \arrow[ur, dotted, "\tilde{g}_{\alpha\beta}"] & SO(4) \end{tikzcd}$$

SATISFYING THE COCYCLE CONDITION $\tilde{g}_{\alpha\beta}\, \tilde{g}_{\beta\gamma}\, \tilde{g}_{\gamma\alpha} = id.$

PROOF : SUPPOSE FIRST THAT A SPIN STRUCTURE EXISTS. LET $\{U_\alpha\}_{\alpha\in A}$ BE A TRIVIALIZING COVER OF M FOR BOTH $S(M)$ AND $F_{SO}(M)$. SELECT LOCAL SECTIONS

$$\tilde{s}_\alpha : U_\alpha \longrightarrow \pi_S^{-1}(U_\alpha) = \lambda^{-1}(\pi_{SO}^{-1}(U_\alpha))$$

AND LET THE CORRESPONDING TRANSITION MAPS BE

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SPIN(4).$$

(THESE NECESSARILY SATISFY THE COCYCLE CONDITION SINCE THEY ARE TRANSITION MAPS).

DEFINE

$$s_\alpha : U_\alpha \rightarrow \pi_{SO}^{-1}(U_\alpha)$$

$$s_\alpha = \lambda \circ \tilde{s}_\alpha$$

$\forall \alpha \in A$. THESE ARE SECTIONS OF $F_{SO}(M)$ SINCE $\pi_{SO} \circ s_\alpha = (\pi_{SO} \circ \lambda) \circ \tilde{s}_\alpha = \pi_S \circ \tilde{s}_\alpha = id$. THUS, THEY DETERMINE TRIVIALIZATIONS OF $F_{SO}(M)$ AND WE CLAIM THAT THE CORRESPONDING TRANSITION MAPS $g_{\alpha\beta}$ ARE GIVEN BY $g_{\alpha\beta} = SPIN \circ \tilde{g}_{\alpha\beta}$. THIS FOLLOWS FROM

$$\begin{aligned} s_\beta(x) = \lambda(\tilde{s}_\beta(x)) &= \lambda(\tilde{s}_\alpha(x)\cdot \tilde{g}_{\alpha\beta}(x)) \\ &= \lambda(\tilde{s}_\alpha(x)) \cdot SPIN(\tilde{g}_{\alpha\beta}(x)) \\ &= s_\alpha(x) \cdot (SPIN \circ \tilde{g}_{\alpha\beta})(x). \end{aligned}$$

FOR THE CONVERSE, SUPPOSE $F_{SO}(M)$ HAS A TRIVIALIZING COVER $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ FOR WHICH THE TRANSITION FUNCTIONS $g_{\alpha\beta}$ LIFT

$$\begin{tikzcd} & SPIN(4) \arrow[d, "SPIN"] \\ U_\alpha \cap U_\beta \arrow[r, "g_{\alpha\beta}"'] \arrow[ur, dashed, "\tilde{g}_{\alpha\beta}"] & SO(4) \end{tikzcd}$$

TO MAPS $\tilde{g}_{\alpha\beta}$ WHICH SATISFY THE COCYCLE CONDITION. THIS CONDITION INSURES THAT $\exists$ UNIQUE (UP TO EQUIVALENCE) BUNDLE

$$SPIN(4) \hookrightarrow S(M) \xrightarrow{\pi_S} M$$

TRIVIALIZED OVER $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ WITH TRANSITION MAPS $\tilde{g}_{\alpha\beta}$.

LET $s_\alpha : U_\alpha \to \pi_{SO}^{-1}(U_\alpha)$ AND $\tilde{s}_\alpha : U_\alpha \to \pi_S^{-1}(U_\alpha)$ BE THE SECTIONS ASSOCIATED WITH THE TRIVIALIZATIONS OF $F_{SO}(M)$ AND $S(M)$, RESPECTIVELY. THUS, THE TRIVIALIZATIONS ARE GIVEN BY

$$\psi_\alpha : \pi_{SO}^{-1}(U_\alpha) \to U_\alpha \times SO(4)$$

$$\psi_\alpha(s_\alpha(x)\cdot h) = (x,h)$$

$$\tilde{\psi}_\alpha : \pi_S^{-1}(U_\alpha) \to U_\alpha \times SPIN(4)$$

$$\tilde{\psi}_\alpha(\tilde{s}_\alpha(x)\cdot g) = (x,g)$$

NOW DEFINE

$$\lambda_\alpha : \pi_S^{-1}(U_\alpha) \longrightarrow \pi_{SO}^{-1}(U_\alpha)$$

$$\lambda_\alpha = \psi_\alpha^{-1} \circ (id_{U_\alpha} \times SPIN) \circ \tilde{\psi}_\alpha$$

THEN

$$\lambda_\alpha(\tilde{s}_\alpha(x) \cdot g) = \psi_\alpha^{-1} \circ (id_{U_\alpha} \times SPIN)(x, g)$$

$$= \psi_\alpha^{-1}(x, SPIN(g))$$

$$= s_\alpha(x) \cdot SPIN(g)$$

SO

$$\pi_{SO} \circ \lambda_\alpha = \pi_S$$

AND $\lambda_\alpha(p \cdot g) = p \cdot SPIN(g)$. MOREOVER, λ_α AND λ_β AGREE ON $U_\alpha \cap U_\beta$ WHENEVER THIS IS NONEMPTY SO THESE MAPS DETERMINE $\lambda : S(M) \to F_{SO}(M)$ WITH THE REQUIRED PROPERTIES. $\square$

TRANSLATED INTO THE LANGUAGE OF ČECH COHOMOLOGY WITH $\mathbb{Z}_2$ COEFFICIENTS THIS BECOMES

> AN ORIENTED RIEMANNIAN 4-MANIFOLD M ADMITS A SPIN STRUCTURE IF AND ONLY IF ITS 2ND STIEFEL-WHITNEY CLASS $w_2(M) \in \check{H}^2(M; \mathbb{Z}_2)$ IS TRIVIAL.

(SEE PAGES 388 - 404 OF REFERENCE [41] FOR THE DETAILS)

IN EXACTLY THE SAME WAY ONE SHOWS THAT M ADMITS A $SPIN^c$ STRUCTURE

$$SPIN^c(4) \hookrightarrow S^c(M) \xrightarrow{\pi_{S^c}} M$$

$$\Lambda : S^c(M) \to F_{SO}(M)$$

IF AND ONLY IF THE TRANSITION MAPS $g_{\alpha\beta}$ FOR SOME TRIVIALIZING COVER FOR $F_{SO}(M)$ LIFT

$$\begin{tikzcd} & SPIN^c(4) \arrow[d, "\pi"] \\ U_\alpha \cap U_\beta \arrow[r, "g_{\alpha\beta}"'] \arrow[ur, dotted, "\tilde{\tilde{g}}_{\alpha\beta}"] & SO(4) \end{tikzcd}$$

TO MAPS SATISFYING THE COCYCLE CONDITION.

SUCH A LIFTING, TOGETHER WITH THE MAP

$$\delta : SPIN^c(4) \longrightarrow U(1)$$

$$\delta(\xi) = \delta\begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} = \delta\begin{pmatrix} e^{\theta i}U_1 & 0 \\ 0 & e^{\theta i}U_2 \end{pmatrix}$$
$$= \det U_+ = \det U_- = e^{2\theta i}$$

GIVES THE PRINCIPAL $U(1)$-BUNDLE, AND CORRESPONDING HERMITIAN COMPLEX LINE BUNDLE, WITH TRANSITION MAPS

$$\delta \circ \tilde{\tilde{g}}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$$

NOTE: IF THE $SPIN^c$ STRUCTURE IS DENOTED $\mathcal{L}$, THEN THESE ARE $L^0(\mathcal{L})$ AND $L(\mathcal{L})$.

CAN SHOW THAT $w_2(M)$ IS THE MOD 2 REDUCTION OF THE 1^{ST} CHERN CLASS OF THIS $U(1)$-BUNDLE. MOREOVER,

SPIN STRUCTURE + HERMITIAN LINE BUNDLE WHOSE 1^{ST} CHERN CLASS REDUCES MOD 2 TO $w_2(M)$

$\Longrightarrow$ $SPIN^c$ STRUCTURE

$$\begin{tikzcd} & SPIN(4) \arrow[d, "SPIN"] \\ U_\alpha \cap U_\beta \arrow[r, "g_{\alpha\beta}"'] \arrow[ur, dotted, "\tilde{g}_{\alpha\beta}"] & SO(4) \end{tikzcd} \quad + \quad U_\alpha \cap U_\beta \xrightarrow[h_{\alpha\beta}]{} U(1)$$

$$\Longrightarrow \quad U_\alpha \cap U_\beta \xrightarrow[\tilde{\tilde{g}}_{\alpha\beta} = h_{\alpha\beta}\tilde{g}_{\alpha\beta}]{} SPIN^c(4)$$

NOW LET'S SEE WHAT HAPPENS IF

$$\begin{array}{c} \text{SPIN STRUCTURE} \\ + \\ \text{LINE BUNDLE} \end{array} \Longrightarrow SPIN^c \text{ STRUCTURE} \Longrightarrow \text{LINE BUNDLE}$$

$$\begin{array}{c}\tilde{g}_{\alpha\beta}\\ +\\ h_{\alpha\beta}\end{array} \Rightarrow \tilde{\tilde{g}}_{\alpha\beta} = h_{\alpha\beta}\,\tilde{g}_{\alpha\beta} \Rightarrow \delta\circ\tilde{\tilde{g}}_{\alpha\beta} = \delta\circ(h_{\alpha\beta}\,\tilde{g}_{\alpha\beta}) = h^2_{\alpha\beta}$$

WHICH ARE THE TRANSITION MAPS FOR THE (TENSOR) SQUARE OF THE ORIGINAL LINE BUNDLE.

IN THIS CASE THE DETERMINANT LINE BUNDLE L FOR THE RESULTING $SPIN^c$ STRUCTURE HAS A (TENSOR) SQUARE ROOT (THE HERMITIAN LINE BUNDLE SELECTED TO SUPPLEMENT THE SPIN STRUCTURE) WHICH ONE CAN WRITE $L^{\frac{1}{2}}$.

<u>NOTE</u>: THE SPIN STRUCTURE $SPIN(4) \hookrightarrow S(M) \xrightarrow{\pi_S} M$ AND THE REPRESENTATIONS $\Delta^{\pm}_{\mathbb{C}} : SPIN(4) \longrightarrow SU(S^{\pm}_{\mathbb{C}})$ GIVE ASSOCIATED POSITIVE AND NEGATIVE (REAL) SPINOR BUNDLES WHICH ONE OFTEN SEES DENOTED $W^{\pm}$. THE CORRESPONDING (COMPLEX) SPINOR BUNDLES ASSOCIATED WITH THE $SPIN^c$ STRUCTURE CONSTRUCTED FROM $L^{\frac{1}{2}}$ AND THE SPIN STRUCTURE ARE THEN

$$W^{\pm} \otimes L^{\frac{1}{2}}$$

REGRETTABLY, ONE OFTEN SEES $\mathcal{S}^{\pm}$ WRITTEN THIS WAY EVEN WHEN THE $SPIN^c$ STRUCTURE DOES NOT COME FROM A SPIN STRUCTURE (SO THAT W^+, W^- AND $L^{\frac{1}{2}}$ DO NOT EXIST).

2. DIRAC OPERATOR INDEPENDENT OF ORIENTED, ORTHONORMAL FRAME FIELD

RECALL : ON THE $SPIN^c$-BUNDLE $SPIN^c(4) \hookrightarrow S^c(M) \to M$ WE HAVE A $SPIN^c$-CONNECTION ω_A WHICH GIVES A COVARIANT DERIVATIVE

$$\nabla = \nabla_A \quad : \Gamma(S(\mathcal{L})) \to \Omega^1(M) \otimes \Gamma(S(\mathcal{L}))$$

ON SPINOR FIELDS. CHOOSE A LOCAL, ORIENTED, ORTHONORMAL FRAME FIELD $\{E_1, E_2, E_3, E_4\}$ ON M. DEFINE

$$\tilde{\not{D}}_A \Psi = \sum_{i=1}^{4} E_i \cdot \nabla_A \Psi(E_i)$$

NOW LET $\{\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4\}$ BE ANOTHER ORIENTED, ORTHONORMAL FRAME FIELD (W.L.O.G., ON THE SAME OPEN SET IN M AS $\{E_1, E_2, E_3, E_4\}$). POINTWISE,

$$\hat{E}_i = \sum_{j=1}^{4} B_{ij} E_j \ ,$$

WHERE $(B_{ij}) \in SO(4)$. THEN, FOR ANY SECTION $\Delta \in \Gamma(S(\mathcal{L}))$

$$\sum_{i=1}^{4} \hat{E}_i \cdot (\nabla_A \Psi(\hat{E}_i))(\Delta) = \sum_{i=1}^{4} \Big(\sum_{j=1}^{4} B_{ij} E_j \Big) \cdot \Big(\Big(\sum_{k=1}^{4} B_{ik} \nabla_A \Psi(E_k) \Big)(\Delta) \Big)$$

SINCE $\nabla_A \Psi(\cdot)$ IS LINEAR

$$= \sum_{i,j,k=1}^{4} B_{ij} B_{ik} E_j \cdot \nabla_A \Psi(E_k)(\Delta)$$

$$= \sum_{j,k=1}^{4} \delta_{jk} E_j \cdot \nabla_A \Psi(E_k)(\Delta)$$

$$= \sum_{j=1}^{4} E_j \cdot \nabla_A \Psi(E_j)(\Delta) \quad \text{AS REQUIRED.}$$

3. SEIBERG-WITTEN GAUGE GROUP

M = COMPACT, CONNECTED, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD

g = RIEMANNIAN METRIC ON M

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M$$

SPIN^c STRUCTURE $\mathcal{L}$:

$$\begin{array}{ccccc} SPIN^c(4) & \hookrightarrow & S^c(M) & \xrightarrow{\pi_{S^c}} & M \\ & & \downarrow \Lambda & & \\ SO(4) & \hookrightarrow & F_{SO}(M) & \xrightarrow[\pi_{SO}]{} & M \end{array}$$

A : CONNECTION ON DETERMINANT LINE BUNDLE $U(1) \hookrightarrow L^o(\mathcal{L}) \to M$

$\psi \in \Gamma(S^+(\mathcal{L}))$ A POSITIVE SPINOR FIELD

RECALL : $S^c(M)$ DOUBLE COVERS THE FIBER PRODUCT $F_{SO}(M) \dot{\times} L^o(\mathcal{L})$ VIA THE MAP SPIN^c, GIVEN LOCALLY BY

$$(x, \xi) = (x, e^{\theta i}\mu) \to ((x,x), (\pi(\xi), \delta(\xi)))$$

$$= ((x,x), (SPIN(\mu), e^{2\theta i}))$$

LET Pr_F AND Pr_{L^o} BE THE PROJECTIONS OF $F_{SO}(M) \times L^o(\mathcal{L})$ ONTO $F_{SO}(M)$ AND $L^o(\mathcal{L})$, RESPECTIVELY.

THEN WE HAVE

$$\begin{tikzcd} SPIN^c(4) \arrow[r, hook] & S^c(M) \arrow[r] \arrow[d, "Pr_F \circ SPIN^c"] & M \\ SO(4) \arrow[r, hook] & F_{SO}(M) \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} SPIN^c(4) \arrow[r, hook] & S^c(M) \arrow[r] \arrow[d, "Pr_{L^0} \circ SPIN^c"] & M \\ U(1) \arrow[r, hook] & L^0(\mathcal{L}) \arrow[r] & M \end{tikzcd}$$

AN AUTOMORPHISM $\sigma : S^c(M) \longrightarrow S^c(M)$ (DIFFEOMORPHISM SATISFYING $\pi_{S^c} \circ \sigma = \pi_{S^c}$ AND $\sigma(p \cdot \xi) = \sigma(p) \cdot \xi$) IS SAID TO <u>COVER THE IDENTITY ON</u> $F_{SO}(M)$ IF

$$Pr_F \circ SPIN^c \circ \sigma = Pr_F \circ SPIN^c$$

$\mathcal{G}(\mathcal{L})$ = GROUP (UNDER COMPOSITION) OF ALL SUCH.

<u>EXAMPLES</u>: FOR ANY SMOOTH MAP $\gamma \in C^\infty(M, U(1))$ OF M TO $U(1) \hookrightarrow SPIN^c(4)$ DEFINE

$$\sigma_\gamma : S^c(M) \longrightarrow S^c(M)$$

$$\sigma_\gamma(p) = p \cdot \gamma(\pi_{S^c}(p))$$

σ_γ IS A DIFFEOMORPHISM AND (SINCE $U(1) \subseteq Z(SPIN^c(4))$)

$$\begin{aligned} \sigma_\gamma(p \cdot \xi) &= (p \cdot \xi) \cdot \gamma(\pi_{S^c}(p \cdot \xi)) \\ &= (p \cdot \xi) \cdot \gamma(\pi_{S^c}(p)) \\ &= p \cdot (\xi \gamma(\pi_{S^c}(p))) \\ &= p \cdot (\gamma(\pi_{S^c}(p)) \xi) \\ &= (p \cdot \gamma(\pi_{S^c}(p))) \cdot \xi = \sigma_\gamma(p) \cdot \xi \end{aligned}$$

SO σ_γ IS AN AUTOMORPHISM. IT COVERS THE IDENTITY ON $F_{SO}(M)$ BECAUSE, LOCALLY, $Pr_F \circ SPIN^c$ IS GIVEN BY

$$p = (x, \xi) = (x, e^{\theta i}\mu) \longrightarrow (x, SPIN(\mu))$$

SO p AND $p \cdot e^{\phi i}$ ALWAYS HAVE THE SAME IMAGE.

IN FACT, <u>EVERY</u> ELEMENT OF $\mathcal{G}(\mathcal{L})$ IS σ_γ FOR SOME $\gamma \in C^\infty(M, U(1))$

<u>PROOF</u>: FIRST OBSERVE THAT TWO ELEMENTS OF $S^c(M)$ WITH THE SAME IMAGE UNDER $Pr_F \circ SPIN^c$ CAN DIFFER ONLY BY THE ACTION OF SOMETHING IN $U(1)$ (IF $p_1 = (x_1, \xi_1)$ AND $p_2 = (x_2, \xi_2)$ HAVE THE SAME IMAGE, THEN $x_1 = x_2$ AND $SPIN(\mu_1) = SPIN(\mu_2)$ SO $\mu_1 = \pm\mu_2$ AND

$$\begin{aligned} \xi_1 = e^{\theta_1 i}\mu_1 &= (\pm e^{\theta_1 i})\mu_2 \\ &= (\pm e^{(\theta_1 - \theta_2)i})e^{\theta_2 i}\mu_2 \\ &= \xi_2(\pm e^{(\theta_1-\theta_2)i}) \end{aligned}$$

SO $p_1 = p_2 \cdot (e^{(\theta_1-\theta_2)i})$ OR $p_1 = p_2 \cdot (e^{(\theta_1 - \theta_2 + \pi)i})$).

THUS, IF AN AUTOMORPHISM $\sigma : S^c(M) \longrightarrow S^c(M)$ COVERS THE IDENTITY ON $F_{SO}(M)$, THEN

$$\sigma(p) = p \cdot (\text{SOMETHING IN } U(1))$$

$\forall p \in S^c(M)$. WE CLAIM THAT THIS "SOMETHING" MUST BE THE

SAME FOR ALL POINTS IN THE SAME FIBER OF π_{S^c}. INDEED, $\pi_{S^c}(p_1) = \pi_{S^c}(p_2)$ IMPLIES $p_2 = p_1 \cdot \xi$ FOR SOME $\xi \in SPIN^c(4)$ AND IF $\sigma(p_1) = p_1 \cdot e^{\phi i}$, THEN

$$\begin{aligned} \sigma(p_2) &= \sigma(p_1 \cdot \xi) = \sigma(p_1) \cdot \xi \\ &= (p_1 \cdot e^{\phi i}) \cdot \xi = (p_1 \cdot \xi) \cdot e^{\phi i} \\ &= p_2 \cdot e^{\phi i} \end{aligned}$$

AS WELL. THUS, $\sigma(p) = p \cdot \gamma(\pi_{S^c}(p))$ FOR SOME $\gamma \in C^\infty(M, U(1))$ SO $\sigma = \sigma_\gamma$. □

THUS,

$$\boxed{\mathcal{G}(\mathcal{L}) \cong C^\infty(M, U(1))}$$

WHERE THE GROUP OPERATION ON $C^\infty(M, U(1))$ IS POINTWISE MULTIPLICATION IN $U(1)$.

WE WILL USE WHICHEVER VIEW OF THE GAUGE GROUP $\mathcal{G}(\mathcal{L})$ IS MOST CONVENIENT IN ANY GIVEN SITUATION.

NOW WE DESCRIBE THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON (A, ψ):

<u>THE SPINOR FIELD</u> $\psi \in \Gamma(S^+(\mathcal{L}))$:

FOR THIS WE IDENTIFY ψ WITH AN EQUIVARIANT $S^+_{\mathbb{C}}$-VALUED

MAP ON $S^c(M)$ $(\psi(p\cdot\xi) = \xi^{-1}\cdot\psi(p))$ AND DEFINE THE ACTION OF σ_γ ON ψ BY PULLBACK:

$$\psi\cdot\sigma_\gamma = \psi\cdot\gamma = \sigma_\gamma^{*}\psi = \psi\circ\sigma_\gamma .$$

THUS, AT EACH $p \in S^c(M)$,

$$(\psi\cdot\sigma_\gamma)(p) = \psi(\sigma_\gamma(p)) = \psi(p\cdot\gamma(\pi_{S^c}(p)))$$

$$= (\gamma(\pi_{S^c}(p)))^{-1}\cdot\psi(p)$$

THEN IF WE THINK OF ψ AGAIN AS A SECTION OF $\mathcal{S}^{+}(\mathcal{L})$,

$$\psi\cdot\gamma = \gamma^{-1}\psi$$

<u>THE</u> <u>CONNECTION</u> A ON $U(1) \hookrightarrow L^{0}(\mathcal{L}) \longrightarrow M$:

NOTE THAT THE AUTOMORPHISM σ_γ OF $S^c(M)$ INDUCES AN AUTOMORPHISM σ_γ' OF $L^{0}(\mathcal{L})$:

$$\begin{array}{ccc} S^c(M) & \xrightarrow{\ \sigma_\gamma\ } & S^c(M) \\ {\scriptstyle Pr_{L^0}\circ SPIN^c}\downarrow & & \downarrow{\scriptstyle Pr_{L^0}\circ SPIN^c} \\ L^{0}(\mathcal{L}) & \overset{}{\underset{\sigma_\gamma'}{\dashrightarrow}} & L^{0}(\mathcal{L}) \end{array}$$

$$\sigma_\gamma'\circ Pr_{L^0}\circ SPIN^c = Pr_{L^0}\circ SPIN^c\circ\sigma_\gamma$$

(WE WILL WRITE OUT AN EXPLICIT LOCAL EXPRESSION FOR σ_γ' IN A MOMENT)

NOW DEFINE THE ACTION OF σ_γ ON A BY

$$A \cdot \sigma_\gamma = A \cdot \gamma = (\sigma_\gamma')^* A$$

WE WILL NEED A LOCAL EXPRESSION FOR COMPUTING THIS ACTION SO LET s BE A LOCAL SECTION OF $L^0(\mathcal{L})$ AND WRITE

$$\mathcal{A} = s^* A$$

FOR THE CORRESPONDING LOCAL GAUGE POTENTIAL. ALSO DEFINE

$$\mathcal{A} \cdot \gamma = s^*(A \cdot \gamma) = s^*((\sigma_\gamma')^* A) = (\sigma_\gamma' \circ s)^* A$$

SINCE $Pr_{L^0} \circ SPIN^c$ IS LOCALLY GIVEN BY

$$(x, \xi) = (x, e^{\theta i} u) \longrightarrow (x, \delta(\xi)) = (x, e^{2\theta i})$$

IT SATISFIES

$$(Pr_{L^0} \circ SPIN^c)(p \cdot \xi_0) = ((Pr_{L^0} \circ SPIN^c)(p)) \cdot \delta(\xi_0)$$

SO

$$\begin{aligned} \sigma_\gamma'((Pr_{L^0} \circ SPIN^c)(p)) &= (Pr_{L^0} \circ SPIN^c)(\sigma_\gamma(p)) \\ &= (Pr_{L^0} \circ SPIN^c)(p \cdot \gamma(\pi_{S^c}(p))) \\ &= ((Pr_{L^0} \circ SPIN^c)(p)) \cdot \delta(\gamma(\pi_{S^c}(p))) \end{aligned}$$

SINCE $Pr_{L^0} \circ SPIN^c$ MAPS ONTO $L^0(\mathcal{L})$ WE CAN WRITE THIS AS

$$\sigma_\gamma'(x) = x \cdot \delta(\gamma(\pi_{L^0}(x))) = x \cdot (\gamma(\pi_{L^0}(x)))^2$$

$\forall\ x \in L^0(\mathcal{L})$.

IN PARTICULAR,

$$\sigma_\gamma' \circ s = s \cdot \gamma^2$$

AT EVERY POINT OF M.

<u>NOTE</u> : s IS A SECTION OF $L^0(\mathcal{L})$ AND SO IS $\sigma_\gamma' \circ s$. THE LAST EQUALITY IDENTIFIES THE TRANSITION MAP THAT RELATES THE TWO SECTIONS AS γ^2.

SINCE
$$a = s^* A \Rightarrow a \cdot \gamma = (\sigma_\gamma' \circ s)^* A$$

WE CONCLUDE THAT

$$a \cdot \gamma = (\gamma^2)^{-1} a (\gamma^2) + (\gamma^2)^{-1} d(\gamma^2)$$
$$= a + (\gamma^2)^{-1}(2\gamma d\gamma)$$
$$a \cdot \gamma = a + 2\gamma^{-1} d\gamma$$

WHICH IS OUR LOCAL EXPRESSION FOR THE ACTION OF THE GAUGE GROUP ON THE CONNECTION.

APPLYING $\pi_{L^0}^*$ TO BOTH SIDES GIVES

$$A\cdot\gamma = A + \pi_{L^0}^*(2\gamma^{-1}d\gamma)$$

NOW WE HAVE THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON SEIBERG-WITTEN CONFIGURATIONS:

$$(A,\psi)\cdot\sigma_\gamma = (A,\psi)\cdot\gamma = ((\sigma_\gamma')^*A,\ \sigma_\gamma^*\psi)$$

OR, LOCALLY ON M,

$$(\mathcal{A},\psi)\cdot\gamma = (\mathcal{A}+2\gamma^{-1}d\gamma,\ \gamma^{-1}\psi)$$

BEFORE PROVING OUR MAJOR RESULT ON SOLUTIONS (A,ψ) TO (SW) (THAT THE ACTION OF $\mathcal{G}(\mathcal{L})$ CARRIES SOLUTIONS TO SOLUTIONS) WE NEED TO OBSERVE THAT THE SPIN^c CONNECTION CORRESPONDING TO $A\cdot\gamma$ IS THE PULLBACK BY σ_γ OF THAT CORRESPONDING TO A:

$$\omega_{A\cdot\gamma} = \sigma_\gamma^*\,\omega_A$$

HERE'S THE PROOF: BY DEFINITION,

$$\sigma_\gamma^*\,\omega_A = (\text{SPIN}^c \circ \sigma_\gamma)^*(\Pr_F^*\,\omega_{LC} + \Pr_{L^0}^*A).$$

MOREOVER,

$$\omega_{A\cdot\gamma} = (\mathrm{SPIN}^c)^*(Pr_F^*\,\omega_{LC} + Pr_{L^0}^*((\sigma_\gamma')^*A))$$

$$= (Pr_F \circ \mathrm{SPIN}^c)^*\omega_{LC} + (\sigma_\gamma' \circ Pr_{L^0} \circ \mathrm{SPIN}^c)^*A$$

$$= \underbrace{(Pr_F \circ \mathrm{SPIN}^c \circ \sigma_\gamma)^*\omega_{LC}}_{\substack{\sigma_\gamma \text{ COVERS} \\ \text{THE IDENTITY} \\ \text{ON } F_{SO}(M)}} + \underbrace{(Pr_{L^0} \circ \mathrm{SPIN}^c \circ \sigma_\gamma)^*A}_{\substack{\text{DEFINITION OF} \\ \sigma_\gamma'}}$$

$$= (\mathrm{SPIN}^c \circ \sigma_\gamma)^*(Pr_F^*\,\omega_{LC} + Pr_{L^0}^*A)$$

$$= \sigma_\gamma^*\,\omega_A$$

AS REQUIRED.

ANOTHER COMPUTATION OF $\omega_{A\cdot\gamma}$ (USING $A\cdot\gamma = A + \pi_{L^0}^*(2\gamma^{-1}d\gamma)$) GIVES

$$\omega_{A\cdot\gamma} = (\mathrm{SPIN}^c)^*(Pr_F^*\,\omega_{LC} + Pr_{L^0}^*(A + \pi_{L^0}^*(2\gamma^{-1}d\gamma)))$$

$$= \omega_A + (\mathrm{SPIN}^c)^*(Pr_{L^0}^*(\pi_{L^0}^*(2\gamma^{-1}d\gamma)))$$

$$= \omega_A + (\pi_{L^0} \circ Pr_{L^0} \circ \mathrm{SPIN}^c)^*(2\gamma^{-1}d\gamma)$$

$$= \omega_A + \pi_{S^c}^*(2\gamma^{-1}d\gamma)$$

NOW FOR OUR MAJOR RESULT :

THEOREM: THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON (A, ψ) CARRIES SOLUTIONS TO (SW) ONTO OTHER SOLUTIONS TO (SW). MORE PRECISELY, IF (A, ψ) SATISFIES

$$\begin{cases} \not{\partial}_A \psi = 0 \\ F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \end{cases}$$

THEN, FOR ANY $\sigma_\gamma \in \mathcal{G}(\mathcal{L})$, $(A, \psi)\cdot\gamma = (A\cdot\gamma, \psi\cdot\gamma)$ SATISFIES

$$\begin{cases} \not{\partial}_{A\cdot\gamma}(\psi\cdot\gamma) = 0 \\ F_{A\cdot\gamma}^+ = \sigma^+\big(((\psi\cdot\gamma)\otimes(\psi\cdot\gamma)^*)_0\big) \end{cases}$$

PROOF: THE CURVATURE EQUATION IS EASY. WE MAY PROVE THE EQUALITY LOCALLY SO LET s BE A SECTION OF $L^0(\mathcal{L})$ AND $\mathcal{A} = s^* A$. WE HAVE SEEN THAT $\mathcal{A}\cdot\gamma = (\sigma_\gamma' \circ s)^* A$ SO $\mathcal{A}$ AND $\mathcal{A}\cdot\gamma$ ARE GAUGE POTENTIALS FOR THE SAME CONNECTION ON $L^0(\mathcal{L})$. WE HAVE ALSO SEEN THAT THE TRANSITION MAP RELATING THE TWO SECTIONS IS γ^2. CONSEQUENTLY, THE (SELF-DUAL PARTS OF THE) CURVATURES ARE RELATED BY

$$F_{\mathcal{A}\cdot\gamma}^+ = \gamma^2 F_{\mathcal{A}}^+ (\gamma^2)^{-1} = F_{\mathcal{A}}^+$$

(BECAUSE $U(1)$ IS ABELIAN). SIMILARLY, COMMUTATIVITY OF $U(1)$ GIVES

$$(\psi\cdot\gamma)\otimes(\psi\cdot\gamma)^{*} = (\gamma^{-1}\psi)\otimes(\gamma^{-1}\psi)^{*}$$
$$= (\gamma^{-1}\psi)\otimes(\gamma\psi^{*})$$
$$= (\gamma^{-1}\gamma)\,\psi\otimes\psi^{*}$$
$$= \psi\otimes\psi^{*}.$$

THUS, $F_A^+ = \sigma^+((\psi\otimes\psi^*)_0)$ IMPLIES $F_{A\cdot\gamma}^+ = \sigma^+(((\psi\cdot\gamma)\otimes(\psi\cdot\gamma)^*)_0)$.

TO VERIFY THE ANALOGOUS STATEMENT FOR THE DIRAC EQUATION IT WILL SURELY BE ENOUGH TO SHOW THAT

$$\not{D}_{A\cdot\gamma}(\psi\cdot\gamma) = (\not{D}_A\psi)\cdot\gamma\ .$$

FOR THIS WE IDENTIFY ψ WITH AN EQUIVARIANT $S_{\mathbb{C}}^+$ - VALUED MAP ON $S^c(M)$ AND COMPARE THE COVARIANT EXTERIOR DERIVATIVES $d_A\psi$ AND $d_{A\cdot\gamma}(\psi\cdot\gamma)$. STANDARD FORMULAS FOR SUCH DERIVATIVES (E.G., THEOREM 3.1.5 OF REFERENCE [9]) GIVE

$$d_A\psi = d\psi + \tfrac{1}{2}\omega_A\psi$$

AND

$$d_{A\cdot\gamma}(\psi\cdot\gamma) = d(\psi\cdot\gamma) + \tfrac{1}{2}\omega_{A\cdot\gamma}(\psi\cdot\gamma)$$

WHERE, E.G., ω_A TAKES VALUES IN $\mathfrak{spin}^c(4)$, IDENTIFIED WITH A SUBSET OF $Cl(4)\otimes\mathbb{C}$, AND SO $\omega_A\psi$ IS A MATRIX PRODUCT.

NOW WE COMPUTE

$$
\begin{aligned}
d_{A\cdot\gamma}(\psi\cdot\gamma) &= d_{A\cdot\gamma}\left((\gamma\circ\pi_{sc})^{-1}\psi\right) \\
&= d\left((\gamma\circ\pi_{sc})^{-1}\psi\right) + \tfrac{1}{2}\,\omega_{A\cdot\gamma}\left((\gamma\circ\pi_{sc})^{-1}\psi\right) \\
&= (\gamma\circ\pi_{sc})^{-1}d\psi - (\gamma\circ\pi_{sc})^{-2}d(\gamma\circ\pi_{sc})\psi \\
&\quad + \tfrac{1}{2}\left(\omega_A + 2(\gamma\circ\pi_{sc})^{-1}d(\gamma\circ\pi_{sc})\right)\left((\gamma\circ\pi_{sc})^{-1}\psi\right) \\
&= (\gamma\circ\pi_{sc})^{-1}\left(d\psi + \tfrac{1}{2}\,\omega_A\psi\right) - (\gamma\circ\pi_{sc})^{-2}d(\gamma\circ\pi_{sc})\psi \\
&\qquad + (\gamma\circ\pi_{sc})^{-2}d(\gamma\circ\pi_{sc})\psi \\
&= (\gamma\circ\pi_{sc})^{-1}(d_A\psi)
\end{aligned}
$$

THUS,

$$d_{A\cdot\gamma}(\psi\cdot\gamma) = (\gamma\circ\pi_{sc})^{-1}(d_A\psi)$$

AND SO

$$\nabla_{A\cdot\gamma}(\psi\cdot\gamma) = (\gamma\circ\pi_{sc})^{-1}\,\nabla_A\psi\ .$$

NOW, FOR THE DIRAC OPERATORS WE HAVE

$$
\begin{aligned}
\not{\partial}_{A\cdot\gamma}(\psi\cdot\gamma) &= \sum_{i=1}^{4} E_i\cdot\nabla_{A\cdot\gamma}\psi(E_i) \\
&= \sum_{i=1}^{4} E_i\cdot(\gamma\circ\pi_{sc})^{-1}\,\nabla_A\psi(E_i) \\
&= (\gamma\circ\pi_{sc})^{-1}\,\not{\partial}_A\psi = (\not{\partial}_A\psi)\cdot\gamma
\end{aligned}
$$

AS REQUIRED. $\square$

BEFORE LEAVING THE SUBJECT OF THE GAUGE ACTION ON (A, ψ) LET US DETERMINE THE REDUCIBLES :

<u>THEOREM</u> : (A, ψ) IS LEFT FIXED BY SOME NON-IDENTITY ELEMENT σ_γ OF $\mathcal{G}(\mathcal{L})$ IF AND ONLY IF $\psi \equiv 0$ AND, IN THIS CASE, $\gamma : M \to U(1)$ MUST BE A CONSTANT MAP.

<u>PROOF</u> : SUPPOSE $(A, \psi) \cdot \sigma_\gamma = (A, \psi)$. THEN,

$$(\mathcal{A} + 2\gamma^{-1} d\gamma \, , \, \gamma^{-1}\psi) = (A, \psi)$$

SO

$$2\gamma^{-1} d\gamma = 0 \quad \text{AND} \quad \gamma^{-1}\psi = \psi$$

SINCE $\gamma \neq 1$, THE SECOND OF THESE IMPLIES $\psi \equiv 0$. THE FIRST GIVES $d\gamma = 0$ AND, SINCE M IS CONNECTED, γ IS CONSTANT. CONVERSELY, IF $\psi \equiv 0$, THEN (A, ψ) IS LEFT FIXED BY ANY σ_γ WITH γ CONSTANT. $\square$

4. A PRIORI BOUNDS (AND COMPACTNESS)

HERE WE WILL SKETCH THE PROOF THAT ANY SOLUTION (A, ψ) TO

$$(SW) \qquad \begin{aligned} \not{D}_A \psi &= 0 \\ F_A^+ &= \sigma^+((\psi \otimes \psi^*)_0) \end{aligned}$$

EITHER HAS $\psi \equiv 0$ ((A,ψ) REDUCIBLE) OR

$$\|\psi(x)\|^2 \le \mathcal{K}(M) := \max \{ -\tfrac{1}{2}\mathcal{K}(x) : x \in M \}$$

WHERE $\mathcal{K}(x)$ IS THE SCALAR CURVATURE OF M AT x.

> NOTE: IN PARTICULAR, IF THE METRIC HAS POSITIVE SCALAR CURVATURE, THEN ALL SOLUTIONS (A,ψ) HAVE $\psi \equiv 0$.

WE WRITE THE SECOND (SW) EQUATION AS

$$\rho^+(F_A) = (\psi \otimes \psi^*)_0$$

(ADDENDUM 14, PAGE 25).

WE WILL ALSO APPEAL TO THE FAMOUS WEITZENBÖCK FORMULA FROM DIFFERENTIAL GEOMETRY WHICH, IN OUR PRESENT CIRCUMSTANCES, ASSERTS THAT

(1) $$\not{D}_A^* \circ \not{D}_A \psi = \nabla_A^* \circ \nabla_A \psi + \tfrac{1}{4} \mathcal{K} \psi + \rho^+(F_A)\psi$$

WHERE THE $*$ INDICATES FORMAL ADJOINT AND $\mathcal{K}$ IS THE SCALAR CURVATURE OF M.

BECAUSE (A, ψ) SATISFIES (SW) THIS REDUCES TO

(2) $$0 = \nabla_A^* \circ \nabla_A \psi + \tfrac{1}{4} \mathcal{K} \psi + (\psi \otimes \psi^*)\psi .$$

TAKE THE POINTWISE INNER PRODUCT OF BOTH SIDES OF (2) WITH ψ TO GET

(3) $$0 = \langle \nabla_A^* \circ \nabla_A \psi(x), \psi(x) \rangle + \tfrac{1}{4} \mathcal{K}(x) \|\psi(x)\|^2 + \tfrac{1}{2} \|\psi(x)\|^4$$

(FOR THE LAST TERM USE (44) OF ADDENDUM 14 TO COMPUTE $(\psi \otimes \psi^*)_0 \psi$).

NOW, $\|\psi\|^2$ IS A CONTINUOUS FUNCTION ON THE COMPACT SPACE M SO THERE IS AN $x_0 \in M$ AT WHICH IT ACHIEVES AN ABSOLUTE MAXIMUM VALUE.
WE CLAIM THAT, AT THIS POINT

(4) $$\langle \nabla_A^* \circ \nabla_A \psi(x_0), \psi(x_0) \rangle \geq 0$$

(NOTE THAT (3) IMPLIES THAT $\langle \nabla_A^* \circ \nabla_A \psi(x), \psi(x) \rangle$ IS REAL FOR ALL x).

THE PROOF OF (4) DEPENDS ON THE FOLLOWING IDENTITY:

(5) $$\Delta_g \|\psi\|^2 = -2\|\nabla_A \psi\|^2 + 2\langle \nabla_A^* \circ \nabla_A \psi, \psi \rangle$$

WHERE $\Delta_g = \delta \circ d$ IS THE SCALAR HODGE LAPLACIAN OF g. THIS CAN BE VERIFIED BY WRITING OUT THE LAPLACIAN IN A LOCAL ORTHONORMAL FRAME FIELD E_1, E_2, E_3, E_4 ON M:

$$\Delta_g \|\psi\|^2 = -\sum_{i=1}^{4} (\partial_i \partial_i \|\psi\|^2 + \mathrm{div}(E_i)\partial_i \|\psi\|^2$$

RECALL : THE (COVARIANT) DIVERGENCE $\mathrm{div}\, V$ OF A VECTOR FIELD V IS DEFINED TO BE THE TRACE OF THE ENDOMORPHISM $X \to \nabla_X V : TM \to TM$ AND, FOR A LOCAL ORTHONORMAL FRAME

$$\mathrm{div}(E_i) = -\sum_j T^i_{jj}$$

WHERE $T^k_{ij} = g(\nabla_{E_i} E_j, E_k)$.

$$= -2\sum_i (\partial_i \mathrm{Re}\langle \psi, \nabla_i \psi \rangle + \mathrm{div}(E_i)\mathrm{Re}\langle \psi, \nabla_i \psi \rangle)$$

$$= -2\sum_i \|\nabla_i \psi\|^2 - 2\sum_i \mathrm{Re}\langle \psi, \nabla_i \nabla_i \psi + \mathrm{div}(E_i)\nabla_i \psi \rangle$$

$$= -2\|\nabla_A \psi\|^2 + 2\,\mathrm{Re}\langle \psi, \nabla_A^* \circ \nabla_A \psi \rangle$$

BECAUSE $\nabla_i^* = -\nabla_i - \mathrm{div}(E_i)$

$$= -2\|\nabla_A \psi\|^2 + 2\langle \psi, \nabla_A^* \circ \nabla_A \psi \rangle$$

FOR SOLUTIONS TO (SW) BY (3).

NOW, TO PROVE (4) WE APPLY (5) AT X_0 TO OBTAIN

$$2 \langle \nabla_A^* \circ \nabla_A \psi(x_0) \rangle = \Delta_g \|\psi\|^2(x_0) + 2\|\nabla_A \psi\|^2(x_0).$$

THE SECOND TERM IS OBVIOUSLY NONNEGATIVE. SINCE $\|\psi\|^2$ ACHIEVES A MAXIMUM AT X_0 (AND SINCE THE HODGE LAPLACIAN HAS THAT ANNOYING EXTRA MINUS SIGN) THE SAME IS TRUE OF THE FIRST TERM SO (4) IS PROVED.

NOW EVALUATE (3) AT X_0 AND USE (4) TO CONCLUDE THAT

$$\frac{1}{4} \mathcal{K}(x_0) \|\psi(x_0)\|^2 + \frac{1}{2} \|\psi(x_0)\|^4 \leq 0,$$

THUS,

$$\|\psi(x_0)\|^4 \leq -\frac{1}{2} \mathcal{K}(x_0) \|\psi(x_0)\|^2 .$$

THERE ARE TWO POSSIBILITIES. EITHER $\|\psi(x_0)\| = 0$, IN WHICH CASE $\psi \equiv 0$, OR

$$\|\psi(x_0)\|^2 \leq -\frac{1}{2} \mathcal{K}(x_0)$$

AND CONSEQUENTLY

$$(6) \qquad \|\psi(x)\|^2 \leq -\frac{1}{2} \mathcal{K}(x_0)$$

$\forall\, x \in M$.

NOTE : WE ARE LOOKING FOR A UNIFORM BOUND ON $\|\psi(x)\|^2$ FOR ALL SOLUTIONS (A, ψ) TO (SW) AND (6) WILL NOT DO SINCE x_0 GENERALLY DEPENDS ON ψ.

NOW, $-\frac{1}{2}\kappa(x)$ IS A CONTINUOUS FUNCTION ON THE COMPACT SPACE M SO WE MAY LET

$$\kappa(M) = \max\left\{-\tfrac{1}{2}\kappa(x) : x \in M\right\}$$

AND CONCLUDE THAT

$$\|\psi(x)\|^2 \leq \kappa(M) \qquad \forall x \in M.$$

THUS, FOR ANY METRIC AND ANY SPIN^c STRUCTURE AND ANY SOLUTION (A, ψ) TO (SW), THE SPINOR PART ψ IS UNIFORMLY BOUNDED BY (0 IF THE SOLUTION IS REDUCIBLE AND, OTHERWISE) THE GEOMETRICAL CONSTANT $\kappa(M)$.

THE SECOND (SW) EQUATION NOW GIVES A UNIFORM BOUND ON THE SELF-DUAL PART OF THE CURVATURE FOR ANY SOLUTION. A BIT MORE WORK THE GIVES A BOUND ON THE ANTI-SELF-DUAL PART OF THE CURVATURE, AND THUS ON THE CURVATURE ITSELF.

THESE UNIFORM BOUNDS ON Ψ AND F_A ARE, BY THEMSELVES, NOT SUFFICIENT TO PROVE THE COMPACTNESS OF THE MODULI SPACE.

> NOTE: THEY ARE, HOWEVER, SUFFICIENT TO PROVE THAT, FOR A GIVEN g, THERE ARE AT MOST FINITELY MANY EQUIVALENCE CLASSES OF SPIN^c STRUCTURES FOR WHICH THE MODULI SPACE IS NONEMPTY AND HAS NONNEGATIVE FORMAL DIMENSION.

FOR THIS ONE MUST ALSO BOUND THE CONNECTION PART A OF A SOLUTION (A, Ψ) "UP TO GAUGE".

AT THIS POINT, HOWEVER, THE "SMOOTH" ARGUMENTS WE HAVE RELIED UPON THUS FAR FAIL US AND ONE MUST PROCEED BY WAY OF A "BOOTSTRAPING" ARGUMENT THROUGH SOBOLEV SPACES OF VARYING INDEX. SINCE THIS USES THE ENTIRE ARSENAL OF SOBOLEV EMBEDDING AND MULTIPLICATION THEOREMS, THE RELLICH THEOREM, ETC., I WILL NOT ATTEMPT A QUICK SYNOPSIS. A REASONABLY CONCISE DISCUSSION OF THE ARGUMENT IS AVAILABLE ON PAGES 80 - 85 OF REFERENCE [36].