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The principles and concepts of geometric analysis

by

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The present article is based on a talk that I gave at the Chinese Academy of Sciences in Beijing on Oct.9, 2001, on the occasion of the inauguration of the Partner Group on Geometric Analysis for the Max Planck Institute for Mathematics in the Sciences in Leipzig at the CAS. It aims at providing a perspective on the present state and future research lines of geometric analysis through a survey of the basic principles and concepts. These are based on ideas from both Riemannian geometry and quantum field theory, and in many cases, they lead to variational problems for minimizing a physical action or optimizing a geometric quantity. Particular emphasis is placed on previous work conducted by the author and his partners. We do not provide a complete bibliography for which the reader is referred to the author' s book [6] on "Riemannian Geometry and Geometric Analysis". There, also all notions and concepts not fully defined in the present article can be found.

We consider a Riemannian manifold, for the sake of simplicity of our discussion assumed to be compact, with its metric g, written in local coordinates as $g_{ij}dx^i dx^j$. As this expression gets transformed under changes of local coordinates $y(x)$ via

$$g_{ij}\frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^l}dy^k dy^l, \tag{1}$$

the basic idea of Riemann was to seek invariants, i.e. expressions that can be computed from the local coordinate representations of the metric, but that remain invariant under coordinate changes. Riemann found that there exist no such invariants involving only first derivatives of the metric as all such first derivatives can be made 0 at a given point by an appropriate choice of coordinates, the so-called Riemann normal coordinates. However, there do exist second order invariants given by the Riemann curvature tensor and the scalar expressions formed from it, most notably the sectional, Ricci, and scalar curvatures. Suitable curvature expressions, when integrated over the manifold M, yield topological invariants, i.e. invariants that do not even depend on the particular chosen metric g, but only on the topological type or the differentiable structure of M, the so-called characteristic classes. The fundamental task of

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global Riemannian geometry then is to explore the relation between such locally computed expressions and the topological structure of the underlying manifold. The most important question in this regard is how sign conditions or even more stringent quantitative restrictions on curvature expressions that are valid across the whole manifold constrain its topological type.

1 Functions

Geometric analysis offers tools for global geometry through the investigation of analytic objects on the manifold. The simplest such objects are functions

$$f : M \to \mathbb{R}. \tag{2}$$

1.1 Critical points

The first attempt in this direction takes a more or less arbitrary such function f and studies its critical points. This is the idea of Morse theory and its generalizations. Here, "more or less arbitrary" means "generic", or more precisely that the function has only isolated critical points and that at all such critical points, its Hessian, i.e. the matrix of its second derivatives, does not have any vanishing eigenvalue. This is indeed a generic property in the sense that it can always be achieved by an arbitrary small perturbation of a given function, and, in fact, the term "generic" signifies a technical refinement of the requirement that some property be open and dense in a space of objects like smooth functions.
As we pointed out that there are no first order invariants for a Riemannian manifold, we also do not expect that the critical points themselves carry any significant information, e.g. through their position or number. Rather, we again need second order invariants, and in the present case, they are given by the index of the Hessian at each critical point p, or, equivalently, by the number $i(p)$ of negative eigenvalues of this Hessian (this does not depend on the choice of the metric g). In particular, the basic formula of Morse theory says that

$$\sum_p (-1)^{i(p)} = \chi(M), \tag{3}$$

the Euler characteristic of the manifold M, where the sum extends over all critical points of the function f. In fact, one can even recover the entire homology of the manifold M from f by studying the latter's gradient flow, as discovered by Floer [3]. For that purpose, a Riemannian metric is needed, in order to define the gradient of f. Again, however, the resulting structural information does not depend on the particular choice of that metric, as long as it is chosen generically; here this means that the stable and unstable manifolds of the individual critical points for that gradient flow intersect transversally. The gradient flow is the dynamical system defined by

$$\dot{x}(t) = -\mathrm{grad} f(x(t)) \tag{4}$$

for curves

$$x : \mathbb{R} \to M. \tag{5}$$

For each such gradient flow line $x(t)$, there exist the limits $x(-\infty)$ and $x(\infty)$, and these limits are critical points of f. In physical terminology, such gradient flow lines thus are instantons or tunneling paths between the critical points or ground states. The homology groups of M (with $\mathbb{Z}_2$ coefficients[1]) can then be simply obtained by considering the vector spaces C_j over $\mathbb{Z}_2$ generated by the critical points of index j together with the boundary operator

$$\partial p = \sum_q n(p,q) q \tag{6}$$

where the sum extends over all critical points q with index $i(q) = i(p) - 1$ and the coefficient $n(p,q)$ counts the number of flow lines from p to q mod 2, i.e. is 0 if the number of such flow lines is even and 1 if it is odd (under our generic conditions, this number is always finite).

1.2 Optimization

Another approach is to not take an essentially arbitrary function, but rather one that is optimized according to some geometrical condition and study its properties. Such optimization schemes are given by variational problems, and the simplest such variational problem is the minimization of the Dirichlet integral

$$\int_M |df(x)|^2 dvol(x). \tag{7}$$

Minimizers are harmonic functions,

$$\Delta f = 0 \tag{8}$$

where Δ denotes the Laplace-Beltrami operator of the metric g, but this variational problem is too simple as the only solutions (on a compact M) are the constants.

[1] Homology groups with integer coefficients can also be obtained from the gradient flow of such a Morse function f when one defines orientations with the help of which he can then assign signs to the flow lines. For $\mathbb{Z}_2$ coefficients, we have the parity of the numbers $n(p,q)$ defined after (6) and therefore do not need signs

1.3 Eigenvalues

More generally, one may study the eigenvalue spectrum of Δ, i.e. those real numbers λ for which there exists a nontrivial function f with

$$\Delta f = \lambda f. \tag{9}$$

Such functions are critical points of the variational integral

$$\int |df|^2 \; - \; \lambda \int f^2 \tag{10}$$

Since the Laplace-Beltrami operator is positive, all eigenvalues are nonnegative. Of course, 0 is an eigenvalue with eigenfunction given by any constant function on M. The rest of the spectrum encodes geometric properties of M. Here, we only give a glimpse and compute for an eigenfunction f, i.e. a solution of (9)

$$-\Delta|df|^2 = \langle D_i d_j f, D_i d_j f\rangle + \langle d_j(-\lambda f), d_j f\rangle + R_{ij}\langle d_i f, d_j f\rangle \tag{11}$$

where R_{ij} is the Ricci tensor of M. Integrating this formula over M leads to an eigenvalue estimate, because the integral over the left hand side of (11) vanishes and the first term on the right hand side is nonnegative, namely

$$\int R_{ij}\langle d_i f, d_j f\rangle \leq \lambda \int |df|^2 \tag{12}$$

and so, a lower bound on the Ricci curvature of M leads to a lower bound for the first nonzero eigenvalue of Δ. In fact, the estimate can be somewhat refined (so that it becomes optimal for the case where M is a sphere with its standard metric) by using

$$\langle D_i d_j f, D_i d_j f\rangle \geq \frac{1}{n}|\Delta f|^2 \tag{13}$$

with n the dimension of M (a consequence of Schwarz' inequality), and for a solution of (9),

$$\int |\Delta f|^2 = \lambda \int \langle f, \Delta f\rangle = \lambda \int |df|^2, \tag{14}$$

to obtain altogether

$$\frac{n}{n-1} \int R_{ij}\langle d_i f, d_j f\rangle \leq \lambda \int |df|^2, \tag{15}$$

i.e.

$$\lambda \geq \frac{n}{n-1}\rho \tag{16}$$

when $\rho > 0$ is a lower bound for the Ricci curvature of M.

2 Generalizations

2.1 Differential forms

Instead of functions, one may consider differential forms. The simplest such forms are 1-forms. If a 1-form ω is closed, $d\omega = 0$, it is locally given by the differential of a function, $\omega = df$, but the function f in general is not defined globally. f is only defined up to the addition of a constant, i.e. it is a multi-valued function. ω is harmonic, i.e. satisfies not only

$$d\omega = 0, \tag{17}$$

but also

$$d^\star \omega = 0 \tag{18}$$

where $d^\star$ is the L^2 adjoint of the exterior derivative d with respect to the metric g, precisely if f is harmonic,

$$d^\star df = 0. \tag{19}$$

Performing the same computation as before for a harmonic ω, we obtain

$$-\Delta|\omega|^2 = -\Delta|df|^2 = \langle D_i\omega_j, D_i\omega_j\rangle + R_{ij}\langle\omega_i,\omega_j\rangle \tag{20}$$

with $\omega_i = d_i f$ and integrating as before, we conclude

$$\int R_{ij}\langle\omega_i,\omega_j\rangle \leq 0. \tag{21}$$

This implies that for a manifold M of positive Ricci curvature, all harmonic 1-forms vanish, and as these represent the first cohomology of M, the first cohomology group then is 0. This is the approach of Bochner.

More generally, by the theorem of Hodge, the de Rham cohomology of M is given by the harmonic differential forms on M, and so, in principle, one can also carry out the preceding analysis for harmonic forms of higher degree. Here, however, the curvature expressions appearing in the formula analogous to (20) become more involved.

2.2 Morse theory and de Rham cohomology

Naturally, there exist relations between the concepts discussed so far. Witten [12] discovered a basic relationship between Morse theory and de Rham cohomology. Given a Morse function f as above, he introduced operators obtained by twisting the exterior derivative d and its adjoint $d^\star$, namely

$$d_s = e^{-sf} d e^{sf} \tag{22}$$

and its adjoint

$$d_s^\star = e^{sf} d^\star e^{-sf} \tag{23}$$

and the corresponding Laplacian

$$\Delta_s = d_s^\star d_s + d_s d_s^\star \tag{24}$$

for a positive parameter s. For $s = 0$, of course, this is the standard Laplacian acting on differential forms. Witten found that for $s \to \infty$, the j-forms ω solving

$$\Delta_s \omega = 0 \tag{25}$$

correspond precisely to the critical points of f of index j. The Floer curves, i.e. the gradient flow lines, are minima of

$$\int (|\dot{x}(t)|^2 + s^2 |\frac{df}{dx}(x(t))|^2) = \int (|\dot{x}(t) + s(\mathrm{grad} f)(x(t))|^2) - \int (f(x(t)))^{\dot{}} \tag{26}$$

This formula allows us to make some observations which will become important in the sequel. Namely, the last term on the right hand side depends only on the limits $x(-\infty)$ and $x(\infty)$ of the flow line, and if we keep those limits fixed, it becomes a constant term. Therefore, under that constraint, absolute minimizers are characterized by the first order equation (4) instead of the second order equation that solutions of variational problems involving first derivatives usually satisfy. Of course, the first order equation implies the second one, but a first order equation typically leads to much stronger properties for its solutions than a second order one. This is the mechanism of self-duality. Secondly, there is a parameter s involved here. In the present case, it can be accounted for by a simple rescaling of the curves $x(t)$), but in some examples to be discussed below, it interpolates in a nontrivial manner between the limiting or asymptotic problems for $s = 0$ and $s = \infty$.

2.3 Novikov theory

Novikov developed a critical point theory for closed 1-forms. Since, as explained above, closed 1-forms are given by multi-valued functions, this constitutes a generalization of the critical point theory for functions as discussed above. Here, if we consider the gradient flow, while there need not exist a maximum or a minimum anymore as for a single-valued function on a compact manifold, in addition to flow lines between critical points, there may also exist closed flow lines. Such flow lines also contribute to the homology. Fan-Jost [4] used the general framework of Conley theory to describe how this flows encodes the homology of the underlying manifold.

3 Sections of vector bundles

Another generalization of a function on a manifold M is given by a section of a vector bundle W over M. The simplest such bundles are line bundles, i.e. vector bundles with fiber $\mathbb{R}$ or $\mathbb{C}$. In that case, again a section is locally represented by a function, and in fact, we have a global function in the special case where the bundle is the trivial bundle $M \times \mathbb{R}$. In general, however, the bundle W is not a product, but twisted. We assume that the bundle carries a metric, and we denote the associated norm by $|.|$. In theoretical physics, W represents a matter particle. The semi-classical states of the particle are determined by field equations coming from a Lagrangian. In the simplest case, we have the Lagrangian

$$\int (|d\phi|^2 - m^2|\phi|^2) \tag{27}$$

and the critical points solve the field equation

$$(\Delta + m^2)\phi = 0. \tag{28}$$

Of course, $\phi = 0$ is a solution, but this Lagrangian is not bounded from below. In order to rectify this, one introduces a higher order correction term. The simplest such term is of fourth order. So, one considers the potential

$$V(\phi) = -m^2|\phi|^2 + \frac{1}{2}|\phi|^4 \tag{29}$$

and the corresponding Lagrangian

$$\int (|d\phi|^2 + V(\phi)). \tag{30}$$

This leads to the confinement

$$|\phi| \leq m \tag{31}$$

for the solutions of the field equations.
If the bundle W is nontrivial, however, there is no canonical derivative d on it. It is therefore natural to also select a covariant derivative or connection A on W by some variational principle. This connection should preserve some structure on W, at least the linear structure, but usually also the metric. A thus has to be an orthogonal or unitary (in the complex case) connection. The corresponding Lie algebra, i.e. the infinitesimal invariance group for the structure under consideration, represents a gauge. A then is locally given by a 1-form with values in that Lie algebra. Physically, such a connection A then represents the state of a gauge field, and its semi-classical state is determined by the corresponding gauge field equations. Again, as the connection is gauged, in order to find a gauge invariant Lagrangian, we need to look at derivatives of the connection, i.e. consider its curvature

$$F_A = D_A A \tag{32}$$

where D_A is the covariant derivative defined by the connection A. As before, the simplest Lagrangian is quadratic,

$$\int |F_A|^2. \tag{33}$$

In the case of a line bundle where the invariance is Abelian, the critical points of this Lagrangian are the solutions of the Maxwell equations. In the general case, we obtain so-called Yang-Mills fields, and the functional (33) is the Yang-Mills functional.

Now, we have neglected the matter field ϕ, however. In order to include it, we now couple the matter and gauge fields in the Lagrangian

$$\int (\gamma_1 |F_A|^2 + \gamma_2(|D_A\phi|^2 + V(\phi)). \tag{34}$$

Of course, only particular choices of the potential V and the coupling parameters $\gamma_1,\ \gamma_2$ lead to physically meaningful and geometrically interesting theories. An example is constituted by the Ginzburg-Landau functional for a section ϕ of a complex line bundle L with a Hermitian metric $\langle .,. \rangle$ over a Riemann surface Σ and a unitary connection A on L,

$$GL(\phi, A) = \int_\Sigma (|F_A|^2 + |D_A\phi|^2 + \frac{1}{4}(1 - |\phi|^2)^2). \tag{35}$$

The Riemann surface structure of Σ allows to split the covariant derivative D_A as

$$D_A = \partial_A + \bar{\partial}_A. \tag{36}$$

The functional can be rewritten as

$$GL(\phi, A) = \int_\Sigma (2|\bar{\partial}_A\phi|^2 + (\star F - \frac{1}{2}(1 - |\phi|^2)^2)) \ + \ 2\pi \deg L \tag{37}$$

where the metric star operator $\star$ converts the 2-form F into a function. Here, we can then see the important self-duality mechanism at work. Namely while the vanishing of all the quadratic terms occurring in (35) leads to a rather trivial and constrained situation (for example, the curvature F_A can only vanish identically if the bundle L is flat), after rewriting it as in (37) and splitting off the topological term $2\pi \deg L$, we can now require that the remaining quadratic terms vanish. Thus ϕ and A then need to solve the first order, so-called self-duality equations

$$\bar{\partial}_A\phi = 0 \tag{38}$$

and

$$\star F = \frac{1}{2}(1 - |\phi|^2) \tag{39}$$

instead of only solving the weaker second order Euler-Lagrange equations of $GL(\phi, A)$. In general, we cannot achieve that ϕ minimizes the potential $(1 - |\phi|^2)^2$ everywhere, i.e. satisfies $|\phi|^2 = 1$ identically, because any section ϕ of a line bundle L over a compact Riemann surface needs to have at least $\deg L$ zeroes. In fact, it was shown by Taubes [10] that for any choice of $\deg L =: d$ points $p_1, ..., p_d$ on Σ, the so-called vortices, there exists a unique solution ϕ, A of (38,39) for which ϕ vanishes precisely at those prescribed points. The self-duality mechanism that allows to rewrite the Ginzburg-Landau functional as in (37) depends on rewriting the mixed term that occurs when we evaluate the last square in the integrand in (37), namely using

$$\int \langle F\phi, \phi\rangle \; = \; \int (|\partial_A \phi|^2 - |\bar{\partial}_A \phi|^2) \tag{40}$$

and playing it off against

$$|D_A\phi|^2 = |\partial_A\phi|^2 + |\bar{\partial}_A\phi|^2. \tag{41}$$

(The topological term arises from $\int F$.) This mechanism remains valid if we consider the more general functional

$$GL_\epsilon(\phi, A) = \int_\Sigma (\epsilon^2 |F_A|^2 + |D_A\phi|^2 + \frac{1}{4\epsilon^2}(1 - |\phi|^2)^2) \tag{42}$$

with a positive parameter ϵ. The corresponding self-duality equations now are

$$\bar{\partial}_A \phi = 0 \tag{43}$$

and6

$$\epsilon^2 \star F = \frac{1}{2}(1 - |\phi|^2). \tag{44}$$

As ϵ tends to 0, the potential term becomes more and more dominant, and so one expects that for a solution of the equations $1 - |\phi|^2$ gets closer and closer to 0. We remember, however, that in general, i.e. if $\deg L$ does not vanish, this expression cannot vanish identically as ϕ needs to have $\deg L$ zeroes, and these can be prescribed at arbitrary points $p_1, ..., p_d$. It was then shown by Hong-Jost-Struwe [5] that as ϵ tends to 0, for a subsequence of the solutions with these prescribed zeroes, $|\phi|$ tends to 1 and $D_A\phi$ and F_A tend to 0 on any compact subset of the complement of these points. Moreover, $\star F_A$ tends to $2\pi \sum_{j=1}^d \delta(p_j)$, i.e. the curvature concentrates at the vortices. This means that we degenerate L into a flat line bundle with d singular points and with a covariantly constant section.
More generally, the self-duality mechanism still works if in place of a constant factor like ϵ^2, we use a function of the coordinate z on Σ. Again, only certain choices lead to rich mathematical structures. We choose here $\frac{\epsilon^2}{|\phi|^2}$. This leads to the Chern-Simons-Higgs functional

$$GL_{\frac{\epsilon}{|\phi|}}(\phi, A) = \int_\Sigma (\frac{\epsilon^2}{|\phi|^2}|F_A|^2 + |D_A\phi|^2 + \frac{|\phi|^2}{4\epsilon^2}(1 - |\phi|^2)^2) \tag{45}$$

introduced by Hong-Kim-Pac and Jackiw-Weinberg, and whose mathematical investigation was started by Caffarelli-Yang [1] and Tarantello [9]. We now have a sixth order potential term in place of a fourth order one, and this potential now has two minimal basins, one at $\phi = 0$, and the other one as before at $|\phi|^2 = 1$. So, if we let ϵ tend to 0 as before, there exist two preferred states, and as a cumulation of research efforts by several people, it was shown by Ding-Jost-Li-Peng-Wang [2], that one obtains two different asymptotic states with rather different properties corresponding to those two basins. While previous work in this direction used a variational method based on refinements of the Moser-Trudinger embedding inequality, the final method was of a nonvariational nature and therefore also extends to higher dimensions although the Lagrangian with such a sixth order potential does not stay renormalizable anymore, or, in mathematical terms, the potential term cannot be controlled anymore by the square of the first derivative of the field ϕ through the Sobolev embedding theorem.

The interesting higher dimensional analogue of the Ginzburg-Landau functional is the Seiberg-Witten functional

$$SW(\phi, A) = \int_M (|D_A\phi|^2 + |F_A^+|^2 + \frac{R}{4}|\phi|^2 + |\phi|^4) \tag{46}$$

where M now is a four dimensional manifold with scalar curvature R, ϕ a section of a spinor bundle for a spin^c structure on M, A a connection on the associated determinant line bundle, F_A^+ the self dual part of its curvature according to the decomposition of the space of two-forms on a four dimensional manifold into the self-dual ($\star\omega = \omega$) and the anti-self-dual ones ($\star\omega = -\omega$). This functional was introduced by Seiberg-Witten [8] as an asymptotic limit of the Yang-Mills functional, and it has a simpler structure than the latter because the connection A now lives on a line bundle and thus the underlying gauge group is Abelian, while it retains the same mathematical power as the Yang-Mills functional. Again, a self-duality mechanism works here, and the functional can be rewritten as

$$SW(\phi, A) = \int_M (|\mathcal{D}_A\phi|^2 + |F_A^+ - \frac{1}{4}\langle e_i e_j \phi, \phi\rangle e^i \wedge e^j|^2) \tag{47}$$

where $\mathcal{D}$ is the Dirac operator and the product inside $\langle\ ,\ \rangle$ denotes Clifford multiplication with the orthonormal tangent vectors e_i dual to the 1-forms e^i, but the details need not concern us here. The important point is that the potential is again of 4th order, and the self-duality mechanism works as before. In particular, we may again introduce the factor ϵ^2 as before, without destroying self-duality, and consider

$$SW_\epsilon(\phi, A) = \int_M (|D_A\phi|^2 + \epsilon^2|F_A^+|^2 + \frac{R}{4}|\phi|^2 + \frac{1}{\epsilon^2}|\phi|^4). \tag{48}$$

The limiting analysis for $\epsilon \to 0$ here was performed by Taubes, and he found that, if M is a symplectic manifold, the limiting zero set of the corresponding sections ϕ and the curvature F_A concentrate on pseudoholomorphic curves. Pseudoholomorphic curves here are generalizations to the symplectic case of the holomorphic curves in a Kähler manifold. Taubes [11] was thus able to relate the Seiberg-Witten invariants coming from the solutions of the self-duality

equations for the Seiberg-Witten functional with the Gromov-Witten invariants of a symplectic manifold that are encoded in pseudoholomorphic curves. The asymptotic analysis was carried in a more general context and with different methods by Lin-Rivière [7].
More generally, as above, instead of the constant ϵ, we may again use a function on M, and as there, $\frac{\epsilon}{|\phi|}$ is a natural choice. We are thus led to consider the functional

$$SW_{\frac{\epsilon}{|\phi|}}(\phi, A) = \int_M (|D_A\phi|^2 + \frac{\epsilon^2}{|\phi|^2}|F_A^+|^2 + \frac{R}{4}|\phi|^2 + \frac{|\phi|^2}{\epsilon^2}|\phi|^4). \tag{49}$$

Again, the potential is now of 6th order, and so, as the underlying manifold now is of dimension 4, it cannot be controlled by the squared derivative anymore through some generalized Sobolev embedding. Physically, the resulting theory is no longer renormalizable. Nevertheless, the method of Ding-Jost-Li-Peng-Wang, which is based on maximum principle arguments, applies as before,if M is assumed to be a Kähler manifold, to show the existence of two different asymptotic limits for $\epsilon \to 0$. The Kähler assumption is needed for converting the problem into a scalar equation. The behavior in the general symplectic case for this functional is open.

4 Mappings

A third generalization of a function on a Riemannian manifold M is given by a mapping f from M into another Riemannian manifold N. Again, among all possible such maps, we may select special ones by a variational principle. In this context, we again have a natural generalization of the Dirichlet integral, namely

$$\int |df|^2 \tag{50}$$

where the norm $|.|$ now also involves the metric of N. More precisely, if the metric of N is locally given by $h_{\alpha\beta}df^\alpha df^\beta$, then we have in local coordinates

$$|df|^2 = g^{ij}h_{\alpha\beta}\frac{\partial f^\alpha}{dx^i}\frac{\partial f^\beta}{dx^j}. \tag{51}$$

The critical points are called harmonic maps as a generalization of the harmonic functions on M. The corresponding Euler-Lagrange equations now become nonlinear, due to the curvature of the target N. Such mappings arise for example from representations

$$\rho : \pi_1(M) \to G \tag{52}$$

of the fundamental group of M into a linear algebraic group G. In that situation, N is a locally symmetric space of noncompact type.
In physics, this variational problem arises as the nonlinear sigma model in which case N is a symmetric space of compact type. A satisfactory analytic theory of

harmonic maps can only be obtained if either the domain M is of low dimension, namely 1 (in which case the critical points are simply geodesics in N) or 2 (where such harmonic maps from S^2 give rise to quantum cohomology, i.e. higher order corrections to the standard cohomology of N) or if the target N has nonpositive sectional curvature. The latter condition is satisfied for symmetric spaces of noncompact type, but not for those of compact type. Thus, harmonic maps can be used to study representations of the fundamental group of M and to construct a nonlinear generalization of the first de Rham cohomology group which arises from harmonic 1-forms as discussed above, i.e. from Abelian representations of the fundamental group.

Th eigenvalue problem encoded in the variational problem (10) also admits a nonlinear generalization to the present context. Namely, one may study the functional

$$\int |df(x)|^2 - \lambda \int d^2(f(x), g(x)) \tag{53}$$

for some given map $g : M \to N$ where $d^2(.,.)$ denotes the squared distance between points in N, as systematically investigated by Jost-Kourouma.

From the physical interpretation of harmonic maps into symmetric spaces of compact type, in particular into spheres, complex projective spaces, or Grassmannians as nonlinear sigma models, i.e. as a nonlinear field theory, it then is natural to couple it with gauge or other fields. A particularly important instance of this is constituted by the supersymmetric nonlinear sigma models.

5 Metrics

So far, we have assumed that the Riemannian metric on M is given and fixed. In order to understand the relationship between geometry and topology, one may also try to vary the Riemannian metric within a given class, for example among all that are compatible with an underlying differentiable structure. Again, it is natural to attempt to select a metric according to a variational principle. The most basic one seems to be the Hilbert-Einstein functional

$$\int R(g) dvol(g), \tag{54}$$

the integral of the scalar curvature for a metric on our manifold. Because in dimensions larger than 2, the functional is not invariant under scaling the metric with a constant factor, one usually fixes the total volume of the metric to be one. Under this constraint, the critical points of this functional are called Einstein metrics, and they have constant Ricci curvature; more precisely, they satisfy

$$R_{ij} = \lambda g_{ij} \tag{55}$$

where the constant λ is given by the scalar curvature R. Although much is known about them, it is at present not clear how general the class of compact manifolds is on which such Einstein metrics can be found.
More general such functionals are proposed in quantum and supergravity; in particular, here the metric as the carrier of gravity is coupled to other fields.

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