ADDENDUM 9

UNIVERSAL THOM CLASS FOR $\mathbb{R}^2$

AS IN APPENDIX 2 : $V = \mathbb{R}^2$ (USUAL ORIENTATION AND INNER PRODUCT)

$\{\psi^1, \psi^2\}$ = STANDARD BASIS

$\{u_1, u_2\}$ = DUAL BASIS (COORDINATE FUNCTIONS)

$SO(V) = SO(2)$

$\{\xi_1\} = \{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \}$

$\{x^1\}$ = DUAL BASIS

WE DERIVED THE UNIVERSAL THOM FORM IN APPENDIX 2 :

$$\nu = \underbrace{(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1\, du_2}_{\mathbb{C}^0[so(2)] \otimes \Omega^2(\mathbb{R}^2)} + \underbrace{(2\pi)^{-1} x^1 e^{-\frac{1}{2}(u_1^2+u_2^2)}}_{\mathbb{C}^1[so(2)] \otimes \Omega^0(\mathbb{R}^2)}$$

WE WILL VERIFY THE FOLLOWING GENERAL PROPERTIES OF THE UNIVERSAL THOM FORM :

1. ν IS A NONHOMOGENEOUS ELEMENT OF $\Omega^{2k}_{SO(V)}(V) = \Omega^2_{SO(2)}(\mathbb{R}^2)$ WITH

$$\int_V \nu = \int_{\mathbb{R}^2} \nu = 1$$

2. $d_{SO(V)}\nu = d_{SO(2)}\nu = 0$ SO ν DETERMINES A $[\nu] \in H^{2k}_{SO(V)}(V) = H^2_{SO(2)}(\mathbb{R}$

ν IS OBVIOUSLY NONHOMOGENEOUS AND (BECAUSE WE DOUBLED THE DEGREES IN $\mathbb{C}[so(V)]$) EACH TERM HAS DEGREE 2. TO SHOW THAT ν IS

SO(2)-INVARIANT WE EXAMINE EACH TERM SEPARATELY. RECALL THAT FOR HOMOGENEOUS ELEMENTS $\alpha = \rho \otimes \varphi$ OF $\mathbb{C}[so(2)] \otimes \Omega^*(\mathbb{R}^2)$, SO(2)-INVARIANCE MEANS

$$\rho(g^{-1}\xi g)\,\sigma_{g^{-1}}^{*}\varphi = \rho(\xi)\varphi$$

$\forall\, g \in SO(2)\ \forall\, \xi \in so(2)$. FOR

$$(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\, du_1\, du_2 \in \mathbb{C}^0[so(2)] \otimes \Omega^2(\mathbb{R}^2)$$

THE ρ-PART (BEING CONSTANT) OBVIOUSLY SATISFIES $\rho(g^{-1}\xi g) = \rho(\xi)\ \forall\, \xi\ \forall\, g$. MOREOVER, THE φ-PART IS A ROTATIONALLY INVARIANT MULTIPLE OF THE VOLUME FORM ON $\mathbb{R}^2$ SO IT SATISFIES $\sigma_{g^{-1}}^{*}\varphi = \varphi\ \ \forall\, g$. THUS, THIS FIRST PIECE IS OBVIOUSLY SO(2)-INVARIANT. FOR

$$(2\pi)^{-1} x^1 e^{-\frac{1}{2}(u_1^2+u_2^2)} \in \mathbb{C}^1[so(2)] \otimes \Omega^0(\mathbb{R}^2)$$

THE φ-PART $(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}$ IS AGAIN ROTATIONALLY INVARIANT SO $\sigma_{g^{-1}}^{*}\varphi = \varphi$. THE ρ PART IS x^1, BUT IT DOESN'T REALLY MATTER WHAT IT IS SINCE SO(2) IS ABELIAN SO $\rho(g^{-1}\xi g) = \rho(\xi)$ FOR ANY ρ.

RECALLING THE DEFINITION OF THE INTEGRAL

$$\int_V : \Omega^*_{SO(V)}(V) \to \mathbb{C}[so(V)]^{SO(V)}$$

$$\left(\int_V \alpha\right)(\xi) = \int_V \alpha(\xi) := \int_V \alpha(\xi)_{[2k]}$$

WE HAVE

$$\left(\int_{\mathbb{R}^2} \nu\right)(\xi) = \int_{\mathbb{R}^2} \nu(\xi)_{[2]}$$

$$= \int_{\mathbb{R}^2} (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2$$

$$= 1$$

$\forall\ \xi \in \mathfrak{so}(2)$ SO

$$\int_{\mathbb{R}^2} \nu = 1 \quad \left(\text{THE CONSTANT FUNCTION IN } \mathbb{C}[\mathfrak{so}(2)]^{SO(2)}\right)$$

ALL THAT REMAINS IS TO SHOW

$$d_{SO(2)} \nu = 0.$$

RECALL THAT

$$(d_{SO(2)} \nu)(\xi) = d(\nu(\xi)) - \iota_{\xi^\#}(\nu(\xi))$$

$\forall\ \xi \in \mathfrak{so}(2)$, LET

$$\xi = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} = \lambda\, \xi_1$$

BE AN ARBITRARY ELEMENT OF $\mathfrak{so}(2)$. THEN $X'(\xi) = \lambda$ SO

$$\nu(\xi) = \underset{\Omega^2(\mathbb{R}^2)}{\underset{\uparrow}{(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2}} + \underset{\Omega^0(\mathbb{R}^2)}{\underset{\uparrow}{(2\pi)^{-1}\lambda\, e^{-\frac{1}{2}(u_1^2+u_2^2)}}}$$

THUS,

$$d(\nu(\xi)) = 0 + (2\pi)^{-1}\lambda\, d\left(e^{-\frac{1}{2}(u_1^2+u_2^2)}\right)$$

$$d(\nu(\xi)) = (2\pi)^{-1}\lambda\, e^{-\frac{1}{2}(u_1^2+u_2^2)}(-u_1 du_1, -u_2 du_2)$$

NEXT,

$$\iota_{\xi^\#}(\nu(\xi)) = \iota_{\xi^\#}\left((2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2 + (2\pi)^{-1}\lambda\, e^{-\frac{1}{2}(u_1^2+u_2^2)}\right)$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\, \iota_{\xi^\#}(du_1 du_2) + 0$$

(THE CONTRACTION OF ANY 0-FORM IS 0.)

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\left(\iota_{\xi^\#}(du_1)du_2 + (-1)^1 du_1\, \iota_{\xi^\#}(du_2)\right)$$

(APPENDIX 3, PAGE 10, # 2)

NOW WE CLAIM THAT

$$\iota_{\xi^\#}(du_1) = -\lambda u_2 \quad \text{AND} \quad \iota_{\xi^\#}(du_2) = \lambda u_1 .$$

TO SEE THIS NOTE THAT

$$\iota_{\xi^\#}(du_1) = u_1(\xi^\#) \quad \text{AND} \quad \iota_{\xi^\#}(du_2) = u_2(\xi^\#)$$

AND $\forall\, v \in \mathbb{R}^2$, $v = v_1\psi^1 + v_2\psi^2$,

$$\xi^\#(v) = \frac{d}{dt}(\exp(-t\xi)\cdot v)\Big|_{t=0}$$

$$= \frac{d}{dt}\left((1 - t\xi + \tfrac{1}{2}t^2\xi^2 - \cdots)\cdot v\right)\Big|_{t=0}$$

$$= -\xi v = -\lambda \xi_1 v = -\lambda\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} v_1 \\ v_2\end{pmatrix}$$

$$= \begin{pmatrix} -\lambda v_2 \\ \lambda v_1\end{pmatrix}$$

THUS,

$$u_1(\xi^{\#})(v) = -\lambda v_2 = -\lambda u_2(v)$$

AND

$$u_2(\xi^{\#})(v) = \lambda v_1 = \lambda u_1(v)$$

$\forall\, v \in \mathbb{R}^2$ SO THE CLAIM FOLLOWS.

THUS,

$$\iota_{\xi^{\#}}(\nu(\xi)) = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} \left(-\lambda u_2\, du_2 - du_1\, (\lambda u_1)\right)$$

$$= (2\pi)^{-1} \lambda\, e^{-\frac{1}{2}(u_1^2+u_2^2)} \left(-u_1\, du_1 - u_2\, du_2\right)$$

$$= d(\nu(\xi))$$

SO

$$(d_{SO(2)}\nu)(\xi) = d(\nu(\xi)) - \iota_{\xi^{\#}}(\nu(\xi)) = 0$$

$\forall\, \xi \in \mathfrak{so}(2)$, I.E.,

$$d_{SO(2)}\nu = 0.$$