ADDENDUM 17 : PHYSICS BEHIND WITTEN'S CONJECTURE

TAKEN FROM

SEIBERG-WITTEN GAUGE THEORY

MATILDE MARCOLLI

HINDUSTAN BOOK AGENCY

1999

ISBN 81-85931-22-4

### 5.2.1 The Physics way: $S$-duality

The physicists' approach to the equivalence of Seiberg–Witten and Donaldson theory is based on Witten's interpretation of Donaldson's theory as a twisted supersymmetric Quantum Field Theory [64] and on the concept of electro-magnetic duality. We attempt here a very rough overview of some of these topics. From the mathematician's point of view this concept of "duality" is rather mysterious; however, we'll try to present the basic ideas, mainly based on [2], [63], and on the exposition [8]. We especially recommend the very nice introduction to $S$-duality given in [18].

**Maxwell equations**

The first appearance of electromagnetic duality is in Maxwell equations. It is well known that the Maxwell equations in vacuum can be written as

$$dF = 0 \quad d^*F = 0,$$

where $F = dA$ is an imaginary 2-form, the curvature of a $U(1)$ bundle with connection $A$. In Physics notation one would write

$$E_k = -iF_{k4}$$

for the electric field, and

$$B^k = -i\frac{1}{2}\epsilon^{kpq}F_{pq}$$

for the magnetic field. The symbol $\epsilon^{kpq}$ is $\pm 1$ according to the sign of the permutation $\{k, p, q\}$ of $\{1, 2, 3\}$ and zero if any two indices are equal.

It is clear that there is a symmetry given by the Hodge $*$-operator

$$F \mapsto *F$$

that preserves the equations and interchanges electric and magnetic fields.

The Maxwell equations are no longer invariant under the $*$-operator if one considers the presence of electric charges and electromagnetic currents, unless one postulates the existence of isolated magnetic charges, namely magnetic monopoles.

Magnetic monopoles satisfy a quantisation condition which states that the magnetic and electric charges are related by

$$m = \frac{2\pi}{e}.$$

There is an elegant topological motivation for this quantisation condition which is beautifully explained by Raoul Bott in [11].

There is an analogue of electromagnetic duality for monopoles in non-abelian field theory, where again one can see that electric and magnetic charges live in dual lattices and the magnetic charge can be given a topological meaning.

The electric charge enters the Lagrangian as a coupling constant (as we are going to discuss in a moment). Thus, one can see how electromagnetic duality interchanges weak and strong coupling (a small with a large coupling constant). Interchanging a weak with a strong coupling means to exchange the range in which perturbative theory can be applied with one in which it cannot. This will be discussed in the following.

### Modular forms

In the abelian context, that is, with structure group $U(1)$, we can write the Lagrangian density on a four-manifold $X$ as

$$\mathcal{L} = \frac{1}{8\pi} \int_X (\frac{4\pi}{e^2} F \wedge *F + \frac{i\theta}{2\pi} F \wedge F).$$

The second part of the Lagrangian density is a topological term,

$$\frac{i\theta}{2\pi} c_1(L)^2,$$

where $L$ is the chosen line bundle on which the Maxwell equations are considered. The angle $\theta$ is the $U(1)$-symmetry of the vacuum state.

Upon setting

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2},$$

one can rewrite $\mathcal{L}$ in terms of $\tau$,

$$\mathcal{L} = \frac{1}{8\pi}\int_X (\bar{\tau}(F^+)^2 - \tau(F^-)^2)dv.$$

The partition function, formally written as an infinite dimensional integral

$$Z \sim \int e^{-\mathcal{L}}\mathcal{D}A,$$

is invariant under the transformation

$$\tau \mapsto \tau + 2.$$

Under the transformation

$$\tau \mapsto \tau + 1$$

we have

$$Z \mapsto Z \cdot e^{\pi i c_1(L)^2}.$$

In the case of a *Spin*-manifold $c_1(L)^2$ is an even integer, hence there is an invariance under $\tau \mapsto \tau + 1$. A more complicated computation with the formal rules of infinite dimensional integrals "shows" that there is also an invariance under

$$\tau \mapsto -\frac{1}{\tau}.$$

This can be viewed as a consequence of a Poisson summation formula applied formally to the infinite dimensional integrals, which leads to the result

$$Z(-\frac{1}{\tau}) = \tau^{\frac{1}{4}(\chi(X)-\sigma(X))}\bar{\tau}^{\frac{1}{4}(\chi(X)+\sigma(X))}Z(\tau).$$

This implies that $Z(\tau)$ behaves like a modular form under the action of $SL(2,\mathbb{Z})$. This fact is an appearance of the phenomenon known as Montonen-Olive duality. It is related to electromagnetic duality, since the transformation

$$\tau \mapsto -\frac{1}{\tau}$$

corresponds to

$$F \mapsto *F,$$

in the sense that all the expectation values are preserved under the combined action of the transformations together. Thus, the modularity can be thought of as a refined version of the Hodge duality which manifests itself at the quantum level.

We should remark, however, that the picture presented here is quite incomplete. In fact it ignores the essential role of supersymmetry.

In the case of non-abelian monopoles the analogous phenomenon happens if one considers the Lagrangian density

$$\mathcal{L} = \frac{1}{g^2} \int_X Tr(F \wedge *F) + \frac{i\theta}{8\pi^2} \int_X Tr(F \wedge F).$$

The presence of the coefficient $\frac{1}{g^2}$ depends on the fact that the Killing form on the compact Lie group $G$ is only defined up to a scalar multiple which is usually set equal to one in the mathematical literature, while it appears in Physics as a coupling constant. The second term represents the second Chern class of the vector bundle $E$ on $X$ on which the connection and curvature $F = dA + A \wedge A$ are considered. The fact that this topological term appears explicitly in the Lagrangian is already an effect of the presence of $N = 1$ supersymmetry. In fact also other terms appear in the partition function that contain the "auxiliary fields" introduced by the supersymmetry. These are the analogue of the elements of the algebra $\Lambda[w]$ in our definition of the fermionic integral in relation to the Mathai-Quillen formalism. The fact that the vacuum state (that is, the minimum of the classical potential) has a $U(1)$-symmetry which explains the presence of the angle $\theta$ is also an effect of the presence of the "unbroken" supersymmetry.

Thus the partition function can be formally written as

$$\int e^{-\mathcal{L}} \mathcal{D}A = \sum_{r=c_2(E)} e^{ir\theta} \int e^{-\mathcal{L}_r} \mathcal{D}A,$$

where $\mathcal{L}_r = \frac{1}{g^2} \int_X Tr(F \wedge *F)$ on the fixed bundle $E$.

Again one can introduce the variable

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

The modularity in this context can be formulated in a different way, which leads to an interesting conjecture [61].

**Conjecture 5.2.5** *consider the expression*

$$Z_G(\tau) = q^c \sum_{r=0}^{\infty} \chi_r q^r,$$

*where* $q = e^{2\pi i\tau}$ *and* $\chi_r$ *is some suitable regularised Euler characteristic of the moduli space of* $G$ *instantons on the four-manifold* $X$ *with instanton number* $r = c_2(E)$. *There is an action of* $SL(2,\mathbb{Z})$ *and* $Z_G$ *transforms like*

$$Z_G(-\frac{1}{\tau}) \sim Z_{\tilde{G}}(\tau),$$

*where* $\tilde{G}$ *is the Langlands dual of* $G$.

We do not discuss this statement any further, but just mention that the Langlands dual interchanges the torus lattice with its dual. It is thus related to electromagnetic duality for non-abelian monopoles.

### Weak and strong coupling

As we have seen in the example of Maxwell theory, the interchanging of electric and magnetic charges due to Hodge duality also interchanges weak and strong coupling in the action. When the coupling constant is small, one can formally compute the infinite dimensional integral by means of a stationary phase approximation [11], [63]. The model is the finite dimensional situation in which one has a function $F(x)$ with an isolated minimum at $x = 0$. We can write $F(x) = F(0) + \frac{1}{2}Q(x) + \cdots$ and for a small coupling constant we can approximate the integral

$$Z = \int e^{-\frac{1}{\lambda}F(x)} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}} \sim Z_0 = \int e^{-\frac{1}{\lambda}(F(0)+\frac{1}{2}Q(x))} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}}.$$

The latter can be computed exactly and it gives

$$Z_0 = e^{-\frac{1}{\lambda}F(0)} \frac{\lambda^{n/2}}{\det(Q)^{1/2}}.$$

The finite dimensional computation can be easily related to the computation of the Pfaffian that we presented in relation to the Mathai-Quillen formalism, with the only difference that the matrix $Q$ is symmetric instead of antisymmetric. This explains why one gets $\det(Q)^{-1/2}$ instead of $\det(Q)^{1/2} = Pf(Q)$.

In order to generalise this argument to the infinite dimensional context, the problem is reformulated in terms of a functional $F$ with non-degenerate minima. The approximation of the partition function in this case can be taken to be the well defined mathematical object $\det(Q)^{-1/2}$, where $Q$ is a positive elliptic operator (the Hessian of the functional $F$ at a minimum) and the determinant is the Ray-Singer determinant [55], [56]. If the coupling constant is large this approximation method no longer works and the partition function is in general no longer computable.

**The $u$-plane**

In the case of $N = 2$ supersymmetry, the auxiliary fields that are introduced can be described as two independent variables of the type of the $\Lambda[w]$ used in the definition of the fermionic integral, and a field $\phi$ which is a section of the adjoint bundle of $E$. The classical potential can be written as a function $V(\phi)$ and as mentioned before the supersymmetry imposes that in the vacuum state $V(\phi) = 0$. This allows for certain symmetries of the vacuum. This means that the vacuum state is not an isolated point but there is some parametrisation of a certain manifold of possible vacuum states. In our case the parameter that classifies inequivalent vacua is $Tr(\phi^2)$.

This is better said by introducing a variable $u = \langle Tr(\phi^2)\rangle$ which is the expectation value (with respect to the partition function $Z$) of $Tr(\phi^2)$. The expectation value of the field $\phi$ is proportional to a variable $a$, $\langle\phi\rangle \sim a$. In the classical limit, that is, when the coupling is weak, one has the relation $u \sim \frac{1}{2}a^2$. In the strong coupling range the relation is more complicated.

In terms of the parameter $a$ one has the corresponding modulus

$$\tau(a) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)}.$$

The symmetry of the action under the transformation $\tau \mapsto \frac{-1}{\tau}$ can be formally described in terms of a Legendre transformation over a potential (called *prepotential* in the Physics literature) $\mathcal{F}$. In fact, a dual variable $a_D$ is introduced by the relation

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}, \tag{5.13}$$

and a dual field $\phi_D$ is defined by $\langle\phi_D\rangle \sim a_D$. Here "dual" is intended in analogy to coordinates and moments in classical mechanics that are related by a Legendre transform similar to (5.13). The transformation $\tau \mapsto \frac{-1}{\tau}$ exchanges the action $Z$ with a dual action $Z_D$ where the field $\phi$ is replaced with $\phi_D$ and $a$ with $a_D$. This exchanges weak and strong coupling.

The reason why this can be still thought of as electromagnetic duality is that one thinks of the purely electric or purely magnetic charge as quantities $q_e = n_e a$ and $q_m = n_m a_D$, for a pair of integers $(n_e, n_m)$. One can also consider states (which are called *dyons* in the literature) that have both electric and magnetic charge $q = n_e a + n_m a_D$. The group

$SL(2;\mathbb{Z})$ acts by mixing the electric and the magnetic charge

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}.$$

If one wants to express the variables $a$ and $a_D$ as functions of the parameter that determines the vacuum state, $a(u)$ and $a_D(u)$, one gets two multivalued functions, defined for $u \in \mathbb{C}$ with branch cuts. In particular one can compute the monodromy at the branch points [8]. One point is certainly the one at infinity, where the weak coupling range is attained. In this case the prepotential takes the form $\mathcal{F}(a) \sim \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}$ and as $u \mapsto e^{2\pi i} u$ one has $a \mapsto -a$ and $a_D \mapsto -a_D + 2a$. Thus the monodromy at $u = \infty$ is

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

An argument depending on the factorisation of the matrix

$$M_\infty \in SL(2;\mathbb{Z})$$

shows that there are other two branching points. Up to the choice of a normalising constant these can be taken to be $u = \pm 1$. As $u \to \pm 1$ the strong coupling range is attained. The corresponding monodromies [8] are

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

and

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

The physical interpretation of the eigenvalues of the monodromy matrices leads to interpreting these branch points as the vacua at which a magnetic monopole (when $u = 1$) or a $(1, -1)$-dyon (when $u = -1$) become massless.

### Elliptic curves

Given the data obtained above by physical arguments, namely the punctured sphere $\mathbb{C}P^1 - \{\infty, 1, -1\}$ (or $u$-plane) and the prescribed monodromies at the punctures, it is possible to proceed with a rigorous construction. The monodromies obtained above span a subgroup $\Gamma(2)$ in $SL(2;\mathbb{Z})$. The $u$-plane is equivalent to the quotient of the upper half plane with respect to the group $\Gamma(2)$. This gives the moduli of the family of elliptic curves

$$y^2 = (x^2 - 1)(x - u)$$

that becomes singular at the points $u = \pm 1$.

The functions $a(u)$ and $a_D(u)$ can be interpreted within this geometric picture as the periods

$$a = \int_{\gamma_1} \lambda \qquad a_D = \int_{\gamma_2} \lambda \ ,$$

with

$$\lambda = \frac{\sqrt{2(x-u)}}{2\pi\sqrt{x^2-1}} dx.$$

The relation of all this with the Witten conjecture comes when one reads the weak coupling limit of $Z$ as Donaldson theory (that is twisted $N = 2$ supersymmetric Yang-Mills theory) and the strong coupling limit of $Z$ as the Seiberg–Witten theory. Then the idea that leads to the equivalence of the two theories is that the geometric data encoded in this family of elliptic curves should provide "the gluing instructions" of how to interpolate for all values of $u$ knowing the asymptotic behaviour at the singular points. The relation obtained would then be in the form given by Kronheimer and Mrowka, as in theorem 5.2.2.

More recently, the conjecture 5.2.3 has been extended by Moore and Witten [50] to the case of manifolds with $b_2^+(X) = 1$. In this case a correction term to the relation 5.2.3 comes from integration over the $u$-plane.