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Geometry of super Yang–Mills and supergravity

by

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Preface

During the academic years 1997-1998, a joint seminar of the University of Leipzig and the Max-Planck-Institut für Mathematik in den Naturwissenschaften was organsized, with the title "Geometry and Physics". Among other things, we presented some of the aspects of string theory, and then decided to commit ourselves to supergravity. With none of us being experts on this topic, we decided to use the book [6], in which supergravity is discussed in a geometric formulation in the last chapter. One semester was devoted to the basics, essentially just the first chapter. In the following semester (WS 1998-1999) we tried to study supergravity and super Yang-Mills. This meant reviewing much of the second and parts of the third and fourth chapters of the book, which is the origin of these notes. It would have been impossible for us to attempt to understand all that material including the proofs, so I set myself the goal of writing a manuscript in which the theory could be *understood*, without going into any of the proofs. The goal of this part of the seminar was to understand how one formulates super Yang-Mills geometrically, and how the Radon-Penrose transform is generalized to this case.

In the last part of that seminar we went on to a more detailed description of supergravity. In addition to explaining how the physisists view it, we gave a detailed description of the geometric presentation of [6]. In this part we in fact derived the proofs of the book in more detail, giving some more background information where necessary.

Bruce Hunt
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1 Gravity & supergravity

We recall some facts already discussed in this seminar.

1.1 Field theory

- *Path Integrals:* the object of quantum field theory is to make sense of objects of the following kind:

  $$\langle A \rangle = \int A(\mathbf{\Phi}) \mathrm{e}^{iS(\mathbf{\Phi})} D(\mathbf{\Phi}).$$

  This is a formal expression, and the object is to make it rigorous. The symbols have the following meanings:

  - $\mathbf{\Phi}$ denotes the set of fields of the theory.
  - $S(\mathbf{\Phi}) = \int_M \mathscr{L}(\mathbf{\Phi}) d^4x$ is the action functional, the integral being extended over space-time M. The function $\mathscr{L}$ is called the *Lagrange density.*
  - A is an operator, and $\langle A \rangle$ is interpreted to be its average value (expectation).
  - the expression $\mathrm{e}^{iS(\mathbf{\Phi})} D(\mathbf{\Phi})$ is the formal part, and is heuristically a measure on the space of all fields $\mathbf{\Phi}$.

- *Field content:* We are given a set of vector bundles $\{E_i\}$ on M. Then the fields are of two kinds:

  - Sections ψ_i of E_i or tensor products of these. These are *matter* fields, which may be fermionic (if spinors are involved) or bosonic (no spinors).
  - Connections ∇_i on E_i. These are the *gluons* (which are always bosonic).

- *Lagrangian:* We look for the simplest function of the fields and their derivatives which is invariant under the symmetries of the theory. In general this means that the Lagrangian density is a function of the fields and their *covariant* derivatives, $\mathscr{L} = \mathscr{L}(\mathbf{\Phi}, \mathbf{\nabla\Phi})$. The terms $\nabla_i\psi_i$ describe the interaction of the matter field ψ_i with the gauge field ∇_i. In section 2.3 below we will describe the contributions of various fields to the Lagrangian.

- *Dynamic equations:* These are the Euler-Lagrange equations one gets from the Lagrangian, i.e., solutions of $\delta \int \mathscr{L}(\mathbf{\Phi}) d^4x = 0$. Examples of these equations are:

  - Maxwell equations
  - Klein-Gordon equation
  - Dirac equation
  - Yang-Mills equation
  - Einstein equation

In section 2.3 below we will discuss these equations in more detail.

1.2 Gravity

Here one of the vector bundles E_i is the *tangent bundle* TM of M. The corresponding gauge group is the Lorentz group. Hence in gravity we are looking for *local* Lorentz invariance, as opposed with *global* Lorentz invariance as in QED. The theory is *pure* gravity if the tangent bundle is the only bundle of the theory, otherwise we speak of *gravity with background*. We recall that the Lagrangian of this theory is just (up to a constant) the scalar curvature R. The dynamic equation is then the Einstein equation (discussed in more detail in section 2.3 below). The *field* here is the metric g, while the *gluon* is the Levi-Cevita connection.

1.3 Supergravity

We now start with a supermanifold M (which one can take as a complex supermanifold with a real structure), TM is the super-tangent bundle, and the gauge group is now the super-Poincaré group. In what follows we consider only *simple supersymmetry*, i.e., $N = 1$.

Remark: Since $\mathscr{L}(\mathbf{\Phi})d^4x$ is a volume form on M, finding the Lagrangian amounts to finding a volume form invariant under the action of the super-Poincaré group.

- *Lagrangian:* In the case of supergravity, the result is that the Lagrangian is a linear combination of the usual Hilbert-Einstein action and a so-called *Rarita-Schwinger* action. This will be discussed below. The result of Manin's anlysis is that in a proper gauge, this Lagrangian is proportional to a Berezinian.

- *Fields:* These are determined by the irreducible representations of the super-Poincaré group $\mathcal{SP}$. Each such representation determines a vector bundle which we may use, sections of which will be the matter fields. We briefly recall the classification of irreducible representations of the super-Poincaré group. Details can be found in [2].

  - Irreducible unitary represetations of the Poincaré group $\mathcal{P}$ ([2], section 1.5):
    * Massive representations: These are characterized by pairs (m, s), where m is the mass, and $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ is the *spin*.
    * Massless representations: These are characterized by an invariant λ known as the *helicity*, $\lambda \in \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots\}$.
  - Irreducible unitary representations of the super-Poincaré group ([2], section 2.3):
    * Massive representations: these are classified by mass m and *superspin* $Y \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$. The corresponding field of such a representation (i.e., section of the vector bundle defined by that representation) is a *superfield*. This can be decomposed into ordinary fields, which amounts to decomposing the irreducible representation under $\mathcal{SP}$ under the subgroup $\mathcal{P}$. The result is as follows. There is a natural decomposition of the Hilbert space $\mathcal{H}$ of the irreducible representation of $\mathcal{SP}$ with numbers (m, Y):

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-,$$

the factors of which are invariant under $\mathcal{P}$. Moreover, $\mathcal{H}_+$ and $\mathcal{H}_-$ are irreducible representations of $\mathcal{P}$, which have (m, s) equal to (m, Y). The subspace $\mathcal{H}_0$ splits into two $\mathcal{P}$-irreducible representations, which have numbers $(m, Y + \frac{1}{2})$ and $(m, Y - \frac{1}{2})$.

* Massless representations: these are classified by *superhelicity* $\kappa \in \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots\}$. The Hilbert space $\mathcal{H}$ of an irreducible representation of $\mathcal{SP}$ with superhelicity κ splits:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where both factors are invariant under $\mathcal{P}$. These are in fact irreducible massless representations of $\mathcal{P}$, with helicity values for $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively, equal to $\lambda = \kappa + \frac{1}{2}$ and $\lambda = \kappa$, respectively.

As a consequence of this classification, we see that the simplest irreducible representation of $\mathcal{SP}$ is the massless one with superhelicity $\kappa = \frac{3}{2}$. This gives us fields of spin 2 (the graviton g) and of spin $\frac{3}{2}$ (the Rarita-Schwinger spin-$\frac{3}{2}$ field). The later is the super partner of the former, i.e., a *gravitino*. Usually it is denoted by χ.

2 Complex space-time

We recall a few notions introduced in Chapter 2 of [6] which are needed in the sequel.

2.1 GS Manifolds

Here we recall some notions presented in the seminar last semester.

- A *GS manifold* is defined to be a tuple $(M, S, \widetilde{S}, \sigma)$, where M is a complex manifold, S and $\widetilde{S}$ are locally free sheaves (sheaves of sections of vector bundles) and

$$\sigma : S \otimes \widetilde{S} \xrightarrow{\sim} \Omega^1 M$$

  is an isomorphism.

- A GS manifold $(M, S, \widetilde{S}, \sigma)$ has natural 1, c, d-conic structures, where $c = \mathrm{rank}(S)$, $d = \mathrm{rank}(\widetilde{S})$. Let $\pi_c : F_c \longrightarrow M$ (resp. $\pi_d : F_d \longrightarrow M$) be these conic structures, h_c (resp. h_d) a conic connection on them. If h_c (resp. h_d) is integrable, then M is said to be *self-dual* or *left-flat* (resp. *anti self-dual* or *right-flat*). A differential equation for the coefficients of h_c (resp. h_d) expressing this is called the *self-duality* (resp. *anti self-duality*) equations.

- Let $E \longrightarrow M$ be a vector bundle over a GS manifold, $\nabla : E \longrightarrow E \otimes \Omega^1 M$ a connection and $F(\nabla) = F_+(\nabla) + F_-(\nabla) \in End(E) \otimes \Omega^2 M$ it curvature. The equation for the coefficients of ∇ so that $F_+(\nabla) = 0$ (resp. $F_-(\nabla) = 0$) are called *Yang-Mills self-duality* (resp. *anti self-duality*) equations. These equations are equivalent to the integrability of $\pi_c^*(\nabla)$ (resp. $\pi_d^*(\nabla)$) along the fibers of a fibering tangent to h_c (resp. h_d). This is I.7.9 in [6].

2.2 Complex space-time

Definition 2.1 A *complex space-time* is a GS manifold $(M, S, \widetilde{S}, \sigma)$, where M has dimension 4, with the following additional stucture:

a) spinor connections $\nabla_l : S \longrightarrow S \otimes \Omega^1 M$, $\nabla_r : \widetilde{S} \longrightarrow \widetilde{S} \otimes \Omega^1 M$.

b) spinor metrics $\varepsilon \in \mathrm{H}^0(M, \Lambda^2 S)$, $\widetilde{\varepsilon} \in \mathrm{H}^0(M, \Lambda^2 \widetilde{S})$.

c) a real structure $\varrho : M \longrightarrow M$, compatible with ε and $\widetilde{\varepsilon}$.

We recall that a real structure is an involution on the set of points of M such that $\varrho^*(\mathscr{O}_M)$ coincides with the sheaf of anti-holomorphic functions (so it is locally $z \mapsto \overline{z}$). The condition of being compatible with the spinor metrics means the following. The metric g on M can be described by setting

$$g := \varepsilon \otimes \widetilde{\varepsilon} \in \mathrm{H}^0(M, \Lambda^2 S \otimes \Lambda^2 \widetilde{S}) \subset \mathrm{H}^0(M, S^2(\Omega^1 M)).$$

Then we assume that ϱ leaves g invariant. It follows from this that g is a real analytic metric on M^ϱ, the set of real points (i.e., those fixed under ϱ), and its signature depends on the action of ϱ on spinors. If $\varepsilon^\varrho = \widetilde{\varepsilon}$, then we have Minkowski signature, if $\varepsilon^\varrho = \varepsilon$, $\widetilde{\varepsilon}^\varrho = \widetilde{\varepsilon}$, then we have Euclidean signature.

All of this can be expressed in local coordinates, in which guise it is well-known in physics liturature. Choose local coordinates (x^a), $a = 0, 1, 2, 3$ on M which take real values on M^ϱ. As a basis of sections in these local coordinates of S and $\widetilde{S}$ we take w^0, w^1 (for S) and $z^{\dot{0}}$, $z^{\dot{1}}$ (for $\widetilde{S}$). The isomorphism σ is given in local coordinates by

$$\sigma(w^\alpha \otimes z^{\dot{\beta}}) = e_a^{\alpha\dot{\beta}} dx^a.$$

The spinor connections are defined by symbols ω, and we have

$$\nabla_l w^\alpha = \omega^\alpha_{\beta a} w^\beta \otimes dx^a, \quad \nabla_r z^{\dot{\alpha}} = \omega^{\dot{\alpha}}_{\dot{\beta} a} z^{\dot{\beta}} \otimes dx^a.$$

Moreover, the spinor sections and metric are given in these local coordinates as follows:

$$\varepsilon = 2w^0 \wedge w^1 = \varepsilon_{\alpha\beta} w^\alpha \otimes w^\beta, \quad \widetilde{\varepsilon} = 2z^{\dot{0}} \wedge z^{\dot{1}} = \varepsilon_{\dot{\alpha}\dot{\beta}} z^{\dot{\alpha}} \otimes z^{\dot{\beta}}.$$
$$g_{ab} dx^a dx^b = \varepsilon_{\alpha\beta} \varepsilon_{\dot{\gamma}\dot{\delta}} e_a^{\alpha\dot{\gamma}} e_b^{\beta\dot{\delta}} dx^a dx^b.$$

2.3 Fields and Lagrangians on complex space-time

Classical fields on M are sections of vector bundles or connections on these bundles. The Lagrangian is a *volume form* on M, which is a function of fields and their derivatives with proper symmetries. This is explained in Chapter 2, §1, 14-19 of [6].

2.3.1 The gravitational field

We sketch the ingredients.

- *Bundle:* TM, the tangent bundle. The metric is given by $g = \varepsilon \otimes \widetilde{\varepsilon}$, and this determines a volume form (not yet the Lagrangian!), $\boldsymbol{\nu} := \sqrt{|\det(g)|}d^4x$. The Lagrangian is of the form $\mathscr{L}(\boldsymbol{\Psi})\boldsymbol{\nu}$, where $\boldsymbol{\Psi}$ is a set of fields containing at least the metric g.

- *Contribution to the Lagrangian:* κR, where R is the scalar curvature. This is the Hilbert-Einstein action.

- *Euler-Lagrange equations:* $R_{ij} - \frac{1}{2}Rg_{ij} =: G = 0$. The expression G is the *Einstein tensor.* This equation is for vacuum (no background).

- *This equation is a self-duality equation:* $G = 0 \iff \mathrm{Ric}^0 = R = 0$, where $\mathrm{Ric}^0_{ij} = R_{ij} - \frac{1}{4}Rg_{ij}$ is the traceless Ricci tensor. We can decompose the curvatures $\Phi_l := \Phi(\nabla_l)$ and $\Phi_r := \Phi(\nabla_r)$ (where Φ denotes curvature) into left and right components, $\Phi_l = \Phi_{ll} + \Phi_{lr}$, $\Phi_r = \Phi_{rl} + \Phi_{rr}$. We have $\mathrm{Ric}^0 = 0 \iff \Phi_{rl} = 0$ or $\Phi_{lr} = 0$. In this sense, the equations are self-duality equations (assuming $R = 0$).

- *Gravity in background:* In this case the equation is $G = T(\boldsymbol{\Psi})$, where T is the *energy-momentum* tensor.

2.3.2 The electromagnetic field

Again we just sketch the ingredients.

- *Bundle:* E, a one-dimensional bundle, with connection ∇. This connection is the *electromagnetic field.*

- *Contribution to the Lagrangian:* This is $(\Phi(\nabla), \Phi(\nabla))$, where $\Phi(\nabla)$ is the curvature of ∇. In local coordinates this is the familiar term $F_{\mu\nu}F^{\mu\nu}$ in the Lagrangian of QED. Note however, that this depends on the metric g (through the scalar product), so if g is a dynamical variable, this term also describes the interaction of the electromagnetic and gravitational field.

- *Euler-Lagrange equations:* These are the well known Maxwell equations

$$\begin{cases} d\Phi &= 0 \quad \text{(Bianchi identity)} \\ d * \Phi &= \begin{cases} 0 & \text{in vacuum} \\ J(\Psi) & \text{in the presence of charged fields} \end{cases} \end{cases}$$

- *These are also self-duality equations:* If $\Phi = \Phi_l + \Phi_r$ is the decomposition into self-dual and anti self-dual parts, then the Maxwell equations are equivalent to $\Phi_{rl} = 0$ in vacuum.

2.3.3 Yang-Mills fields

- *Bundle:* The vector bundle E is now of higher rank ≥ 2. The *Yang-Mills field* is a connection ∇ on this bundle. The curvature is now $\Phi(\nabla) \in End(E) \otimes \Omega^2 M$.

- *Contribution to the Lagrangian:* This is again of the form $(\Phi(\nabla), \Phi(\nabla))$, where now one uses also the scalar product $< a, b >= tr(a \cdot b)$ in $End(E)$. As above, this also depends on the metric and so also describes the interaction of the Yang-Mills field with the gravitational field.

- *Euler-Lagrange equations:* These are now the Yang-Mills equations:

$$\begin{cases} \widetilde{\nabla}(\Phi) &= 0 \quad \text{(Bianchi identity)} \\ \widetilde{\nabla}(*\Phi) &= \begin{cases} 0 & \text{in vacuum} \\ J(\mathbf{\Psi}) & \text{in the presence of sources} \end{cases} \end{cases},$$

  where $\widetilde{\nabla}$ is the extension of ∇ to $End(E) \otimes \Omega^{\cdot} M$.

- *Solutions of the self-dual equations give special solutions of the Euler-Lagrange equations.* The curvature is self-dual if $*\Phi = \Phi$ (Euclidean case) or $*\Phi = i\Phi$ (Minkowski case). If this equation is satisfied, then the second equation above follows from the first (Bianchi identity). As is well known, these solutions of the second equation correspond to *absolute minima* of the action. Such solutions are referred to as *instantons*.

2.3.4 Matter fields of Spin 0

- *Bundle:* A vector bundle with connection (E, ∇), not composed of spinor bundles. The field is a section ϕ of E.

- *Contribution to Lagrangian:* These are of two types.
  - A *potential* term, $V(\phi)$, often a polynomial in ϕ.
  - A *kinetic* term of the form $(\nabla\phi, \nabla\phi)$. This describes the interaction of ϕ (matter) with the connection field ∇ (the gluon), as well as with the gravitational field (through the definition of the scalar product).

If the bundle E is of rank one (real or complex), then ϕ is a scalar and there is also a mass term. The Euler-Lagrange equation in this case is the Klein-Gordon equation $(\Box + m^2)\phi = 0$, and ϕ is then a Klein-Gordon particle.

2.3.5 Matter fields of Spin $\frac{1}{2}, \frac{3}{2}, \ldots$

- *Bundle:* This is now of the form $(E \otimes (S \oplus \widetilde{S}), D)$, where $D : E \otimes (S \oplus \widetilde{S}) \longrightarrow \widetilde{S} \oplus S$ is the Dirac operator. The fields are sections, the spin is $\frac{1}{2}$. More generally we have tensor products of an odd number of spinor bundles with E.

- *Contribution to Lagrangian:* There are again two types:
  - A *mass* term (ψ^ϱ, ψ).
  - A *kinetic* term of the form $Re(\psi^\varrho, D\psi)$.
- *Euler-Lagrange equation:* This amounts to the Dirac equation in this case, $(D-m)\psi = 0$, where D denotes the Dirac operator, and m is the mass.

2.3.6 Examples

- The Lagrangian of Quantum Electrodynamics (QED) is

$$\mathscr{L}(A_\mu, \psi) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\overline{\psi}\gamma^\mu d_\mu \psi - m\overline{\psi}\psi - e\overline{\psi}\gamma^\mu \psi A_\mu .$$

  Here ψ is a Dirac spinor (the electron), A is the electromagnetic field, F its curvature, m the mass of the electron, and e is the charge, the coupling parameter for the *interaction term* $\overline{\psi}\gamma^\mu\psi A_\mu$. More invariantly, this can be written:

$$\mathscr{L}(\nabla, \psi) = -\frac{1}{4}(\Phi(\nabla), \Phi(\nabla)) + Re(\overline{\psi}, D\psi) - m(\overline{\psi}, \psi).$$

  Note that the interaction term, which physically is put in by hand, is just the covariant part of the covariant derivative $D\psi = (d - eiA_\mu)\psi$.

- The Lagrangian of Quantum Chromodynamics (QCD) is ([3], (12.1.9))

$$\mathscr{L}(V_\mu, \mathbf{\Psi}) = -\frac{1}{4}\left\{\partial_\mu V_\nu^m - \partial_\nu V_\mu^m - \frac{g}{2} f_{npm}(V_\mu^n V_\nu^p - V_\nu^n V_\mu^p)\right\}^2 + i\overline{\psi}^a \gamma_\mu \partial_\mu \psi^a - M\overline{\psi}^a \psi^a - \frac{g}{2}\overline{\psi}^a \gamma_\mu (\lambda_m)^{ab} V_\mu^m \psi^b .$$

  Here $\mathbf{\Psi} = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}$ is a $SU(3)$-triplet of spinors ψ^a, the λ_m are the *Gell-Mann matrices* (cf. *loc. cit.*, (1.2.13)), which are generators of the Lie algebra of $SU(3)$, so these fields are the *quarks*, the connection on the bundle (of the form $\mathscr{E} \otimes S$) is given by V_μ; these are the *gluons*, and g is the coupling constant for the theory. Again, this can be written more invariantly

$$\mathscr{L}(\nabla, \mathbf{\Psi}) = -\frac{1}{4}(\Phi(\nabla), \Phi(\nabla)) + Re(\mathbf{\Psi}^\varrho, D\mathbf{\Psi}) - M(\mathbf{\Psi}^\varrho, \mathbf{\Psi}).$$

3 The Radon-Penrose transform and Yang-Mills equations

The Radon-Penrose tranform was introduced in Chapter 2 of [6]. It is used for two purposes:

- Construction of instantons in terms of algebraic geometry (Atiyah, Hitchin, Ward, Drinfeld, Manin).
- Characterization of solutions of Yang-Mills equations (not YM self-duality equations!) in terms of algebraic geometric structures (Witten, Manin).

The second of these is generalized in Chapter 5 to the super context. Therefore we begin by describing the transform and the second result.

3.1 The Radon-Penrose transform & sheaves trivial along fibrations

We consider, in any of our categories, diagrams of the type

$$\begin{array}{ccc} & F & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ Z & & M, \end{array} \tag{1}$$

where π_1, π_2 are assumed to be submersive morphisms in the category. The kind of transform to be discussed in the following is (pull-back)–(push-down), in either direction, i.e., $(\pi_1)_* \circ \pi_2^* : M \longrightarrow Z$ or $(\pi_2)_* \circ \pi_1^* : Z \longrightarrow M$, and the goal is to make sense of this. For example, in any of the categories we may apply this to differential forms: pulling back is the usual pull-back of differential forms, while push-down is done by integrating along the fibers of the other projection. Since this transformation induces a map between cohomology groups, it is a *linear* transformation. For a transformation like this to work one may need assumption on the fibrations π_1 and π_2, for example, connectivity or connectedness of the fibers. We will apply a transformation in the category of complex manifolds (or superspaces), and instead of differential forms, we will pull back and push down *locally free sheaves.* Since the spaces of these sheaves are non-linear manifolds, this is an essentially *non-linear* transformation, the *Radon-Penrose* transform.

Definition 3.1 A locally free sheaf $\mathscr{E}_Z$ on Z is *M-trivial*, if the restriction $\pi_1^*(\mathscr{E}_Z)$ to any fiber $\pi_2^{-1}(x)$ of π_2 is free, i.e., $\pi_1^*(\mathscr{E}_Z)_{|\pi_2^{-1}(x)} = \mathscr{O}^a_{\pi_2^{-1}(x)}$ is a trivial bundle.

Recall that $\Omega^1 F/Z$ denotes the sheaf of relative differentials, and that the restriction of this sheaf to any fiber $\pi_1^{-1}(z)$ is just the sheaf of differentials of the fiber, $\Omega^1\pi_1^{-1}(z)$. Pulling back any 1-form ω on M and resticting to the fiber defines the restriction map at z, and varying z

we get res : $\pi_2^*(\Omega^1 M) \longrightarrow \Omega^1 F/Z$, the restriction to the vector fields tangent to the fibers of π_1. The subsheaf $\mathscr{N} := \mathrm{Ker}(\mathrm{res})$ of $\pi_2^*(\Omega^1 M)$ is locally free and there is an exact sequence

$$0 \longrightarrow \mathscr{N} \longrightarrow \pi_2^*(\Omega^1 M) \stackrel{\mathrm{res}}{\longrightarrow} \Omega^1 F/Z \longrightarrow 0. \tag{2}$$

Applying the functor $R(\pi_2)_*$ to this, one gets the following:

$$0 \longrightarrow (\pi_2)_*\mathscr{N} \longrightarrow (\pi_2)_*\pi_2^*(\Omega^1 M) = \Omega^1 M \stackrel{(\pi_2)_*\mathrm{res}}{\longrightarrow} (\pi_2)_*\Omega^1 F/Z \longrightarrow R^1(\pi_2)_*\mathscr{N} \longrightarrow \cdots \tag{3}$$

Proposition 3.2 *Assume the fibers of π_2 are compact and connected, and that $(\pi_2)_*\mathscr{N} = R^1(\pi_2)_*\mathscr{N} = 0$. Then there is a well-defined functor*

$$\begin{array}{ccc} (M\text{-trivial sheaves on } Z) & \longrightarrow & (\text{Sheaves with connection on } M) \\ \mathscr{E}_Z & \mapsto & (\mathscr{E}, \nabla), \end{array}$$

where $(\mathscr{E}, \nabla)$ is defined below.

$\mathscr{E}$ is the image of $\mathscr{E}_Z$ under the Penrose transform, $\mathscr{E} = (\pi_2)_*\pi_1^*(\mathscr{E}_Z)$. To define the connection ∇, we start with a connection $\nabla_{F/Z}$ on the bundle $\pi_2^*(\mathscr{E}) =: \mathscr{E}_F$, which is defined as the relative connection

$$\nabla_{F/Z} : \mathscr{E}_F \longrightarrow \mathscr{E}_F \otimes \Omega^1 F/Z$$

which annihilates the subsheaf $\pi_1^{-1}(\mathscr{E}_Z) \subset \mathscr{E}_F$. This is then pushed down to M, $\nabla := (\pi_2)_*(\nabla_{F/Z}) : (\pi_2)_*\mathscr{E}_F = \mathscr{E} \longrightarrow \mathscr{E} \otimes \Omega^1 M$. In this way one constructs a connection ∇ on $\mathscr{E}$ without pulling back a connection, by using the fibration π_2 instead to define "horizontal". The proof of 3.2 is based on the fact that by (3) and the assumption $(\pi_2)_*\mathscr{N} = R^1(\pi_2)_*\mathscr{N} = 0$, we have an isomorphism

$$\Omega^1 M \stackrel{\sim}{\longrightarrow} (\pi_2)_*\Omega^1 F/Z,$$

from which one concludes easily that $\nabla = (\pi_2)_*(\nabla_{F/Z})$ is a connection.

Conversely, given $(\mathscr{E}, \nabla)$ on M, we can lift ∇ to $\pi_2^*(\mathscr{E}) = \mathscr{E}_F$, then restrict the connection to $\Omega^1 F/Z$ and define a $\nabla_{F/Z}$:

$$\nabla_{F/Z} : \mathscr{E}_F \stackrel{\pi_2^*(\nabla)}{\longrightarrow} \mathscr{E}_F \otimes \pi_2^*(\Omega^1 M) \stackrel{\mathrm{id}\otimes\mathrm{res}}{\longrightarrow} \mathscr{E}_F \otimes \Omega^1 F/Z.$$

Let $\mathscr{E}_F'$ denote the kernel of $\nabla_{F/Z}$. The sheaf we are looking for, the $\mathscr{E}_Z$ of the above, should be $\mathscr{E}_Z := (\pi_1)_*(\mathscr{E}_F')$. For this sheaf to be *locally free*, we need some assumptions: the fibers of π_1 should be connected, the curvature and holonomy of ∇ along the fibers of π_1 should be trivial.

Theorem 3.3 (2.2.3) [1] *Assume: i) the fibers of π_1 are connected, ii) the fibers of π_2 are compact and connected, iii) $(\pi_2)_*\mathscr{N} = R^1(\pi_2)_*, \mathscr{N} = 0$. Then the transformations above give (quasi-inverse) equivalences of the following categories:*

[1] Numbers like this give reference to the numbering in Manin's book [6], where **(x.y.z)** is a reference to Chapter x, §y No. z

(a) *the category of M-trivial locally free sheaves on Z,*

(b) *the category of pairs $(\mathscr{E}, \nabla)$, whre $\mathscr{E}$ is a locally free sheaf on M and ∇ is a connection on $\mathscr{E}$ with trivial curvature and monodromy along the fibers of π_1.*

Recalling that pairs $(\mathscr{E}, \nabla)$ on M define the basic fields of the theory, we see that this "translates" physical fields into purely algebraic geometric objects (locally free sheaves $\mathscr{E}_Z$ on Z).

3.2 The self-dual diagram and the diagram of null geodesics

In this section we introduce the two basic diagrams which will later be generalized to the super context. With their help, the *equations* defining the physical fields (Maxwell...) will be "translated" into problems in algebraic geometry, namely, conditions on elements of certain cohomology groups. The calculation of these groups with the methods of algebraic geometry then "encode" the *solutions* of the physical equations, i.e., the physical fields.

3.2.1 The self-dual diagram

Definition 3.4 A *self-duality diagram* is a double fibration as in (1) with the following properties:

(a) The dimensions of Z, F and M are 3, 5 and 4, respectively.

(b) F/M is a relative projective line: $F = \mathbb{P}_M(S^*)$, where S^* is a locally free rank two sheaf on M.

(c) The fibers of π_1 are transversal to the fibers of π_2. More precisely, the map $d\pi_2 : \mathscr{T}F/Z \longrightarrow \pi_2^*(\mathscr{T}M)$ is a local direct sum embedding, and the corresponding morphism $i : F \longrightarrow G_M(2, \mathscr{T}M)$ is a closed imbedding.

Starting with a complex space-time M, we have an integrable conic connection h on the standard 2-conic structure $F \xrightarrow{\pi_2} M$. Assume that this conic structure integrates to give a fibration (see Chapter 1, §5 for this notion) $F \xrightarrow{\pi_1} Z$. Then the double fibration

$$Z \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M$$

is a self-duality diagram. In particular, this is the case for M the Grassmannian. This situation is refered to in [6] as the "flat case". In this case, the self-dual diagram is

$$\begin{array}{ccc} & F(1,2;T) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}(T) & & G(2;T), \end{array}$$

and the fibers of π_1 are projective planes, the fibers of π_2 are projective lines, so to see that Theorem 3.3 applies, it suffices to check that $(\pi_2)_*\mathscr{N} = R^1(\pi_2)_*\mathscr{N} = 0$. By the calculations of Chapter 1, presented in the last semester, it follows that $\mathscr{N} = \Omega^1 F/\mathbb{P}(1) \otimes \pi_2^*(\widetilde{S})$, and it follows from this that the assumption is satisfied (see 2.2.7 in [6] for details). So Theorem 3.3 applies in this situation.

One of the original ideas of Penrose was that starting with *real* space-time M_0, one can form its complexification M, which is a space-time in the sense of Definition 2.1 and hence, assuming the assumption above is met, Theorem 3.3 can be used. In an extension of this, the seminal paper [1] starts with an arbitrary compact, Riemannian four-manifold M_0, constructs a space Z "locally" in terms of the spinors on M_0, as $Z := \mathbb{P}(S_-)$, and then shows that Z always has an almost complex structure, which is *integrable* exactly when M_0 is self-dual in the sense of Riemannian geometry. The manifold Z in this case is called the *twistor space* of M_0. This situation applies to our standard self-dual GS manifold, if one considers the real structure of Chapter 1, §3, 17-19 of [6]. In this case the set of real points of the real structure is just a four-sphere, and the construction of [1] to S^4 yields $Z = \mathbb{P}^3$. Hitchin in [4] proved that starting with M_0 as above and constructing the twistor space Z, that if Z is Kähler, then either M_0 is S^4 (and $Z = \mathbb{P}^3$) or M_0 is $\mathbb{P}^2$ (and $Z = F(1,2;\mathbb{C}^3)$).

In Chapter 2, §3-§4 of [6], the Radon-Penrose transform is applied to get solutions of the *instanton equations*, i.e., the *self-dual Yang-Mills equations*, by means of constructing certain bundles on $Z = \mathbb{P}^3$ in the case $M_0 = S^4$. The rest of the Chapter is concerned with the study of the full Yang-Mills equations, for which the self-duality diagram is replaced by the diagram of null geodesics.

3.2.2 The diagram of null geodesics

The self-duality diagram introduced above only exists if M has a three-dimensional family of null planes (the α- or β-planes in the flat case). For a general space-time in the sense of Definition 2.1 this does not hold. As a substitute, one uses the diagram which will be introduced in this section, and applies the Radon-Penrose transform to it. This is what is considered in [6] in Chapter 2, §5-§9, and it can be used to characterize solutions of the full Yang-Mills equations on M_0. This generalization was discovered independently by Isenberg, Yasskin and Green [5] and Witten [10].

The diagram: Let M be a complex space-time, an let in this section S_+ and S_- denote the spin bundles. The choice of metric given by the spin decomposition $\Omega^1 M \cong S_+ \otimes S_-$ determines a 1-conic connection on $F := \mathbb{P}(S_+^*) \times_M \mathbb{P}(S_-^*)$; thus F is the 1-conic structure of all null directions.

Assumption: The foliation of F by lifted null geodesics integrates to a fibration.

Under this assumption we get a double fibration $L \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M$, which is called the *digram of null geodesics.* The dimensions of L, F and M are 5, 6 and 4, respectively. The fibers of π_1 are sheets of the fibering of lifted null geodesics, the fibers of π_2 are two-dimensional quadrics. In the flat case, the diagram is

$$\begin{array}{ccc} & F = F(1,2,3;T) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ L = F(1,3;T) & & M = G(2;T) \end{array}.$$

The space F is the *total* flag manifold, in other words, viewing it as a Kähler homogenous space, it is $SU(4)$ modulo a maximal torus. Viewed as a homogenous space of the *complex* Lie group $GL(4,\mathbb{C})$, it is that group modulo a Borel subgroup (minimal parabolic).

The Radon-Penrose transform: We next check that for a diagram of null geodesics, the assumptions of Theorem 3.3 are satisfied. For this we must identify the sheaf $\mathscr{N}$ and verify $(\pi_2)_*\mathscr{N} = R^1(\pi_2)_*\mathscr{N} = 0$. We have the following commutative diagram

$$\begin{array}{ccc} \pi_2^*(\Omega^1 M) & \xrightarrow{\text{res}} & \Omega^1 F/L \\ \cong\uparrow \pi_2^*(\sigma) & & \cong\uparrow \\ \pi_2^*(S_+)\otimes\pi_2^*(S_-) & \xrightarrow{j_+\otimes j_-} & \mathscr{O}_F(1,1), \end{array}$$

where σ is the spinor decompostion of $\Omega^1 M$, and $j_\pm$ are the morphisms in the following sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_2^*(\wedge^2 S_+)(-1,0) & \xrightarrow{i_+} & \pi_2^*(S_+) & \xrightarrow{j_+} & \mathscr{O}_F(1,0) & \longrightarrow & 0, \\ 0 & \longrightarrow & \pi_2^*(\wedge^2 S_-)(0,-1) & \xrightarrow{i_-} & \pi_2^*(S_-) & \xrightarrow{j_-} & \mathscr{O}_F(0,1) & \longrightarrow & 0. \end{array} \quad (4)$$

This is a sequence of sheaves on F, and the sheaf $\mathscr{O}_F(a,b)$ is the sheaf on F corresponding to the realization of F as a relative quadric $\mathbb{P}(S_+^*)\times_M \mathbb{P}(S_-^*)$. It follows that we have the following exact sequence including $\mathscr{N}$:

$$0 \longrightarrow \pi_2^*(\wedge^2 S_+\otimes\wedge^2 S_-)(-1,-1) \longrightarrow \\ \longrightarrow \pi_2^*(\wedge^2 S_+\otimes S_-)(-1,0)\oplus\pi_2^*(S_+\otimes\wedge^2 S_-)(0,-1) \longrightarrow \mathscr{N} \longrightarrow 0. \quad (5)$$

Now the clever observation is that by Theorem 1.2.2 in [6] (we have a product of relative $\mathbb{P}^1$'s), the first two sheaves in this sequence are *relatively* acyclic over M, so that

$$R^i(\pi_2)_*(\mathscr{N}) = 0 \quad \text{for all} \quad i \geq 0.$$

Consequently, if the fibers of π_1 are connected, we get an equivalence of categories:

Proposition 3.5 *For a general diagram of null geodesics* $L \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M$*, assuming the fibers of* π_1 *are connected, we have an equivalence of the following categories:*

(a) M*-trivial locally free sheaves* $\mathscr{E}_L$ *on* L*, and*

(b) pairs $(\mathscr{E}, \nabla)$, where $\mathscr{E}$ is locally free on M and ∇ is a connection with trivial monodromy along the null geodesics.

Note that one does not assume the triviality of the *curvature* of ∇ along geodesics; this follows automatically.

The next step is to interpret the physical equations (in this case the full Yang-Mills equations) in terms of the sheaves $\mathscr{E}_L$. For this one require more structure than for the simple Radon-Penrose transform. This structure is the following. In the flat case one has some additional structure, namely an imbedding $L \subset \mathbb{P}(T) \times \mathbb{P}(T^*)$. In the general case such an embedding does not exist. The very interesting fact which follows from the analysis is *that one can essentially replace the embedding above by using infinitesimal neighborhoods $L^{(i)}$ of L.* To make this precise, one must first find a sheaf on L which can serve as conormal sheaf (for the hypothetical embedding), then construct an infinitesimal neighborhood with respect to it. The sheaf to use is described next.

The sheaf $\mathscr{I}$: Let $\mathscr{I} \subset \Omega^1 L$ denote the sheaf of holomorphic forms ω such that: ω vanishes on every tangent vector in each quadric $L(x) := \pi_1\pi_2^{-1}(x),\ x \in M$. This is well-defined given an arbitrary diagram of null geodesics. We note that in the flat case, the sheaf $\mathscr{I}$ can be described as $\mathscr{O}_L(-1,-1)$, which means that with respect to the embedding $L \subset \mathbb{P}(T) \times \mathbb{P}(T^*)$, restrict the sheaf $\mathscr{O}_{\mathbb{P}(T)}(-1) \boxtimes \mathscr{O}_{\mathbb{P}(T^*)}(-1)$ to L. This is in fact the conormal sheaf of the embedding, so looks like the correct sheaf to use in general. If $\mathscr{I}$ were a conormal sheaf, then it should certainly be locally free of rank 1 (this is the codimension). The sheaf $\mathscr{I}$, defined for general diagrams of null geodesics, fulfills this condition, by the following result of Le Brun (2.5.3 in [6]):

$\mathscr{I} \subset \Omega^1 L$ is a rank one local direct summand; its restriction to any quadric $L(x)$ is $\mathscr{O}_{L(x)}(-1,-1)$.

Given such a sheaf on L, one can form the infinitesimal neighborhoods. These are defined generally as follows. Let $Y \subset X$ be an embedding of schemes, with $\mathscr{J}$ the ideal defining Y in X. Then the conormal sheaf of Y is $\mathscr{N}_X Y^* = \mathscr{J}/\mathscr{J}^2$. The *$n$-th infinitesimal neighborhood of Y in X*, denoted $Y^{(n)}$, is the ringed space $(Y, \mathscr{O}_X/\mathscr{J}^{n+1})$. Note that the 0-th neighborhood is just Y itself, as $\mathscr{O}_Y = \mathscr{O}_X/\mathscr{J}$. Hence there are natural inclusions $Y^{(i)} \subset Y^{(i+1)}$, or put differently, $Y^{(i+1)}$ is an *extension* of $Y^{(i)}$. Finally note that from the identity

$$\mathrm{Gr}\mathscr{O}_X = \bigoplus_{n=0}^{\infty} \mathscr{O}_X/\mathscr{J}^n \cong S^{\cdot}_{\mathscr{O}_Y}(\mathscr{J}/\mathscr{J}^2),$$

we can define the infinitesimal neighborhoods in terms of the conormal sheaf.

The spinor decomposition We next interpret the spinor decomposition of $\Omega^1 M$ in terms of the sheaf $\mathscr{I}$ on L above.

Definition 3.6 A *small* space-time M is one for which M is Stein and convex-geodesic (for a suitable metric in the conformal class), and the fibers of π_1 are connected and simply connected.

Any space-time has a basis consisting of small open sets.

The spinor decomposition of $\Omega^1 M$ can be interpreted via the Radon-Penrose transform on L. To do this, one works on a small M.

Theorem 3.7 (2.5.6) *Let M be a small space-times, L its space of null geodesics. The following structures on M and on L are equivalent:*

(a) a spinor decomposition $\Omega^1 M \cong S_+ \otimes S_-$ and a pair of nowhere vanishing spinor metrics $\varepsilon_\pm \in \Gamma(\wedge^2 S_\pm)$;

(b) a decomposition $\mathscr{I} = \mathscr{I}_+ \otimes \mathscr{I}_-$, where $\mathscr{I}_\pm$ are invertible sheaves whose restrictions to any quadric $L(x)$ are isomorphic to $\mathscr{O}_L(-1,0)$ and $\mathscr{O}_L(0,-1)$, respectively, and two cohomology classes $(\varepsilon_\pm)_L \in \mathrm{H}^1(L, \mathscr{I}_\pm^2)$ which do not vanish when restricted to any of the $L(x)$.

The correspondence fulfills the following condition. Let $(S_\pm)_L$ denote the Radon-Penrose transform for $(S_\pm, \nabla_\pm)$, where $\nabla_\pm \varepsilon_\pm = 0$. Then one has exact sequences on L

$$0 \longrightarrow \mathscr{I}_\pm \longrightarrow (S_\pm)_L \longrightarrow \mathscr{I}_\pm^{-1} \longrightarrow 0, \tag{6}$$

whose extension classes coincide with $(\varepsilon_\pm)_L$.

The proof of this theorem is based on the observation that pushing down the sequence (4) to L, we get the sequence (6).

3.2.3 Extensions and obstructions

Before procedding, we need a few notions from the theory of extensions of locally free sheaves to extensions of spaces. Later this will help us define conditions which under the Radon-Penrose transform are equivalent to a pair $(\mathscr{E}, \nabla)$ on M yielding a solution of the (full) Yang-Mills equations.

Extending sheaves Let $Y \subset X$ be an extension of Y. If $\mathscr{E}$ is a sheaf on Y, an *extension* to X is a locally free sheaf $\mathscr{F}$ on X such that $\mathscr{F}_{|Y} = \mathscr{E}$. This is related to the usual notion of an extension of a sheaf by means of the isomorphism $\mathscr{F}/\mathscr{J}\mathscr{F} = i_*\mathscr{E}$ (where $i : Y \hookrightarrow X$ is the inclusion and as above $\mathscr{J}$ denotes the ideal sheaf of Y) and $\mathrm{H}^k(Y, \mathscr{E}) \cong \mathrm{H}^k(X, i_*(\mathscr{E}))$. Namely, we have the exact sequence

$$0 \longrightarrow \mathscr{J}\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{J}\mathscr{F} = i_*\mathscr{E} \longrightarrow 0, \tag{7}$$

which is an extension of $i_*\mathscr{E}$ by $\mathscr{J}\mathscr{F}$.

Suppose we are given a sheaf $\mathscr{E}$ on Y with an extension $\mathscr{F}$ on X. It then makes sense to speak of extensions of cohomology classes, i.e., given $\eta \in \mathrm{H}^k(Y, \mathscr{E})$, a cohomology class $\xi \in \mathrm{H}^k(X, \mathscr{F})$ is said to be an extension of η if $i^*(\xi) = \eta$. Extending cohomology classes is a *linear* problem, which can be seen as follows. From the exact sequence (7) the following long exact sequence:

$$\cdots \longrightarrow \mathrm{H}^{k-1}(Y,\mathscr{E}) \longrightarrow \mathrm{H}^{k}(X,\mathscr{J}\mathscr{F}) \longrightarrow \mathrm{H}^{k}(X,\mathscr{F}) \longrightarrow \\ \xrightarrow{i^*} \mathrm{H}^{k}(Y,\mathscr{E}) \xrightarrow{\delta} \mathrm{H}^{k+1}(X,\mathscr{J}\mathscr{F}) \longrightarrow \cdots \quad (8)$$

Since this sequence is exact, if follows that η has an extension if and only if $\delta(\eta) = 0$. This class is referred to as the *obstruction to extending* η. In case this class does vanish, the group $\mathrm{H}^k(X, \mathscr{J}\mathscr{F})$ acts transitively on the set of extensions. In special cases other extension problems can also be dealt with cohomologically, giving "linear approximations". This will be important for us in particular concerning our infinitesimal neighborhoods $Y^{(i)}$ of Y.

Simple extensions We introduce a special type of extension.

Definition 3.8 An extension $Y \subset X$ of analytic spaces is called *simple*, if the following hold:

(a) $\mathscr{J}^2 = 0$, where $\mathscr{O}_Y = \mathscr{O}_X/\mathscr{J}$.

(b) The sequence of sheaves of $\mathscr{O}_Y$-modules

$$0 \longrightarrow \mathscr{J} \xrightarrow{d\otimes 1} \Omega^1 X \otimes_{\mathscr{O}_X} \mathscr{O}_Y \longrightarrow \Omega^1 Y \longrightarrow 0 \quad (9)$$

is exact.

We remark that the sequence is defined and is a complex if (a) is satisfied. Thus the condition (b) is whether $d \otimes 1$ is injective. It turns out that for simple extensions, one can describe extensions and obstructions particularly effectively. The sequence (9) determines a class

$$b = b(X,Y) \in \mathrm{Ext}^1_{\mathscr{O}_Y}(\Omega^1 Y, \mathscr{J}),$$

which one calls the *characteristic class* of the extension. If Ω^1 is locally free (Y is a manifold), then using well-known isomorphisms we may view b as an element of $\mathrm{H}^1(Y, \mathscr{T}Y \otimes \mathscr{J})$. Also, any extension given by a class $\mathrm{H}^1(Y, \mathscr{T}Y \otimes \mathscr{J})$ yields a simple extension $Y \subset X$ (see Chapter 2, §6.4 for this). The importance for us of the notion of simple extension is given by

Lemma 3.9 *Let $Y \subset X$ be a closed embedding of one manifold into another. Then $Y^{(n+1)}$ is a simple extension of $Y^{(n)}$ for all $n \geq 0$.*

The result mentioned above, describing the extensions and obstructions for simple extensions is the following.

Theorem 3.10 (2.6.6) *Assume that $Y \subset X$ is a simple extension. Then* *(a) A necessary and sufficient condition for a locally free sheaf $\mathscr{E}$ on Y to have an exension to X is that a certain obstruction*

$$\omega(\mathscr{E}) \in \mathrm{H}^2(Y, End(\mathscr{E}) \otimes \mathscr{J})$$

vanish.

(b) If $\omega(\mathscr{E}) = 0$, then the group $\mathrm{H}^1(Y, End(\mathscr{E}) \otimes \mathscr{J})$ acts transitively on the set of isomorphism classes of extensions. This action is effective if there exists an extension $\mathscr{F}$ of the sheaf $\mathscr{E}$ such that any section of $End(\mathscr{E})$ extends to a section of $End(\mathscr{F})$.

3.3 Yang-Mills fields and infinitesimal neighborhoods

In this section we formulate the result which gives a characterization in terms of algebraic geometric data for a pair $(\mathscr{E}, \nabla)$ on M to satisfy the full Yang-Mills equations. This characterization will be in terms of the extendability of the Radon-Penrose transform $\mathscr{E}_L$ of $(\mathscr{E}, \nabla)$ to certain infinitesimal neighborhoods of L, defined formally in terms of the sheaf $\mathscr{I}$. Throughout this section, this sheaf on L will be fixed.

3.3.1 Cohomological calculations

As we have seen above, the question as to whether a sheaf $\mathscr{F}$ on Y extends to infinitesimal neighborhoods $Y^{(i)}$ of Y will depend on certain cohomology groups. By Theorem 3.10, If the group of obstructions vanish, there is an extension, which, moreover, is unique if the corresponding H^1 vanishes.

In the formulation of the next result, F will be identified with $\mathbb{P}(S_+) \times_M \mathbb{P}(S_-)$ instead of with $\mathbb{P}(S_+^*) \times_M \mathbb{P}(S_-^*)$. For a sheaf $\mathscr{F}$ on L, $\mathscr{F}(a, b)$ will denote the sheaf $\mathscr{F} \otimes \mathscr{I}_+^{-a} \otimes \mathscr{I}_-^{-b}$, so that $\pi_1^*(\mathscr{I}_+) = \mathscr{O}_F(-1, 0)$ and $\pi_1^*(\mathscr{I}_-) = \mathscr{O}_F(0, -1)$. It follows that $\pi_1^*(\mathscr{F}(a, b)) = \pi_1^*(\mathscr{F})(a, b)$. Let $(\mathscr{E}, \nabla)$ be pair on M consisting of a locally free sheaf and a connection with no monodromy along geodesics, and let $\mathscr{E}_L$ be its Radon-Penrose transform on L. For a small M, the calculation of the cohomology groups on L can be reduced to calculations on M. The following theorem calculates the cohomology groups required in the sequel:

Theorem 3.11 (2.8.3) *Let M be a small space-time (Definition 3.6). Then for all values of $(i; a, b)$ in the table below we have*

$$\mathrm{H}^i(L, \mathscr{E}_L(a, b)) = \Gamma(M, \mathscr{E} \otimes S(i; a, b)).$$

The sheaf $S(i; a, b)$ is given in the table below.

i	(a, b)			
	$(-1, 0)$	$(-1, -1)$	$(-2, 0)$	$(-2, -1)$
1	$S_- \otimes \wedge^2 S_+$	$\wedge^2 S_+ \otimes \wedge^2 S_-$	$\wedge^2 S_+$	0
2	0	0	0	0

i	(a,b)		
	$(-2,-2)$	$(-3,-1)$	$(-3,-2)$
1	0	0	0
2	$\wedge^2 S_+ \otimes \wedge^2 S_-$	$(\wedge^2 S_+)^2 \otimes \wedge^2 S_-$	$S_+ \otimes \wedge^2 S_+ \otimes \wedge^2 S_-$

For $(a,b) = (-3,-3)$ *we have* $\mathrm{H}^1(L, \mathscr{E}_L(-3,-3)) = 0$ *and*

$$\mathrm{H}^2(L, \mathscr{E}_L(-3,-3)) = \mathrm{Ker}(\nabla_3 : \Gamma(M, \mathscr{E} \otimes \Omega^3 M) \longrightarrow \Gamma(M, \mathscr{E} \otimes \Omega^4 M)),$$

where ∇_3 *denotes the differential in the de Rham sequence of* $\mathscr{E}$ *induced by* ∇.

3.3.2 Regular towers of extensions

We assume that we have the sheaf $\mathscr{I}$ on L, from which we have constructed the infinitesimal neighborhoods $L^{(i)}$ for $i = 1, 2, 3$. By Lemma 3.9, if these embedding come from a closed embedding $L \subset X$ of manifolds, then at each step the extension is *simple* (Definition 3.8), and Theorem 3.10 applies. Hence in this case, the obstructions to extending a sheaf $\mathscr{E}_L$ on L to $L^{(1)}$ lie in $\mathrm{H}^2(L, End(\mathscr{E}_L) \otimes \mathscr{I})$, the set of isomorphism classes of such extensions is acted on by the group $\mathrm{H}^1(L, End(\mathscr{E}_L) \otimes \mathscr{I})$, and similarly with extensions from $L^{(i)}$ to $L^{(i+1)}$. Using the fact that for a pair $(\mathscr{E}, \nabla)$ on M and its Radon-Penrose transform $\mathscr{E}_L$ on L, we have $End(\mathscr{E}_L) = (End(\mathscr{E}))_L$, we can then use the above theorem to check which of these groups vanish. This is done as follows.

I The group of obstructions of extending the sheaf $\mathscr{E}_L$ from L to $L^{(1)}$ is $\mathrm{H}^2(L, End(\mathscr{E}_L) \otimes \mathscr{I})$. By Theorem 3.11, this group vanishes (this corresponds to $(i; a, b) = (2; -1, -1)$). Hence there exist extensions. By Theorem 3.10, the group $\mathrm{H}^1(L, End(\mathscr{E}_L) \otimes \mathscr{I})$ parametrizes all such extensions. By Theorem 3.11 on the other hand this group is isomorphic to the group of sections $\Gamma(M, End(\mathscr{E}) \otimes \wedge^2 S_+ \otimes \wedge^2 S_-)$ on M. Choose some class η_0 as an origin, then any class is of the form $\eta_0 + h$ for some section h.

II The group of obstructions of extending from $L^{(1)}$ to $L^{(2)}$ is the group $\mathrm{H}^2(L, End(\mathscr{E}_L) \otimes \mathscr{I}^2)$, which by Theorem 3.11 is not zero (this is $(i; a, b) = (2; -2, -2)$). Hence extensions need not exist. However, if they do, since $\mathrm{H}^1(L, End(\mathscr{E}_L) \otimes \mathscr{I}^2) = 0$ by Theorem 3.10, then this extension will in fact be *unique*. Note that there is a natural map $h \mapsto \omega(\eta_0 + h)$ from the H^1 of I to the H^2 here, since by Theorem 3.11, there is an isomorphism of sheaves $S(1; -1, -1) \cong S(2; -2, -2)$, so H^1 and H^2 can be identified with one and the same space of sections over M. We now make the assumption:

Assume: For the tower $L^{(i)}$, the map $\omega : \mathrm{H}^1 \longrightarrow \mathrm{H}^2$ is a bijection.

Then there is a unique extension $\mathscr{E}_L^{(1)}$ of $\mathscr{E}_L$ from L to $L^{(1)}$ which has an extension $\mathscr{E}_L^{(2)}$ to $L^{(2)}$.

III Under the assumption just made, we have extended $\mathscr{E}_L$ to a unique sheaf $\mathscr{E}_L^{(2)}$ on $L^{(2)}$; let $\eta^{(2)}$ denote its extension class. The obstruction of extending $\mathscr{E}_L^{(2)}$ from $L^{(2)}$ to $L^{(3)}$ is an element of the group $\mathrm{H}^2(End(\mathscr{E}_L) \otimes \mathscr{I}^3)$. By Theorem 3.11, this is the kernel of the map

$$\nabla_3 : \Gamma(M, End(\mathscr{E}) \otimes \Omega^3 M) \longrightarrow \Gamma(M, End(\mathscr{E}) \otimes \Omega^4 M).$$

There is an element of this kernel which is directly related to our pair $(\mathscr{E}, \nabla)$, namely, if $\Phi(\nabla) = \Phi_+(\nabla) + \Phi_-(\nabla)$ is the decomposition of the curvature of ∇ into self-dual and anti self-dual parts, then

$$j := \widetilde{\nabla}\Phi_+(\nabla) = -\widetilde{\nabla}\Phi_-(\nabla)$$

is an element in $\Gamma(End(E) \otimes \Omega^3 M)$ and $\widetilde{\nabla} j = 0$, i.e., this element lies in the kernel. The element j is called the *axial flow* of the Yang-Mills field ∇.

We can now define a regular tower of extensions.

Definition 3.12 A tower of extensions $L \subset L^{(i)}$, $i \leq 3$ is called *regular*, if any Yang-Mills sheaf $\mathscr{E}_L$ on L has a unique extension to $L^{(2)}$, and if the obstruction to extending it to $L^{(3)}$ coincides (up to normalizations) with the axial flow j of the field $(\mathscr{E}, \nabla)$.

Note that the *homogenous* Yang-Mills equations are that the obstruction above *vanish*. But as opposed with the self-dual equations and the self-dual Radon-Penrose transform, here we can also consider the *non-homogenous* equations, i.e., with sources. In this case the Yang-Mills sheaf will extend to $L^{(2)}$, and the obstruction to extending to $L^{(3)}$ is the axial flow.

The question of existence and uniqueness of a regular tower for an arbitrary space of null geodesics is an unsolved problem. The case considered in [6] is the flat case, where the infinitesimal neighborhoods $L^{(i)}$ correspond to the embedding $L \hookrightarrow \mathbb{P}(T) \times \mathbb{P}(T^*)$.

3.3.3 The Yang-Mills equations

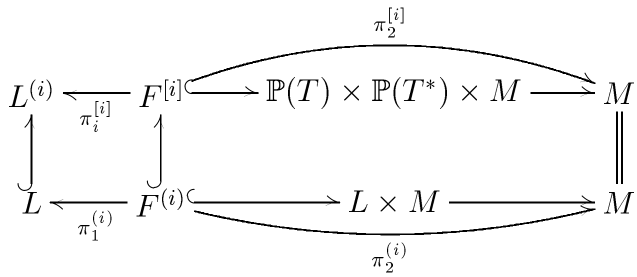
The following result is proved in [6], Chapter 2, §9. The proof is long and difficult. The main interest is in the Corollary.

Theorem 3.13 *Let $U \subset M = G(2;T)$ be a small space time in the flat case. Let $L(U) = \pi_1\pi_2^{-1}(U) \subset L$. Then the neighborhoods $L(U)^{(i)} = (L(U), \mathscr{O}_{L^{(i)}|L(U)})$ form a regular tower.*

The proof is based on using not only the infinitesimal neighborhoods of L, but also of F. Note that F can be viewed as a subset of $L \times M$, and since L itself is a relative quadric in $\mathbb{P}(T) \times_M \mathbb{P}(T^*)$, F also may be viewed as a subset of $\mathbb{P}(T) \times_M \mathbb{P}(T^*) \times M$. One sets:

- $F^{[i]}$ is the i-th infinitesimal neighborhood of F in $\mathbb{P} \times \mathbb{P}^* \times M$.
- $F^{(i)}$ is the i-th infinitesimal neighborhood of F in $L \times M$.

Depending on which cohomology groups one wishes to calculate, the one or the other of these two might be more advantageous. At any rate, one gets the following diagram.



The proof proceeds in the following steps:

1. One shows that there is a one to one correspondence between the set of extensions $\mathscr{E}_L^{(i)}$ to $L(U)^{(i)}$ and extensions $(\mathscr{E}_F^{[i]}, \nabla_{F/L}^{[i]})$ to $F(U)^{[i]}$.

2. Extending to $F(U)^{[i]}$ is done by explicit calculation of the relevant cohomology groups. First one extends to $F^{(i)}$, from here to $F^{[i]}$. It turns out there is a unique extension $(\mathscr{E}_F^{[2]}, \nabla_{F/L}^{[2]})$.

3. This pair determines by 1. a unique extension $\mathscr{E}_L^{(2)}$ to $L(U)^{(2)}$. The class $\omega(\mathscr{E}_L^{(2)})$ is calculated. It is shown that $\omega(\mathscr{E}_L^{(2)})$ is equal to the class $\left[\mathscr{E}_{F|F^{(3)}}^{[3]}\right] - \left[\mathscr{E}_F^{(3)}\right]$ of a certain extension $\mathscr{E}_F^{[3]}$. A rather complicated diagram chase then shows that this class is the image $\widetilde{\nabla}\Phi_+(\nabla)$, proving the theorem.

An immediate corollary of this, mentioned above, is of direct physical interest.

Corollary 3.14 (2.9.4) *A Yang-Mills field $(\mathscr{E}, \nabla)$ defined in some region of a flat space-time is a solution to the system of Yang-Mills equations with no sources*

$$\widetilde{\nabla}\Phi_\pm(\nabla) = 0$$

if and only if every point in its domain of definition has a neighborhood U such that the Radon-Penrose transform $\mathscr{E}_L$ on $L(U)$ of the pair $(\mathscr{E}, \nabla)_{|U}$ has an extension to $L^{(3)}$.

The construction of a class of YM sheaves on arbitrary neighborhoods $L(U)^{(i)}$ by the monad method is done in Chapter 5, directly in the super context.

4 Some supergeometry

In this section we recall a few basic facts and fix some notations.

4.1 Differential forms

The purpose of this subsection is to elaborate on the exact sequence

$$0 \longrightarrow \Omega_0^1 M \longrightarrow \Omega^1 M \longrightarrow \Omega_l^1 M \oplus \Omega_r^1 M \longrightarrow 0$$

on page 238 of [6].

4.1.1 Parity change

Let $A = A_0 \oplus A_1$ be a supercommutative ring, the base ring, and let $S = S_0 \oplus S_1$ be a left or right module. The module ΠS is defined by: (a) $(\Pi S)_0 = S_1$ and $(\Pi S)_1 = S_0$; (b) addition and *right* multiplication in ΠS are the same as in S, but *left* multiplication differs by sign: $a(\Pi s) = (-1)^{\deg(a)}\Pi(as)$. Exchanging right and left in this definition defines $S\Pi$. Then $S \mapsto \Pi S$ and $S \mapsto S\Pi$ are functors.

4.1.2 Even and odd

A free A-module of rank $p|q$ is an A-module isomorphic to $A^{p|q} = A^p \oplus (\Pi A)^q$. Note that the decomposition of A into an *evenly generated* part $A^{p|0}$ and an *oddly generated* part $A^{0|q}$ does *not* coincide with the decomposition into even and odd parts; the latter is $(A^{p|q})_0 \oplus (A^{p|q})_1 = [A_0^p \oplus (\Pi A_1)^q] \oplus [A_1^p \oplus (\Pi A_0)^q]$.

4.1.3 Homomorphisms

Let A be a supercommutative ring, S an A-module, $S = S_0 \oplus S_1$, so that S_0 and S_1 are A_0-submodules. Let T be a module. A map $f : S \longrightarrow T$ is an *even* (homo-)morphism if it preserves the $\mathbb{Z}_2$-grading and is A-linear, and an *odd* (homo-)morphism if it reverses the $\mathbb{Z}_2$-grading and is A-linear. We have two additive groups

$$\mathrm{Hom}_0(S,T) := \{\text{even homomorphisms}\}, \quad \mathrm{Hom}_1(S,T) := \{\text{odd homomorphisms}\}.$$

Then the direct sum

$$\mathrm{Hom}(S,T) := \mathrm{Hom}_0(S,T) \oplus \mathrm{Hom}_1(S,T)$$

can be given the structure of A-module in the usual way ($(af)(s) = a(f(s))$). From the point of view of the ordinary algebra of categories, $\mathrm{Hom}(S,T)$ should be considered the intrinsic Hom functor.

4.1.4 Differentials

Let M be an A-module. A *derivation* is an additive map $X : A \longrightarrow M$ with

$$X(ab) = (Xa)b + (-1)^{\tilde{a}\tilde{X}} a(Xb), \quad \forall_{a,b \in A}.$$

If $\widetilde{X} = 0$, then X is an even derivation, if $\widetilde{X} = 1$ it is an odd derivation. Just as in the usual commutative case there is a universal module of derivations, unique up to isomorphism. There are two privledged realizations of this module,

$$d_{\text{ev}} : A \longrightarrow \Omega^1_{\text{ev}} A, \quad d_{\text{odd}} : A \longrightarrow \Omega^1_{\text{odd}} A,$$

such that D_{ev} (respectively d_{odd}) is an even (repectively odd) map. Then

$$\Omega^1_{\text{odd}} A = \Pi \Omega^1_{\text{ev}} A, \quad d_{\text{ev}} = \Pi d_{\text{odd}}.$$

Now let S, T be A-modules, and suppose we are given morphisms $\sigma_a : S \longrightarrow S$ and $\tau_a : T \longrightarrow T$, $a \in I$. Recall that the *supercommutator* is defined by $[a, b] := ab - (-1)^{\widetilde{a}\widetilde{b}} ba$, and the *superanticommutator* is defined by $\{a, b\} := ab + (-1)^{\widetilde{a}\widetilde{b}} ba$. We assume

(a) The degree $\widetilde{\sigma}_a + \widetilde{\tau}_b$ does not depend on $a \in I$,

(b) For all $a, b \in I$, either σ_a and σ_b supercommute or superanticommute, and the same for τ_a and τ_b.

Under these assumption define a differential

$$d := \sum_{a \in I} \sigma_a \otimes \tau_a : S \otimes T \longrightarrow S \otimes T.$$

Then (3.4.3) $d^2 = 0$ if one of the following hold:

(I) d is even, and for all $(a, b) \in I \times I$, the commutation of (σ_a, σ_b) and (τ_a, τ_b) are opposite (i.e., the σ supercommute $\iff$ the τ superanticommute and vice versa).

(II) d is odd, and for all $(a, b) \in I \times I$, the commutation of (σ_a, σ_b) and (τ_a, τ_b) are the same.

4.1.5 De Rham complexes

Let A be a supercommutative B-algebra over a commutative ring B, and let $\{X_a\}$, $a \in I$ be a finite family of homogenous superderivations of A over B, such that $[X_a, X_b] = 0$ for all a, b. Furthermore, let $\{\omega_a\}$, $a \in I$ be a family of variables which are free over B. The construction just described leads in this case to a de Rham complex of even differentials, if the parities of ω_a and X_a coincide, and a de Rham complex of odd differentials if the parities are different. Explicitly:

$$\Omega^{\cdot}_{\text{ev}}(A/B; \{X_a\}) = B\{\omega_a\} \otimes_B A, \quad \widetilde{\omega}_a = \widetilde{X}_a, \quad \{\omega_a, \omega_b\} = 0, \quad d_{\text{ev}} = \sum \omega_a \otimes X_a,$$

and

$$\Omega^{\cdot}_{\text{odd}}(A/B; \{X_a\}) = B\{\omega_a\} \otimes_B A, \quad \widetilde{\omega}_a = \widetilde{X}_a + 1, \quad [\omega_a, \omega_b] = 0, \quad d_{\text{odd}} = \sum \omega_a \otimes X_a.$$

In supergeometry, this constuction is used as follows:

$$B = \mathbb{R}, \quad A = C^\infty(x_1, \dots, x_m)[\xi_1, \dots, \xi_N], \quad \{X_a\} = \left\{ \frac{\partial}{\partial x_b}, \frac{\partial}{\partial \xi_\beta} \right\}, \quad \omega_a = (dx_b, d\xi_\beta),$$

for the smooth case and similarly for the complex analytic case.

4.1.6 One-forms

Let $(M, \mathscr{O}_M)$ be a supermanifold. A locally free sheaf of rank $p|q$ is defined to be a sheaf of $\mathscr{O}_M$-modules $\mathscr{F}$, naturally $\mathbb{Z}_2$-graded, which is locally isomorphic to $\mathscr{O}_M^{p|q} = \mathscr{O}_M^p \oplus (\Pi\mathscr{O}_M)^q$. All the formalism of tensor algebra, in particular the functor Π can be carried over to general quasi-coherent sheaves of $\mathscr{O}_M$-modules. To fix ideas let us emphasize again that

$$(\mathscr{O}_M^{p|q})_0 = (\mathscr{O}_M)_0^p \oplus (\Pi(\mathscr{O}_M)_1)^q, \quad (\mathscr{O}_M^{p|q})_1 = (\mathscr{O}_M)_1^p \oplus (\Pi(\mathscr{O}_M)_0)^q.$$

Let $\mathscr{T}M$ denote the sheaf of local vector fields, i.e., superderivations of the ring of functions. This sheaf is locally free of rank equal to the dimension of M. If (x^i, ξ^j) are local coordinates, then $\mathscr{T}M$ is freely generated by sections $(\partial/\partial x^1, \partial/\partial\xi^j)$. The cotangent sheaf $\Omega^1 M$ can be defined in two ways (cf. book, 4.1.10):

$$\Omega^1 M_{\text{ev}} := (\mathscr{T}M)^* = Hom_{\mathscr{O}_M}(\mathscr{T}M, \mathscr{O}_M), \quad \Omega^1 M_{\text{odd}} := Hom_{\mathscr{O}_M}(\mathscr{T}M, \Pi\mathscr{O}_M).$$

We again emphasize that the Hom groups consists of all homomorphisms, both even and odd, so that both sheaves are sheaves of $\mathscr{O}_M$-modules. In purely even geometry one uses $\Omega^1 M = \Omega^1 M_{\text{ev}}$, but in supergeometry, $\Omega^1 M = \Omega^1 M_{\text{odd}}$ is used. Hence later the sheaf denoted $\Omega^1 M$ means the odd one. (Although not stated very clearly in the text, this is clear from the remark in 5.7.4).

4.1.7 Chiral superspaces

We now pass to the situation of Chapter 5, where $T = T^{4|N}$, the manifold M is the flag space $M = F(2|0, 2|N; T)$. We recall that M has two projections onto the left and right superspaces, each of which specifies *half* of the odd coordinates, but is an isomorphism on the even coordinates. These are defined as $M_l := G(2|0; T)$ and $M_r := G(2|N; T) \cong G(2|0; T^*)$. The coordinates on big cells of these chiral superspaces are given in §1.6, and are as follows. On the left superspace, we have coordinates $(x_l^{\alpha\dot\beta}|\theta_l^{\alpha j})$; here $\alpha, \dot\beta$ are spinor indices (i.e., each is equal to 0 or 1), while $j = 1, \ldots, N$ is the index of supersymmetry. Similarly, on the right superspace we have $(x_r^{\dot\beta\alpha}|\theta_{rj}^{\dot\beta})$. Note that each has four even dimensions but only $2N$ odd ones; the left and right odd coordinates are also characterized by the fact that the left ones have undotted indices while the right ones have dotted indices. Finally, on a big cell in M, these are combined to give coordinates

$$x^{\alpha\dot\beta} = \frac{1}{2}\left(x_l^{\alpha\dot\beta} + x_r^{\dot\beta\alpha}\right).$$

Then the projections to the left and right superspaces are given by:

$$M \ni (x^{\alpha\dot\beta}|\theta_l^{\alpha j}; \theta_{rj}^{\dot\beta})$$

$$\swarrow \qquad \searrow$$

$$(x_l^{\alpha\dot\beta} = x^{\alpha\dot\beta} + i\theta_l^{\alpha j}\theta_{rj}^{\dot\beta}|\theta_l^{\alpha j}) \in M_l \qquad M_r \ni (x_r^{\dot\beta\alpha} = x^{\alpha\dot\beta} - i\theta_l^{\alpha j}\theta_{rj}^{\dot\beta}|\theta_{rj}^{\dot\beta})$$

The space M is a relative Grassmanian over each of the chiral superspaces of relative dimension $0|2N$; one then defines

$$\mathscr{T}_l/M := \mathscr{T}M/M_r, \quad \mathscr{T}_r/M := \mathscr{T}M/M_l.$$

Sections of $\mathscr{T}_lM$ (respectively $\mathscr{T}_rM$) are superderivations on a big cell of M which take (x_r, θ_r) (respectively (x_l, θ_l)) to zero. In terms of the coordinates used above, sections of these tangent spaces have basis:

$$\mathscr{T}_lM: \; D_{l\alpha j} = \frac{\partial}{\partial \theta_l^{\alpha j}} + i\theta_{rj}^{\dot\beta}\frac{\partial}{\partial x^{\alpha\dot\beta}}, \quad \mathscr{T}_rM: \; D_{r\dot\beta}^{j} = \frac{\partial}{\partial \theta_{rj}^{\dot\beta}} + i\theta_l^{\alpha j}\frac{\partial}{\partial x^{\alpha\dot\beta}},$$

where $\alpha = 0, 1$, $\dot\beta = \dot 0, \dot 1$ and $j = 1, \ldots, N$ is the supersymmetry index.

4.1.8 Spinor decomposition on the superspace M

We now let $\mathcal{S}$, $\widetilde{\mathcal{S}}$ denote the superspinor sheaves, and notations like $\mathcal{S}^{a|b}$ will mean taking only that part of the sheaves. The tangent sheaf $\mathscr{T}M$ fits, by Proposition 1.4 (in Chapter 5 of [6]), into the following exact sequence:

$$0 \longrightarrow \mathscr{T}_lM \oplus \mathscr{T}_rM \longrightarrow \mathscr{T}M \longrightarrow \mathscr{T}_0M \longrightarrow 0, \tag{10}$$

where $\mathscr{T}_0M \cong (\mathcal{S}^{2|0})^* \otimes (\widetilde{\mathcal{S}}^{2|0})^*$. Reducing odd coordinates this is the usual isomorphism on $M_{\rm rd} = G(2;T)$: $\mathscr{T}M_{\rm rd} = (\mathcal{S}^{2|0}_{\rm rd})^* \otimes (\widetilde{\mathcal{S}}^{2|0}_{\rm rd})^* = S^* \otimes \widetilde{S}^*$. This isomorphism can be interpreted to mean: $\mathscr{T}_0M$ is isomorphic to the standard tangent sheaf of $M_{\rm rd}$, tensored over $\mathscr{O}_{M_{\rm rd}}$ by $\mathscr{O}_M$.

Now set $\Omega^1M := \Omega^1M_{\rm odd} = \Pi\mathscr{T}^*M$; note the extra Π, which occurs simply because Manin takes the odd space as the space of differential forms. We now dualize the sequence (10) and get the following sequence:

$$0 \longrightarrow \Omega_0^1M \longrightarrow \Omega^1M \xrightarrow{b_l \oplus b_r} \Omega_l^1M \oplus \Omega_r^1M \longrightarrow 0, \tag{11}$$

where we now have an isomorphism $\Omega_0^1M = \Pi(\mathcal{S}^{2|0} \otimes \widetilde{\mathcal{S}}^{2|0})$ (again, the extra Π from the fact that Ω^1M is odd, not even), and this can again be interpreted as meaning that this sheaf is isomorphic to the sheaf on the reduced space. Hence, this sequence can be seen as the super analog of the *isomorphism* $\Omega^1M \cong S \otimes \widetilde{S}$ in the purely even case; the “stuff” which is different here is described by the (purely odd) sum $\Omega_l^1M \oplus \Omega_r^1M$.

On the sheaf of two forms, we have a filtration, one piece of which is, as above, isomorphic to the usual two forms of the purely even case. This part is $\Omega_0^2M := S^2(\Omega_0^1M)$, and the filtration is

$$\Omega_0^1M \subset \Omega_1^2M \subset \Omega_2^2M = \Omega^2M,$$

where $\Omega_1^2 = \Omega_0^1M \cdot \Omega^1M$. The quotients of this filtration are

$$\begin{aligned} \Omega_1^2/\Omega_0^2 &= \Omega_0^1M \otimes (\Omega_l^1M \oplus \Omega_r^1M), \\ \Omega_2^2/\Omega_1^2 &= \Omega_l^2M \oplus \Omega_r^2M \oplus (\Omega_l^1M \otimes \Omega_r^1M). \end{aligned}$$

4.1.9 Berezinians

For a supermatrix $B \in \mathrm{GL}(p|q;A) = (\mathrm{Hom}_0(A^{p|q}, A^{p|q}))^\times$ in the previous notation, one sets

$$\mathrm{Ber}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \det(B_1 - B_2 B_4^{-1} B_3)\det B_4^{-1}.$$

This defines a group homomorphism $\mathrm{Ber} : \mathrm{GL}(p|q;A) \longrightarrow \mathrm{GL}(1|0;A_0)$ which agrees with the determinant if $q = 0$. The Berezinian of a matrix has the following properties

1) $\mathrm{Ber}(\Pi B) = \mathrm{Ber}(B\Pi) = (\mathrm{Ber}B)^{-1}$.

2) Let $B \oplus C$ denote the direct sum of two matrices, i.e., the block-diagonal matrix with B and C along the diagonal. Then

$$\mathrm{Ber}(B \oplus C) = \mathrm{Ber}B \cdot \mathrm{Ber}C. \tag{12}$$

3) If $B \in \mathrm{GL}(n_0|n_1;A)$ and $C \in \mathrm{GL}(m_0|m_1;A)$, then

$$\mathrm{Ber}(B \otimes C) = \mathrm{Ber}(B)^{m_0-m_1} \cdot \mathrm{Ber}(C)^{n_0-n_1}. \tag{13}$$

Now let S be a free A-module of rank $p|q$. There is a free A-module $\mathrm{Ber}(S)$ defined, which has rank $1|0$ (if q is even) or $0|1$ (if q is odd). It is defined in the following, abstract way. If $\{s_i\}$ is a basis of S, then this defines an element $D(\{s_i\}) \in \mathrm{Ber}(S)$, with the stipulation that for any other basis $\{s_i'\}$, related to $\{s_i\}$ by a matrix B: $\{s_i'\} = \{s_i\}B$, then $D(\{s_i'\}) = D(\{s_i\})\mathrm{Ber}\ B$. It is further required that for any *even* isomorphism $b : S \longrightarrow S'$ of A-modules, there is a corresponding isomorphism $\mathrm{Ber}\ b : \mathrm{Ber}(S) \longrightarrow \mathrm{Ber}(S')$ which carries $D(\{s_i\})$ to $D(\{b(s_i)\})$.

It is then clear how this definition can be applied to sheaves of $\mathscr{O}_M$-modules. The properties listed above then carry over to this situation. In particular, for any exact sequence of locally free sheaves

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

we have

$$\mathrm{Ber}(\mathscr{F}) = \mathrm{Ber}(\mathscr{E}) \otimes \mathrm{Ber}(\mathscr{G}). \tag{14}$$

For a supermanifold M one *defines* $\mathrm{Ber}(M) := \mathrm{Ber}(\Omega^1_{\mathrm{odd}}M)^*$.

We now apply this to our case at hand; from (11) and (14) we get

$$\mathrm{Ber}(\Omega^1 M) = \mathrm{Ber}(\Omega^1_0 M) \otimes \mathrm{Ber}(\Omega^1_l M) \otimes \mathrm{Ber}(\Omega^1_r M). \tag{15}$$

By definition we have $\Omega^1_0 M := \Pi(\mathcal{S}^{2|0} \otimes \widetilde{\mathcal{S}}^{2|0})$, so that $(\mathrm{Ber}(\Omega^1_0 M))^* = \mathrm{Ber}(\mathcal{S}^{2|0} \otimes \widetilde{\mathcal{S}}^{2|0})$. Since the latter sheaves have ranks $2|0$, it follows from the rule 3) concerning Berezinians of tensor products that

$$(\mathrm{Ber}(\Omega^1_0 M))^* = (\mathrm{Ber}(\mathcal{S}^{2|0}))^2 \otimes (\mathrm{Ber}(\widetilde{\mathcal{S}}^{2|0}))^2. \tag{16}$$

The sheaves $\mathcal{E}_l := \Pi(\widetilde{\mathcal{S}}^{2|N}/\widetilde{\mathcal{S}}^{2|0})$ and $\mathcal{E}_r := \Pi(\mathcal{S}^{2|N}/\mathcal{S}^{2|0}) = \mathcal{E}_l^*$ have rank $N|0$, so from the definitions

$$\Omega^1_l M = \mathcal{S}^{2|0} \otimes \mathcal{E}_l, \qquad \Omega^1_r M = \widetilde{\mathcal{S}}^{2|0} \otimes \mathcal{E}_r,$$

we have the equalities

$$\begin{aligned} \mathrm{Ber}(\Omega^1_l M) &= (\mathrm{Ber}(\mathcal{S}^{2|0}))^N \otimes (\mathrm{Ber}(\mathcal{E}_l))^2 && (17)\\ \mathrm{Ber}(\Omega^1_r M) &= (\mathrm{Ber}(\widetilde{\mathcal{S}}^{2|0}))^N \otimes \mathrm{Ber}(\mathcal{E}_r)^2 = (\mathrm{Ber}(\widetilde{\mathcal{S}}^{2|0}))^N \otimes \mathrm{Ber}(\mathcal{E}_l)^{-2}. && (18)\end{aligned}$$

Combining this with the definition of the Berezinian of M and (15), we have the formula

$$\begin{aligned} \mathrm{Ber}M := (\mathrm{Ber}(\Omega^1_{\mathrm{odd}}))^*) &= \mathrm{Ber}(\Omega^1_0 M)^* \otimes \mathrm{Ber}(\Omega^1_l M)^* \otimes \mathrm{Ber}(\Omega^1_r M)^* \\ \text{(by (16),} \quad &= (\mathrm{Ber}(\mathcal{S}^{2|0}))^2 \otimes (\mathrm{Ber}(\widetilde{\mathcal{S}}^{2|0}))^2 \otimes \\ \text{(17) and (18))} \quad & \quad (\mathrm{Ber}(\mathcal{S}^{2|0}))^{-N} \otimes \mathrm{Ber}(\mathcal{E}_l)^{-2} \otimes (\mathrm{Ber}(\widetilde{\mathcal{S}}^{2|0}))^{-N} \otimes \mathrm{Ber}(\mathcal{E}_l)^2 \\ &= (\mathrm{Ber}(\mathcal{S}^{2|0}))^{2-N} \otimes (\mathrm{Ber}(\widetilde{\mathcal{S}}^{2|0}))^{2-N}. && (19)\end{aligned}$$

Note that this equation for $N = 0$ (reducing the odd coordinates) expresses the fact that $\sqrt{|\det(g)|}$ is an expression in the spinor metrics ε and $\widetilde{\varepsilon}$. Note also that for $N = 2$, the bundle is trivial, hence there is a *canonical* volume form on M.

4.1.10 Real structures

To make contact with the physics liturature, it is important to understand the real structure in terms of the local coordinates introduced above. This is (we consider here the Minkowski involution, which is determined by its action on the spinor bundles, as described in the paragraph following Definition 2.1) on the big cell in M:

$$(x^{\alpha\dot\beta})^\varrho = {}^t(x^{\alpha\dot\beta}), \quad (\theta_l^{\alpha j}, \theta_{rk}^{\dot\beta})^\varrho = (\theta_{rj}^{\dot\alpha}, \theta_l^{\beta k}).$$

It follows that at real points the values of the coordinates (x, θ_l, θ_r) satisfy the following conditions:

$${}^t(x^{\alpha\dot\beta}) = (\overline{x}^{\alpha\dot\beta}), \quad (\theta_{rj}^{\dot\alpha}, \theta_l^{\beta k}) = (\overline{\theta}_l^{\alpha j}, \overline{\theta}_{rk}^{\dot\beta}).$$

In particular the coordinates $x^a := \sigma^a_{\alpha\dot\beta} x^{\alpha\dot\beta}$ are *real* (here σ^a are the Pauli matrices, see 1.3.16), and these are used in the physics liturature. Moreover, the subscripts $_r$ and $_l$ are redundant, as the dotted and undotted indices already describe the chirality of the odd coordinates. The usual choice of coordinates in the physics liturature is:

$$(x^a, \theta^{\alpha j}, \theta_{\dot\beta k}) \quad \text{with} \quad \theta_{\dot\beta k} = \overline{\theta^{\beta k}} \quad \text{at real points.} \qquad (20)$$

4.2 Superfields

We fix the space M as above, and consider the natural "bundles" (actually sheaves) on this space: the super spinor bundles $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, the super tangent bundle $\mathscr{T}M$, the super differential sheaf $\Omega^1 M$, the two sheaves of *chirality*, $\mathcal{E}_l$ and $\mathcal{E}_r$, which are defined as follows:

$$\mathcal{E}_l = \Pi\left(\widetilde{\mathcal{S}}^{2|N}/\widetilde{\mathcal{S}}^{2|0}\right), \quad \mathcal{E}_r = \Pi\left(\mathcal{S}^{2|N}/\mathcal{S}^{2|0}\right).$$

Definition 4.1 A *superfield* is a section of or connection on one of the natural bundles on M.

The symmetry group of the theory is the *super-Poicaré group* $\mathcal{SP}$, which can be described in this geometric setup as follows: it is the ϱ-invariant part of $SL(T)$ (this is the super special linear group) which carries the light cone at infinity into itself; this is the cone which compactifies the big cell in M which we chose as a model of Minkowski superspace. Its irreducible representations are called supermultiplets. The Lagrangian of the theory is a volume form on M which depends on the superfields. In many cases, such a super-Lagrangian is not known.

The superfields can be decomposed into irreducible parts with respect to the ordinary Poincaré group. This is known as *component decomposition.* There is not a unique way of doing this, and depending on the way it is done, some component fields will be set to zero or expressed in terms of others. This is a sort of gauging of the theory.

4.2.1 Chiral superfields

A *scalar field* $\mathbf{\Phi}$ is just a section of the structure sheaf of M; we assume that its domain of definition includes the big cell given by the local coordinates used above. The innocent looking creature $\mathbf{\Phi}$ changes dramatically if we write it down in components. We do this only in the case $N=1$. Recall that all odd coordinates square to zero; hence, when we develop $\mathbf{\Phi}$ in terms of the even and odd coordinates, for $N=1$ the longest product of odd coordinates occuring is of length four. Expanding, we get:

$$\begin{aligned}\mathbf{\Phi}(x^a,\theta^\alpha,\theta_{\dot\beta}) = A(x^a) + \theta^\alpha\psi_\alpha(x^a) + \overline{\phi}^{\dot\beta}\theta_{\dot\beta} + \theta^\alpha\theta^\beta F_{\alpha\beta}(x^a)\\ + \theta_{\dot\alpha}\theta_{\dot\beta}\overline{G}^{\dot\alpha\dot\beta}(x^a) + \sigma_b^{\alpha\dot\beta}\theta^\alpha\theta_{\dot\beta}B^b(x^a) + \theta^\alpha\theta^\beta\overline{\kappa}_{\alpha\beta}^{\dot\gamma}(x^a)\theta_{\dot\gamma}\\ + \theta_{\dot\alpha}\theta_{\dot\beta}\theta^\gamma\lambda_\gamma^{\dot\alpha\dot\beta}(x^a) + \varepsilon_{\alpha\beta}\varepsilon_{\dot\gamma\dot\delta}\theta^\alpha\theta^\beta\theta_{\dot\gamma}\theta_{\dot\delta}D(x^a).\end{aligned} \quad (21)$$

In this expression, the fields A, F, G, B and D are the boson components, while ψ, ϕ, κ and λ are the fermion components. Note that there is a single *vector*: B^b. The condition for reality of this field is

$$\mathbf{\Phi}^\varrho = \mathbf{\Phi} \iff \phi^{\dot\alpha} = \psi_\alpha,\ F_{\alpha\beta} = G^{\dot\alpha\dot\beta},\ \overline{B}^b = B^b,\ \kappa_{\alpha\beta}^{\dot\gamma} = \lambda_\gamma^{\dot\beta\dot\alpha},\ \overline{D} = D. \quad (22)$$

Definition 4.2 The scalar superfield $\mathbf{\Phi}$ is *chiral*, if $d_r\mathbf{\Phi}=0$ or $d_l\mathbf{\Phi}=0$, where $d_{l,r}=b_{l,r}\circ d$, and $b_{l,r}$ is the map in sequence (11).

This just states that $\mathbf{\Phi}$ belongs is a section of one of the chiral structure sheaves, $\mathscr{O}_{M_l}$ or $\mathscr{O}_{M_l}$. One finds such scalar fields simply by passing to left or right coordinates, in the expansion of $\mathbf{\Phi}$ above. For example

$$\mathbf{\Phi}_l = A(x_l^{\gamma\dot{\delta}}) + \theta^\alpha \psi_\alpha(x_l^{\gamma\dot{\delta}}) + \theta^\alpha\theta^\beta F_{\alpha\beta}(x_l^{\gamma\dot{\delta}}) \tag{23}$$

is left chiral.
Remark: The physisists refer to chiral fields as *scalar* fields, somewhat clashing with the notation above.
In terms of physics, one starts with an ordinary scalar field A, applies supersymmetry transformations, and gets ψ; then one again applies supersymmetry transformations to ψ, and the result is a linear combination of a space derivative of A and the field F. Finally, appling a supersymmetry transformation to F, one gets a space derivative of ψ, and the multiplet (A, ψ, F) “closes”. This description can be found in [8], III. An invariant action is constructed there, of the form

$$\mathscr{L} = \left(i\partial_a\overline{\psi}\overline{\sigma}^a\psi + A^*\Box A + F^*F\right) + m\left(AF + A^*F^* - \frac{1}{2}\left(\psi\psi + \overline{\psi\psi}\right)\right),$$

giving rise to the following field equations:

$$\begin{aligned} i\overline{\sigma}^a\partial_a\psi + m\overline{\psi} &= 0 \\ F + mA^* &= 0 \\ \Box A + mF^* &= 0. \end{aligned} \tag{24}$$

Note that the second equation can be used to eliminate F – a typical situation in supersymmetry. The remaining equations are simply an equation for a Weyl spinor ψ and a Klein-Gordon equation for A. Written in terms of the superfield $\mathbf{\Phi}$, this Lagrangian is

$$\mathscr{L} = \mathbf{\Phi}^\varrho\mathbf{\Phi}_{|\theta\theta\overline{\theta}\overline{\theta}\text{component}} + m\frac{1}{2}\left(\mathbf{\Phi}\mathbf{\Phi} + \mathbf{\Phi}^\varrho\mathbf{\Phi}^\varrho\right).$$

Thus, using chiral superfields we already get our basic matter fields, spinors and Klein-Gordon particles.

4.2.2 Vector superfields

It turns out that all supersymmetric renormalizable Lagrangians can be constructed in terms of chiral fields and *vector superfields*, which are a kind of supersymmetric analog of usual gauge fields (see [8], VI).

Definition 4.3 A superfield $\mathbf{V}$ (hier conforming to physics notation; this superfield is just as the $\mathbf{\Phi}$ before) is called a *vector superfield*, if $\mathbf{V}^\varrho = \mathbf{V}$.

A typical example of vector superfield is just a sum $\mathbf{\Phi} + \mathbf{\Phi}^{\varrho}$ of chiral superfields. The conditions on the components of a vector superfield were given in (22). Let $\mathbf{\Phi}_l$ be left chiral. Then, in terms of the x^a-coordinates on M, it can be written (note that (23) is written in terms of the chiral coordinates)

$$\mathbf{\Phi}_l = A(x) + i\theta^{\alpha}\sigma^a_{\alpha\dot{\beta}}\theta^{\dot{\beta}}\partial_a A(x) + \frac{1}{4}\theta^{\alpha}\theta^{\beta}\theta_{\dot{\gamma}}\theta_{\dot{\delta}}\Box A(x) \\ + \theta^{\alpha}\psi_{\alpha}(x) - i\theta^{\alpha}\theta^{\beta}\partial_a\psi_{\alpha}(x)\sigma^a_{\beta\dot{\gamma}}\theta^{\dot{\gamma}} + \theta^{\alpha}\theta^{\beta}F_{\alpha\beta}(x), \quad (25)$$

and for the sum $\mathbf{\Phi}^{\varrho}_l + \mathbf{\Phi}_l$ we get

$$\mathbf{\Phi}^{\varrho}_l + \mathbf{\Phi}_l = A + A^* + (\theta\psi + \overline{\theta\psi}) + \theta\theta F + \overline{\theta\theta}F^* + i\theta\sigma^a\overline{\theta}\partial_a(A - A^*) \\ + i\theta\theta\overline{\theta}\overline{\sigma}^a\partial_a\psi + i\overline{\theta\theta}\theta\sigma^a\partial_a\overline{\psi} + \frac{1}{4}\theta\theta\overline{\theta\theta}\Box(A + A^*), \quad (26)$$

where we are suppressing the indices, i.e., $\theta\theta$ means $\theta^{\alpha}\theta^{\beta}$, $\overline{\theta}$ denotes $\theta^{\dot{\alpha}}$ and so on. We note that the vector component is $i(\partial_a(A - A^*))$, i.e., it is *pure gauge*. Because of this, one defines as a supersymmetric analog of gauge transformation the following *supergauge transformation*:

$$\mathbf{V} \mapsto \mathbf{V} + \mathbf{\Phi} + \mathbf{\Phi}^{\varrho}$$

for some (left or right) chiral field $\mathbf{\Phi}$.

4.2.3 Gauge field strength

Now, if $\mathbf{V}$ is the super analog of the gauge field, what is the analog of the field strength, i.e., the curvature? In terms of components, a vector superfield can be written as follows, where we again suppress the α indices:

$$\mathbf{V}(x,\theta,\overline{\theta}) = C(x) + i(\theta\chi(x) - \overline{\theta}\overline{\chi}(x)) + \frac{i}{2}\left(\theta\theta[M(x) + iN(x)] - \overline{\theta\theta}[M(x) - iN(x)]\right) \\ - \theta\sigma^a\overline{\theta}v_a(x) + i\left(\theta\theta\overline{\theta}\left[\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^a\partial_a\chi(x)\right] - \overline{\theta\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^a\partial_a\overline{\chi}(x)\right]\right) \\ + \frac{1}{2}\theta\theta\overline{\theta\theta}\left[D(x) + \frac{1}{2}\Box C(x)\right]. \quad (27)$$

Under the gauge transformation above, the term λ is *invariant*, as the curvature should be. To find an acceptable supersymmetric analog of the curvature, we need an expression which: is invariant under gauge transformations, is super-Poincaré invariant, and finally, which is chiral (or anti-chiral). First we recall the differential operators

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + i\theta^{\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\beta}}}, \qquad D_{\dot{\beta}} = \frac{\partial}{\partial\theta^{\dot{\beta}}} + i\theta^{\alpha}\frac{\partial}{\partial x^{\alpha\dot{\beta}}}, \quad (28)$$

where there is no supersymmetry index j since we are assuming $N = 1$. These operators formed a basis of the sheaves $\mathscr{T}_l M$ and $\mathscr{T}_r M$. From these we form operators

$$D^2 := \varepsilon^{\alpha\beta} D_\beta D_\alpha, \qquad \overline{D}^2 := D_{\dot\alpha} \varepsilon^{\dot\alpha\dot\beta} D_{\dot\beta}.$$

These operators are quite useful and fulfill a bunch of identities (cf. [2], (2.5.30)). Of particular interest is the fact that given a tensor superfield V, the object $\overline{D}^2 V$ is chiral and the object $D^2 V$ is anti-chiral. We can now define the object of interest:

$$W_\alpha := -\frac{1}{4}\overline{D}^2 D_\alpha \mathbf{V}, \qquad \overline{W}_{\dot\alpha} := -\frac{1}{4} D^2 \overline{D}_\alpha \mathbf{V}, \tag{29}$$

which may be though of as the chiral and anti-chiral field strengths, respectively. They fulfill the conditions mentioned above, and in addition the following *reality constraint*:

$$D^\alpha W_\alpha = D_{\dot\alpha} \overline{W}^{\dot\alpha}. \tag{30}$$

We will write down this term in a special gauge.

Definition 4.4 The *Wess-Zumino* or WZ gauge is when one sets C, χ, M and N in (27) all equal to zero.

In this gauge, we have the following supersymmetric generalization of the field strength of the vector superfield $\mathbf{V}$.

$$W_\alpha = -i\lambda_\alpha(x_l) + \left[\delta_\alpha^\beta D(x_l) - \frac{i}{2}(\sigma^a \overline{\sigma}^b)_\alpha^\beta \left(\partial_a v_b(x_l) - \partial_b v_a(x_l)\right)\right] \theta_\beta + \theta\theta\sigma^a_{\alpha\dot\alpha} \partial_a \overline{\lambda}^{\dot\alpha}(x_l), \tag{31}$$

and a similar expression for $\overline{W}_{\dot\alpha}$. These superfields only contain the gauge invariant fields D, λ_α and $v_{ab} = \partial_a v_b - \partial_b v_a$. To get the supersymmetric generalization of the Lagrangian, we need an expression which transforms under supersymmetry transformations into a space derivative. Since W_α is chiral, so is the product $W^\alpha W_\alpha$, and its highest component is the $\theta\theta$ component. A general fact about superfields is that the highest component always transforms as a space derivative. Hence,

$$\mathscr{L} = \frac{1}{4}(W^\alpha W_{\alpha|\theta\theta} + \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha}_{|\overline{\theta\theta}})$$

is the candidate for a supersymmetric Lagrangian of the super gauge field. We will meet this condition again later. We should remark here that the projection $_{|\theta\theta}$ can be affected by taking the derivatives with respect to θ. It is a fact about integration of odd coordinates that the same thing can be achieved by *integrating* over $d\theta^2$:

$$F_{|\theta\theta} = \int F d\theta^2 = \partial^2 F / \partial\theta^2.$$

Our superspace has has $4+4$ coordinates, and the integral over these coordinates is denoted (in the physics liturature) by $\int d^8 z$. The integral over M_l (resp. M_r) is denoted $\int d^6 z$ (resp. $\int d^6 \overline{z}$). Note that then $\int_{M_{\mathrm{rd}}} d^4 x \mathscr{L} = \int_M d^8 z \left(\mathscr{L}_{|\theta\theta} + \mathscr{L}_{|\theta\theta}\right)$, so this "projection" is just a kind of bookkeeping device.

5 YM fields and integrability equations along super-geodesics

The first object is to generalize Propostition 3.5 to the supergeometric case. This involves defining the super diagram of null geodesics, and determining the necessary conditions on that fibration for a super version of the Radon-Penrose transform to work.

5.1 The diagram of super null geodesics

We will consider the following diagram:

$$\begin{array}{ccc} & F^{6|4N} = F(1|0,2|0,2|N,3|N;T^{4|4N}) & \\ & \pi_1 \swarrow \quad \searrow \pi_2 & \\ L^{5|2N} = F(1|0,3|N;T^{4|4N}) & & M^{4|4N} = F(2|0,2|N;T^{4|4N}) \end{array}$$

Here we have $F = \mathbb{P}(\mathcal{S}_M^{2|0}) \times_M \mathbb{P}(\widetilde{\mathcal{S}}_M^{2|0})$, which is a relative two-dimensional quadric. $F|L$ has relative dimension $1|2N$, and its fibers are called *light supergeodesics* of M. As a relative flag manifold, $F|L$ is the following:

$$F = F_L(1|0,1|N;\mathcal{S}_L^{3|N}/\mathcal{S}_L^{1|0}),$$

where $\mathcal{S}_L^{1|0} \subset \mathcal{S}_L^{3|N}$ is the universal (tautological) flag on L. Then, just as in the case for M, we can define the left and right chiral spaces F_l, F_r as the following relative Grassmannians.

$$F_l := G_L(1|0;\mathcal{S}_L^{3|N}/\mathcal{S}_L^{1|0}), \qquad F_r := G_L(1|N;\mathcal{S}_L^{3|N}/\mathcal{S}_L^{1|0}).$$

Again, it is easily seen that these two spaces have dimensions $6|2N$, and in the diagram

$$\begin{array}{ccc} & F & \\ \pi_l \swarrow & & \searrow \pi_r \\ F_l & & F_r \end{array}$$

each of the projections $\pi_{l,r}$ have relative dimension $0|2N$. Hence, again the chiral spaces have just left- (respectively right-) handed odd dimensions. Accordingly, we get again left and right-handed differentials

$$\Omega_l^1 F/L = \Omega^1 F/F_r, \qquad \Omega_r^1 F/L = \Omega^1 F/F_l,$$

and a corresponding exact sequence

$$0 \longrightarrow \Omega_0^1 F/L \longrightarrow \Omega^1 F/L \longrightarrow \Omega_l^1 F/L \oplus \Omega_r^1 F/L \longrightarrow 0. \qquad (32)$$

On F we have the tautological flag $\mathcal{S}_F^{1|0} \subset \mathcal{S}_F^{2|0} \subset \mathcal{S}_F^{2|N} \subset \mathcal{S}_F^{3|N}$, and in terms of this the individual terms above are

$$\begin{aligned}
\Omega^1_l F/L &= \mathcal{S}_F^{2|0}/\mathcal{S}_F^{1|0} \otimes \Pi\left(\mathcal{S}_F^{2|N}/\mathcal{S}_F^{2|0}\right)^*, \\
\Omega^1_r F/L &= \Pi\left(\mathcal{S}_F^{2|N}/\mathcal{S}_F^{2|0}\right) \otimes \widetilde{\mathcal{S}}_F^{2|0}/\widetilde{\mathcal{S}}_F^{1|0}, \\
\Omega^1_0 F/L &= \Pi\left(\mathcal{S}_F^{2|0}/\mathcal{S}_F^{1|0} \otimes \widetilde{\mathcal{S}}_F^{2|0}/\widetilde{\mathcal{S}}_F^{1|0}\right).
\end{aligned} \tag{33}$$

5.2 The super-Radon-Penrose transform

Recall that the necessary assumptions for the Radon-Penrose transform to work was a condition on the sheaf $\mathcal{N}$ of the sequence (2), as well as assumptions on the fibers of the two projections π_i, as well as assumptions on the curvature of ∇, restricted to the fibers of F/L. We need super analogs of these here. First, the definition of the sheaf $\mathcal{N}$ is the same as in (2). We have just seen that for $\Omega^1 M$ and $\Omega^1 F/L$ we have exact sequences (11) and (32), and fitting these together we get the following exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_0 & \longrightarrow & \pi_2^*(\Omega^1_0 M) & \xrightarrow{\mathrm{res}_0} & \Omega^1_0 F/L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N} & \longrightarrow & \pi_2^*(\Omega^1 M) & \xrightarrow{\mathrm{res}} & \Omega^1 F/L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_l \oplus \mathcal{N}_r & \longrightarrow & \pi_l^*(\Omega^1_l \oplus \Omega^1_r) & \xrightarrow{\mathrm{res}_l \oplus \mathrm{res}_r} & \Omega^1_l F/L \oplus \Omega^1_r F/L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Using the fact that $\pi_2^*(\mathcal{S}_M^{2|0}) = \mathcal{S}_F^{2|0}$, etc., we see that the factors $\mathcal{N}_{l,r}$ have the following structure:

$$\mathcal{N}_l = \mathcal{S}_F^{1|0} \otimes \Pi\left(\mathcal{S}_F^{2|N}/\mathcal{S}_F^{2|0}\right)^*, \qquad \mathcal{N}_r = \widetilde{\mathcal{S}}_F^{1|0} \otimes \Pi\left(\mathcal{S}_F^{2|N}/\mathcal{S}_F^{2|0}\right).$$

Finally, using the fact that $\pi_2^*(\bigwedge^2 \mathcal{S}_M^{2|0})(-1,0) = \mathcal{S}_F^{1|0}$, and similarly for $\widetilde{\mathcal{S}}_F$, we get the following analog of the sequence (5):

$$0 \longrightarrow \Pi\left(\mathcal{S}_F^{1|0} \otimes \widetilde{\mathcal{S}}_F^{1|0}\right) \longrightarrow \Pi\left(\mathcal{S}_F^{2|0} \otimes \widetilde{\mathcal{S}}_F^{1|0} \oplus \mathcal{S}_F^{1|0} \otimes \widetilde{\mathcal{S}}_F^{2|0}\right) \longrightarrow \mathcal{N}_0 \longrightarrow 0.$$

Now using the fact that the sheaf $\mathcal{O}_F(-1,-1)$ is acyclic, and that $R^i\pi_{2*}(\pi_2^*\mathcal{E}(a,b)) = 0$ for all $i \geq 0$ if $a = -1$ or $b = -1$, we get in the same way as in the non-supersymmetric case the vanishing of $\pi_{2*}(\mathcal{N}) = R^1\pi_{2*}(\mathcal{N}) = 0$, and with it, the first part of the following theorem

5.2. In the supersymmetric case, there is still the issue of what we should understand under "vanishing curvature along the fibers of F/L". Again, we consider the

$$\pi_{2*}(\mathrm{res}) : \Omega^2 M \longrightarrow \pi_{2*}\Omega^2 F/L,$$

which in the non-supersymmetric case is trivial (since there F/L is a fibration of $\mathbb{P}^1$'s). Here it is not the case.

Definition 5.1 $\Omega^2_{\mathrm{con}} M := \mathrm{Ker}(\pi_{2*})$ is called the sheaf of 2-forms *satisfying constraints.*

These constraints become increasingly restrictive for growing N. In fact, as Witten observed in [10], the constraint equations for $N = 3$ are *equivalent* to the Euler-Lagrange equations (the $N = 3$ supersymmetric Yang-Mills equations), while for $N = 4$ they are overdetermined, i.e., the $N = 4$ supersymmetric Yang-Mills *plus* some unphysical equations. In order to understand these constraints, one considers a filtration of $\Omega^2_{\mathrm{con}} M$. The filtration $\Omega^2_0 M \subset \Omega^2_1 M \subset \Omega^2_2 M = \Omega^2 M$ discussed above induces a similar filtration of both $\Omega^2_{\mathrm{con}} M$ and of $\Omega^2 F/L$. This is also described in the following theorem.

Theorem 5.2 *a)* $\pi_{2*}(\mathrm{res}) : \Omega^1 M \longrightarrow \pi_{2*}\Omega^1 F/L$ *is an isomorphism.*

b) On 2-forms, the mapping $\pi_{2*}(\mathrm{res})$ *is surjective, and the filtration on its kernel is the following:*

$$\begin{aligned}
\Omega^2_{\mathrm{con}} M &= R^1\pi_{2*}\left(\bigwedge^2 \mathscr{N}\right),\\
\Omega^2_{\mathrm{con}} M/\Omega^2_{1,\mathrm{con}} M &= \bigwedge^2 \left(\mathcal{S}^{2|0}_M\right) \otimes \bigwedge^2 \mathcal{E}_l \oplus \bigwedge^2 \left(\widetilde{\mathcal{S}}^{2|0}_M\right) \otimes \bigwedge^2 \mathcal{E}_r,\\
\Omega^2_{1,\mathrm{con}} M &= \Pi\left(\bigwedge^2 \left(\mathcal{S}^{2|0}_M\right) \otimes \widetilde{\mathcal{S}}^{2|0}_M \otimes \mathcal{E}_l \oplus \bigwedge^2 \left(\widetilde{\mathcal{S}}^{2|0}_M\right) \otimes \mathcal{S}^{2|0}_M \otimes \mathcal{E}_r\right),\\
\Omega^2_{0,\mathrm{con}} &= 0.
\end{aligned}$$

We just mentioned how part a) is proved. The surjectivity of the map in b) is reduced to showing $R^2\pi_{2*}(\bigwedge^2 \mathscr{N}) = 0$, which is easily seen from the explicit structure of $\mathscr{N}$ described above. As to the filtrations, since $\mathrm{rank}(\Omega^1_0 F/L) = 0|1$, it follows that $S^2(\Omega^1_0 F/L) = 0$ and hence, by definition, that the part $\Omega^2_0 F/L$ is trivial. Hence the filtration is two-term, or equivalently, given by an exact sequence, namely

$$0 \longrightarrow \Omega^1_0 F/L \otimes \left(\Omega^1_l F/L \oplus \Omega^1_r F/L\right) \longrightarrow \Omega^2 F/L \longrightarrow S^2\left(\Omega^1_l F/L \oplus \Omega^1_r F/L\right) \longrightarrow 0,$$

which, when pushed down with π_2, becomes

$$\begin{aligned}
0 \longrightarrow \pi_{2*}\left[\Omega^1_0 F/L \otimes \left(\Omega^1_l F/L \oplus \Omega^1_r F/L\right)\right] &\longrightarrow \pi_{2*}\Omega^2 F/L\\
&\longrightarrow S^2\left(\mathcal{S}^{2|0}_M\right) \otimes S^2\mathcal{E}_l \oplus \Omega^1_l M \otimes \Omega^1_r M \oplus S^2\left(\widetilde{\mathcal{S}}^{2|0}_M\right) \otimes S^2\mathcal{E}_r \longrightarrow 0. \quad (34)
\end{aligned}$$

The two-term filtration one gets for $\Omega^2_{\text{con}}M$ is easily seen to be the following:

$$0 \longrightarrow \Omega^2_{1,\text{con}}M \longrightarrow \Omega^2_{\text{con}}M \longrightarrow \bigwedge^2 \left(\mathcal{S}^{2|0}_M\right) \otimes \bigwedge^2 \mathcal{E}_l \oplus \bigwedge^2 \left(\widetilde{\mathcal{S}}^{2|0}_M\right) \otimes \bigwedge^2 \mathcal{E}_r \longrightarrow 0,$$

verifying the first two parts of the filtration. Finally, $\Omega^2_{1,\text{con}}M$ can be calculated as the kernel of the composite map

$$\Omega^1_1 M \longrightarrow \Omega^2_1 M/\Omega^2_0 M = \Omega^1_0 M \otimes \left(\Omega^1_l M \oplus \Omega^1_r M\right) \longrightarrow \pi_{2*}\left[\Omega^1_0 F/L \otimes \left(\Omega^1_l F/L \oplus \Omega^1_r F/L\right)\right].$$

It is then easy, using the descriptions above, to see that the kernel is that given in the Theorem.

Let $\mathscr{E}$ be a locally free sheaf on some $U \subset M$ and $\nabla : \mathscr{E} \longrightarrow \mathscr{E} \otimes \Omega^1 M$ a connection.

Definition 5.3 The Yang-mills field $(\mathscr{E}, \nabla)$ is *null-integrable*, if one of the following two equivalent conditions is satisfied:

(a) $\nabla^2_{F/L} = 0$,

(b) $\Phi(\nabla) \in \Gamma(\text{End}(\mathscr{E}) \otimes \Omega^2_{\text{con}}M)$, where $\Phi(\nabla)$ is the curvature form of ∇.

The fact that these two conditions are equivalent follows directly from the Theorem above. Note that in the non-supersymmetric case, the second condition is vacuous, hence *automatically fulfilled*. This remark was made above, and is in contrast to the case of a self-dual diagram. We now see the reason: for a self-dual diagram the fibers of F/Z are two dimensional, while for a diagram of null geodesics they are one-dimensional hence $\Omega^2 F/L$ vanishes and any 2-form is trivial along the fibers.

Definition 5.4 Set $L(U) := \pi_1\pi_2^{-1}(U)$. A locally free sheaf $\mathscr{E}_L$ on $L(U)$ is said to be U-*trivial*, if its restriction to any of the quadrics $L(x)$, $x \in U$ is a free (trivial) sheaf. A locally free sheaf $\mathscr{E}_L$ is called a *YM-sheaf*, if it is defined on an open $V \subset L$ such that $L(U) \subset V$ and $\mathscr{E}_{L|L(U)}$ is U-trivial.

Then, just as in the non-supersymmetric case, we now get

Theorem 5.5 *Let the nonempty intersections of light geodesics with U be connected. Then the following categories are equivalent:*

(a) The category of null-integrable YM-fields $(\mathscr{E}, \nabla)$ on U with trivial monodromy along the fibers.

(b) The category of U-trivial YM-sheaves on $L(U)$.

6 Super Yang-Mills equations

In this section we will define what is to be meant by a solution of the super Yang-Mills equations, and give an interpretation in terms of algebraic geometric data, utilizing the (super) diagram of null geodesics.

6.1 Constraints and Lagrangians

We have seen above that the notion of 2-forms fulfilling constraints is a purely odd phenomenon. We compare, following section 5.3.4 of [6], the description given above in coordinate form. For a null-integrable Yang-Mills field $(\mathscr{E}, \nabla)$ the curvature $\Phi(\nabla)$ is contained in the 2-forms with constraints, i.e., $\Phi(\nabla) \in \Gamma(End(\mathscr{E}) \otimes \Omega^2_{\text{con}} M)$. It follows that it can be written, on a big cell, in terms of the basis of $\Omega^1 M$ which is dual to the $D_{\alpha,j}$ and $D^k_{\dot\beta}$ introduced above in (28). One can also take the elements

$$D_a = \frac{1}{N} \sigma_a^{\alpha\dot\beta} \left[D_{\alpha,j}, D^j_{\dot\beta} \right].$$

For what follows, consider the elements dual to the $\{D_{\alpha,i}, D_{\beta,j}\}$, $\{D_{\alpha,i}, D_{\dot\beta,j}\}$ and $\{D_{\dot\alpha,i}, D_{\dot\beta,j}\}$, and let $\Phi_{\alpha,i,\beta,j}$ etc. denote the coefficients of the curvature form with respect to these elements.

6.1.1 $N = 1$

If $N = 1$, then the sheaves $\mathcal{E}_{l,r}$ have rank $1|0$, hence $\bigwedge^2 \mathcal{E}_{l,r} = 0$. Then by Theorem 5.2, $\Omega^2_{\text{con}} M = \Omega^2_{1,\text{con}} M$. It follows that in this case the curvature must lie in $\Omega^2_{1,\text{con}} M$. In particular, its components in the first filtration level $\Omega^2_{\text{con}}/\Omega^2_{1,\text{con}}$ must vanish. These conditions in local coordinates take the form:

$$\Phi_{\alpha\beta} = \Phi_{\dot\alpha\dot\beta} = \Phi_{\alpha\dot\beta} = 0.$$

In fact, this is just the constraint (30), [8], Chapter VI, Exercise (4).

It now follows that the curvature must lie in $End(\mathscr{E}) \otimes \Omega^2_{1,\text{con}} M$. On a big cell, we can trivialize the rank one sheaves $\mathcal{E}_{l,r}$ and $\bigwedge^2 \mathcal{S}^{2|0}_M$, $\bigwedge^2 \widetilde{\mathcal{S}}^{2|0}_M$, and then $\Omega^2_{1,\text{con}} M$ may be identified with $\Pi(\mathcal{S}^{2|0}_M \oplus \widetilde{\mathcal{S}}^{2|0}_M) = \mathcal{S}_l \oplus \mathcal{S}_r$. The components of the curvature on this big cell in the left, respectively right half are the prepotentials $W_\alpha \in End(\mathscr{E}) \otimes \mathcal{S}_l$, $\overline{W}_{\dot\alpha} \in End(\mathscr{E}) \otimes \mathcal{S}_r$. Note that this description is, with no extra work, valid for any gauge group.

The Lagrangian must be real section of Ber$M$. The easiest expression is by setting $\Phi = \Phi_l \oplus \Phi_r$, the decomposition into left and right-handed parts, a section of $\Pi(\mathcal{S}^{2|0}_M \oplus \widetilde{\mathcal{S}}^{2|0}_M)$. Then set $\mathscr{L} = \text{Tr}(\Phi_l \oplus \Phi_r)$, where Tr denotes the trace in $End(\mathscr{E})$, combined with the contraction of $\mathcal{S}_l$ and $\mathcal{S}_r$ by means of $\varepsilon_{\alpha\beta}$ and the real structure. Note that this is a global expression, valid on all of M. In terms of the superpotentials, this can be written on a big cell as

$$\mathscr{L} = \text{Tr}\left(W^\alpha W_\alpha + \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha} \right).$$

6.1.2 $N = 2$

We also briefly sketch this case. The constraint condition is

$$\Phi \text{mod} \Omega^1_{1,\text{con}} M \in \bigwedge^2 \left(\mathcal{S}^{2|0}\right) \otimes \bigwedge^2 \mathcal{E}_l \oplus \bigwedge^2 \left(\widetilde{\mathcal{S}}^{2|0}\right) \otimes \bigwedge^2 \mathcal{E}_r.$$

The constraints in coordinate notation are

$$\Phi_{\alpha i,\beta j} = -\Phi_{\beta i,\alpha j}, \quad \Phi^{ij}_{\dot\alpha\dot\beta} = -\Phi^{ij}_{\dot\beta\dot\alpha}, \quad \Phi^{j}_{\alpha i,\dot\beta} = 0,$$

which are easily seen to follow from the invariant constraints. Again one gets prepotentials W_l, W_r, using the trivializations of the two invertible sheaves as above, one can write on a big cell

$$\Phi \mathrm{mod} \Omega^2_{1,\mathrm{con}} M = W_l \oplus W_r,$$

for scalar functions W_l, W_r. In this $N = 2$ case, we have $\Phi \mathrm{mod} \Omega^2_{1,\mathrm{con}} M \in End(\mathscr{E}) \otimes \Omega^2_{\mathrm{con}} M / \Omega^2_{1,\mathrm{con}} M$, but this does not automatically imply that Φ belongs to $End(\mathscr{E}) \otimes \Omega^2_{\mathrm{con}}$. In fact, this statement does not hold in general, but: since Φ is the curvature, it satisfies the Bianchi identity, and from this one can show that Φ can be expressed completely by the scalars W_l and W_r, an from this in turn it does follow that Φ lies in $End(\mathscr{E}) \otimes \Omega^2_{\mathrm{con}} M$. In terms of these the Lagrangian can be written as

$$\mathscr{L} = \mathrm{Tr}\left(W_l^2 \otimes W_r^2\right).$$

In this case, since the Berezinian is trivial, any scalar function will serve as Lagrangian.

6.1.3 $N = 3$

This is perhaps the most interesting case, discussed in detail in [10]. This is the case where the constraints are actually *equivalent* to the supersymmetric Yang-Mills equations. Hence, in this case, beyond satisfying the constraints, there are no further conditions on the sheaf $\mathscr{E}_L$ for it to correspond to a solution of the Yang-Mills-equations.

6.2 Supersymmetric Yang-Mills equations

We first want to define what a solution of supersymmetric Yang-Mills equations should be. But before we come to this, we first recall the notion of infinitesimal neighborhood of an algebraic variety. We had worked in the situation where $Y \subset X$ is an embedding of a smooth variety Y into a smooth space X. By definition, this implies in particular that the structure sheaf of Y has no nilpotent elements. This is why this notion is somewhat strange in supergeometry, since, unless a supermanifold Y is purely even, its structure sheaf definitly has nilpotent elements. In particular, if $(Y, \mathscr{O}_Y)$ is a supermanifold, then, since $\mathscr{O}_Y$ has nilpotent elements, even the 0th infinitesimal embedding, is not trivial, in the following sense: letting $\mathscr{I}_Y$ denote the ideal generated by the odd elements, we can consider the embeddings $Y_{\mathrm{rd}} = (Y, \mathscr{O}_{Y,0}) = (Y, \mathscr{O}_Y/\mathscr{I}_Y) =: Y^{[0]} \subset (Y, \mathscr{O}_Y/\mathscr{I}_Y^2) =: Y^{[1]} \subset \ldots \subset (Y, \mathscr{O}_Y) = Y^{(0)}$. We have used the square brackets as superscripts to emphasize the difference to the usual infinitesimal neighborhoods. In the sense of purely even geometry, these still correspond to the zeroth infinitesimal neighborhood. There is a slight difference, however, in that here the *order of nilpotency*, i.e., the number m such that $\mathscr{I}_Y^m = 0$, is finite. Hence, there are only finitely many of these "supersymmetric infinitesimal neighborhoods".

With these remarks out of the way, suppose we have a YM sheaf $\mathscr{E}_L$ on an open set $L(U)$ of L, which is U-trivial. Then we will call $\mathscr{E}_L$ *M-extendable*, if it extends to a locally free sheaf $\mathscr{E}_L^{(m)}$ on the m-th infinitesimal neighborhood $L^{(m)}(U)$ in $\mathbb{P}(T) \times \mathbb{P}(T^*) = \mathbb{P} \times \widehat{\mathbb{P}}$. Again, this condition can be expressed by a supplementary system of differential equations on the Yang-Mills field.

Definition 6.1 A field $(\mathscr{E}, \nabla)$ on M is said to be a *solution of the supersymmetric Yang-Mills equations*, if (locally on M) it corresponds to a $(3 - N)$-extendable YM-sheaf $\mathscr{E}_L$.

Remark: Note that for $N = 3$ this is an empty condition. But as $\mathscr{E}_L$ was obtained by means of the super Radon-Penrose transform, it fulfills the constraints. This is again the statement mentioned above that for $N = 3$ the constraints are equivalent to the YM equations.

6.3 Purely odd extensions

Let M be an analytic supermanifold, and consider the ideal $\mathscr{I} = \mathscr{O}_{M,1} + \mathscr{O}_{M,1}^2$, the part of the structure sheaf generated by the odd elements. Set $M^{[\nu]} = (M, \mathscr{O}_M/\mathscr{I}^{\nu+1})$. Note that for ν greater that the odd dimension of M, the power $\mathscr{I}^{\nu+1} = 0$. We let G denote a connected analytic supergroup (this will play the role of gauge group), and let $\rho : G \longrightarrow \mathrm{GL}(p|q)$ denote a finite dimensional superrepresentation of G. Just as in the purely even case, it holds that the cohomology space $\mathrm{H}^1(M, G(\mathscr{O}_M))$ (Čech cohomology) parametrizes the G-torsors, i.e., sheaves with structure group reduced to G. Together, the pair (e, ρ) can be considered as a rank $p|q$ sheaf with structure group reduced to G. We may also consider this for $M^{[\nu]}$ in place of M: $\mathrm{H}^1(M, G(\mathscr{O}_M/\mathscr{I}^{\nu+1}))$ is the space of G-torsors on $M^{[\nu]}$.

Now let a class $e^{[\nu]} \in \mathrm{H}^1(M, G(\mathscr{O}_M/\mathscr{I}^{\nu+1}))$ be given. We can reduce $e^{[\nu]}$ modulo $\mathscr{I}$, giving us a class $e^{(0)} \in \mathrm{H}^1(M, G(\mathscr{O}_M))$. Let Ad denote the adjoint representation of G; then $(e^{(0)}, \mathrm{Ad})$ determines a locally free sheaf on M_{rd}, which will be denoted in what follows by $\mathcal{G}$. Also, let $\mathcal{G}^{[\nu]}$ denote the sheaf defined by the pair $(e^{[\nu]}, \mathrm{Ad})$. Then we have the following result.

Theorem 6.2 (§5.3.10) *Given the class $e^{[\nu]} \in \mathrm{H}^1(M, G(\mathscr{O}_M/\mathscr{I}^{\nu+1}))$ and the sheaf $\mathcal{G}$ just defined, the following facts hold.*

(1) There is an obstruction class $\omega(e^{[\nu]}) \in \mathrm{H}^2(M, (\mathcal{G} \otimes \mathscr{I}^{\nu+1}/\mathscr{I}^{\nu+2})_0)$ defined, such that $\omega(e^{[\nu]}) = 0$ if and only if $e^{[\nu]}$ extends to a class $e^{(\nu+1)} \in \mathrm{H}^1(M, G(\mathscr{O}_M/\mathscr{I}^{\nu+2}))$.

(2) If $\omega(e^{[\nu]}) = 0$, then the group $\mathrm{H}^1(M, (\mathcal{G} \otimes \mathscr{I}^{\nu+1}/\mathscr{I}^{\nu+2})_0)$ acts transitively on the set of extensions of $\omega(e^{[\nu]})$.

(3) This action is effective if the mapping $\mathrm{H}^0(M, \mathcal{G}^{(\nu+1)}) \longrightarrow \mathrm{H}^0(M, \mathcal{G}^{[\nu]})$ is a surjection for one of the extensions.

6.4 Cohomological calculations

One can apply this to the following situation: on the supermanifold L, we have the sheaf $\mathscr{I} = \mathscr{O}_{L,1} + \mathscr{O}_{L,1}^2$. Since the odd dimension of L is $2N$, it follows that $\mathscr{I}_L^{\nu+1} = 0$ for any $\nu \geq 2N$. We think of the inclusions

$$L_{\mathrm{rd}} = L^{[0]} \subset \cdots \subset L^{[2N]} = L^{(0)},$$

and starting with a sheaf $(\mathscr{E}_L)_{\mathrm{rd}}$ on L_{rd}, consider its extension to $L^{(0)}$. In this way, the constraints $\Phi(\nabla) \in End(\mathscr{E}) \otimes \Omega^2_{\mathrm{con}} M$ are translated into the question of extending a sheaf $(\mathscr{E}_L)_{\mathrm{rd}}$ on L_{rd} (the existence of which follows from the ordinary Radon-Penrose transform, without any condition on the curvature) to L, which again can be described in terms of certain cohomology groups. Starting with an element $e \in \mathrm{H}^1(L(U), G(\mathscr{O}_L))$, we can consider extensions $e^{[\nu]}$ of e, and then their reductions $e^{[0]}$. Letting ρ denote a representation of G as above, the pair $(e^{[0]}, \rho)$ determines a YM sheaf $\mathscr{E}_L^{[0]}$ on L_{rd} with structure group reduced to G. Corresponding to this is an ordinary Yang-Mills field on $U \subset M_{\mathrm{rd}}$ (if $\mathscr{E}_L^{[0]}$ is U-trivial). Just as was done in the purely even case, if $e^{[\nu]}$ has been constructed and $\omega(e^{[\nu]}) = 0$, an extension exists; we choose one $\widetilde{e}^{(\nu+1)}$, and parametrize the other extensions by elements $h^{(\nu+1)}$ of the corresponding cohomology groups. The elements $h^{(\nu+1)}$ are turned into fields on U_{rd} by the Radon-Penrose transform.

The case $N = 3$, $m = 0$ (i.e., the extension for $N = 3$ to the 0th infinitesimal neighborhood) is considered explicitly in [6], and a table of cohomology groups is derived. This is on page 254. The basic observation of the table is that, as we move upward through the levels $L(U)^{[\nu]}$, we first collect fields (up to $\nu \leq 3$) and then equations for $3 \leq \nu \leq 6$. The last obstruction is the current for the ordinary Yang-Mills field which is obtained by putting odd coordinates equal to zero.

7 Geometry of simple ($N = 1$) supergravity

There are different formulations of simple supergravity, each of which has certain advantages and disadvantages. The perhaps most important are the so-called old minimal formulation of the physisists and the one given in [6]. Roughly speaking, both have certain difficult points. In the case of the minimal formulation, a Lagrangian is postulated, and a rather difficult calculation shows that it is locally super Poincaré invariant. In the geometrical formulation of [6], the Lagrangian is *derived*, this by a rather (in fact quite) complicated calculation, but there is no issue about supersymmety invariance; it is built in from the start. Moreover, the minimal formulation is only defined on $\mathbb{R}^4$, while the geometrical formulation is something *globally defined* on any space-time with superconformal symmetry.

7.1 The minimal formulation

We already know from the study of the representations of the super Poincaré group what the simplest irreducible representation is which contains a graviton; it has fields of spin 2 and spin $\frac{3}{2}$. From these we must write down an invariant Lagrangian. We start by recalling the basic constituents. Much of what follows is taken (literally) from [7].

7.1.1 The Vielbein formalism

In the usual formulation of gravitational theories the metric tensor $g_{\mu\nu}(x)$ $(\mu, \nu = 0, 1, \cdots, d-1)$ is used to describe gravity. Our signature convention of $g_{\mu\nu}$ is $(+, -, \cdots, -)$. The Einstein Lagrangian in this formulation is

$$\mathscr{L} = -\frac{1}{16\pi G}\sqrt{-g}R, \tag{35}$$

where G is the gravitational constant and $g = \det g_{\mu\nu}$. In the following we will put $4\pi G = 1$ for simplicity. In the usual manner one introduces the Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right),$$

as well as the scalar curvature R, Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ by

$$R = g^{\mu\nu}R_{\mu\nu}, \qquad R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu},$$
$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}.$$

To couple gravity to spinor fields it is more convenient to use the vielbein formulation of gravity. In this formulation we introduce d vectors $e_a{}^{\mu}(x)$ $(a = 0, 1, \cdots, d-1)$ at each point of space-time, which are orthogonal to each other and have a unit length

$$e_a{}^{\mu}(x)e_b{}^{\nu}(x)g_{\mu\nu}(x) = \eta_{ab}, \tag{36}$$

where $\eta_{ab} = \mathrm{diag}(+1, -1, \cdots, -1)$ is a flat Minkowski metric. We also introduce inverse matrices $e_\mu{}^{a}(x)$, which satisfy

$$e_\mu{}^{a}(x)e_a{}^{\nu}(x) = \delta^{\nu}_{\mu}, \qquad e_a{}^{\mu}(x)e_\mu{}^{b}(x) = \delta^{b}_{a}. \tag{37}$$

The fields $e_\mu{}^{a}(x)$ are called vielbein (vierbein or tetrad in four dimensions, fünfbein in five dimensions, etc.). From eqs. (36) and (37) we can express the metric in terms of the vielbein

$$g_{\mu\nu}(x) = e_\mu{}^{a}(x)e_\nu{}^{b}(x)\eta_{ab}. \tag{38}$$

Therefore, we can use the vielbein $e_\mu{}^{a}(x)$ as dynamical variables representing gravitational degrees of freedom.

For a given metric $g_{\mu\nu}$ the vielbein $e_\mu{}^{a}$ satisfying eq. (38) is not uniquely determined. If $e_\mu{}^{a}$ satisfies eq. (38), then another vielbein

$$e'_\mu{}^{a}(x) = e_\mu{}^{b}(x)\Lambda_b{}^{a}(x) \qquad \left(\Lambda_a{}^{c}(x)\Lambda_b{}^{d}(x)\eta_{cd} = \eta_{ab}\right) \tag{39}$$

also satisfies eq. (38) with the same $g_{\mu\nu}$. The metric tensor has $\frac{1}{2}d(d+1)$ independent components, while the vielbein has d^2 components. The difference $\frac{1}{2}d(d-1)$ is the number of independent components of $\Lambda_a{}^{b}$. The transformation (39) is called a local Lorentz transformation. Since the theories are originally formulated by using only $g_{\mu\nu}$, they should be invariant under the local Lorentz transformations. Thus, gravitational theories in the vielbein formulation have two local symmetries: the general coordinate symmetry and the local Lorentz symmetry.

We have now two kinds of vector indices: $\mu, \nu, \cdots$ and $a, b, \cdots$. To distinguish them the indices $\mu, \nu, \cdots$ are called 'world indices', while $a, b, \cdots$ are called 'local Lorentz indices'. These two kinds of indices are converted into each other by using the vielbein and its inverse, e.g.,

$$A_a(x) = e_a{}^\mu(x) A_\mu(x), \qquad A_\mu(x) = e_\mu{}^a(x) A_a(x).$$

Tensor fields with local Lorentz indices transform under the local Lorentz transformations as in eq. (39). They also transform under the general coordinate transformations as tensor fields determined by the world indices they have.

To construct an action of spinor fields invariant under the local Lorentz transformations we need a gauge field. It is called a spin connection $\omega_\mu{}^a{}_b(x)$ ($\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$). The local Lorentz transformation of the spin connection should be

$$\delta_L \omega_\mu{}^{ab} = D_\mu \lambda^{ab} \equiv \partial_\mu \lambda^{ab} + \omega_\mu{}^a{}_c \lambda^{cb} + \omega_\mu{}^b{}_c \lambda^{ac}$$

so that the covariant derivative of a spinor field ψ

$$D_\mu \psi = \left(\partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \right) \psi \tag{40}$$

transforms covariantly. The spinor Lagrangian invariant under the general coordinate and the local Lorentz transformations is

$$\mathscr{L} = ie \bar{\psi} \gamma^\mu D_\mu \psi, \tag{41}$$

where $e = \det e_\mu{}^a = \sqrt{-g}$ and $\gamma^\mu = \gamma^a e_a{}^\mu$.

The spin connection is completely determined by the vielbein if we impose the torsionless condition

$$D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 0 \qquad (D_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b). \tag{42}$$

(The metricity condition corresponds to the antisymmetry property $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$.) The solution of eq. (42) is $\omega_{\mu ab} = \omega_{\mu ab}(e)$, where

$$\omega_{\mu ab}(e) = \frac{1}{2} \left(e_a{}^\nu \Omega_{\mu\nu b} - e_b{}^\nu \Omega_{\mu\nu a} - e_a{}^\rho e_b{}^\sigma e_\mu{}^c \Omega_{\rho\sigma c} \right), \quad \Omega_{\mu\nu a} = \partial_\mu e_{\nu a} - \partial_\nu e_{\mu a}. \tag{43}$$

The spin connection (43) is related to the Christoffel symbols by

$$\partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\mu{}^b - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a = 0.$$

The field strength of the spin connection

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b.$$

is shown to be related to the Riemann tensor and the scalar curvature as

$$R_{\mu\nu}{}^\rho{}_\sigma = R_{\mu\nu}{}^a{}_b e_a{}^\rho e_\sigma{}^b, \qquad R = e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}.$$

7.1.2 The minimal formulation

The field content of the $d = 4$, $N = 1$ supergravity is the vierbein (tetrad) $e_\mu{}^a(x)$ and a Majorana Rarita-Schwinger field $\psi_\mu(x)$. The vierbein is related to the metric as $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$ is the flat Minkowski metric. The Rarita-Schwinger field satisfies the Majorana condition $\psi^c_\mu (\equiv C\bar{\psi}^T_\mu) = \psi_\mu$, where C is the charge conjugation matrix satisfying

$$C^{-1}\gamma^a C = -\gamma^{aT}, \qquad C^T = -C.$$

The Lagrangian consists of the Einstein term and the Rarita-Schwinger term

$$\mathscr{L} = -\frac{1}{4}e\widehat{R} - \frac{1}{2}ie\bar{\psi}_\mu\gamma^{\mu\nu\rho}\widehat{D}_\nu\psi_\rho, \tag{44}$$

where $e = \det e_\mu{}^a$ and γ's with multiple indices are antisymmetrized products of gamma matrices with unit strength

$$\gamma^{\mu\nu\rho} = \frac{1}{3!}\left(\gamma^\mu\gamma^\nu\gamma^\rho \pm \text{permutations of } \mu\nu\rho\right).$$

The curvature and the covariant derivative are defined by

$$\begin{aligned}
\widehat{R} &= e_a{}^\mu e_b{}^\nu \widehat{R}_{\mu\nu}{}^{ab}, \\
\widehat{R}_{\mu\nu}{}^{ab} &= \partial_\mu\widehat{\omega}_\nu{}^{ab} - \partial_\nu\widehat{\omega}_\mu{}^{ab} + \widehat{\omega}_\mu{}^a{}_c\widehat{\omega}_\nu{}^{cb} - \widehat{\omega}_\nu{}^a{}_c\widehat{\omega}_\mu{}^{cb}, \\
\widehat{D}_\nu\psi_\rho &= \left(\partial_\nu + \frac{1}{4}\widehat{\omega}_\nu{}^{ab}\gamma_{ab}\right)\psi_\rho.
\end{aligned}$$

The spin connection $\widehat{\omega}_\mu{}^{ab}$ used here is given by

$$\widehat{\omega}_{\mu ab} = \omega_{\mu ab} - \frac{1}{2}i\bar{\psi}_a\gamma_\mu\psi_b - \frac{1}{2}i\bar{\psi}_\mu\gamma_a\psi_b + \frac{1}{2}i\bar{\psi}_\mu\gamma_b\psi_a, \tag{45}$$

where $\omega_{\mu ab}$ is the spin connection without torsion given in eq. (43). The spin connection (45) has a torsion depending on the Rarita-Schwinger field

$$\widehat{D}_\mu e_\nu{}^a - \widehat{D}_\nu e_\mu{}^a = -i\bar{\psi}_\mu\gamma^a\psi_\nu. \tag{46}$$

If one wishes, it is also possible to express the Lagrangian using the torsionless spin connection $\omega_{\mu ab}$ but with explicit 4-fermi terms

$$\mathscr{L} = -\frac{1}{4}eR - \frac{1}{2}ie\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho + (\text{4-fermi terms}), \tag{47}$$

where R and D_ν are defined by using the torsionless spin connection.

The Lagrangian (44) is invariant under three kinds of local symmetries up to total divergences:

(i) general coordinate transformations

$$\begin{aligned}\delta_G(\xi)e_\mu{}^a &= -\xi^\nu\partial_\nu e_\mu{}^a - \partial_\mu\xi^\nu e_\nu{}^a,\\ \delta_G(\xi)\psi_\mu &= -\xi^\nu\partial_\nu\psi_\mu - \partial_\mu\xi^\nu\psi_\nu,\end{aligned} \tag{48}$$

(ii) local Lorentz transformations

$$\begin{aligned}\delta_L(\lambda)e_\mu{}^a &= -\lambda^a{}_b e_\mu{}^b,\\ \delta_L(\lambda)\psi_\mu &= -\frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu,\end{aligned} \tag{49}$$

(iii) local supertransformations

$$\begin{aligned}\delta_Q(\epsilon)e_\mu{}^a &= -i\bar{\epsilon}\gamma^a\psi_\mu,\\ \delta_Q(\epsilon)\psi_\mu &= \widehat{D}_\mu\epsilon \equiv \left(\partial_\mu + \frac{1}{4}\widehat{\omega}_\mu{}^{ab}\gamma_{ab}\right)\epsilon,\end{aligned} \tag{50}$$

where the transformation parameters $\xi^\mu(x)$, $\lambda^a{}_b(x)$ $(\lambda^{ab} = -\lambda^{ba})$ and $\epsilon_\alpha(x)$ $(\epsilon^c = \epsilon)$ are arbitrary functions of the space-time coordinates x^μ. The invariance under the bosonic transformations (i), (ii) is manifest. The invariance under the local supertransformations (iii) is shown in the following subsection.

7.1.3 Local supersymmetry invariance

As mentioned, one of the more difficult parts here is to show the invariance of the Lagrangian. Here we sketch this briefly. The Lagrangian of $d = 4$, $N = 1$ supergravity consists of two terms

$$\begin{aligned}\mathscr{L} &= \mathscr{L}_{\rm E} + \mathscr{L}_{\rm RS},\\ \mathscr{L}_{\rm E} &= -\frac{1}{4}e\, e_a{}^\mu e_b{}^\nu \widehat{R}_{\mu\nu}{}^{ab} = \frac{1}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e_\rho{}^c e_\sigma{}^d\widehat{R}_{\mu\nu}{}^{ab},\\ \mathscr{L}_{\rm RS} &= -\frac{1}{2}iee_a{}^\mu e_b{}^\nu e_c{}^\rho\bar{\psi}_\mu\gamma^{abc}\widehat{D}_\nu\psi_\rho = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5\widehat{D}_\rho\psi_\sigma,\end{aligned} \tag{51}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon^{0123} = +1$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The Riemann tensor $\widehat{R}_{\mu\nu}{}^{ab}$ and the covariant derivative $\widehat{D}_\mu$ depend on the vierbein $e_\mu{}^a$ only through the spin connection $\widehat{\omega}_\mu{}^{ab}$. When the action is viewed as a functional of $e_\mu{}^a$, ψ_μ and $\widehat{\omega}_\mu{}^{ab}$, the spin connection (45) satisfies an equation

$$\frac{\delta}{\delta\widehat{\omega}_{\mu ab}}\int d^4x\, \mathscr{L}(e, \psi, \widehat{\omega}) = 0. \tag{52}$$

(To show this it is convenient to use the second form of $\mathscr{L}_{\rm E}$ and $\mathscr{L}_{\rm RS}$ in eq. (51).) Therefore, when we compute a variation of the Lagrangian under supertransformations, the spin connection need not be varied.

To show the local supersymmetry invariance we need the following formulae involving spinors. For four arbitrary spinors ψ, χ, λ and ϕ the Fierz identity

$$\bar{\psi}\chi\bar{\lambda}\phi = -\frac{1}{4}\left[\bar{\psi}\phi\bar{\lambda}\chi + \bar{\psi}\gamma^a\phi\bar{\lambda}\gamma_a\chi - \frac{1}{2}\bar{\psi}\gamma^{ab}\phi\bar{\lambda}\gamma_{ab}\chi - \bar{\psi}\gamma^a\gamma_5\phi\bar{\lambda}\gamma_a\gamma_5\chi + \bar{\psi}\gamma_5\phi\bar{\lambda}\gamma_5\chi\right] \qquad (53)$$

is satisfied. Bilinears of two arbitrary Majorana spinors ψ and χ have symmetry properties

$$\begin{aligned}
\bar{\psi}\chi &= \bar{\chi}\psi, \\
\bar{\psi}\gamma^a\chi &= -\bar{\chi}\gamma^a\psi, \\
\bar{\psi}\gamma^{ab}\chi &= -\bar{\chi}\gamma^{ab}\psi, \\
\bar{\psi}\gamma^a\gamma_5\chi &= \bar{\chi}\gamma^a\gamma_5\psi, \\
\bar{\psi}\gamma_5\chi &= \bar{\chi}\gamma_5\psi.
\end{aligned} \qquad (54)$$

Let us now compute the variation of the Lagrangian (51) under the supertransformation (50). Using the first form in eq. (51) the variation of the Einstein term is

$$\begin{aligned}
\delta_Q\mathscr{L}_{\rm E} &= -\frac{1}{4}\delta_Q(e\, e_a{}^\mu e_b{}^\nu)\widehat{R}_{\mu\nu}{}^{ab} \\
&= -\frac{1}{2}ie\bar{\epsilon}\gamma^\mu\psi_a\left(e_b{}^\nu\widehat{R}_{\mu\nu}{}^{ab} - \frac{1}{2}e_\mu{}^a\widehat{R}\right).
\end{aligned} \qquad (55)$$

On the other hand, using the second form in eq. (51) the variation of the Rarita-Schwinger term is

$$\begin{aligned}
\delta_Q\mathscr{L}_{\rm RS} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\delta_Q\bar{\psi}_\mu\gamma_\nu\gamma_5\widehat{D}_\rho\psi_\sigma + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5\widehat{D}_\rho\delta_Q\psi_\sigma \\
&\quad +\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\delta_Q e_\nu{}^a\ \bar{\psi}_\mu\gamma_a\gamma_5\widehat{D}_\rho\psi_\sigma.
\end{aligned} \qquad (56)$$

By partial integration the first term becomes

$$\begin{aligned}
\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widehat{D}_\mu\bar{\epsilon}\gamma_\nu\gamma_5\widehat{D}_\rho\psi_\sigma &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\epsilon}\gamma_\nu\gamma_5\widehat{D}_\mu\widehat{D}_\rho\psi_\sigma - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widehat{D}_\mu e_\nu{}^a\,\bar{\epsilon}\gamma_a\gamma_5\widehat{D}_\rho\psi_\sigma \\
&\quad + \text{total derivative terms.}
\end{aligned} \qquad (57)$$

By using eq. (46), the Fierz identity (53) and the symmetry properties (54) the second term in eq. (57) is shown to cancel the third term in eq. (56). Then, eq. (56) becomes

$$\begin{aligned}
\delta_Q\mathscr{L}_{\rm RS} &= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\epsilon}\gamma_\nu\gamma_5[\widehat{D}_\mu, \widehat{D}_\rho]\psi_\sigma + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5[\widehat{D}_\rho, \widehat{D}_\sigma]\epsilon \\
&= \frac{1}{2}ie\bar{\epsilon}\gamma^\mu\psi_a\left(e_b{}^\nu\widehat{R}_{\mu\nu}{}^{ab} - \frac{1}{2}e_\mu{}^a\widehat{R}\right),
\end{aligned} \qquad (58)$$

up to total derivative terms. In the last equality we have used

$$[\widehat{D}_\mu, \widehat{D}_\nu]\epsilon = \frac{1}{4}\widehat{R}_{\mu\nu}{}^{ab}\gamma_{ab}\epsilon \qquad (59)$$

and the properties (54). Thus, the variations of $\mathscr{L}_{\mathrm{E}}$ and $\mathscr{L}_{\mathrm{RS}}$ cancel each other and the total Lagrangian (44) is invariant under the local supertransformation (50) up to total derivative terms.

Finally, one must show invaraince under the supercommutators of the generators discussed up to now. The interesting remark is that these commutators "close only on-shell", meaning the equations of motion must be utilized to get the invariance.

7.2 The geometric formulation

We now sketch the geometric formulation as given in [6]. The basic advantages of this formulation are: (i) the Lagrangian is *derived*, not postulated, (ii) the supersymmetry of the Lagrangian is manifest by construction and need not be checked, and (iii) the construction is *global*, working on an arbtrary supermanifold of dimension $4|4$.

7.2.1 Basic structures

We begin with the definition of a complex space-time for simple supergravity.

Definition 7.1 A *complex superspace for simple supergravity* is a complex supermanifold M of dimension $4|4$ with the following three structures:

1. A superconformal structure: two integrable complex distributions $\mathscr{T}_l M,\ \mathscr{T}_r M \subset \mathscr{T} M$ of rank $0|2$ whose sum in $\mathscr{T} M$ is direct and which satisfy: the Frobenius form

$$\begin{array}{rcl}\phi : \mathscr{T}_l M \otimes \mathscr{T}_r M & \longrightarrow & \mathscr{T}_0 M := \mathscr{T} M/(\mathscr{T}_l M \oplus \mathscr{T}_r M) \\ X \otimes Y & \mapsto & [X,Y]\mathrm{mod}(\mathscr{T}_l M \oplus \mathscr{T}_r M)\end{array}$$

   is an isomorphism.

2. A real structure ϱ which has a four-dimensional real manifold of fixed points on M_{rd} and which exchanges $\mathscr{T}_l M$ with $\mathscr{T}_r M$.

3. Two even nondegenerate volume forms $v_{l,r} \in \mathrm{H}^0(M_{l,r}, \mathrm{Ber}(M_{l,r}))$ with the condition $v_l^{\varrho} = v_r$.

Given the structure in 1., define $\mathscr{O}_{M_l}$ (respectively $\mathscr{O}_{M_r}$) to be the subsheaf of $\mathscr{O}_M$ annihilated by all vector fields in $\mathscr{T}_r M$ (resp. $\mathscr{T}_l M$) and set $M_l := (M, \mathscr{O}_{M_l}),\ M_r := (M, \mathscr{O}_{M_r})$. Then the two distrubutions are the relative tangent bundles $\mathscr{T}_{l,r} = \mathscr{T} M/M_{l,r}$.

We now describe how the structures 1.-3. in the definition give the structure of complex space-time on M_{rd}, which also shows how supergravity is related to gravity. We start with the spinor bundles and metrics. First one *defines* $\Omega^1_{l,r} M := \Pi(\mathscr{T}_{l,r} M)^*$ and $\Omega^1_0 M := \Pi(\mathscr{T}_0 M)^*$. Set

$$S := \Pi(\Omega^1_l M)_{\mathrm{rd}}, \qquad \widetilde{S} := \Pi(\Omega^1_r M)_{\mathrm{rd}}. \tag{60}$$

Then these are spinor bundles for a complex space-time, i.e., fulfill

$$S \otimes \widetilde{S} \cong \Omega^1 M_{\mathrm{rd}}$$

(where here Ω^1 denotes Ω^1_{ev} for the purely even manifold M_{rd}). This follows from the following two facts: first, the dual of the Frobenius isomorphism in the definition gives an isomorphism

$$\Pi(\Omega^1_l M) \otimes \Pi(\Omega^1_r M) \xrightarrow{\sim} \Pi(\Omega^1_0 M). \tag{61}$$

Secondly, the composition of the inclusion $\Omega^1_0 \subset \Omega^1 M$ and the reduction of odd coordinates produces a mapping $\Pi(\Omega^1_0 M)_{\mathrm{rd}} \longrightarrow \Omega^1(M_{\mathrm{rd}})$, which can be shown to be an isomorphism (book, 5.7.12, note the slight difference in notation).

Before defining the spinor metrics, we require a result about the relation between the Berezinian of M and those of $M_{l,r}$.

Proposition 7.2 (5.7.2) *There is a canonical isomorphism of sheaves*

$$(\mathrm{Ber}(M))^3 \cong \pi_l^* \mathrm{Ber}(M_l) \otimes \pi_r^* \mathrm{Ber}(M_r).$$

Proof: We have already seen how one makes computations with the Berezinians. From (61) we know that

$$\mathrm{Ber}(\Omega^1_0 M)^* = (\mathrm{Ber}(\Omega^1_l M))^2 \otimes (\mathrm{Ber}(\Omega^1_r M))^2, \tag{62}$$

since both sheaves $\Omega^1_{l,r} M$ have rank $0|2$, and just as in the situation of complex space time we get the exact sequence

$$0 \longrightarrow \Omega^1_0 M \longrightarrow \Omega^1 M \xrightarrow{b_l \oplus b_r} \Omega^1_l M \oplus \Omega^1_r M \longrightarrow 0, \tag{63}$$

from which we get

$$\begin{aligned} \mathrm{Ber}(M) = \mathrm{Ber}(\Omega^1 M)^* &= (\mathrm{Ber}(\Omega^1_0 M))^* \otimes (\mathrm{Ber}(\Omega^1_l M))^* \otimes (\mathrm{Ber}(\Omega^1_r M))^* \\ \text{by (62)} \quad &= \mathrm{Ber}(\Omega^1_l M) \otimes \mathrm{Ber}(\Omega^1_r M). \end{aligned} \tag{64}$$

We now utilize the following exact sequences, which are proved to exist in 5.7.11 of [6], see Lemma 7.6 below:

$$0 \longrightarrow \Omega^1_0 M \longrightarrow \pi^*_{l,r}(\Omega^1 M_{l,r}) \longrightarrow \Omega^1_{l,r} M \longrightarrow 0,$$

from which we get the relations

$$\begin{aligned} \pi_l^* \mathrm{Ber}(M_l) &= \mathrm{Ber}(\pi_l^*(\Omega^1 M_l))^* = \mathrm{Ber}(\Omega^1_l M) \otimes (\mathrm{Ber}(\Omega^1_r M))^2, \\ \pi_r^* \mathrm{Ber}(M_r) &= \mathrm{Ber}(\pi_r^*(\Omega^1 M_r))^* = (\mathrm{Ber}(\Omega^1_l M))^2 \otimes \mathrm{Ber}(\Omega^1_r M). \end{aligned} \tag{65}$$

Comparing (64) with (65), we get the statement of the proposition. □

Note that according to this proposition, the section $\pi_l^* v_l \otimes \pi_r^* v_r$ of $\pi_l^*\mathrm{Ber}(M_l) \otimes \pi_r^*\mathrm{Ber}(M_r)$ is actually a third power, i.e., it is of the form ξ^3 for some section ξ of $\mathrm{Ber}(M)$. Hence the cube roots are well-defined, and one now defines the *Lagrangian* of the theory by

$$\mathscr{L} := w = (\pi_l^* v_l \otimes \pi_r^* v_r)^{1/3}. \tag{66}$$

The spinor metrics are defined by

$$\varepsilon_l := (\pi_l^* v_l)^{1/3} \otimes (\pi_r^* v_r)^{-2/3}, \qquad \varepsilon_r := (\pi_l^* v_l)^{-2/3} \otimes (\pi_r^* v_r)^{1/3}. \tag{67}$$

From (65) we see that the cube ε_l^3 is a section of

$$\pi_l^*\mathrm{Ber}(M_l) \otimes \pi_r^*\mathrm{Ber}(M_r)^{-2} \cong \mathrm{Ber}(\Omega_l^1 M)^{-3},$$

and similarly for ε_r, so we get

$$\varepsilon_{l,r} \in (\mathrm{Ber}(\Omega_{l,r}^1 M))^{-1} = \mathrm{Ber}(\Pi(\Omega_{l,r}^1 M)). \tag{68}$$

This means that after reducing the odd coordinates, the $\varepsilon_{l,r}$ reduce to spinor metrics denoted ε, $\widetilde{\varepsilon}$ (see (60) and Definition 2.1 b)) on M_{rd}. These are sections of $\wedge^2 S$ and $\wedge^2 \widetilde{S}$, and $g = \varepsilon \otimes \widetilde{\varepsilon}$ is a holomorphic metric on M_{rd}. Finally, we can introduce superconnections on $\Omega^1 M$ as well as on $\Omega_{l,r}^1 M$, and reducing the latter to M_{rd}, we get usual spin connections on M_{rd}. Thus, we have on M_{rd} the structure of complex space-time, and our supermanifold M is a supersymmetrization of complex space-time.

7.2.2 The formalism of Ogietvetskii and Sokachev

We now will interpret the geometric set-up in terms of superfields. This makes contact with the physics liturature, as well as giving a more explicit form to the super Lagrangian (66). First we will define local coordinates, then the Ogievetskii-Sokachev prepotential, and in terms of these, there will be different (local) basis of $\mathscr{T}M$,

$$(\partial_a, \Delta_\alpha, \Delta_{\dot\alpha}), \qquad \left(\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot\alpha}, \frac{i}{2}\left[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot\alpha}\right]\right),$$

and if E_B^A denote the change of basis matrix from one to the other, it will turn out that our Lagrangian can (locally!) be identified as (cf. [6], 5.7.16)

$$\mathscr{L} = \frac{1}{16}\mathrm{Ber}(E_B^A) D^*(d\theta^\alpha, d\theta^{\dot\alpha}, dx^a),$$

where $D^*(d\theta^\alpha, d\theta^{\dot\alpha}, dx^a)$ denotes the element in the Berezinian which is determined by that basis of $\Omega^1 M$. Calculating this Berezinian then yields the Lagrangian, expressed in terms of superfields.

Adapted coordinates In the following we will assume we are given a complex superspace for simple supergravity M. From the conditions which define M, we can find local coordinates of the following kind, valid on a big open set.

- We have local coordinates (x_l^a, θ^α) on M_l; we set $(x_r^a, \theta^{\dot\alpha}) := ((x_l^a)^\varrho, (\theta^\alpha)^\varrho)$.
- The functions $(x_l^a)_{\mathrm{rd}}$ on M_{rd} are real (ϱ-invariant).
- The functions $(x^a := \frac{1}{2}(x_l^a + x_r^a), \theta^\alpha, \theta^{\dot\alpha})$ form a local system of coordinates on M.

Such a set of (local!) coordinates are called adapted coordinates. We will consider as coordinate transformations such which map adapted coordinates into adapted coordinates, and these will be refered to as *gauge transformations* (of the local super-Poincaré symmetry). For example, using our coordinates $x_l^{\alpha\dot\beta}$, we get such real coordinates by setting $x_l^a = \sigma^a_{\alpha\dot\beta} x_l^{\alpha\dot\beta}$, and similarly for x_r^a.

O-S prepotential Note that in our flat situation, we had the relations

$$x_l^{\alpha\dot\beta} = x^{\dot\beta\alpha} + i\theta^\alpha\theta^{\dot\beta}, \qquad x_r^{\alpha\dot\beta} = x^{\alpha\dot\beta} - i\theta^\alpha\theta^{\dot\beta},$$

from which it follows that $(x_l^{\alpha\dot\beta})_{\mathrm{rd}} = (x_r^{\dot\beta\alpha})_{\mathrm{rd}}$. Hence the difference is purely odd, i.e., nilpotent; in particular the expression

$$H^a = \frac{1}{2i}(x_l^a - x_r^a) \tag{69}$$

defines four real nilpotent functions, which are collectively called the *Ogievetskii-Sokachev prepotential.* These are essentially just products of spinors, but with vector indices. For example, in the flat case, From $x_r^{\dot\beta\alpha} = x^{\alpha\dot\beta} - i\theta^\alpha\theta^{\dot\beta}$ and $x_r^a = \sigma^a_{\alpha\dot\beta} x_r^{\dot\beta\alpha}$, we get $H^a = \sigma^a_{\alpha\dot\beta}\theta^\alpha\theta^{\dot\beta}$. One can however define more general H^a, and on an arbitrary $4|4$ manifold X with local coordinates $(X^a, \theta^\alpha, \theta^{\dot\beta})$, one can satisfy the first two conditions in the definition of complex superspace for simple gravity by setting

$$(X^a)^\varrho := X^a + 2iH^a, \qquad (\theta^{\dot\alpha})^\varrho := \theta^\alpha.$$

One then defines $\mathscr{O}_l$ to consist of functions in X^a and θ^α, $\mathscr{O}_r$ to consist of functions in $(X^a)^\varrho$ and $\theta^{\dot\alpha}$. A particular case is that of the *Wess-Zumino gauge*, where one sets

$$H^a(x, \theta^\alpha, \theta^{\dot\alpha}) = \theta^\alpha\theta^{\dot\alpha} e^a_{\alpha\dot\alpha} + \varepsilon_{\dot\alpha\dot\beta}\theta^{\dot\alpha}\theta^{\dot\beta}\theta^\gamma\psi^a_\gamma + \varepsilon_{\alpha\beta}\theta^\alpha\theta^\beta\theta^{\dot\gamma}\psi^a_{\dot\gamma} + \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\theta^\alpha\theta^\beta\theta^{\dot\alpha}\theta^{\dot\beta}A^a. \tag{70}$$

In this expression, the $e^a_{\alpha\dot\alpha}$ is a vierbein, and turns the metric into a *dynamical variable.* Recallin our formula $g = \varepsilon \otimes \widetilde{\varepsilon}$ in terms of the reductions of $\varepsilon_{l,r}$ (see (68) in terms of $M_{l,r}$, we note that making g dynamical can be achieved by making the notions of left-right (i.e., $M_{l,r}$) dynamical, which is what is achieved by H^a as in (70). For example, we keep our variables x^a and *define* left and right in terms of H^a:

$$x_r^a := x^a - iH^a, \quad x_l^a := x^a + iH^a \quad \Rightarrow x_l^a = x_r^a + 2iH^a,$$

as above. We can think of this as the analog here of (38) in the minimal formulation of supergravity.

A gauge transformation changes these functions also. We next recall that $D^*(d\theta^\alpha, dx_l^a)$ denotes the element in the Berezinian of $\Omega^1 M_l$ determined by the basis $d\theta^\alpha, dx_l^a$ on the open set where the θ^α and x_l^a are local coordinates. The third condition for a complex superspace for supergravity is the existence of section $v_{l,r}$ of the corresponding Berezinians. It follows that on this open set, these sections are just some functions times the above elements, i.e., the following *defines* the functions Φ_l, Φ_r:

$$v_l = \Phi_l^3 \pi_l^* D^*(d\theta^\alpha, dx_l^a), \qquad v_r = \Phi_r^3 \pi_r^* D^*(d\theta^{\dot\alpha}, dx_r^a). \tag{71}$$

In the sequel it will be shown that in these local coordinates, all objects of interest can be calculated in terms of the objects H^a and $\Phi_{l,r}$. The Wess-Zumino gauge mentioned above takes the form $\Phi_l = \Phi_r = 1$ on these functions.

The fundamental axiom of a superconformal structure is the maximal non-degeneracy of the Frobenius form, which as we will see below amounts to the invertibility of the matrix of second spinor derivatives of the H^a. Before describing the Frobenius form in terms of our local coordinates, we introduce a number of basis in the various sheaves of interest.

Some basis We have the following local basis of $\mathscr{T}M$,

$$\partial_a = \frac{\partial}{\partial x^a}, \quad \partial_\alpha = \frac{\partial}{\partial \theta^\alpha}, \quad \partial_{\dot\alpha} = \frac{\partial}{\partial \theta^{\dot\alpha}}.$$

One forms the matrix $(\partial H/\partial x)_b^a = (\partial_b H^a)$, and sets

$$\begin{aligned} X_\alpha^a &= i \left[\left(\mathbf{1} - i \frac{\partial H}{\partial x} \right)^{-1} \right]_b^a \partial_\alpha H^b, \\ X_{\dot\alpha}^a &= -i \left[\left(\mathbf{1} + i \frac{\partial H}{\partial x} \right)^{-1} \right]_b^a \partial_{\dot\alpha} H^b. \end{aligned}$$

The X_α^a and $X_{\dot\alpha}^a$ are odd, hence nilpotent functions. Our goal now is to describe everything in terms of the H^a. Since we have changed the notion of left-right, the ∂_α (resp. $\partial_{\dot\alpha}$) are no longer a basis of $\mathscr{T}_l$ (resp. $\mathscr{T}_r$). Instead, we have the following elements:

$$\Delta_\alpha := \partial_\alpha + X_\alpha^a \partial_a, \qquad \Delta_{\dot\alpha} := -\partial_{\dot\alpha} - X_{\dot\alpha}^a \partial_a. \tag{72}$$

The condition that the so defined elements lie in $\mathscr{T}_l M$ (respectively in $\mathscr{T}_r M$) yields the expressions above for the X_α^a (respectively the $X_{\dot\alpha}^a$) (recall that elements of $\mathscr{T}_l M$ must annihilate $\theta^{\dot\alpha}$ and $x_r^a = x^a - iH^a$, and similarly for $\mathscr{T}_r M$):

$$\Delta_\alpha(x^b - iH^b) = (\partial_\alpha + X_\alpha^a \partial_a)(x^b - iH^b) = 0 \iff (\delta_a^b - i\partial_a H^b) X_\alpha^a = i\partial_\alpha H^b,$$

and analogously for $\Delta_{\dot\alpha}$. The latter equation is easily seen to be equivalent to the definition of the X_α^a. Moreover,

Lemma 7.3 *The Δ_α (respectively $\Delta_{\dot\alpha}$) form a local basis of $\mathscr{T}_l M$ (respectively of $\mathscr{T}_r M$). The two basis*

$$(\partial_a, \partial_\alpha, \partial_{\dot\alpha}), \qquad (\partial_a, \Delta_\alpha, \Delta_{\dot\alpha})$$

represent the same element in the Berezinian of $\mathscr{T} M$.

Proof: We have seen that the mentioned elements are contained in the corresponding sheaves; to show they form a local basis, it suffices to show that any element of $\mathscr{T}_l M$ (respectively $\mathscr{T}_r M$) modulo the span of the Δ_α (respectively $\Delta_{\dot\alpha}$) vanishes. This is the calculation: let $D = A^a\partial_a + B^\alpha\partial_\alpha + C^{\dot\alpha}\partial_{\dot\alpha} \in \mathscr{T}_l M$. Then, subtracting $B^\alpha\Delta_\alpha$ we can assume $B^\alpha = 0$. Applying D to $\theta^{\dot\alpha}$, we see $C^{\dot\alpha} = 0$. Hence $D = A^a\partial_a$, and applying this to x_r^b, we get $A^a(\delta_a^b - i\partial_a H^b) = 0$, and since $\mathbf{1} - (\partial H/\partial x)$ is invertible, it follows that $A^a = 0$.

Next we note that

$$\begin{pmatrix} \partial_a \\ \Delta_\alpha \\ \Delta_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_4 & \mathbf{0}_{4,2} & \mathbf{0}_{4,2} \\ X_\alpha^a & \mathbf{1}_2 & \mathbf{0}_2 \\ X_{\dot\alpha}^a & \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \partial_a \\ \partial_\alpha \\ \partial_{\dot\alpha} \end{pmatrix},$$

from which we get by definition of the Berezinian that the two elements of the Berezinian determined by the two basis differ by the Berezinian of the change of base matrix. But note that, written as a block matrix, it has the form $\begin{pmatrix} \mathbf{1}_4 & \mathbf{0}_4 \\ \mathbf{X} & \mathbf{1}_4 \end{pmatrix}$, from which it follows that this matrix has Berezinian equal to unity. Hence,

$$D(\partial_a, \partial_\alpha, \partial_{\dot\alpha}) = D(\partial_a, \Delta_\alpha, \Delta_{\dot\alpha}), \tag{73}$$

completing the proof. □

From this we get

Corollary 7.4 *The forms*

$$\omega^a := dx^a - X_\alpha^a d\theta^\alpha - X_{\dot\alpha}^a d\theta^{\dot\alpha}$$

form a basis for the sheaf $\Omega_0^1 M$.

Proof: By our exact sequence (10), it will suffice to show that the inner product of the elements Δ_α and $\Delta_{\dot\alpha}$ with ω^a vanish. But

$$\Delta_\alpha \lrcorner \omega^a = (\partial_\alpha + X_\alpha^a\partial_a)\lrcorner(dx^a - X_\alpha^a d\theta^\alpha - X_{\dot\alpha}^a d\theta^{\dot\alpha}) = \partial_\alpha\lrcorner(-X_\alpha^a d\theta^\alpha) + X_\alpha^a\partial_a\lrcorner dx^a = 0,$$

and similarly for $\Delta_{\dot\alpha}\lrcorner\omega^a$. □

Finally, we introduce the elements $\mathscr{D}_a$ of $\mathscr{T}_0 M$, which are dual to the ω^a of $\Omega_0^1 M$, i.e.,

$$\mathscr{D}_a \in \mathscr{T}_0 M : \quad (\mathscr{D}_a, \omega^b) = \delta_a^b. \tag{74}$$

The Frobenius form We now calculate the Frobenius isomorphism in the local coordinates, with the goal of getting an expression in terms of the H^a.

$$\begin{array}{rcl} \phi : \mathscr{T}_l M \otimes \mathscr{T}_r M & \longrightarrow & \mathscr{T}_0 M \\ \Delta_\alpha \otimes \Delta_{\dot\beta} & \mapsto & \phi^a_{\alpha\dot\beta}\mathscr{D}_a, \end{array} \tag{75}$$

while by the definition of ϕ, we have $\phi(\Delta_\alpha \otimes \Delta_{\dot\beta}) = [\Delta_\alpha, \Delta_{\dot\beta}]\mathrm{mod}(\mathscr{T}_l M \oplus \mathscr{T}_r M)$. Combining these facts with description of H^a, we get

Lemma 7.5 $\phi^a_{\alpha\dot\beta} = -i\left\{\Delta_\alpha, \Delta_{\dot\beta}\right\}H^a$, *where* $\{\ ,\ \}$ *denotes the supercommutator.*

Proof: From the definitions

$$\left(\phi(\Delta_\alpha \otimes \Delta_{\dot\beta}), \omega^b\right) = \left[\Delta_\alpha, \Delta_{\dot\beta}\right] \lrcorner \omega^b = \phi^a_{\alpha\dot\beta}\mathscr{D}_a \lrcorner \omega^b = \phi^b_{\alpha\dot\beta},$$

which implies

$$\phi^b_{\alpha\dot\beta} = \left[\Delta_\alpha, \Delta_{\dot\beta}\right] \lrcorner \omega^a = \left[\Delta_\alpha, \Delta_{\dot\beta}\right] \lrcorner \left(dx^a - d\theta^\gamma X^a_\gamma - d\theta^{\dot\delta} X^a_{\dot\delta}\right). \tag{76}$$

Since the ∂_a form a basis of $\mathscr{T}_0 M$, we can also write

$$\left[\Delta_\alpha, \Delta_{\dot\beta}\right] = \nu^a_{\alpha\dot\beta}\partial_a,$$

and comparison with (76), we see that $\nu^a_{\alpha\dot\beta} = \phi^a_{\alpha\dot\beta}$. To turn this into an expression in terms of H^a, recall that $x^a_r = x^a - iH^a$, and that Δ_α annihilates it, and similar statements for x^a_l. Hence,

$$\Delta_\alpha x^b = i\Delta_\alpha H^b \Rightarrow \Delta_{\dot\beta}\Delta_\alpha x^b = i\Delta_{\dot\beta}\Delta_\alpha H^b, \quad \Delta_{\dot\beta} x^b = -i\Delta_{\dot\beta} H^b \Rightarrow \Delta_\alpha\Delta_{\dot\beta} x^b = -i\Delta_\alpha\Delta_{\dot\beta} H^b.$$

Combining these, we get

$$\phi^a_{\alpha\dot\beta}\partial_a x^b = \left[\Delta_\alpha, \Delta_{\dot\beta}\right] x^b = -i\left[\Delta_\alpha, \Delta_{\dot\beta}\right] H^a \quad \Rightarrow \quad \phi^a_{\alpha\dot\beta} = -i\left\{\Delta_\alpha, \Delta_{\dot\beta}\right\} H^a,$$

where $\{\ ,\ \}$ denotes the supercommutator, since both Δ_α and $\Delta_{\dot\beta}$ are odd. $\Box$

Two exact sequences

Lemma 7.6 *There are exact sequences*

$$0 \longrightarrow \Omega^1_0 M \longrightarrow \pi^*_{l,r}(\Omega^1 M_{l,r}) \longrightarrow \Omega^1_{l,r} M \longrightarrow 0;$$

the forms (dx^a_l) *and* (dx^a_r) *form bases of* $\Omega^1_0 M$ *in* $\pi^*_{l,r}(\Omega^1 M_{l,r})$, *and can be expressed in terms of the forms* ω^a *as*

$$dx^a_l = (\delta^a_b + i\partial_b H^a)\omega^b, \qquad dx^a_r = (\delta^a_b - i\partial_b H^a)\omega^b.$$

The dual sequences are

$$0 \longrightarrow \mathscr{T}_{l,r}M \longrightarrow \pi_{l,r}^*(\mathscr{T}M_{l,r}) \longrightarrow \mathscr{T}_0 M \longrightarrow 0;$$

if one defines the vector fields

$$\widetilde{\mathscr{D}}_{l,a} = (\delta_a^b + i\partial_a H^b)\pi_l^*\left(\frac{\partial}{\partial x_l^b}\right), \qquad \widetilde{\mathscr{D}}_{r,a} = (\delta_a^b - i\partial_a H^b)\pi_r^*\left(\frac{\partial}{\partial x_r^b}\right),$$

then the images of both $\widetilde{\mathscr{D}}_{l,a}$ *and* $\widetilde{\mathscr{D}}_{r,a}$ *coincide with the elements* $\mathscr{D}_a$ *of (74).*

Proof: From the definition of Δ_α, one sees that $\Delta_\alpha \lrcorner d\theta^\beta = \delta_\alpha^\beta$. Since $\pi_l^*(\Omega^1 M)$ is freely generated by dx_l^a, $d\theta^\alpha$, it follows that $\pi_l^*(\Omega^1 M_l) \longrightarrow \Omega^1 M$ is surjective with kernel generated by the dx_l^a, verifying the exactness of the first two sequences.

Next note that the expression $dx_l^a - (\delta_b^a + i\partial_b H^a)\omega^b$ is contained in $\Omega_0^1 M$, as both dx_l^a and ω^a are. But

$$\begin{aligned} dx_l^a - (\delta_b^a + i\partial_b H^a)\omega^b = dx_l^a - (\delta_b^a + i\partial_b H^a)(dx^b - X_\alpha^b d\theta^\alpha - X_{\dot\alpha}^b d\theta^{\dot\alpha}) = \\ dx_l^a - (\delta_b^a + i\partial_b H^a)(d(x_l^b + iH^b) - X_\alpha^b d\theta^\alpha - X_{\dot\alpha}^b d\theta^{\dot\alpha}) = \\ -(X_\alpha^b + i\partial_b H^a X_\alpha^b)d\theta^\alpha - (X_{\dot\alpha}^b + i\partial_b H^a X_{\dot\alpha}^b)d\theta^{\dot\alpha}, \end{aligned}$$

which is a linear combination of $d\theta^\alpha$ and $d\theta^{\dot\alpha}$. Since it is also in $\Omega_0^1 M$, it vanishes, yielding the expressions for the dx_l^a; those for the dx_r^a are derived in the same way.

That the second set of sequences is exact follows from the exactness of the first. The image of the $\widetilde{\mathscr{D}}_{l,r,a}$ is $\mathscr{D}_a$, which follows from the relation

$$\widetilde{\mathscr{D}}_{l,a} \lrcorner \omega^b = \delta_a^b = \widetilde{\mathscr{D}}_{r,a} \lrcorner \omega^b,$$

which follows in turn from the previous expression for the $dx_{l,r}^a$ in terms of the ω^b:

$$\widetilde{\mathscr{D}}_{l,a} \lrcorner \omega^b = (\delta_a^b + i\partial_a H^b)\pi_l^*\left(\frac{\partial}{\partial x_l^b}\right) \lrcorner (\delta_a^b + i\partial_a H^b)^{-1} dx_l^a = \delta_a^b,$$

and similarly for $\widetilde{\mathscr{D}}_{r,a}$. □

We now have the following basis of $\pi_{l,r}^*(\mathscr{T}M_{l,r})$:

$$\begin{pmatrix} \widetilde{\mathscr{D}}_{l,a} \\ \Delta_\alpha \end{pmatrix} = \left(\begin{array}{c|c} l_a^b & 0 \\ \hline X_\alpha^b & \delta_\alpha^\beta \end{array}\right) \begin{pmatrix} \pi_l^*(\partial/\partial x_l^b) \\ \pi_l^*(\partial/\partial \theta_l^\beta) \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{\mathscr{D}}_{r,a} \\ \Delta_{\dot\alpha} \end{pmatrix} = \left(\begin{array}{c|c} r_a^b & 0 \\ \hline X_{\dot\alpha}^b & \delta_{\dot\alpha}^{\dot\beta} \end{array}\right) \begin{pmatrix} \pi_r^*(\partial/\partial x_r^b) \\ \pi_r^*(\partial/\partial \theta_r^{\dot\beta}) \end{pmatrix},$$

where $l_a^b = \delta_a^b + i\partial_a H^b$ and $r_a^b = \delta_a^b - i\partial_a H^b$. For the elements in $\mathrm{Ber}(\pi_{l,r}^*(\mathscr{T}M_{l,r}))$ we have

$$D(\widetilde{\mathscr{D}}_{l,a}, \Delta_\alpha) = \det(l_a^b)\pi_l^* D\left(\frac{\partial}{\partial x_l^a}, \frac{\partial}{\partial \theta^\alpha}\right), \qquad D(\widetilde{\mathscr{D}}_{r,a}, \Delta_{\dot\alpha}) = \det(r_a^b)\pi_r^* D\left(\frac{\partial}{\partial x_r^a}, \frac{\partial}{\partial \theta^{\dot\alpha}}\right). \quad (77)$$

Note also that

$$\det(l_a^b)\det(r_a^b) = \det\left(1 + \left(\frac{\partial H}{\partial x}\right)^2\right). \quad (78)$$

The superspinor metrics The superspinor metrics $\varepsilon_{l,r}$ are defined in (67), which utilizes the isomorphism of Proposition 7.2. We can express this isomorphism in local coordinates. By the Frobenius isomorphism (75), we have $\mathscr{D}_a = (\phi^{-1})_a^{\alpha\dot\beta}\Delta_\alpha \otimes \Delta_{\dot\beta}$, and by Lemma 7.6, the images of $\widetilde{\mathscr{D}}_{l,a}$ (respectively of $\widetilde{\mathscr{D}}_{r,a}$) are the $\mathscr{D}_a$; hence we can write

$$D(\widetilde{\mathscr{D}}_{l,a}, \Delta_\alpha) = D\left((\phi^{-1})_a^{\alpha\dot\beta}\Delta_\alpha \otimes \Delta_{\dot\beta}, \Delta_\alpha\right) = (\det\, \phi)^{-1} D(\Delta_\alpha)^{-1} D(\Delta_{\dot\beta})^{-2}, \qquad (79)$$

in which we have utilized the dual of the first equation of (65), which is

$$\pi_l^* \mathrm{Ber} \mathscr{T} M_l = (\mathrm{Ber} \mathscr{T}_l M)^{-1} \otimes (\mathrm{Ber} \mathscr{T}_r M)^{-2}.$$

similarly, we get

$$D(\widetilde{\mathscr{D}}_{r,a}, \Delta_{\dot\alpha}) = D\left((\phi^{-1})_a^{\beta\dot\alpha}\Delta_{\dot\alpha} \otimes \Delta_{\beta}, \Delta_{\dot\alpha}\right) = (\det\, \phi)^{-1} D(\Delta_{\dot\alpha})^{-2} D(\Delta_{\beta})^{-1}. \qquad (80)$$

We now insert (79) and (80) into (77), giving

$$\begin{aligned} \pi_l^* D\left(\frac{\partial}{\partial x_l^a}, \frac{\partial}{\partial \theta_l^\alpha}\right) &= (\det(l_a^b))^{-1} (\det\, \phi)^{-1} D(\Delta_\alpha)^{-1} D(\Delta_{\dot\beta})^{-2}, \\ \pi_r^* D\left(\frac{\partial}{\partial x_r^a}, \frac{\partial}{\partial \theta_l^{\dot\alpha}}\right) &= (\det(r_a^b))^{-1} (\det\, \phi)^{-1} D(\Delta_{\dot\alpha})^{-1} D(\Delta_{\beta})^{-2}. \end{aligned} \qquad (81)$$

We can now describe our spinor metrics in these local coordinates:

$$\begin{aligned} \varepsilon_l &= (\pi_l^* v_l)^{1/3} \otimes (\pi_r^* v_r)^{-2/3} && \text{(by (67))} \\ \text{(by (71))} \quad &= \Phi_l \Phi_r^{-2} \pi_l^* D^*(d\theta_l^\alpha, dx_l^a)^{1/3} \pi_r^* D^*(d\theta^{\dot\alpha}, dx_r^a)^{-2/3} \\ \text{(by (81) and the duality of the basis)} \quad &= \Phi_l \Phi_r^{-2} (\det\, l_a^b)^{1/3} (\det\, r_a^b)^{-2/3} (\det\, \phi)^{-1/3} D(\Delta_\alpha)^{-1}, \end{aligned} \qquad (82)$$

and similarly for ε_r:

$$\varepsilon_r = \Phi_l^{-2} \Phi_r (\det\, l_a^b)^{-2/3} (\det\, r_a^b)^{1/3} (\det\, \phi)^{-1/3} D(\Delta_{\dot\alpha})^{-1}. \qquad (83)$$

As a final additional frame, note that upon setting

$$\widetilde{\Delta}_\alpha := F\Delta_\alpha, \qquad \widetilde{\Delta}_{\dot\alpha} = F^\varrho \Delta_{\dot\alpha},$$

that for the corresponding elements of the Berezinians, we have

$$D(\widetilde{\Delta}_\alpha) = F^{-2} \cdot D(\Delta_\alpha), \qquad D(\widetilde{\Delta}_{\dot\alpha}) = (F^\varrho)^{-2} \cdot D(\Delta_{\dot\alpha}),$$

(think of the right hand side as a section of $\mathscr{O}_M \otimes \mathscr{T}_l M$, hence using the formula (13), since the rank of $\mathscr{T}_l$ is $0|2$, F has to be taken to the power -2). Consequently,

$$D(\widetilde{\Delta}_\alpha) = \varepsilon_l^{-1} \Rightarrow F = \Phi_l^{1/2} \Phi_r^{-1} (\det\, l_a^b)^{1/6} (\det\, r_a^b)^{-1/3} (\det\, \phi)^{-1/6},$$

with a similar statement for F^ϱ.

Definition 7.7 A *structure frame* for a superspace M is a local basis of vector fields on M

$$\left(\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot{\alpha}}, \frac{i}{2}\left[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot{\alpha}}\right]\right)$$

such that

$$D(\widetilde{\Delta}_\alpha) = \varepsilon_l^{-1}, \qquad D(\widetilde{\Delta}_{\dot{\alpha}}) = \varepsilon_r^{-1}.$$

7.2.3 The Lagrangian

By the definition of the Lagrangian (66), the definition just made and the formulas (82) and (83), we have the expression for the Lagrangian

$$\begin{array}{rcl} \mathscr{L} & = & (\pi_l^*(v_l) \otimes \pi_r^*(v_r))^{1/3} = \varepsilon_l^{-1} \otimes \varepsilon_r^{-1} \\ & = & D(\widetilde{\Delta}_\alpha) D(\widetilde{\Delta}_{\dot{\alpha}}) \\ & = & \Phi_l \Phi_r (\det\, \phi)^{2/3} \det(l_a^b)^{1/3} \det(r_a^b)^{1/3} D(\Delta_\alpha) D(\Delta_{\dot{\beta}}) \\ \text{(by (78))} & = & \Phi_l \Phi_r (\det\, \phi)^{2/3} \det\left(1 + \left(\frac{\partial H}{\partial x}\right)\right)^{1/3} D(\Delta_\alpha) D(\Delta_{\dot{\beta}}). \end{array} \quad (84)$$

Finally, we now need to express $D(\Delta_\alpha)D(\Delta_{\dot{\alpha}})$ in terms of the element $D^*(d\theta^\alpha, d\theta^{\dot{\alpha}}, dx^a)$ in order to be able to use dirctly the fomula for the Berezinian integral defining the action to write the Euler-Lagrange equations. Note that by the Frobenius isomorphism we have $\partial_a = (\phi^{-1})_a^{\alpha\dot{\alpha}}(\Delta_\alpha \otimes \Delta_{\dot{\alpha}})$ (for this observe that in the exact sequence (10), ∂_a maps to $\mathscr{D}_a$, since $\partial_a \lrcorner \omega^b = \delta_a^b$). This results in

$$\begin{array}{rcl} D(\partial_a, \partial_\alpha, \partial_{\dot{\alpha}}) & = & D(\partial_a, \Delta_\alpha, \Delta_{\dot{\alpha}}) \quad \text{(by Lemma 7.3)} \\ \text{(by the remark just made)} & = & D((\phi^{-1})_a^{\alpha\dot{\alpha}}(\Delta_\alpha \otimes \Delta_{\dot{\alpha}}), \Delta_\alpha, \Delta_{\dot{\alpha}}) \\ (D(\Delta_\alpha \otimes \Delta_{\dot{\alpha}}) = D(\Delta_\alpha)^{-2} D(\Delta_{\dot{\alpha}})^{-2}) & = & (\det\, \phi)^{-1} D(\Delta_\alpha)^{-1} D(\Delta_{\dot{\alpha}})^{-1} \\ & \Rightarrow & \\ D(\Delta_\alpha) D(\Delta_\alpha) & = & (\det\, \phi)^{-1} D^*(d\theta^\alpha, d\theta^{\dot{\alpha}}, dx^a). \end{array}$$

This gives us our finished formula for the Lagrangian

$$\mathscr{L} = \Phi_l \Phi_r (\det\, \phi)^{-1/3} \det\left(1 + \left(\frac{\partial H}{\partial x}\right)^2\right)^{1/3} D^*(d\theta^\alpha, d\theta^{\dot{\alpha}}, dx^a). \quad (85)$$

Let E_B^A denote the matrix changing the basis from the structure frame of Definition 7.7 to the holomorhic frame $(\partial_a, \partial_\alpha, \partial_{\dot{\alpha}})$. Then by definition $D(\partial_a, \partial_\alpha, \partial_{\dot{\alpha}}) = \mathrm{Ber}(E_B^A) D(\frac{i}{2}[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot{\alpha}}], \widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot{\alpha}})$. Furthermore, from the exact sequence (10) and formula (14) together with (12) we have

$$D(\frac{i}{2}[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot{\alpha}}], \widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot{\alpha}}) = D(\frac{i}{2}[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot{\alpha}}]) D(\widetilde{\Delta}_\alpha) D(\widetilde{\Delta}_{\dot{\alpha}}),$$

and writing the equation (85) as $\mathscr{L} = L \cdot D^*(d\theta^\alpha, d\theta^{\dot\alpha}, dx^a)$, we get

$$\mathscr{L} = L \cdot D^*(d\theta^\alpha, d\theta^{\dot\alpha}, dx^a) = L \cdot \mathrm{Ber}(E_B^A)^{-1} \left(D(\frac{i}{2}[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot\alpha}]) D(\widetilde{\Delta}_\alpha) D(\widetilde{\Delta}_{\dot\alpha}) \right)^{-1}. \qquad (86)$$

Next, recall that under the Frobenius isomorphism $[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot\alpha}] = \phi(\widetilde{\Delta}_\alpha \otimes \widetilde{\Delta}_{\dot\alpha})$ hence

$$D\left(\frac{i}{2}[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot\alpha}]\right) = \frac{1}{16} D(\phi(\widetilde{\Delta}_\alpha \otimes \widetilde{\Delta}_{\dot\alpha})) = \frac{1}{16} D(\widetilde{\Delta}_\alpha)^{-2} D(\widetilde{\Delta}_{\dot\alpha})^{-2} \cdot \det(\phi).$$

But note that here, with respect to the basis used here (as opposed with that used in (75) and what follows that) the matrix for ϕ is just the identity, and hence $\det(\phi) = 1$. Consequently

$$D\left(\frac{i}{2}[\widetilde{\Delta}_\alpha, \widetilde{\Delta}_{\dot\alpha}]\right) D(\widetilde{\Delta}_\alpha) D(\widetilde{\Delta}_{\dot\alpha}) = \frac{1}{16} D(\widetilde{\Delta}_\alpha)^{-1} D(\widetilde{\Delta}_{\dot\alpha})^{-1}.$$

Inserting this in (86) and combining with (84), we get

$$L \cdot 16 \cdot \mathrm{Ber}(E_B^A)^{-1} = 1,$$

from which the *Wess-Zumino* formula follows:

$$L = \frac{1}{16} (\mathrm{Ber}(E_B^A)).$$

The difference of factors 1/16 here or 1/8 in [6] is irrelevant, not changing the equations of motion. This formula was "checked" by Wess and Zumino in [9], that is, they derived the Lagrange equations which follows from this Lagrange density and found they were indeed the (known) equations of motion for supergravity. Written in terms of component fields in the Wess-Zumino gauge of (70), this is the Lagrangian (44), and how the Euler-Lagrange equations can be derived from this is explained in [2], section 6.1 as well as in [9]. The equations of motion are the usual Einstein equation for the metric, plus the Rarita-Schwinger equations (cf. [2], (1.8.43))

$$\varepsilon^{abcd} (\widetilde{\sigma}_b)^{\dot\beta\beta} \psi_{cd\beta} = 0, \quad \dot\beta = \dot 0, \dot 1, \quad a = 1, \ldots, 4$$

which amounts to the fact that $\psi_{\alpha\beta\gamma}$ and $\psi_{\dot\alpha\dot\beta\dot\gamma}$ are massless fields with helicity $\pm\frac{3}{2}$.

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