ADDENDUM 7

UNIVERSAL THOM FORM FOR $\mathbb{R}^2$

$V = \mathbb{R}^2$ (USUAL ORIENTATION AND INNER PRODUCT)

$\{\psi^1, \psi^2\}$ = STANDARD BASIS

$\{u_1, u_2\}$ = DUAL BASIS (COORDINATE FUNCTIONS)

$SO(V) \equiv SO(2)$

$\{\xi_1\} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$\{x^1\}$ = DUAL BASIS

WE COMPUTE

$$\nu = (2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^T du - \frac{1}{2}\sum_\ell \psi^\ell x^a M_a \psi^\ell} D\psi .$$

$$-\tfrac{1}{2}\|u\|^2 = -\tfrac{1}{2}(u_1^2 + u_2^2)$$

$$i\psi^T du = i\psi^j du_j = i(\psi^1 du_1 + \psi^2 du_2)$$

$$x^a M_a = x^1 M_1 = x^1 M_{\xi_1} = x^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^1 \\ -x^1 & 0 \end{pmatrix}$$

$$\begin{aligned}
\sum_\ell \psi^\ell x^a M_a \psi^\ell &= \psi^1 x^1 M_1 \psi^1 + \psi^2 x^1 M_1 \psi^2 \\
&= \psi^1 \begin{pmatrix} 0 & x^1 \\ -x^1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi^2 \begin{pmatrix} 0 & x^1 \\ -x^1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \psi^1 \begin{pmatrix} 0 \\ -x^1 \end{pmatrix} + \psi^2 \begin{pmatrix} x^1 \\ 0 \end{pmatrix} \\
&= \psi^1 \left(-x^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + \psi^2 \left(x^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= -\psi^1 x^1 \psi^2 + \psi^2 x^1 \psi^1 \\
&= x^1(-\psi^1\psi^2 + \psi^2\psi^1) \\
&= -2x^1 \psi^1 \psi^2
\end{aligned}$$

$$-\frac{1}{2}\sum_{\ell}\psi^{\ell}\chi^{a}\Pi_{a}\psi^{\ell} = \chi^{1}\psi^{1}\psi^{2}$$

THUS,

$$\nu = (2\pi)^{-1}\int e^{-\frac{1}{2}(u_1^2+u_2^2)+i(\psi^1 du_1+\psi^2 du_2)+\chi^1\psi^1\psi^2}\,\mathcal{D}\psi$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\int e^{i(\psi^1 du_1+\psi^2 du_2)}\, e^{\chi^1\psi^1\psi^2}\,\mathcal{D}\psi$$

(EACH TERM IN THE EXPONENT IS EVEN
SO THEY COMMUTE)

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [1+i(\psi^1 du_1+\psi^2 du_2)-\frac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2][1+\chi^1\psi^1\psi^2]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [1+i(\psi^1 du_1+\psi^2 du_2)-\frac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2 + \chi^1\psi^1\psi^2 + i(\psi^1 du_1+\psi^2 du_2)\chi^1\psi^1\psi^2 - \frac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2\chi^1\psi^1\psi^2]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [-\frac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2+\chi^1\psi^1\psi^2]\,\mathcal{D}\psi$$

(ALL OTHER TERMS ARE ZERO OR DO NOT
CONTRIBUTE TO $\psi^1\psi^2$)

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [\chi^1\psi^1\psi^2 - \frac{1}{2}(\psi^1 du_1\,\psi^2 du_2 + \psi^2 du_2\,\psi^1 du_1)]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [\chi^1\psi^1\psi^2 - \frac{1}{2}(-du_1 du_2\,\psi^1\psi^2 - du_1 du_2\,\psi^1\psi^2)]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\int (\chi^1 + du_1 du_2)\,\psi^1\psi^2\,\mathcal{D}\psi$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(\chi^1 + du_1 du_2)$$