EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN

WITTEN'S MAGICAL FORMULA

$$\mathcal{D}_M(x) = \exp\left(Q_M(x,x)/2\right) \sum_{\mathcal{L} \in \Lambda} 2^{m(M)} SW_0(M,\mathcal{L}) \exp\left(c_1(L^0(\mathcal{L}))(x)\right)$$

DONALDSON INVARIANTS

SEIBERG-WITTEN INVARIANTS

AND THE PHYSICS IT CAME FROM

$$\Phi = (\omega, \phi, \lambda, \eta, \psi, \zeta)$$

$$\begin{aligned} S_{DW}[\Phi] = \int_M Tr \Big\{ & \tfrac{1}{4} F_\omega \wedge {}^*F_\omega + \tfrac{1}{4} F_\omega \wedge F_\omega - \tfrac{1}{2} \psi \wedge [\phi, \psi] \\ & - i\, d^\omega \zeta \wedge \psi - 2i [\zeta, {}^*\zeta] \lambda \\ & + i\, \phi\, d^\omega {*} d^\omega \lambda - \zeta \wedge {}^* d^\omega \eta \Big\} \end{aligned}$$

$$Z_{DW} = \int \exp\left(-S_{DW}[\Phi]/e^2\right) \mathcal{D}\Phi$$

- MATHAI - QUILLEN (1986)

  "SUPERSYMMETRIC" FORMULAS FOR THE EULER CHARACTERISTIC OF FINITE - DIMENSIONAL VECTOR BUNDLES FROM EQUIVARIANT COHOMOLOGY.

- ATIYAH - JEFFREY (1990)

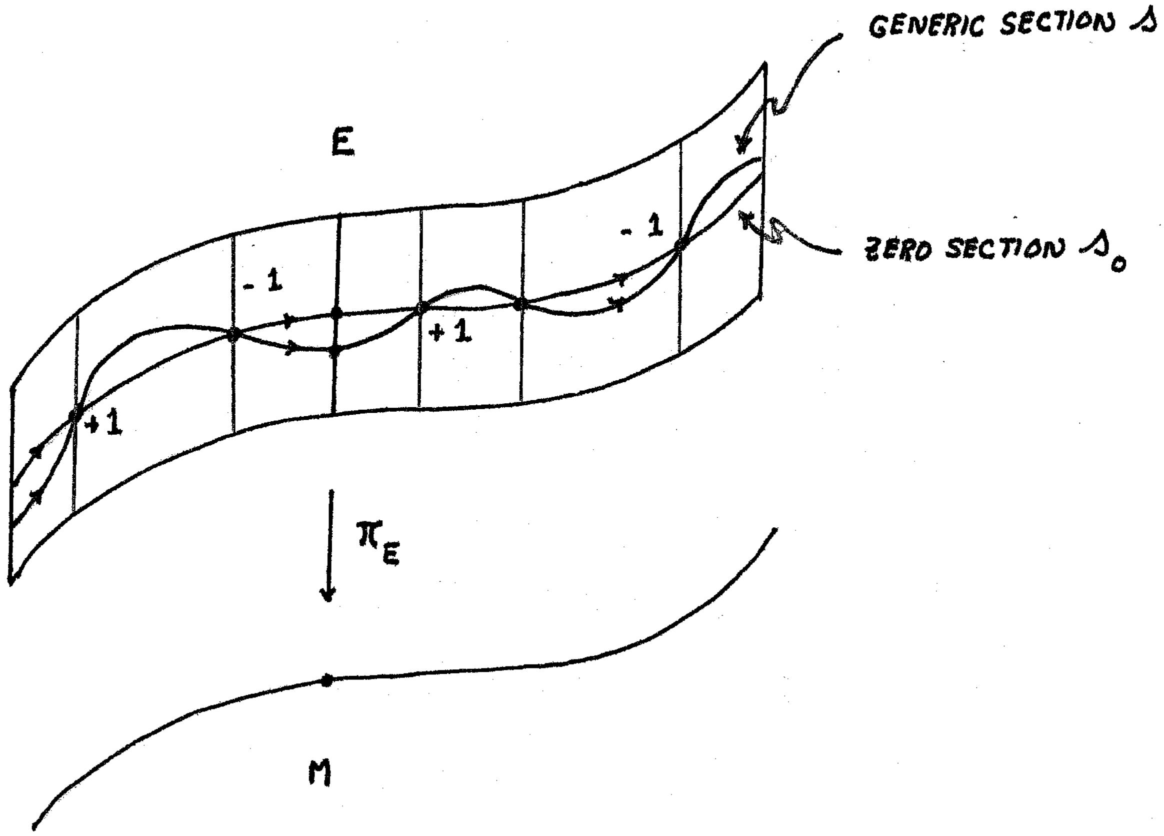
  FORMAL EXTENSION OF MATHAI - QUILLEN TO CERTAIN INFINITE - DIMENSIONAL VECTOR BUNDLE REPRODUCES $S_{DW}[\Phi]$ AND EXHIBITS Z_{DW} AS AN "EULER CHARACTERISTIC".

- MATHAI - QUILLEN FORMALISM FOR TQFT OF "COHOMOLOGICAL TYPE".

<u>EULER</u> <u>CHARACTERISTIC</u> OF AN ORIENTED, REAL VECTOR BUNDLE

$$E \xrightarrow{\pi_E} M$$

OF FIBER DIMENSION $2k$ OVER A COMPACT, ORIENTED MANIFOLD M OF DIMENSION $2k$ IS DEFINED AS FOLLOWS :



$\chi(E)$:= INTERSECTION NUMBER OF ANY GENERIC SECTION

<u>INTEGRAL</u> <u>REPRESENTATION</u> : $\chi(E) = \int_M e(E)$

$e(E)$ = EULER CLASS OF $E \xrightarrow{\pi_E} M$

TWO APPROACHES TO $e(E)$:

1. (<u>CHERN</u>-<u>WEIL</u>) FIBER METRIC ON $E \xrightarrow{\pi_E} M$ GIVES

$$SO(2k) \hookrightarrow F_{SO}(E) \xrightarrow{\pi_{SO}} M.$$

CHOOSE CONNECTION ω WITH CURVATURE Ω. CHOOSE THE ad - INVARIANT POLYNOMIAL

$$\text{PFAF} : \mathfrak{so}(2k) \to \mathbb{R}$$

$$(2\pi)^{-k}\,\text{PFAF}(\Omega) = \frac{1}{2^{2k}\pi^k k!} \sum_{\sigma \in S_{2k}} (-1)^\sigma \Omega_{\sigma(1)\sigma(2)} \wedge \cdots \wedge \Omega_{\sigma(2k-1)\sigma(2k)}$$

IS A CLOSED FORM ON $F_{SO}(E)$ AND IS BASIC ($SO(2k)$-INVARIANT AND HORIZONTAL) SO IT DESCENDS TO A CLOSED $2k$-FORM ON M WHOSE COHOMOLOGY CLASS

$$e(E)$$

DOES NOT DEPEND ON THE CHOICE OF ω.

<u>NOTE</u> : $e(TS^2)$ IS COMPUTED IN ADDENDUM 6.

2. (THOM CLASS) THERE EXISTS A UNIQUE $U(E) \in H^{2k}_{cv}(E;\mathbb{R})$ WHOSE INTEGRAL OVER EACH FIBER $\pi_E^{-1}(p)$ IS 1. THEN

$$e(E) = s^*(U(E))$$

FOR ANY SECTION s OF $E \xrightarrow{\pi_E} M$.

MATHAI-QUILLEN WILL YIELD VARIOUS EXPLICIT REPRESENTATIVES OF THE THOM CLASS $U(E)$.

ANOTHER VIEW OF THE PFAFFIAN PFAF : $so(2k) \longrightarrow \mathbb{R}$:

$$V = \text{TYPICAL FIBER OF } E \xrightarrow{\pi_E} M$$

$\{\psi^1, \dots, \psi^{2k}\}$ = ORIENTED, ORTHONORMAL BASIS FOR V

= ODD GENERATORS OF THE SUPERCOMMUTATIVE SUPERALGEBRA

$$\Lambda V = \bigoplus_{i=0}^{2k} \Lambda^i V = (\Lambda V)_0 \oplus (\Lambda V)_1$$

$$\text{VOL} = \psi^1 \cdots \psi^{2k} \in \Lambda^{2k} V$$

$$A = (A_{ij}) \in so(2k) \longrightarrow \alpha_A = \frac{1}{2} A_{ij}\psi^i\psi^j \in \Lambda^2 V$$

$$= \frac{1}{2}\psi^T A \psi$$

$$= -\frac{1}{2}\sum_{l=1}^{2k} \psi^l A \psi^l$$

$$\frac{1}{k!}\,\alpha_A^{\,k} = \mathrm{PFAF}(A)\ \mathrm{VOL}$$

AS A <u>BEREZIN</u> (OR <u>FERMIONIC</u>) <u>INTEGRAL</u>:

ANY $f \in \Lambda V$ IS A POLYNOMIAL WITH REAL COEFFICIENTS IN THE ODD "VARIABLES" $\psi^1, \dots, \psi^{2k}$. DEFINE

$$\int f\, \mathcal{D}\psi = f_{\mathrm{VOL}} \in \mathbb{R}$$

$$= \text{COEFFICIENT OF } \mathrm{VOL} = \psi^1 \cdots \psi^{2k}$$

THUS,

$$\mathrm{PFAF}(A) = \int e^{\frac{1}{2}\psi^T A \psi}\, \mathcal{D}\psi = \int e^{-\frac{1}{2}\sum_\ell \psi^\ell A \psi^\ell}\, \mathcal{D}\psi$$

SUBSTITUTING FOR A THE CURVATURE Ω OF A CONNECTION ON $SO(2k) \hookrightarrow F_{SO}(E) \longrightarrow M$ GIVES A BEREZIN INTEGRAL REPRESENTATION OF A FORM WHICH DESCENDS TO A REPRESENTATIVE OF THE EULER CLASS ON M.

MATHAI-QUILLEN REPRESENTATIVES OF THE THOM CLASS ARE LIKEWISE GIVEN AS BEREZIN INTEGRALS.

EXTENDED BEREZIN INTEGRATION :

$\mathcal{A}$ = ANOTHER SUPERCOMMUTATIVE SUPERALGEBRA

E.G., $\Omega^*(V)$

$\mathcal{A} \otimes \Lambda V$ = SUPER TENSOR PRODUCT

$\left((a_1 \otimes f_1)(a_2 \otimes f_2) = (-1)^{\deg f_1 \deg a_2} (a_1 a_2) \otimes (f_1 f_2) \right)$

= POLYNOMIALS F WITH COEFFICIENTS IN $\mathcal{A}$ IN THE ODD "VARIABLES" $\psi^1, \ldots, \psi^{2k}$

$$\int F \, \mathcal{D}\psi = F_{\text{VOL}} \in \mathcal{A}$$

EXAMPLES :

1. $\mathcal{A} = \Omega^*(V)$ = COMPLEX-VALUED DIFFERENTIAL FORMS ON V

$$\Omega^*(V) \otimes \Lambda V$$

$\{ \mu_1, \ldots, \mu_{2k} \}$ = BASIS FOR $V^* \subseteq \Omega^0(V)$ DUAL TO $\{ \psi^1, \ldots, \psi^{2k} \}$

OMITING THE "$\otimes$" FOR ELEMENTS OF $\Omega^*(V) \otimes \Lambda V$ WE HAVE

$$-i \, d\mu_j \, \psi^j = i \, \psi^j d\mu_j = i \, \psi^T d\mu \in \Omega^*(V) \otimes \Lambda V$$

$$-\tfrac{1}{2} \|\mu\|^2 1 = -\tfrac{1}{2} \|\mu\|^2 = -\tfrac{1}{2} (\mu_1^2 + \cdots + \mu_{2k}^2) \in \Omega^*(V) \otimes \Lambda V$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2 + i \psi^T d\mu} \in \Omega^*(V) \otimes \Lambda V$$

$$\int (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2 + i\psi^T du} \, \Theta\psi =$$

$$(2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2} e^{i\psi^T du} \, \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int e^{i\psi^1 du_1 + \cdots + i\psi^{2k} du_{2k}} \, \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int e^{i\psi^1 du_1} \cdots e^{i\psi^{2k} du_{2k}} \, \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int (1 + i\psi^1 du_1) \cdots (1 + i\psi^{2k} du_{2k}) \, \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int (i\psi^1 du_1) \cdots (i\psi^{2k} du_{2k}) \, \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} i^{2k} \int (-1)^{\frac{1}{2}(2k)(2k+1)} du_1 \cdots du_{2k} \, \psi^1 \cdots \psi^{2k} \, \Theta\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} du_1 \cdots du_{2k} \quad \in \Omega^*(V)$$

INTEGRATES TO 1 OVER V. THINK OF IT AS A "GAUSSIAN" REPRESENTATIVE OF THE THOM CLASS OF V, REGARDED AS A VECTOR BUNDLE OVER A POINT.

2. $\mathcal{A} = \mathbb{C}[so(V)] \otimes \Omega^*(V)$

$\{\xi_1, \ldots, \xi_n\}$ = BASIS FOR $so(V)$ $\quad (n = k(2k-1))$

$\{x^1, \ldots, x^n\}$ = DUAL BASIS FOR $so(V)^*$, REGARDED AS LINEAR FUNCTIONS ON $so(V)$

$\mathbb{C}[so(V)] = \mathbb{R}[x^1, \ldots, x^n] \otimes \mathbb{C}$

$\mathbb{C}[so(V)] \otimes \Omega^*(V)$ = SUMS OF TERMS

$$\alpha = \rho \otimes \varphi$$

WHICH ARE TO BE REGARDED AS $\Omega^*(V)$-VALUED POLYNOMIALS ON $so(V)$

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

<u>GRADING</u> : $\deg(\rho \otimes \varphi) = 2\deg\rho + \deg\varphi$

WE DESCRIBE AN ELEMENT ν OF $\mathbb{C}[so(V)] \otimes \Omega^*(V)$, CALLED THE <u>UNIVERSAL</u> <u>THOM</u> <u>FORM</u> OF V, AS THE BEREZIN INTEGRAL OF AN ELEMENT OF

$$\mathbb{C}[so(V)] \otimes \Omega^*(V) \otimes \Lambda V.$$

<u>NOTATION</u> : $\xi \in so(V)$ GIVES A LINEAR TRANSFORMATION

$$M_\xi : V \to V$$

$$M_\xi(\psi) = \frac{d}{dt}(\exp(t\xi)(\psi))\Big|_{t=0}$$

IF $M_a = M_{\xi_a}$, THEN

$$M_\xi = x^a(\xi) M_a$$

$$\mathrm{PFAF}(M_\xi) = \int e^{-\frac{1}{2}\sum_\ell \psi^\ell x^a(\xi) M_a \psi^\ell} \mathcal{D}\psi$$

NOW NOTICE THAT

$$-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell} \;\in\; \mathbb{C}[so(V)]\otimes\Omega^{*}(V)\otimes\Lambda V$$

$$e^{-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell}} \;\in\; \mathbb{C}[so(V)]\otimes\Omega^{*}(V)\otimes\Lambda V$$

$$(2\pi)^{-k}\,e^{-\frac{1}{2}\|u\|^{2}+i\,\psi^{T}du-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell}} \;\in\; \mathbb{C}[so(V)]\otimes\Omega^{*}(V)\otimes\Lambda V$$

SO WE CAN DEFINE

$$\nu = (2\pi)^{-k}\int e^{-\frac{1}{2}\|u\|^{2}+i\,\psi^{T}du-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell}}\,\mathcal{D}\psi \;\in\; \mathbb{C}[so(V)]\otimes\Omega^{*}(V$$

<u>EXAMPLE</u> : $V=\mathbb{R}^{2}$ (USUAL ORIENTATION AND INNER PRODUCT)

$\{\psi^{1},\psi^{2}\}$ = STANDARD BASIS

$\{u_{1},u_{2}\}$ = DUAL BASIS (COORDINATE FUNCTIONS)

$SO(V) \equiv SO(2)$

$\{\xi_{1}\} = \left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}$

$\{x^{1}\}$ = DUAL BASIS

$$\nu = \underbrace{(2\pi)^{-1}e^{-\frac{1}{2}(u_{1}^{2}+u_{2}^{2})}du_{1}du_{2}}_{\mathbb{C}^{0}[so(2)]\otimes\Omega^{2}(\mathbb{R}^{2})} + \underbrace{(2\pi)^{-1}x^{1}e^{-\frac{1}{2}(u_{1}^{2}+u_{2}^{2})}}_{\mathbb{C}^{1}[so(2)]\otimes\Omega^{0}(\mathbb{R}^{2})}$$

INTEGRATES TO 1 OVER $\mathbb{R}^{2}$

<u>NOTE</u> : THE PROOF IS IN ADDENDUM 7.

WE WILL SEE SHORTLY HOW THE UNIVERSAL THOM FORM OF V CAN BE USED TO PRODUCE (GAUSSIAN) REPRESENTATIVES OF THE THOM CLASS FOR ANY VECTOR BUNDLE WITH TYPICAL FIBER V.

FIRST, HOWEVER,

WHAT KIND OF "THING" IS $\mathcal{V}$?

BRIEF ASIDE ON THE <u>CARTAN</u> <u>MODEL</u> OF <u>EQUIVARIANT</u> <u>COHOMOLOGY</u>:

M = SMOOTH MANIFOLD G = COMPACT, CONNECTED LIE GROUP

LEFT ACTION : $\sigma : G \times M \to M$

$$\sigma(g, m) = g \cdot m = \sigma_g(m)$$

OBJECTIVE : COHOMOLOGY OF M/G

$\Omega^*(M)$ = GRADED ALGEBRA OF COMPLEX-VALUED DIFFERENTIAL FORMS ON M

$\mathbb{C}[\mathfrak{g}]$ = GRADED ALGEBRA OF COMPLEX-VALUED POLYNOMIALS ON THE LIE ALGEBRA $\mathfrak{g}$ OF G

$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ = SUMS OF $\alpha = \rho \otimes \varphi$: $\Omega^*(M)$-VALUED POLYNOMIALS ON $\mathfrak{g}$

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

GRADING : $\deg(\alpha) = \deg(\rho \otimes \varphi) = 2\deg(\rho) + \deg(\varphi)$

INDUCED ACTION OF G ON $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$:

$$(g\cdot\alpha)(\xi) = (g\cdot(p\otimes\varphi))(\xi) = p(g^{-1}\xi g)\,\sigma_{g^{-1}}^*(\varphi)$$

$$\begin{aligned}\Omega_G^*(M) &= [\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)]^G \\ &= G\text{-INVARIANT ELEMENTS OF } \mathbb{C}[\mathfrak{g}]\otimes\Omega^*(M) \\ &= G\text{-}\underline{\text{EQUIVARIANT}}\ \underline{\text{DIFFERENTIAL}}\ \underline{\text{FORMS}} \text{ ON } M\end{aligned}$$

G-<u>EQUIVARIANT</u> <u>EXTERIOR</u> <u>DERIVATIVE</u> :

$$d_G : \Omega_G^*(M) \to \Omega_G^*(M)$$

$$(d_G\alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi^*}(\alpha(\xi))$$

WHERE

$$\xi^*(m) = \frac{d}{dt}\left(\exp(-t\xi)\cdot m\right)\Big|_{t=0} .$$

$(\Omega_G^*(M), d_G)$ IS A COCHAIN COMPLEX AND ITS COHOMOLOGY

$$H_G^*(M)$$

IS THE <u>CARTAN</u> <u>MODEL</u> OF THE G-<u>EQUIVARIANT</u> <u>COHOMOLOGY</u> OF M.

<u>THEOREM</u> (CARTAN): IF THE ACTION OF G ON M IS FREE, THEN

$$H_G^*(M) \cong H^*_{deRham}(M/G;\mathbb{C}).$$

<u>NOTE</u>: MORE DETAILS AND $H_{S^1}^*(S^3)$ CAN BE FOUND IN ADDENDUM 8.

INTEGRATION OF EQUIVARIANT DIFFERENTIAL FORMS :

$$\int_M : \Omega_G^*(M) \longrightarrow \mathbb{C}[\mathfrak{g}]$$

$$\left(\int_M \alpha \right)(\xi) = \int_M \alpha(\xi) := \int_M \alpha(\xi)_{[2k]}$$

UNIVERSAL THOM FORM

$$\nu = (2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^T du - \frac{1}{2}\sum_\ell \psi^\ell x^a M_a \psi^\ell} \mathcal{D}\psi \in \Omega^*_{SO(V)}(V)$$

SATISFIES

$$d_{SO(V)} \nu = 0$$

AND

$$\int_V \nu = 1$$

NOTE : PROOFS FOR $V = \mathbb{R}^2$ IN ADDENDUM 9.

THOM FORMS FROM THE UNIVERSAL THOM FORM :

ANY ORIENTED REAL VECTOR BUNDLE WITH TYPICAL FIBER V CAN BE VIEWED AS THE VECTOR BUNDLE

$$P \times_\rho V$$

ASSOCIATED TO SOME PRINCIPAL G-BUNDLE

$$G \hookrightarrow P \longrightarrow M$$

BY SOME REPRESENTATION

$$\rho : G \to SO(V) \qquad (g\cdot\psi = \rho(g)(\psi))$$

$$(p\cdot g,\ g^{-1}\cdot\psi) \to [p,\psi] \qquad \begin{tikzcd} P\times V \arrow[r] \arrow[d] & P \arrow[d] \\ P\times_\rho V \arrow[r] & M \end{tikzcd}$$

$$\rho_* : \mathfrak{g} \to so(V)$$

$$\mathcal{P}\otimes\varphi \in [\mathbb{C}[so(V)]\otimes\Omega^*(V)]^{SO(V)}$$

$$\to (\mathcal{P}\circ\rho_*)\otimes\varphi \in [\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(V)]^{G}$$

E.G., $\quad \nu \in \Omega^*_{SO(V)}(V) \to \nu_G \in \Omega^*_G(V)$

CHERN-WEIL MAP :

CHOOSE ω ON P

$$\mathbb{C}[\mathfrak{g}]^G \to \Omega^*(P)_{BASIC} \cong \Omega^*(M)$$

$$\mathcal{P} \to \mathcal{P}(\Omega)$$

MATHAI-QUILLEN MAP :

$$[\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(V)]^G \to \Omega^*(P\times V)_{BASIC} \cong \Omega^*(P\times_\rho V)$$

$$\mathcal{P}\otimes\varphi \to HOR_\omega(\mathcal{P}(\Omega)\wedge\varphi)$$

d_G - CLOSED $\to$ d - CLOSED

NOTE : WE USE THE SAME SYMBOLS FOR FORMS ON P AND V AND THEIR PULLBACKS TO $P\times V$ BY A PROJECTION.

UNDER THE MATHAI-QUILLEN MAP

$$\nu_G = (2\pi)^{-k} \int e^{-\frac{1}{2}\|\mu\|^2 + i\,\psi^T d\mu - \frac{1}{2}\sum_{\ell} \psi^{\ell} (x^a \circ \rho_*) M_a \psi^{\ell}} \, \mathcal{D}\psi$$

GOES TO

$\mho$ = THE HORIZONTAL PROJECTION OF

$$(2\pi)^{-k} \int e^{-\frac{1}{2}\|\mu\|^2 + i\,\psi^T d\mu - \frac{1}{2}\sum_{\ell} \psi^{\ell} (x^a(\rho_*\Omega)) M_a \psi^{\ell}} \, \mathcal{D}\psi$$

$$= (2\pi)^{-k} \int e^{-\frac{1}{2}\|\mu\|^2 + i\,\psi^T d\mu + \frac{1}{2}\psi^T(\rho_*\Omega)\psi} \, \mathcal{D}\psi$$

WHERE $(\rho_*\Omega)$ IS THE SKEW-SYMMETRIC MATRIX IMAGE OF THE G-CURVATURE UNDER ρ_*.

<u>EXAMPLE</u> : $V = \mathbb{R}^2$ AS IN THE EXAMPLE ON PAGE 10.

$$\nu = (2\pi)^{-1} e^{-\frac{1}{2}(\mu_1^2 + \mu_2^2)} (x' + d\mu_1 d\mu_2)$$

VECTOR BUNDLE $= TS^2$ (USUAL STRUCTURE ON S^2)

$G = SO(2)$

$SO(2) \hookrightarrow F_{SO}(TS^2) \longrightarrow S^2$

$\rho = id_{SO(2)}$ (SO $\rho_* = id_{so(2)}$)

$\omega = \omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\qquad$ $\Omega = \Omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mho = (2\pi)^{-1} e^{-\frac{1}{2}(\mu_1^2 + \mu_2^2)} (\Omega' + d\mu_1 d\mu_2 + \omega' \wedge (\mu_1 d\mu_1 + \mu_2 d\mu_2))$$

<u>NOTE</u> : THE PROOF IS IN ADDENDUM 10.

IN GENERAL,

$$U = HOR_{\omega}\left((2\pi)^{-k} \int e^{-\frac{1}{2}\|\mu\|^2 + i\psi^T d\mu + \frac{1}{2}\psi^T(\rho_*\Omega)\psi} \mathcal{D}\psi \right)$$

IS A BASIC FORM ON $P \times V$ WHICH DESCENDS TO A GAUSSIAN REPRESENTATIVE OF THE THOM CLASS ON $P \times_\rho V$.

PULLING BACK TO M BY A SECTION OF $P \times_\rho V \to M$ GIVES A REPRESENTATIVE OF THE EULER CLASS.

BUT EVERY SECTION OF $P \times_\rho V \to M$ IS OF THE FORM

$$m \longrightarrow (s(m), S(s(m))) \to [s(m), S(s(m))]$$

WHERE s IS A SECTION OF $G \hookrightarrow P \to M$ AND $S : P \to V$ IS EQUIVARIANT $\left(S(p \cdot g) = \rho(g^{-1})(S(p)) \right)$.

THUS, ONE CAN PULL U BACK DIRECTLY BY

$$(s, S \circ s)^* = ((1, S) \circ s)^* = s^* \circ (1, S)^*$$

I.E., PULL V-FACTORS BACK BY S TO GET A FORM ON P AND THEN PULL THIS BACK TO M BY s.

<u>EXAMPLE</u>: $TS^2 = F_{SO}(TS^2) \times_{id_{SO(2)}} \mathbb{R}^2$ AS IN THE EXAMPLE ON PAGE 16.

ω = LEVI-CIVITA CONNECTION ON $F_{SO}(TS^2)$

s = SECTION OF $F_{SO}(TS^2)$

$:$ ORIENTED, ORTHONORMAL FRAME FIELD ON S^2

$$s(\phi,\theta) = \left(\phi, \theta, \frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)$$

$S : F_{SO}(TS^2) \to \mathbb{R}^2$ AN EQUIVARIANT MAP

(I.E., A VECTOR FIELD ON S^2)

$$S(s(\phi,\theta)) = (\gamma \sin\theta, \gamma\cos\theta\cos\phi)$$

WHERE $\gamma \in \mathbb{R}$ IS ARBITRARY

THE RESULTING REPRESENTATIVE OF THE EULER CLASS IS

$$(2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta\cos^2\phi)} \sin\phi\,(1+\gamma^2\cos^2\theta\sin^2\phi)\, d\phi\wedge d\theta$$

<u>NOTE</u> : THE PROOF IS IN ADDENDUM II.

IN PARTICULAR, WE OBTAIN THAT, FOR ANY $\gamma \in \mathbb{R}$,

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} e^{-\frac{1}{2}\gamma^2(\sin^2\theta+\cos^2\theta\cos^2\phi)} \sin\phi\,(1+\gamma^2\cos^2\theta\sin^2\phi)\, d\phi\, d\theta$$

$$= \chi(S^2)$$

$$= 2$$

ATIYAH - JEFFREY TRANSFORMATION OF THE UNIVERSAL THOM FORM (SKETCH):

NOTE: DETAILED CALCULATIONS CAN BE FOUND IN ADDENDUM 12.

(1) $$U = (2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \int \exp\left(i\,\psi^T d\mu + \tfrac{1}{2}\psi^T (\rho_* \Omega)\psi \right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

ADDITIONAL ASSUMPTIONS AND CHOICES:

1. $G \hookrightarrow P \xrightarrow{\pi_P} M$ IS ORIENTABLE (I.E., $\exists$ n-FORM Ψ ON P ($n = \dim G$) SUCH THAT, IF $m \in M$ AND $\iota_m : \pi_P^{-1}(m) \hookrightarrow P$, THEN $\iota_m^* \Psi$ IS AN ORIENTATION FOR $\pi_P^{-1}(m) \cong G$). FOLLOWS THAT P IS ORIENTED BY THE "LOCAL PRODUCT ORIENTATION" $\pi_P^*(\mathrm{vol}_M) \wedge \Psi$.

2. THE ACTION OF G ON P IS ORIENTATION PRESERVING ($\sigma_g : P \to P$, $\sigma_g(p) = p \cdot g$, IS AN ORIENTATION PRESERVING DIFFEOMORPHISM $\forall g \in G$).

3. A G-INVARIANT RIEMANNIAN METRIC $\langle\ ,\ \rangle$ HAS BEEN CHOSEN FOR P.

4. THE CONNECTION ω FOR P IS THE ONE WHOSE DISTRIBUTION OF HORIZONTAL SPACES IS THE FAMILY OF $\langle\ ,\ \rangle$- ORTHOGONAL COMPLEMENTS TO THE $G \hookrightarrow P \xrightarrow{\pi_P} M$ VERTICAL SPACES

5. AN ad- INVARIANT INNER PRODUCT $(\ ,\ )$ ON $\mathfrak{g}$ HAS BEEN CHOSEN AND NORMALIZED SO THAT THE VOLUME OF G (ARISING FROM THE CORRESPONDING BI-INVARIANT RIEMANNIAN METRIC ON G) IS 1.

NOW WE BEGIN TO MANIPULATE U, GIVEN BY (1):

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega] \text{ AND}$$

$[\omega,\omega]$ VANISHES ON HORIZONTAL PARTS SO

$$(2) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\,\psi^T du + \tfrac{1}{2}\psi^T(\rho_*(d\omega))\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

<u>TOOLS</u>:

$$C_p : \mathfrak{g} \longrightarrow \mathrm{VERT}_p(P) \subseteq T_p(P)$$

$$C_p(\xi) = \xi^*(p) = \frac{d}{dt}(p \cdot \exp(t\xi))\Big|_{t=0}$$

$$C_p^* : T_p(P) \to \mathfrak{g}$$

$$\langle v_p, C_p(\eta)\rangle_p = (C_p^*(v_p), \eta) \quad \forall v_p \in T_p(P)\ \forall \eta \in \mathfrak{g}$$

$$R_p = C_p^* \circ C_p : \mathfrak{g} \to \mathfrak{g} \quad \text{(INVERTIBLE AND SELF-ADJOINT)}$$

$$R_p^{-1} : \mathfrak{g} \to \mathfrak{g}$$

THEN

$$\omega_p = R_p^{-1} \circ C_p^*$$

AS LIE ALGEBRA - VALUED 1-FORMS,

$$\omega = R^{-1} \circ C^*$$

SO

$$d\omega = dR^{-1} \wedge C^* + R^{-1} dC^*$$

C^* VANISHES ON HORIZONTAL VECTORS AND THEREFORE SO DOES $dR^{-1} \wedge C^*$ SO

$$(3) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^T du + \tfrac{1}{2}\psi^T(\rho_*(R^{-1}dC^*))\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

TO ELIMINATE THE EXPLICIT APPEARANCE OF THE INVERSE, USE THE FOURIER INVERSION FORMULA (ON $\mathfrak{g}$) AND A CHANGE OF VARIABLE.

$\phi = (\phi_1, \ldots, \phi_n)$ AND $\lambda = (\lambda_1, \ldots, \lambda_n)$ COORDINATES ON $\mathfrak{g}$

AND f A REAL-VALUED FUNCTION ON $\mathfrak{g}$

$$(\mathfrak{F} f)(\phi) = (2\pi)^{-n/2} \int_{\mathfrak{g}} e^{-i(\phi,\lambda)} f(\lambda)\, d\lambda$$

$$(\tilde{\mathfrak{F}} f)(\lambda) = (2\pi)^{-n/2} \int_{\mathfrak{g}} e^{i(\lambda,\phi)} f(\phi)\, d\phi$$

NOW, $f = \mathcal{F}(\tilde{\mathcal{F}} f)$ IMPLIES

$$f(\mu) = (2\pi)^{-n} \int_{\mathfrak{g}} \int_{\mathfrak{g}} e^{-i(\mu,\lambda)} e^{i(\lambda,\phi)} f(\phi)\, d\phi\, d\lambda$$

USE THIS TO EVALUATE $f(R^{-1}\mu)$ AND MAKE THE CHANGE OF VARIABLE $\lambda \to R\lambda$ TO OBTAIN

$$f(R^{-1}\mu) = (2\pi)^{-n} \int_{\mathfrak{g}} \int_{\mathfrak{g}} e^{-i(\mu,\lambda)} e^{i(\phi,R\lambda)} f(\phi) \det R \, d\lambda\, d\phi$$

APPLY THIS TO THE MATHAI-QUILLEN FORM WITH $\mu = dC^*$ TO OBTAIN

$$(4) \quad U = (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \iiint \exp\left(\tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T d\mu - i(dC^*,\lambda) + i(\phi,R\lambda)\right) \det R\, \mathcal{D}\psi\, d\phi\, d\lambda$$

(EVALUATED ON HORIZONTAL PARTS)

TO INCORPORATE "EVALUATED ON HORIZONTAL PARTS" DIRECTLY INTO THE INTEGRAL WE NEED:

A <u>NORMALIZED</u> <u>VERTICAL</u> <u>VOLUME</u> <u>FORM</u> FOR A PRINCIPAL G-BUNDLE $G \hookrightarrow Q \xrightarrow{\pi} X$ (WE HAVE IN MIND $G \hookrightarrow P \times V \longrightarrow P \times_\rho V$) IS AN n-FORM ($n = \dim G$) W ON Q SUCH THAT, IF $\iota_x : \pi^{-1}(x) \hookrightarrow Q$, THEN

$$\int_{\pi^{-1}(x)} \iota_x^* W = 1$$

$\forall x \in X$.

ONE CAN ALWAYS CONSTRUCT SUCH A THING AND, IN OUR CASE, IT CAN BE WRITTEN AS A BEREZIN INTEGRAL

$$W = (\det R)^{-1} \int e^{(c^*, \eta)} D\eta$$

WHERE η IS A VARIABLE IN $\mathfrak{g}$ AND (c^*, η) IS THE 1-FORM ON $P \times V$ DEFINED BY

$$(c^*, \eta)(\xi) = (c^*(\xi), \eta) = \langle \xi, c(\eta) \rangle = \langle \xi, \eta^* \rangle$$

FOR ANY VECTOR FIELD ξ.

WEDGING WITH W KILLS ANY VERTICAL PARTS AND LEAVES ONLY TERMS WITH HORIZONTAL PARTS AND A FACTOR OF W. THEN FIBER INTEGRATION (I.E., IGNORING W) LEAVES ONLY HORIZONTAL PARTS.

DOING THIS FOR U IN (4) ADDS (c^*, η) TO THE EXPONENT, CANCELS THE $\det R$ AND INTRODUCES ONE MORE BEREZIN INTEGRAL:

$$(5) \quad U = (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \iiiint \exp\left(\tfrac{1}{2}\psi^T(\rho_* \phi)\psi + i\psi^T d\mu - i(dc^*, \lambda) + i(\phi, R\lambda) + (c^*, \eta)\right) D\eta\, D\psi\, d\phi\, d\lambda$$

PULLING BACK THE V-PARTS OF THIS FORM ON $P \times V$ BY AN EQUIVARIANT MAP

$$s : P \longrightarrow V$$

GIVES A FORM

$$(6) \quad (2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \iiiint \exp\left(\tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T ds - i(dc^*, \lambda) + i(\phi, R\lambda) + (c^*, \eta)\right) \mathcal{D}\eta\, \mathcal{D}\psi\, d\phi\, d\lambda$$

WHOSE INTEGRAL OVER P IS THE EULER NUMBER OF $P \times_\rho V$.

FINALLY, WE ADOPT THE PRACTICE IN (SUPERSYMMETRIC) PHYSICS OF WRITING THE INTEGRAL OF A FORM AS A BEREZIN INTEGRAL FOLLOWED BY A VOLUME INTEGRAL:

$$\int_P \alpha = \int_P \int \alpha(x^i, \xi_i)\, \mathcal{D}\xi\, d\omega$$

APPLY THIS TO (6) AND (FINALLY!) SUPPRESS ALL BUT ONE OF THE INTEGRAL SIGNS TO GET THE ATIYAH-JEFFREY FORMULA FOR THE EULER NUMBER OF $P \times_\rho V$:

$$(7) \quad (2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \int \exp\left(\tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T ds - i(dc^*, \lambda) + i(\phi, R\lambda) + (c^*, \eta)\right) \mathcal{D}\xi\, \mathcal{D}\eta\, \mathcal{D}\psi\, d\lambda\, d\phi\, d\omega$$

NOW WE WANT TO APPLY THIS (FORMALLY) TO AN INFINITE-DIMENSIONAL VECTOR BUNDLE THAT ARISES NATURALLY IN DONALDSON THEORY.

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD WITH $b_2^+(M) > 1$ AND ODD

g = GENERIC RIEMANNIAN METRIC ON M

$SU(2) \hookrightarrow P \xrightarrow{\pi} M$ WITH

$$8c_2(P) - 3(1 + b_2^+(M)) = 0$$

PRINCIPAL BUNDLE :

$$\hat{\mathcal{G}}(P) \hookrightarrow \hat{\mathcal{A}}(P) \longrightarrow \hat{\mathcal{B}}(P)$$

(NOTATION FROM ADDENDUM 2)

$\hat{\mathcal{G}}(P)$- ACTION ON $\Omega_+^2(M, \mathrm{ad}\,P)$ GIVES AN ASSOCIATED VECTOR BUNDLE

$$\hat{\mathcal{A}}(P) \times_{\hat{\mathcal{G}}(P)} \Omega_+^2(M, \mathrm{ad}\,P) \longrightarrow \hat{\mathcal{B}}(P)$$

s_0 = 0-SECTION

s = SELF-DUAL CURVATURE SECTION $= F^+$

$$s([\omega]) = [\omega, F_\omega^+]$$

$$\mathcal{M}(P,g) = \mathcal{S}(\hat{\mathcal{B}}(P)) \cap \mathcal{S}_0(\hat{\mathcal{B}}(P))$$

$$\chi_0(M) = \text{"EULER NUMBER"}$$

ATIYAH-JEFFREY (SKETCH):

NOTE: DETAILS ARE AVAILABLE IN ADDENDUM 13.

REQUIRES A $\hat{\mathcal{G}}$-INVARIANT RIEMANNIAN METRIC FOR $\hat{\mathcal{A}}$ (I WILL DROP THE "(P)" EVERYWHERE FROM THIS POINT ON).

$\hat{\mathcal{A}}$ OPEN IN $\mathcal{A}$, WHICH IS AN AFFINE SPACE

$$T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, \mathrm{ad}\, P)$$

RECALL: ON $\Omega^k(M, \mathrm{ad}\, P)$,

$$\langle \mu, \nu \rangle_k = \int_M -2\,\mathrm{tr}\,(\mu \wedge {*\nu})$$

IS INVARIANT UNDER ADJOINT ACTION.

THUS, $\langle\ ,\ \rangle_1$ ON $T_\omega(\hat{\mathcal{A}})$ GIVES $\hat{\mathcal{G}}$-INVARIANT RIEMANNIAN METRIC ON $\hat{\mathcal{A}}$.

THIS GIVES A CORRESPONDING CONNECTION ON $\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \to \hat{\mathcal{B}}$:

$$T_\omega(\hat{\mathcal{A}}) \cong T_\omega(\omega\cdot\hat{\mathcal{G}}) \oplus \mathrm{KER}(\delta^\omega) \equiv \mathrm{IM}(d^\omega) \oplus \mathrm{KER}(\delta^\omega)$$

$$d^\omega : \Omega^0(M, \mathrm{ad}\,P) \to \Omega^1(M, \mathrm{ad}\,P)$$

$$\delta^\omega : \Omega^1(M, \mathrm{ad}\,P) \to \Omega^0(M, \mathrm{ad}\,P)$$

(SEE ADDENDUM 2)

$$\mathrm{HOR}_\omega(\hat{\mathcal{A}}) \cong \mathrm{KER}(\delta^\omega)$$

NOW EXAMINE EACH TERM IN THE ATIYAH-JEFFREY EXPONENT

$$-\tfrac{1}{2}\|s(\omega)\|^2 + \tfrac{1}{2}\psi^T(P_*\phi)\psi + i\,\psi^T ds_\omega(\xi)$$

$$-i\,(dC_\omega^*(\xi), \lambda) + i\,(\phi, R_\omega\lambda) + (C_\omega^*(\xi), \eta)$$

SEPARATELY.

1. $$-\tfrac{1}{2}\|s(\omega)\|^2 = -\tfrac{1}{2}\|F_\omega^+\|^2 = \int_M \mathrm{tr}(F_\omega^+ \wedge {}^*F_\omega^+)$$

$$= \int_M \mathrm{tr}(F_\omega^+ \wedge F_\omega^+)$$

$$\boxed{-\tfrac{1}{2}\|s(\omega)\|^2 = \underbrace{\tfrac{1}{2}\int_M \mathrm{tr}(F_\omega \wedge {}^*F_\omega)}_{\text{YANG-MILLS TERM}} + \underbrace{\tfrac{1}{2}\int_M \mathrm{tr}(F_\omega \wedge F_\omega)}_{\text{TOPOLOGICAL TERM}}}$$

DONALDSON THEORY ANALOGUES OF C, C^* AND R :

$$\text{LIE ALGEBRA OF } \hat{\mathcal{G}} \cong \Omega^0(M, \mathrm{ad}P)$$

$$T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, \mathrm{ad}P)$$

SO $\forall\, \omega \in \hat{\mathcal{A}}$

$$C_\omega : \Omega^0(M, \mathrm{ad}P) \longrightarrow \Omega^1(M, \mathrm{ad}P)$$

$$C_\omega(\xi) = \frac{d}{dt}(\omega \cdot \exp(t\xi))\Big|_{t=0} = d^\omega \xi$$

$$C_\omega = d^\omega$$

THUS, RELATIVE TO $\langle\ ,\ \rangle_0$ AND $\langle\ ,\ \rangle_1$,

$$C_\omega^* : \Omega^1(M, \mathrm{ad}P) \to \Omega^0(M, \mathrm{ad}P)$$

$$C_\omega^* = \delta^\omega = -{*}d^\omega{*}$$

AND

$$R_\omega : \Omega^0(M, \mathrm{ad}P) \to \Omega^0(M, \mathrm{ad}P)$$

$$R_\omega = C_\omega^* \circ C_\omega = \delta^\omega \circ d^\omega = \Delta_0^\omega = -{*}d^\omega{*}d^\omega$$

$=$ SCALAR LAPLACIAN OF ω

2. $i(\phi, R_\omega \lambda)$

BOTH ϕ AND λ ARE IN THE LIE ALGEBRA OF $\hat{\mathcal{G}}$ SO INTRODUCE TWO "BOSONIC" FIELDS

$$\phi, \lambda \in \Omega^0(M, \mathrm{ad}P)$$

AND INTERPRET $(\ ,\ )$ AS $\langle\ ,\ \rangle_0$. FOR EACH $\omega \in \hat{\mathcal{A}}$,

$$\boxed{i(\phi, R_\omega \lambda) = 2i \int_M \mathrm{tr}\,(\phi\, d^\omega * d^\omega \lambda)}$$

3. $(C_\omega^*(\xi), \eta)$

$C_\omega^* : \Omega^1(M, \mathrm{ad}P) \to \Omega^0(M, \mathrm{ad}P)$ SO INTRODUCE TWO "FERMIONIC" FIELDS

$$\xi \in \Omega^1(M, \mathrm{ad}P)$$

$$\eta \in \Omega^0(M, \mathrm{ad}P)$$

AND IDENTIFY $(\ ,\ ) = \langle\ ,\ \rangle_0$.

$$(C_\omega^*(\xi), \eta) = \langle C_\omega^*(\xi), \eta \rangle_0 = \langle \xi, C_\omega(\eta) \rangle_1$$

$$= \langle \xi, d^\omega \eta \rangle_1$$

$$\boxed{(C_\omega^*(\xi), \eta) = -2 \int_M \mathrm{tr}\,(\xi \wedge d^\omega \eta)}$$

4. $i\,\psi^T ds_\omega(\eta)$

$$s = F^+ : \hat{\mathcal{A}} \longrightarrow \Omega^2_+(M, \mathrm{ad}P)$$

$$ds_\omega = d^\omega_+ : \Omega^1(M, \mathrm{ad}P) \longrightarrow \Omega^2_+(M, \mathrm{ad}P)$$

$$ds_\omega(\eta) = d^\omega_+ \eta$$

INTRODUCE "FERMIONIC" FIELD

$$\psi \in \Omega^2_+(M, \mathrm{ad}P)$$

AND INTERPRET FINITE-DIMENSIONAL EXPRESSIONS

$$A^T B = (A^1 \cdots A^r) \begin{pmatrix} B^1 \\ \vdots \\ B^r \end{pmatrix}$$

AS THE APPROPRIATE INNER PRODUCT ON FIELDS.

$$i\,\psi^T ds_\omega(\eta) = i \langle \psi, d^\omega_+(\eta) \rangle_2$$

$$\boxed{i\,\psi^T ds_\omega(\eta) = -2i \int_M d^\omega \eta \wedge \psi}$$

5. $\frac{1}{2}\psi^T (\rho_+ \phi)\psi$

ρ IS THE REPRESENTATION OF $\hat{\mathcal{G}}$ ON $\Omega^2_+(M, ad\,P)$ GIVING RISE TO THE ASSOCIATED VECTOR BUNDLE. REGARDING $\hat{\mathcal{G}}$ AS SECTIONS OF $Ad\,P$ THIS IS, POINTWISE, THE ADJOINT ACTION OF $SU(2)$ ON $su(2)$. INFINITESIMAL ACTION ρ_* IS BRACKET.

$$(\rho_* \phi)\psi = [\phi, \psi]$$

SO

$$\tfrac{1}{2}\psi^T(\rho_* \phi)\psi = \tfrac{1}{2}\langle [\phi,\psi], \psi\rangle_2$$

$$\boxed{\tfrac{1}{2}\psi^T(\rho_*\phi)\psi = -\int_M tr\,(\psi \wedge [\phi, \psi])}$$

6. $-i\,(dC^*_\omega(\xi), \lambda)$

dC^* IS A 2-FORM ON $\hat{A}$ WITH VALUES IN $\Omega^0(M, ad\,P)$

COMPUTING dC^*_ω REQUIRES A LITTLE WORK (ADDENDUM 13), BUT THE RESULT IS

$$dC^*_\omega(\xi_1, \xi_2) = -2\,{}^*[\xi_1, {}^*\xi_2]$$

$$-i\,(dC^*_\omega(\xi), \lambda) = 2i\,\langle {}^*[\xi, {}^*\xi], \lambda\rangle_0$$

$$\boxed{-i\,(dC_\omega^*(\zeta),\lambda) = -4i\int_M tr\,([\zeta,{}^*\zeta]\lambda)}$$

ADDING TOGETHER ALL OF THE TERMS IN THE BOXES AND (IN DEFERENCE TO THE PHYSICS LITERATURE) SWITCHING TO

$$Tr = -2\,tr$$

GIVES

$$\int_M Tr\,\{-\tfrac{1}{4}F_\omega\wedge{}^*F_\omega - \tfrac{1}{4}F_\omega\wedge F_\omega + \tfrac{1}{2}\psi\wedge[\phi,\psi] + i\,d^\omega\zeta\wedge\psi$$
$$+\,2i\,[\zeta,{}^*\zeta]\lambda - i\phi\, d^{\omega *}d^\omega\lambda + \zeta\wedge{}^*d^\omega\eta\ \}$$

WRITING THE INTEGRAL IN (7) IN THE FORM

$$\int e^{-\int_M Tr\,\mathcal{L}_{DW}}\,\mathcal{D}\Phi$$

WE IDENTIFY THE <u>DONALDSON</u>-<u>WITTEN</u> <u>LAGRANGIAN</u> :

$$\mathcal{L}_{DW}[\Phi] = \tfrac{1}{4}F_\omega\wedge{}^*F_\omega + \tfrac{1}{4}F_\omega\wedge F_\omega - \tfrac{1}{2}\psi\wedge[\phi,\psi] - i\,d^\omega\zeta\wedge\psi$$
$$-\,2i\,[\zeta,{}^*\zeta]\lambda + i\phi\, d^{\omega *}d^\omega\lambda - \zeta\wedge{}^*d$$

<u>ACTION</u> : $S_{DW}[\Phi] = \int_M Tr\,\mathcal{L}_{DW}$

<u>PARTITION</u> <u>FUNCTION</u> : $Z_{DW} = \int e^{-S_{DW}[\Phi]/e^2}\,\mathcal{D}\Phi$