

DONALDSON AND SEIBERG-WITTEN INVARIANTS

THE WITTEN CONJECTURE

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DONALDSON INVARIANTS \hookrightarrow SEIBERG-WITTEN INVARIANTS

$SU(2)$

$U(1)$

$$* F_{\omega} = -F_{\omega}$$

$$\begin{cases} \not{D}_A \psi = 0 \\ F_A^+ = \sigma^+(\psi \otimes \psi^*)_0 \end{cases}$$

$$\mathcal{D}_M(x) = \exp(Q_M(x, x)/2) \sum_{\alpha \in \Lambda} 2^{m(\alpha)} SW_0(M, \alpha) \exp(c_1(L^\alpha(\alpha))(x))$$

- DONALDSON THEORY
- EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN
- SEIBERG-WITTEN INVARIANTS AND THE CONJECTURE
- ADDENDA

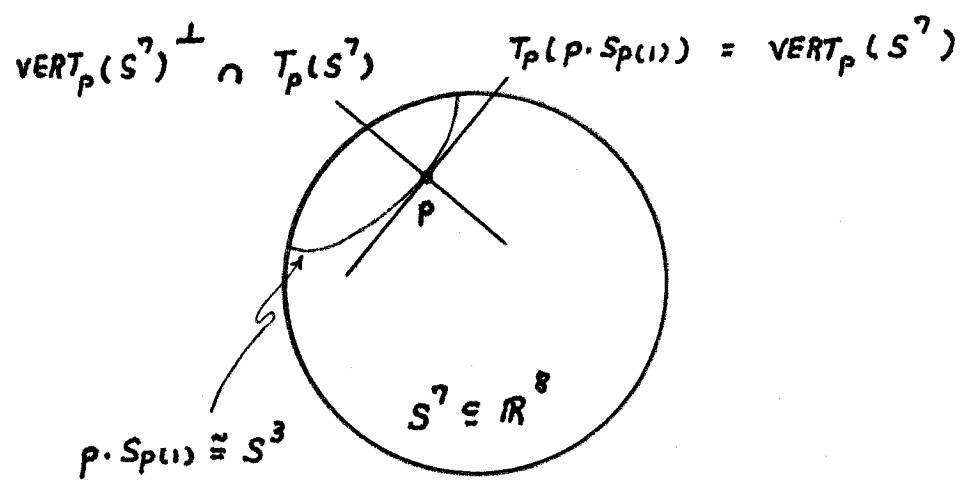
1. A SIMPLE EXAMPLE

QUATERNIONIC HOPF BUNDLE : $Sp(1) \hookrightarrow S^7 \downarrow S^4$

$Sp(1) = \{g \in \mathbb{H} : |g| = 1\} \hookrightarrow S^7 = \{p = (q^1, q^2) \in \mathbb{H}^2 : |q^1|^2 + |q^2|^2 = 1\}$
 $\cong SU(2)$
 $\cong S^3$

$p \cdot g = (q^1, q^2) \cdot g = (q^1 g, q^2 g)$

$S^7 / Sp(1) := \mathbb{H}P^1 \cong S^4$



NATURAL CONNECTION ON $Sp(1) \hookrightarrow S^7 \rightarrow S^4$: $\omega \in \Omega^1(S^7, sp(1))$

$KER \omega_p = VERT_p(S^7)^\perp \cap T_p(S^7)$
 $= HOR_p^\omega(S^7)$

$\omega = IM(\bar{q}^1 dq^1 + \bar{q}^2 dq^2)$ (RESTRICTED TO S^7)

CORRESPONDING GAUGE POTENTIAL ON $\mathbb{H} \cong \mathbb{R}^4$:

$$a = s^* \omega = \text{IM} \left(\frac{\bar{q}}{1+|q|^2} dq \right)$$

MORE GENERALLY, $(\lambda, n) \in (0, \infty) \times \mathbb{H}$ GIVES

$$\omega_{\lambda, n} \in \Omega^1(S^7, \mathfrak{sp}(1))$$

UNIQUELY DETERMINED BY

$$a_{\lambda, n} = s^* \omega_{\lambda, n} = \text{IM} \left(\frac{\bar{q} - \bar{n}}{\lambda^2 + |q-n|^2} dq \right)$$

BPST INSTANTON WITH CENTER n AND SCALE λ ($\omega = \omega_{1,0}$)

CURVATURES

$$\Omega_{\lambda, n} = d\omega_{\lambda, n} + \frac{1}{2} [\omega_{\lambda, n}, \omega_{\lambda, n}]$$

UNIQUELY DETERMINED BY GAUGE FIELD STRENGTHS

$$\mathcal{F}_{\lambda, n} = s^* \Omega_{\lambda, n} = da_{\lambda, n} + \frac{1}{2} [a_{\lambda, n}, a_{\lambda, n}] = \dots = \frac{\lambda^2}{(\lambda^2 + |q-n|^2)^2} d\bar{q} \wedge dq$$

EACH IS ANTI-SELF-DUAL (ASD)

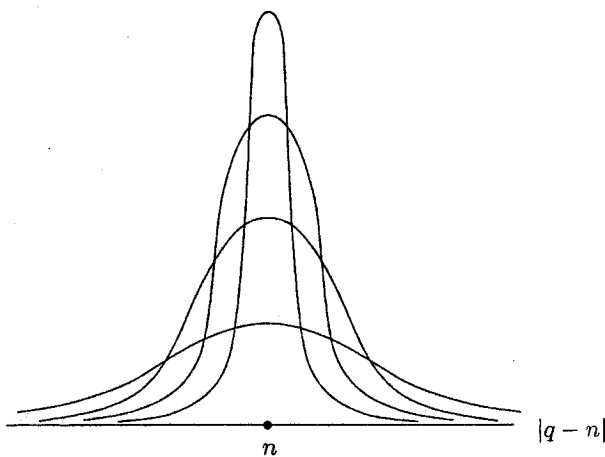
$$* \mathcal{F}_{\lambda, n} = -\mathcal{F}_{\lambda, n}$$

(* = HODGE STAR ON $\mathbb{H} \cong \mathbb{R}^4$ DETERMINED BY THE STANDARD ORIENTATION AND RIEMANNIAN METRIC)

COMPUTE

$$\begin{aligned} \frac{1}{8\pi^2} \int_{\mathbb{R}^4} -\text{Tr}(\mathcal{F}_{\lambda,n} \wedge^* \mathcal{F}_{\lambda,n}) &= \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr}(\mathcal{F}_{\lambda,n} \wedge \mathcal{F}_{\lambda,n}) = \dots \\ &= \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \frac{48\lambda^4}{(\lambda^2 + |q-n|^2)^4} d\text{Vol}_{\mathbb{R}^4} \\ &= 1 = \text{CHERN NUMBER OF } Sp(1) \hookrightarrow S^7 \rightarrow S^4 \end{aligned}$$

ADDENDUM 1 : CHERN-WEIL CHARACTERISTIC CLASSES



- $\text{Tr}(\mathcal{F}_{\lambda,n} \wedge^* \mathcal{F}_{\lambda,n})$ FOR FIXED n
AND VARIOUS λ

INCREASINGLY CONCENTRATED AT
CENTER n AS SCALE $\lambda \rightarrow 0$

GAUGE GROUP : $\mathcal{G} =$ ALL DIFFEOMORPHISMS $S^7 \xrightarrow{f} S^7$
 SATISFYING $f(p \cdot g) = f(p) \cdot g$
 AND COVERING id_{S^4}

$$\begin{array}{ccc}
 S^7 & \xrightarrow{f} & S^7 \\
 \downarrow & & \downarrow \\
 S^4 & \xrightarrow{\dots\dots\dots} & S^4 \\
 & \text{id}_{S^4} &
 \end{array}$$

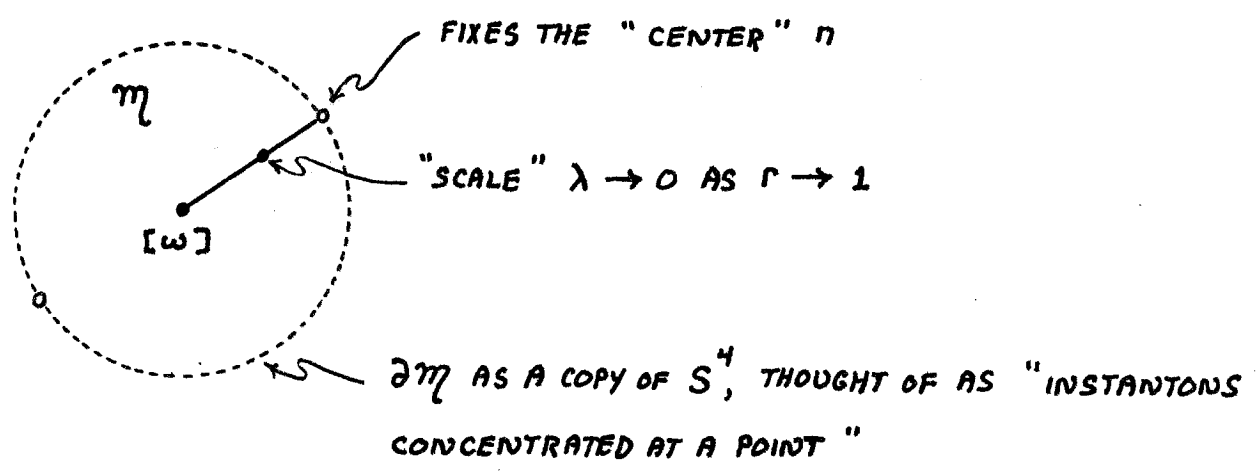
TWO CONNECTIONS ω_1 AND ω_2 ON $Sp(1) \hookrightarrow S^7 \rightarrow S^4$ ARE GAUGE EQUIVALENT IF, FOR SOME $f \in \mathcal{G}$,

$$\omega_2 = f^* \omega_1$$

$$\omega_{\lambda_1, n_1} \text{ GAUGE EQUIVALENT TO } \omega_{\lambda_2, n_2} \iff (\lambda_1, n_1) = (\lambda_2, n_2)$$

$\mathcal{M} =$ MODULI SPACE OF GAUGE EQUIVALENCE CLASSES $[\omega]$ OF ASD CONNECTIONS ω ON $Sp(1) \hookrightarrow S^7 \rightarrow S^4$

ATIYAH-HITCHIN-SINGER : $\mathcal{M} = \{ [\omega_{\lambda, n}] : (\lambda, n) \in (0, \infty) \times \mathbb{R}^4 \}$
 $\cong (0, \infty) \times \mathbb{R}^4$
 $\cong \{ x \in \mathbb{R}^5 : \|x\| < 1 \}$



2. DONALDSON'S GENERALIZATION

$M =$ COMPACT, SIMPLY CONNECTED, ORIENTED,
SMOOTH (C^∞) 4-MANIFOLD WITH

$$" b_2^+(M) = 0 "$$

REMARKS ON $b_2^+(M)$:

$$M \text{ CONNECTED} \Rightarrow H_0(M; \mathbb{Z}) \cong \mathbb{Z}$$

$$\pi_1(M) \cong 0 \Rightarrow H_1(M; \mathbb{Z}) \cong 0$$

$$\text{POINCARÉ DUALITY} \Rightarrow H_4(M; \mathbb{Z}) \cong \mathbb{Z} \text{ AND } H_3(M; \mathbb{Z}) \cong 0$$

$H_2(M; \mathbb{Z})$ IS A FINITELY GENERATED, FREE ABELIAN GROUP

(ISOMORPHIC TO $\mathbb{Z} \oplus \overset{b_2^+(M)}{\dots} \oplus \mathbb{Z}$) EACH GENERATOR OF

WHICH CAN BE REPRESENTED BY A SMOOTHLY EMBEDDED,

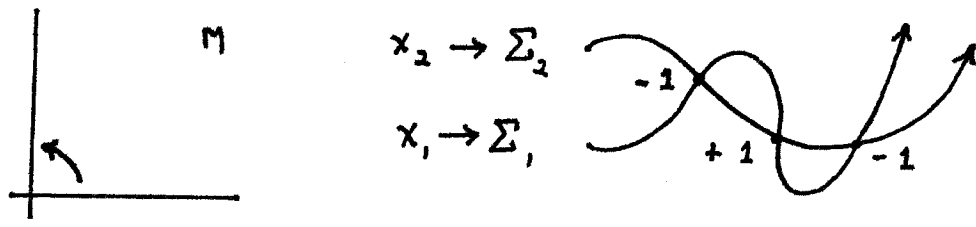
ORIENTED, CLOSED SURFACE Σ IN X

INTERSECTION FORM (ASSUMING $H_2(M; \mathbb{Z}) \neq 0$):

$$Q_M : H_2(M; \mathbb{Z}) \oplus H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

A UNIMODULAR, SYMMETRIC, \mathbb{Z} -VALUED BILINEAR FORM ON $H_2(M; \mathbb{Z})$

DEFINED AS FOLLOWS:



$Q_M(x_1, x_2) = \text{SUM OF SIGNED INTERSECTION POINTS}$

ALTERNATIVE : $Q_M : H^2(M; \mathbb{Z}) \oplus H^2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$

$$Q_M(\alpha_1, \alpha_2) = \int_M \alpha_1 \wedge \alpha_2$$

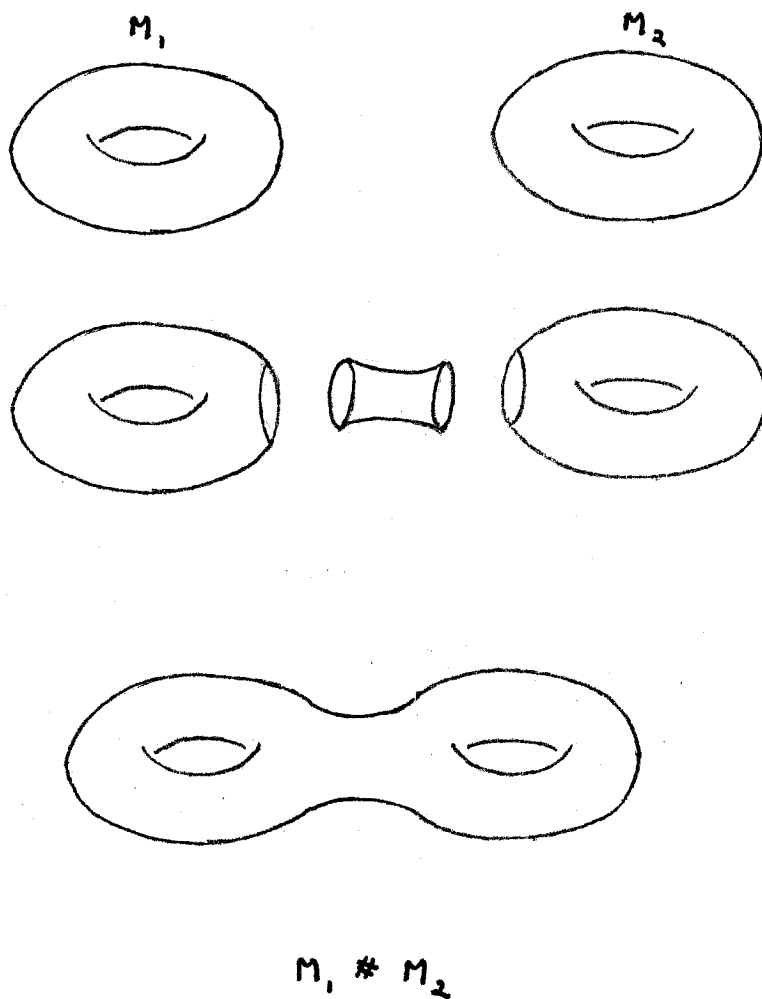
$b_2^+(M) = \text{MAXIMAL DIMENSION OF A SUBSPACE OF } H_2(M; \mathbb{Z}) \text{ ON WHICH } Q_M \text{ IS POSITIVE DEFINITE}$

= DIMENSION OF THE SPACE OF SELF-DUAL HARMONIC 2-FORMS ON M (FOR ANY CHOICE OF RIEMANNIAN METRIC ON M)

EXAMPLES :

M	$H_2(M; \mathbb{Z})$	Q_M	$b_2^+(M)$
S^4	0	ϕ	-
$\mathbb{C}P^2$	\mathbb{Z}	(1)	1
$\overline{\mathbb{C}P^2}$	\mathbb{Z}	(-1)	0
$S^2 \times S^2$	$\mathbb{Z} \oplus \mathbb{Z} = 2\mathbb{Z}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	1
K3	$22\mathbb{Z}$	$3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8)$	3

MORE EXAMPLES FROM CONNECTED SUMS :



$$H_2(M_1 \# M_2; \mathbb{Z}) \cong H_2(M_1; \mathbb{Z}) \oplus H_2(M_2; \mathbb{Z})$$

$$Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$$

E.G.,

$$Q_{m \mathbb{C}P^2 \# n \bar{\mathbb{C}}P^2} = \text{DIAG} (1 \dots 1 \quad -1 \dots -1)$$

$$b_2^+(m \mathbb{C}P^2 \# n \bar{\mathbb{C}}P^2) = m$$

DONALDSON'S THEOREM (1983): M A COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH (C^∞) 4-MANIFOLD. THEN

$$b_2^+(M) = 0 \implies Q_M = -id.$$

REMARK: Q_M CAN ALSO BE DEFINED FOR TOPOLOGICAL 4-MANIFOLDS.

FREEDMAN (1982) PROVED THAT EVERY UNIMODULAR, SYMMETRIC, \mathbb{Z} -VALUED BILINEAR FORM ON A FINITELY GENERATED, FREE ABELIAN GROUP IS THE INTERSECTION FORM OF AT LEAST ONE (AND AT MOST TWO) SUCH MANIFOLDS. COROLLARY OF DONALDSON + FREEDMAN :

THERE ARE OVER 10 MILLION TOPOLOGICALLY DISTINCT COMPACT, ORIENTED, SIMPLY CONNECTED, TOPOLOGICAL 4-MANIFOLDS M WITH $b_2(M) = 32$, NONE OF WHICH ADMIT A SMOOTH STRUCTURE.

" PROOF " $SU(2) \hookrightarrow P_1 \xrightarrow{\pi_1} M$ (CHERN CLASS 1)

CHOOSE RIEMANNIAN METRIC g ON M . GIVES HODGE STAR $*$.

A CONNECTION ω ON P_1 IS g -ANTI-SELF-DUAL (g -ASD) IF

$$* F_\omega = - F_\omega$$

(SOLUTIONS ALSO SATISFY THE YANG-MILLS EQUATIONS)

NOTE : TAUBES HAS PROVED THAT SUCH CONNECTIONS EXIST.
 HE DOES THIS BY LOCALLY "GRAFTING" A BPST INSTANTON
 ONTO M . THE GRAFTING PROCEDURE INTRODUCES A
 SMALL SELF-DUAL PART WHICH, PROVIDED $b_2^+(M) = 0$,
 CAN BE KILLED BY A SMALL PERTURBATION.

GAUGE GROUP :

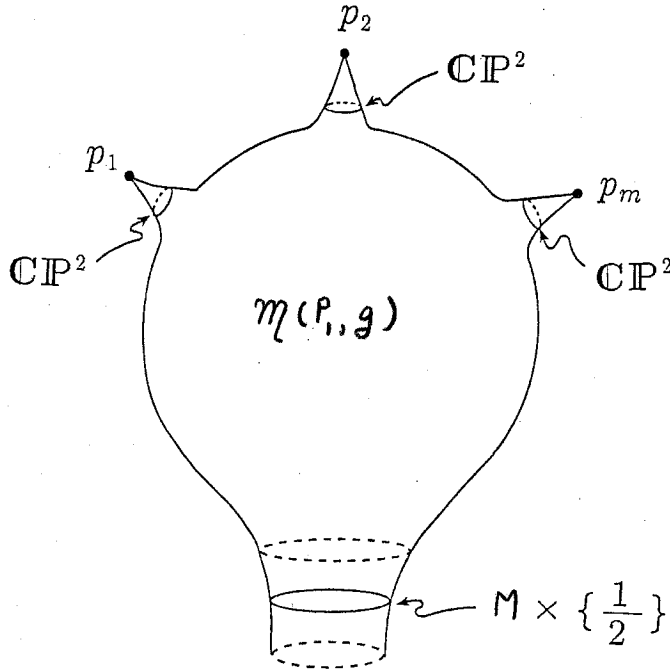
$$\begin{array}{l}
 \mathcal{G}(P, g) = \text{ALL DIFFEOMORPHISMS } P_1 \xrightarrow{f} P_2 \\
 \text{SATISFYING} \\
 f(p \cdot g) = f(p) \cdot g \\
 \text{AND COVERING } id_M. \quad \begin{array}{ccc} P_1 & \xrightarrow{f} & P_2 \\ \downarrow & & \downarrow \\ M & \xrightarrow{id_M} & M \end{array}
 \end{array}$$

TWO CONNECTIONS ω_1 AND ω_2 ARE GAUGE EQUIVALENT IF, FOR SOME $f \in \mathcal{G}(P, g)$,
 $\omega_2 = f^* \omega_1$.

$\mathcal{M}(P, g) =$ MODULI SPACE OF GAUGE EQUIVALENCE
 CLASSES OF g -ASD CONNECTIONS ON P ,

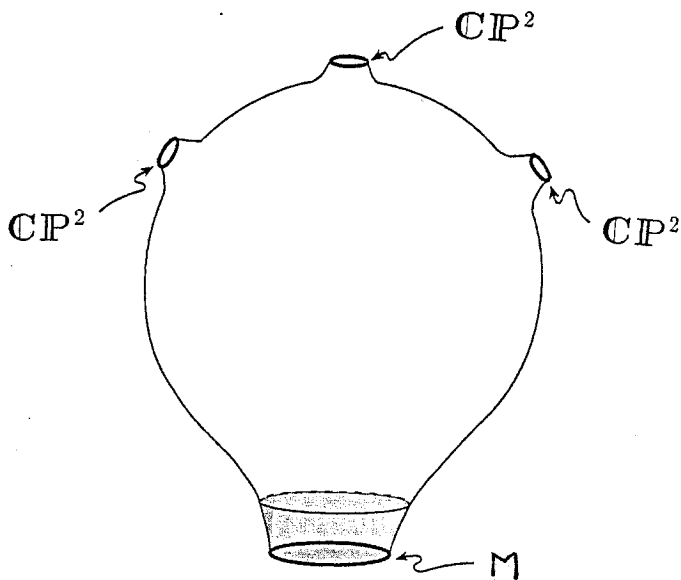
ADDENDUM 2 : THE MODULI SPACES

DONALDSON HAS SHOWN THAT, FOR A "GENERIC" RIEMANNIAN METRIC g ,
 $\mathcal{M}(P, g)$ LOOKS LIKE THIS :



- $\exists p_1, \dots, p_m \in \mathcal{M}(p, g)$
 SUCH THAT
 $\mathcal{M}(p, g) - \{p_1, \dots, p_m\}$
 IS A SMOOTH, ORIENTED
 5-MANIFOLD
- EACH p_i HAS A NEIGHBORHOOD
 HOMEOMORPHIC TO A CONE
 OVER $\mathbb{C}P^2$.
- $\exists K^{\text{COMPACT}} \subseteq \mathcal{M}(p, g)$
 SUCH THAT $\mathcal{M}(p, g) - K$ IS
 DIFFEOMORPHIC TO THE
 CYLINDER $M \times (0, 1)$.

NOW CUT OFF THE TOP HALF OF EACH CONE AND THE BOTTOM HALF OF THE CYLINDER.



- M IS COBORDANT TO A
 DISJOINT UNION OF $\mathbb{C}P^2$ 'S.
 - SIGNATURE OF THE INTERSECTION
 FORM IS A COBORDISM INVARIANT.
- ETC. □

APPENDIX 3 : "ETC."

3. DONALDSON POLYNOMIAL INVARIANTS $\gamma_d(M)$, $d = 0, 1, 2, \dots$

$M =$ COMPACT, SIMPLY CONNECTED, ORIENTED,
SMOOTH (C^∞) 4-MANIFOLD WITH

"VARIOUS ASSUMPTIONS ON
 $b_2^+(M)$ AS THE NEED ARISES."

$$\gamma_0(M) \in \mathbb{Z}$$

$$\gamma_d(M) : H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}[\frac{1}{2}], \quad d = 1, 2, \dots$$

REMARKS ON THE CONSTRUCTION :

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} M \quad (\text{CHERN CLASS } k \gg 0)$$

CHOOSE A RIEMANNIAN METRIC g ON M . GIVES HODGE STAR $*$.

$\mathcal{M}(P_k, g) =$ MODULI SPACE OF GAUGE EQUIVALENCE
CLASSES OF g -ASD CONNECTIONS ON P_k

ADDENDUM 2 : THE MODULI SPACES

EXAMPLES :

1. $M = S^4$, $g =$ STANDARD RIEMANNIAN METRIC, $k = 1$

$\mathcal{M}(P_1, g) =$ MODULI SPACE OF BPST INSTANTONS

\cong OPEN 5-BALL

2. M ARBITRARY, g ARBITRARY, $k \leq 0$

$$k = \frac{1}{8\pi^2} \int_M \text{Tr}(F_\omega \wedge F_\omega) = \frac{1}{8\pi^2} \int_M (|F_\omega^-|^2 - |F_\omega^+|^2) d\text{Vol}_g$$

WHERE $F_\omega^\pm = \frac{1}{2}(F_\omega \pm *F_\omega)$.

$$k < 0 \Rightarrow F_\omega^+ \neq 0 \Rightarrow \omega \text{ CANNOT BE } g\text{-ASD}$$

$$\mathcal{M}(P_i, g) = \mathcal{M}(P_2, g) = \dots = \emptyset$$

$$k = 0 \Rightarrow \text{ANY } g\text{-ASD } \omega (F_\omega^+ = 0) \text{ IS FLAT } (F_\omega^- = 0 \text{ AS WELL})$$

$$\text{CONVERSELY, FLAT} \Rightarrow g\text{-ASD}$$

THE $k = 0$ BUNDLE IS TRIVIAL AND FLAT CONNECTIONS EXIST ON ANY TRIVIAL BUNDLE SO $\mathcal{M}(P_0, g) \neq \emptyset$.

M SIMPLY CONNECTED \Rightarrow ANY TWO FLAT CONNECTIONS ON THE TRIVIAL BUNDLE ARE GAUGE EQUIVALENT SO

$$\mathcal{M}(P_0, g) = \{\text{POINT}\}$$

HENCEFORTH CONSIDER ONLY

$$k \geq 1$$

3. $M = S^2 \times S^2$, $g = \text{STANDARD METRIC}$, $k = 1$

$$\mathcal{M}(P_1, g) = \emptyset$$

$$4. \quad M = \mathbb{C}P^2, \quad g = \text{STANDARD (FUBINI-STUDY) METRIC}, \quad k = 1$$

$$\mathcal{M}(P, g) = \emptyset$$

$$5. \quad M = \overline{\mathbb{C}P^2}, \quad g = \text{STANDARD (FUBINI-STUDY) METRIC}, \quad k = 1$$

$$\mathcal{M}(P, g) \cong \text{OPEN CONE OVER } \overline{\mathbb{C}P^2}$$

REMARK: COMPARE # 4 AND 5. DONALDSON THEORY IS HIGHLY SENSITIVE TO ORIENTATIONS.

ASSUMING

$$b_2^+(M) > 0$$

DONALDSON SHOWS THAT

FOR A "GENERIC" RIEMANNIAN METRIC g ,
 $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR A SMOOTH (C^∞),
 ORIENTABLE MANIFOLD OF DIMENSION

$$8k - 3(1 + b_2^+(M))$$

(AN ORIENTATION IS CANONICALLY DETERMINED BY
 ORIENTING THE VECTOR SPACE $H_+^2(M; \mathbb{R})$).

REMARKS ON $b_2^+(M) > 0$:

THE SINGULARITIES (CONES OVER $\mathbb{C}P^2$) IN THE MODULI SPACE WHEN $b_2^+(M) = 0$ ARISE FROM "REDUCIBLE" CONNECTIONS (LARGE ISOTROPY GROUP UNDER \mathcal{G} -ACTION). IN THE SPACE \mathcal{Q} OF RIEMANNIAN METRICS ON M THE SET OF THOSE g FOR WHICH THERE EXIST REDUCIBLE g -ASD ω IS A COUNTABLE UNION OF SUBMANIFOLDS OF CODIMENSION $b_2^+(M)$.
 $b_2^+(M) > 0 \Rightarrow$ CAN FIND A "GOOD" g .

TO DEFINE INVARIANTS FROM THE MODULI SPACES REQUIRES THE STRONGER ASSUMPTION

$$b_2^+(M) > 1.$$

TO ENSURE INDEPENDENCE OF THE CHOICE OF g , NEED THE SET OF METRICS WHICH INTRODUCE REDUCIBLES TO BE SUFFICIENTLY "THIN" THAT A GENERIC VARIATION OF g (PATH IN \mathcal{Q}) CAN AVOID REDUCIBLES.

DEFINING THE INVARIANTS :

$$\underline{\text{CASE 1}} : \quad 8k - 3(1 + b_2^+(M)) = 0$$

GENERICALLY, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR 0-DIMENSIONAL, ORIENTED, AND COMPACT (THIS IS THE ONLY CASE IN WHICH THE MODULI SPACE IS COMPACT).

$$\chi_0(M) = \begin{cases} \sum_{[\omega] \in \mathcal{M}(P_k, g)} (-1)^{[\omega]} & , \mathcal{M}(P_k, g) \neq \emptyset \\ 0 & , \mathcal{M}(P_k, g) = \emptyset \end{cases}$$

$$\underline{\text{CASE 2}} : \quad 8k - 3(1 + b_2^+(M)) > 0$$

ONE MORE ASSUMPTION REQUIRED. HENCEFORTH,

$$b_2^+(M) > 1 \text{ AND } \underline{\text{ODD}}.$$

WRITE

$$8k - 3(1 + b_2^+(M)) = 2d_k$$

FOR SOME POSITIVE INTEGER d_k .

GENERICALLY, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR $2d_k$ -DIMENSIONAL, ORIENTED, AND NEVER COMPACT.

DONALDSON μ -MAP :

$$\mu : H_2(M; \mathbb{Z}) \rightarrow H^2(\mathcal{M}(P_k, g); \mathbb{Z})$$

ADDENDUM 4 : THE μ -MAP

NAIVE DEFINITION OF

$$\gamma_{d_k}(M) : H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}[\frac{1}{2}]$$

$$x \in H_2(M; \mathbb{Z}) \Rightarrow \mu(x) \wedge \dots \wedge \mu(x) \in H^{2d_k}(\mathcal{M}(P_k, g); \mathbb{Z})$$

$$\gamma_{d_k}(M)(x) = \int_{\mathcal{M}(P_k, g)} \mu(x) \wedge \dots \wedge \mu(x)$$

HERE ARE A FEW OF THE THINGS WRONG WITH THIS "DEFINITION" :

- $\mathcal{M}(P_k, g)$ IS NOT COMPACT SO YOU CAN'T INTEGRATE OVER IT
- DEEP ANALYTICAL RESULTS OF UHLENBECK AND TAUBES LED DONALDSON TO A CANONICAL COMPACTIFICATION $\bar{\mathcal{M}}(P_k, g)$ OF $\mathcal{M}(P_k, g)$ AND AN EXTENSION $\bar{\mu} : H_2(M; \mathbb{Z}) \rightarrow H^2(\bar{\mathcal{M}}(P_k, g); \mathbb{Z})$ OF THE μ -MAP. HOWEVER,

- $\bar{m}(P_k, g)$ IS NOT A MANIFOLD SO INTEGRATION OVER IT MUST BE REPLACED BY PAIRING WITH THE FUNDAMENTAL CLASS.
- REGRETTABLY, $\bar{m}(P_k, g)$ ADMITS A FUNDAMENTAL CLASS ONLY FOR SUFFICIENTLY LARGE k , I.E., IN THE STABLE RANGE

$$k > \frac{3}{4} (1 + b_2^+(M))$$

OR EQUIVALENTLY

$$d_k > \frac{3}{2} (1 + b_2^+(M)).$$

IF d IS AN INTEGER SATISFYING

$$d \equiv -\frac{3}{2} (1 + b_2^+(M)) \pmod{4}$$

AND

$$d > \frac{3}{2} (1 + b_2^+(M))$$

WE HAVE

$$\gamma_d(M) : H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

- REMOVING THE STABLE RANGE CONDITION AND THE MOD 4 CONGRUENCE REQUIRES
 - ANOTHER μ -MAP $\mu : H_0(M; \mathbb{Z}) \rightarrow H^4(\bar{m}(P_k, g); \mathbb{Z})$
 - A DETOUR AROUND THE FACT THIS μ -MAP DOES NOT FULLY EXTEND TO $\bar{m}(P_k, g)$
 - A BLOW-UP FORMULA RELATING $\gamma_d(M)$ AND $\gamma_d(M \# n \mathbb{C}P^2)$

ADDENDUM 5 : DONALDSON POLYNOMIALS

THE END RESULT IS A SEQUENCE

$$\gamma_d(M), \quad d = 0, 1, 2, \dots$$

OF ORIENTATION PRESERVING DIFFEOMORPHISM INVARIANTS OF M .

DONALDSON (FORMAL POWER) SERIES

$$\mathcal{D}_M(x) = \sum_{d=0}^{\infty} \frac{\gamma_d(M)(x)}{d!}$$

E.G., FOR $M = K3$

$$\mathcal{D}_{K3}(x) = \sum_{d=0}^{\infty} \frac{(Q_{K3}(x,x)/2)^d}{d!} = \exp(Q_{K3}(x,x)/2).$$

SPRING, 1994 : A BREAKTHROUGH

KRONHEIMER - MROWKA STRUCTURE THEOREM: IF M IS OF
"D-SIMPLE TYPE", THEN THERE EXIST COHOMOLOGY CLASSES

$$K_1, \dots, K_s \in H^2(M; \mathbb{Z}) \quad (\underline{\text{D-BASIC CLASSES}})$$

AND RATIONAL NUMBERS

$$a_1, \dots, a_s \in \mathbb{Q} \quad (\underline{\text{COEFFICIENTS}})$$

SUCH THAT

$$\mathcal{D}_M(x) = \exp(Q_M(x, x)/2) \sum_{r=1}^s a_r \exp(K_r(x))$$

MOREOVER, EACH K_r IS AN INTEGRAL LIFT OF THE SECOND
STIEFEL-WHITNEY CLASS $w_2(M) \in H^2(M; \mathbb{Z}_2)$.

FALL, 1994 : RENDERED MOOT (DISCOVERY OF SEIBERG-WITTEN
INVARIANTS)