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Donaldson and Seiberg-Witten Invariants:
The Witten Conjecture

by

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DONALDSON AND SEIBERG-WITTEN INVARIANTS

THE WITTEN CONJECTURE

GREG NABER

DONALDSON INVARIANTS $\hookrightarrow$ SEIBERG-WITTEN INVARIANTS

$SU(2)$ $\qquad\qquad$ $U(1)$

$$*F_\omega = -F_\omega \qquad\qquad \begin{cases} \not{D}_A \psi = 0 \\ F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \end{cases}$$

$$\mathcal{D}_M(x) = \exp\left(Q_M(x,x)/2\right) \sum_{\alpha \in \Lambda} 2^{m(M)} SW_0(M,\alpha) \exp\left(c_1(L^0(\alpha))(x)\right)$$

- DONALDSON THEORY
- EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN
- SEIBERG-WITTEN INVARIANTS AND THE CONJECTURE
- ADDENDA

1. A SIMPLE EXAMPLE

QUATERNIONIC HOPF BUNDLE :

$$\begin{array}{ccc} Sp(1) & \hookrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array}$$

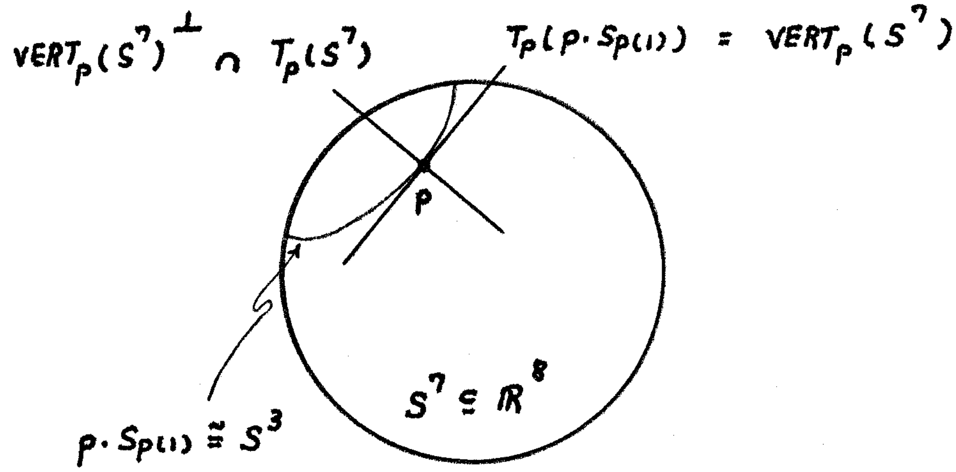
$$Sp(1) = \{ g \in \mathbb{H} : |g| = 1 \} \hookrightarrow S^7 = \{ p = (q^1, q^2) \in \mathbb{H}^2 : |q^1|^2 + |q^2|^2 = 1 \}$$

$$\cong SU(2)$$

$$\cong S^3$$

$$p \cdot g = (q^1, q^2) \cdot g = (q^1 g, q^2 g)$$

$$S^7 / Sp(1) := \mathbb{H}P^1 \cong S^4$$



NATURAL CONNECTION ON $Sp(1) \hookrightarrow S^7 \to S^4$: $\omega \in \Omega^1(S^7, sp(1))$

$$\mathrm{KER}\ \omega_p = \mathrm{VERT}_p(S^7)^{\perp} \cap T_p(S^7)$$

$$= \mathrm{HOR}_p^{\omega}(S^7)$$

$$\omega = \mathrm{IM}\,(\bar{q}^1 dq^1 + \bar{q}^2 dq^2) \qquad (\text{RESTRICTED TO } S^7)$$

CORRESPONDING GAUGE POTENTIAL ON $\mathbb{H} \cong \mathbb{R}^4$:

$$\mathcal{A} = s^* \omega = \operatorname{Im}\left(\frac{\bar{q}}{1+|q|^2}\, dq \right)$$

MORE GENERALLY, $(\lambda, n) \in (0,\infty) \times \mathbb{H}$ GIVES

$$\omega_{\lambda,n} \in \Omega^1(S^7, sp(1))$$

UNIQUELY DETERMINED BY

$$\mathcal{A}_{\lambda,n} = s^* \omega_{\lambda,n} = \operatorname{Im}\left(\frac{\bar{q} - \bar{n}}{\lambda^2 + |q-n|^2}\, dq \right)$$

BPST INSTANTON WITH CENTER n AND SCALE λ $\quad (\omega = \omega_{1,0})$

CURVATURES

$$\Omega_{\lambda,n} = d\omega_{\lambda,n} + \tfrac{1}{2}[\omega_{\lambda,n}, \omega_{\lambda,n}]$$

UNIQUELY DETERMINED BY GAUGE FIELD STRENGTHS

$$\mathcal{F}_{\lambda,n} = s^* \Omega_{\lambda,n} = d\mathcal{A}_{\lambda,n} + \tfrac{1}{2}[\mathcal{A}_{\lambda,n}, \mathcal{A}_{\lambda,n}] = \cdots = \frac{\lambda^2}{(\lambda^2 + |q-n|^2)^2}\, d\bar{q} \wedge dq$$

EACH IS ANTI-SELF-DUAL (ASD)

$$* \mathcal{F}_{\lambda,n} = - \mathcal{F}_{\lambda,n}$$

($*$ = HODGE STAR ON $\mathbb{H} \cong \mathbb{R}^4$ DETERMINED BY THE STANDARD ORIENTATION AND RIEMANNIAN METRIC)

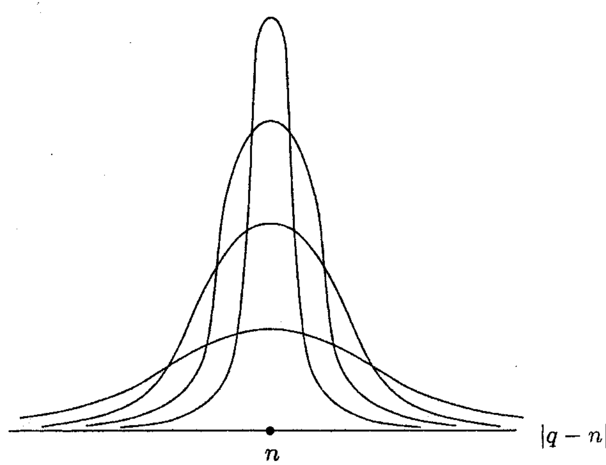
COMPUTE

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} -\mathrm{Tr}\left(\mathcal{F}_{\lambda,n}\wedge {}^*\mathcal{F}_{\lambda,n}\right) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\left(\mathcal{F}_{\lambda,n}\wedge \mathcal{F}_{\lambda,n}\right) = \cdots$$

$$= \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \frac{48\lambda^4}{(\lambda^2+|q-n|^2)^4}\, d\mathrm{Vol}_{\mathbb{R}^4}$$

$$= 1 \quad = \text{CHERN NUMBER OF } Sp(1) \hookrightarrow S^7 \to S^4$$

$\boxed{\text{ADDENDUM 1 : CHERN-WEIL CHARACTERISTIC CLASSES}}$



$-\mathrm{Tr}\left(\mathcal{F}_{\lambda,n}\wedge {}^*\mathcal{F}_{\lambda,n}\right)$ FOR FIXED n

AND VARIOUS λ

INCREASINGLY CONCENTRATED AT

CENTER n AS SCALE $\lambda \to 0$

<u>GAUGE GROUP</u> : $\mathcal{G}$ = ALL DIFFEOMORPHISMS SATISFYING

$$f(p\cdot g) = f(p)\cdot g$$

AND COVERING id_{S^4}

$$\begin{tikzcd} S^7 \arrow[r, "f"] \arrow[d] & S^7 \arrow[d] \\ S^4 \arrow[r, dashed, "id_{S^4}"'] & S^4 \end{tikzcd}$$

TWO CONNECTIONS ω_1 AND ω_2 ON $Sp(1) \hookrightarrow S^7 \to S^4$ ARE <u>GAUGE EQUIVALENT</u> IF, FOR SOME $f \in \mathcal{G}$,

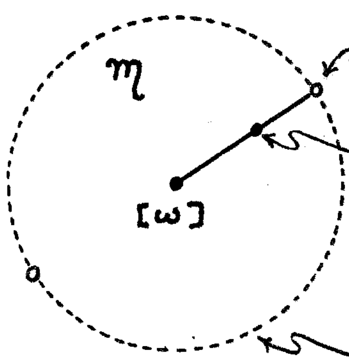
$$\omega_2 = f^*\omega_1$$

ω_{λ_1, n_1} GAUGE EQUIVALENT TO ω_{λ_2, n_2} $\Longleftrightarrow$ $(\lambda_1, n_1) = (\lambda_2, n_2)$

$\mathcal{M}$ = <u>MODULI SPACE</u> OF GAUGE EQUIVALENCE CLASSES $[\omega]$ OF ASD CONNECTIONS ω ON $Sp(1) \hookrightarrow S^7 \longrightarrow S^4$

<u>ATIYAH-HITCHIN-SINGER</u> :

$$\begin{aligned} \mathcal{M} &= \{ [\omega_{\lambda,n}] : (\lambda, n) \in (0,\infty) \times \mathbb{R}^4 \} \\ &\cong (0,\infty) \times \mathbb{R}^4 \\ &\cong \{ x \in \mathbb{R}^5 : \|x\| < 1 \} \end{aligned}$$



FIXES THE "CENTER" n

"SCALE" $\lambda \to 0$ AS $r \to 1$

$\partial\mathcal{M}$ AS A COPY OF S^4, THOUGHT OF AS "INSTANTONS CONCENTRATED AT A POINT"

2. DONALDSON'S GENERALIZATION

M = COMPACT, SIMPLY CONNECTED, ORIENTED,

SMOOTH (C^∞) 4-MANIFOLD WITH

" $b_2^+(M) = 0$ "

REMARKS ON $b_2^+(M)$:

M CONNECTED $\Rightarrow H_0(M;\mathbb{Z}) \cong \mathbb{Z}$ $\qquad$ $\pi_1(M) \cong 0 \Rightarrow H_1(M;\mathbb{Z}) \cong 0$

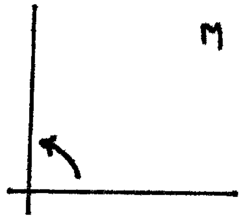
POINCARÉ DUALITY $\Rightarrow H_4(M;\mathbb{Z}) \cong \mathbb{Z}$ AND $H_3(M;\mathbb{Z}) \cong 0$

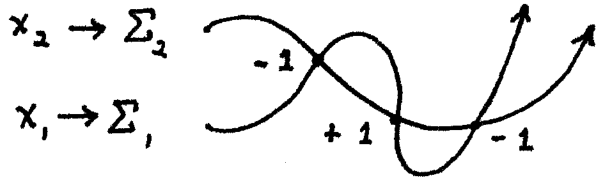
$H_2(M;\mathbb{Z})$ IS A FINITELY GENERATED, FREE ABELIAN GROUP (ISOMORPHIC TO $\mathbb{Z} \oplus \overset{b_2(M)}{\cdots} \oplus \mathbb{Z}$) EACH GENERATOR OF WHICH CAN BE REPRESENTED BY A SMOOTHLY EMBEDDED, ORIENTED, CLOSED SURFACE Σ IN X

INTERSECTION FORM (ASSUMING $H_2(M;\mathbb{Z}) \not\cong 0$) :

$$Q_M : H_2(M;\mathbb{Z}) \oplus H_2(M;\mathbb{Z}) \rightarrow \mathbb{Z}$$

A UNIMODULAR, SYMMETRIC, $\mathbb{Z}$-VALUED BILINEAR FORM ON $H_2(M;\mathbb{Z})$ DEFINED AS FOLLOWS :





$Q_M(x_1, x_2)$ = SUM OF SIGNED INTERSECTION POINTS

<u>ALTERNATIVE</u> :

$$Q_M : H^2(M;\mathbb{Z}) \oplus H^2(M;\mathbb{Z}) \to \mathbb{Z}$$

$$Q_M(\alpha_1, \alpha_2) = \int_M \alpha_1 \wedge \alpha_2$$

$b_2^+(M)$ = MAXIMAL DIMENSION OF A SUBSPACE OF $H_2(M;\mathbb{Z})$ ON WHICH Q_M IS POSITIVE DEFINITE

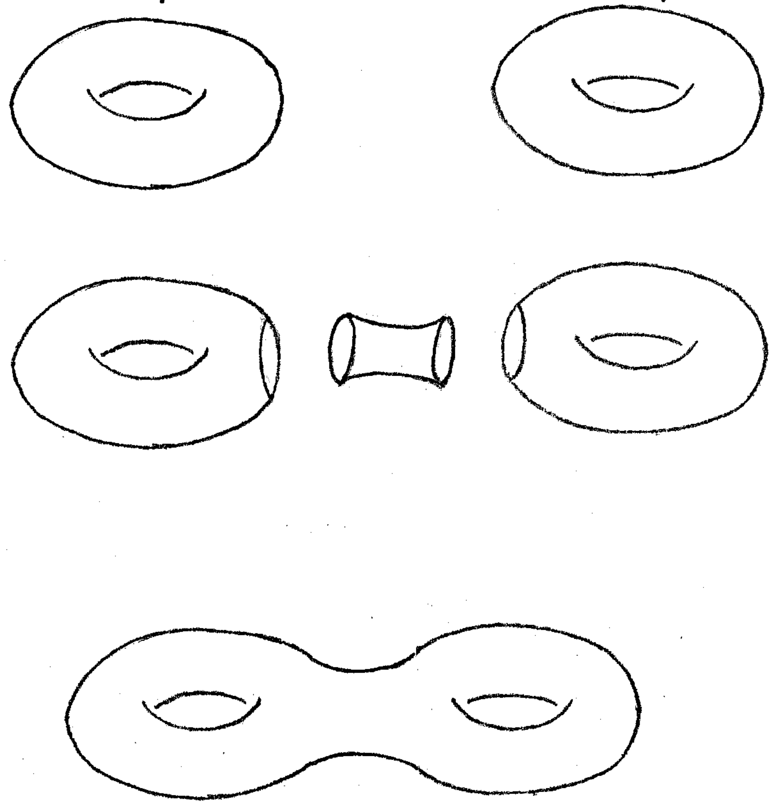
= DIMENSION OF THE SPACE OF SELF-DUAL HARMONIC 2-FORMS ON M (FOR ANY CHOICE OF RIEMANNIAN METRIC ON M)

<u>EXAMPLES</u> :

M	$H_2(M;\mathbb{Z})$	Q_M	$b_2^+(M)$
S^4	0	ϕ	-
$\mathbb{CP}^2$	$\mathbb{Z}$	(1)	1
$\overline{\mathbb{CP}}^2$	$\mathbb{Z}$	(-1)	0
$S^2 \times S^2$	$\mathbb{Z} \oplus \mathbb{Z} = 2\mathbb{Z}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	1
K3	$22\mathbb{Z}$	$3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8)$	3

MORE EXAMPLES FROM CONNECTED SUMS :

M_1 $\qquad\qquad$ M_2



$M_1 \# M_2$

$$H_2(M_1 \# M_2 ; \mathbb{Z}) \cong H_2(M_1 ; \mathbb{Z}) \oplus H_2(M_2 ; \mathbb{Z})$$

$$Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$$

E.G.,

$$Q_{m\,\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}}^2} = \mathrm{DIAG}\,(1 \overset{m}{\cdots} 1 \;\, -1 \overset{n}{\cdots} -1)$$

$$b_2^+(m\,\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}}^2) = m$$

<u>DONALDSON'S</u> <u>THEOREM</u> (1983): M A COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH (C^∞) 4-MANIFOLD. THEN

$$b_2^+(M) = 0 \quad \Rightarrow \quad Q_M = -id.$$

<u>REMARK</u>: Q_M CAN ALSO BE DEFINED FOR TOPOLOGICAL 4-MANIFOLDS. FREEDMAN (1982) PROVED THAT EVERY UNIMODULAR, SYMMETRIC, $\mathbb{Z}$-VALUED BILINEAR FORM ON A FINITELY GENERATED, FREE ABELIAN GROUP IS THE INTERSECTION FORM OF AT LEAST ONE (AND AT MOST TWO) SUCH MANIFOLDS. COROLLARY OF DONALDSON + FREEDMAN:

> THERE ARE OVER 10 MILLION TOPOLOGICALLY DISTINCT COMPACT, ORIENTED, SIMPLY CONNECTED, TOPOLOGICAL 4-MANIFOLDS M WITH $b_2(M) = 32$, NONE OF WHICH ADMIT A SMOOTH STRUCTURE.

"PROOF" $\quad SU(2) \hookrightarrow P_1 \xrightarrow{\pi_1} M \quad$ (CHERN CLASS 1)

CHOOSE RIEMANNIAN METRIC g ON M. GIVES HODGE STAR $*$.

A CONNECTION ω ON P_1 IS g-ANTI-SELF-DUAL (g-ASD) IF

$$* F_\omega = - F_\omega$$

(SOLUTIONS ALSO SATISFY THE YANG-MILLS EQUATIONS)

<u>NOTE</u> : TAUBES HAS PROVED THAT SUCH CONNECTIONS EXIST. HE DOES THIS BY LOCALLY "GRAFTING" A BPST INSTANTON ONTO M. THE GRAFTING PROCEDURE INTRODUCES A SMALL SELF-DUAL PART WHICH, PROVIDED $b_2^+(M) = 0$, CAN BE KILLED BY A SMALL PERTURBATION.

<u>GAUGE GROUP</u> :

$\mathcal{G}(P_1)$ = ALL DIFFEOMORPHISMS SATISFYING $f(p \cdot g) = f(p) \cdot g$ AND COVERING id_M.

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_1 \\ \downarrow & & \downarrow \\ M & \xrightarrow[id_M]{} & M \end{array}$$

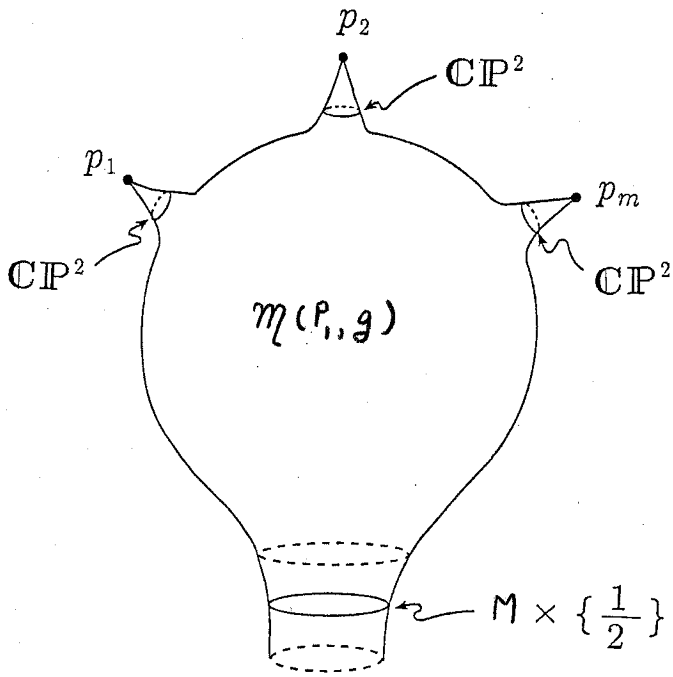
TWO CONNECTIONS ω_1 AND ω_2 ARE <u>GAUGE EQUIVALENT</u> IF, FOR SOME $f \in \mathcal{G}(P_1)$,

$$\omega_2 = f^* \omega_1 .$$

$\mathcal{M}(P_1, g)$ = <u>MODULI SPACE</u> OF GAUGE EQUIVALENCE CLASSES OF g-ASD CONNECTIONS ON P_1

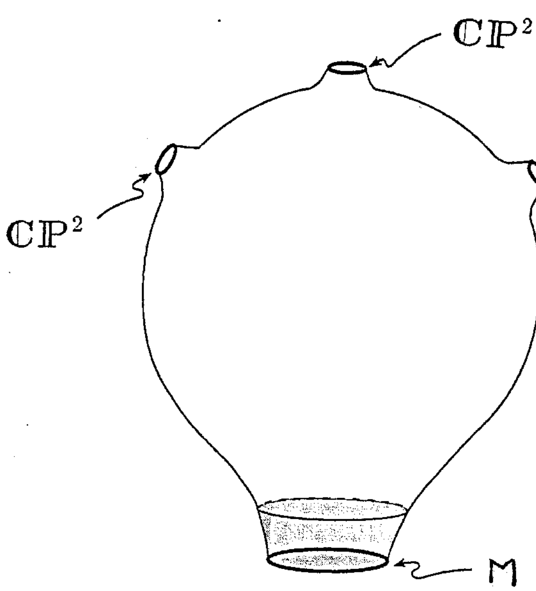
ADDENDUM 2 : THE MODULI SPACES

DONALDSON HAS SHOWN THAT, FOR A "GENERIC" RIEMANNIAN METRIC g, $\mathcal{M}(P_1, g)$ LOOKS LIKE THIS :



- $\exists\ P_1, \ldots, P_m \in \mathcal{M}(P_1, g)$ SUCH THAT $\mathcal{M}(P_1, g) - \{P_1, \ldots, P_m\}$ IS A SMOOTH, ORIENTED 5-MANIFOLD
- EACH P_i HAS A NEIGHBORHOOD HOMEOMORPHIC TO A CONE OVER $\mathbb{CP}^2$.
- $\exists\ K \overset{\text{compact}}{\subseteq} \mathcal{M}(P_1, g)$ SUCH THAT $\mathcal{M}(P_1, g) - K$ IS DIFFEOMORPHIC TO THE CYLINDER $M \times (0, 1)$.

NOW CUT OFF THE TOP HALF OF EACH CONE AND THE BOTTOM HALF OF THE CYLINDER.



- M IS COBORDANT TO A DISJOINT UNION OF $\mathbb{CP}^2$'s.
- SIGNATURE OF THE INTERSECTION FORM IS A COBORDISM INVARIANT.

ETC. $\square$

$\boxed{\text{ADDENDUM 3 : "ETC."}}$

3. DONALDSON POLYNOMIAL INVARIANTS $\gamma_d(M)$, $d = 0, 1, 2, \ldots$

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH (C^∞) 4-MANIFOLD WITH

"VARIOUS ASSUMPTIONS ON $b_2^+(M)$ AS THE NEED ARISES."

$$\gamma_0(M) \in \mathbb{Z}$$

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}], \quad d = 1, 2, \ldots$$

REMARKS ON THE CONSTRUCTION :

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} M \qquad \text{(CHERN CLASS } k \geq 0\text{)}$$

CHOOSE A RIEMANNIAN METRIC g ON M. GIVES HODGE STAR $*$.

$\mathcal{M}(P_k, g)$ = MODULI SPACE OF GAUGE EQUIVALENCE CLASSES OF g-ASD CONNECTIONS ON P_k

$$\boxed{\text{ADDENDUM 2 : THE MODULI SPACES}}$$

EXAMPLES :

1. $M = S^4$, g = STANDARD RIEMANNIAN METRIC, $k = 1$

$\mathcal{M}(P_1, g)$ = MODULI SPACE OF BPST INSTANTONS

$\cong$ OPEN 5-BALL

2. M ARBITRARY , g ARBITRARY , $k \leq 0$

$$k = \frac{1}{8\pi^2} \int_M Tr\,(F_\omega \wedge F_\omega) = \frac{1}{8\pi^2} \int_M (\,|F_\omega^-|^2 - |F_\omega^+|^2\,)\, dVol_g$$

WHERE $F_\omega^{\pm} = \frac{1}{2}(F_\omega \pm {}^*F_\omega)$.

$$k < 0 \Rightarrow F_\omega^+ \neq 0 \Rightarrow \omega \text{ CANNOT BE } g\text{-ASD}$$

$$\mathcal{M}(P_{-1}, g) = \mathcal{M}(P_{-2}, g) = \cdots = \phi$$

$k = 0 \Rightarrow$ ANY g-ASD ω ($F_\omega^+ = 0$) IS FLAT ($F_\omega^- = 0$ AS WELL)

CONVERSELY, FLAT $\Rightarrow$ g-ASD

THE $k = 0$ BUNDLE IS TRIVIAL AND FLAT CONNECTIONS EXIST ON ANY TRIVIAL BUNDLE SO $\mathcal{M}(P_0, g) \neq \phi$.

M SIMPLY CONNECTED $\Rightarrow$ ANY TWO FLAT CONNECTIONS ON THE TRIVIAL BUNDLE ARE GAUGE EQUIVALENT SO

$$\mathcal{M}(P_0, g) = \{\text{POINT}\}$$

HENCEFORTH CONSIDER ONLY

$$k \geq 1$$

3. $M = S^2 \times S^2$, g = STANDARD METRIC , $k = 1$

$$\mathcal{M}(P_1, g) = \phi$$

4. $M = \mathbb{C}P^2$, g = STANDARD (FUBINI-STUDY) METRIC , $k = 1$

$$\mathcal{M}(P_1, g) = \emptyset$$

5. $M = \overline{\mathbb{C}P}^2$, g = STANDARD (FUBINI-STUDY) METRIC , $k = 1$

$$\mathcal{M}(P_1, g) \cong \text{OPEN CONE OVER } \overline{\mathbb{C}P}^2$$

<u>REMARK</u>: COMPARE #4 AND 5. DONALDSON THEORY IS HIGHLY SENSITIVE TO ORIENTATIONS.

ASSUMING

$$b_2^+(M) > 0$$

DONALDSON SHOWS THAT

FOR A "GENERIC" RIEMANNIAN METRIC g, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR A SMOOTH (C^∞), ORIENTABLE MANIFOLD OF DIMENSION

$$8k - 3(1 + b_2^+(M))$$

(AN ORIENTATION IS CANONICALLY DETERMINED BY ORIENTING THE VECTOR SPACE $H_+^2(M; \mathbb{R})$).

REMARKS ON $b_2^+(M) > 0$:

THE SINGULARITIES (CONES OVER $\mathbb{CP}^2$) IN THE MODULI SPACE WHEN $b_2^+(M) = 0$ ARISE FROM "REDUCIBLE" CONNECTIONS (LARGE ISOTROPY GROUP UNDER $\mathcal{G}$-ACTION). IN THE SPACE $\mathcal{R}$ OF RIEMANNIAN METRICS ON M THE SET OF THOSE g FOR WHICH THERE EXIST REDUCIBLE g-ASD ω IS A COUNTABLE UNION OF SUBMANIFOLDS OF CODIMENSION $b_2^+(M)$.
$b_2^+(M) > 0 \Rightarrow$ CAN FIND A "GOOD" g.

TO DEFINE INVARIANTS FROM THE MODULI SPACES REQUIRES THE STRONGER ASSUMPTION

$$b_2^+(M) > 1 .$$

TO ENSURE INDEPENDENCE OF THE CHOICE OF g, NEED THE SET OF METRICS WHICH INTRODUCE REDUCIBLES TO BE SUFFICIENTLY "THIN" THAT A GENERIC VARIATION OF g (PATH IN $\mathcal{R}$) CAN AVOID REDUCIBLES.

DEFINING THE INVARIANTS :

CASE 1 : $8k - 3(1+b_2^+(M)) = 0$

GENERICALLY, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR 0-DIMENSIONAL, ORIENTED, AND COMPACT (THIS IS THE ONLY CASE IN WHICH THE MODULI SPACE IS COMPACT).

$$\gamma_0(M) = \begin{cases} \displaystyle\sum_{[\omega] \in \mathcal{M}(P_k, g)} (-1)^{[\omega]} & , \ \mathcal{M}(P_k, g) \neq \emptyset \\ 0 & , \ \mathcal{M}(P_k, g) = \emptyset \end{cases}$$

CASE 2 : $8k - 3(1+b_2^+(M)) > 0$

ONE MORE ASSUMPTION REQUIRED. HENCEFORTH,

$$b_2^+(M) > 1 \text{ AND ODD.}$$

WRITE

$$8k - 3(1+b_2^+(M)) = 2d_k$$

FOR SOME POSITIVE INTEGER d_k.

GENERICALLY, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR $2d_k$-DIMENSIONAL, ORIENTED, AND NEVER COMPACT.

DONALDSON μ-MAP :

$$\mu : H_2(M;\mathbb{Z}) \longrightarrow H^2(\mathcal{M}(P_k, g);\mathbb{Z})$$

$\boxed{\text{ADDENDUM 4 : THE } \mu\text{-MAP}}$

NAIVE DEFINITION OF

$$\gamma_{d_k}(M) : H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}]$$

$$x \in H_2(M;\mathbb{Z}) \implies \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x) \in H^{2d_k}(\mathcal{M}(P_k, g);\mathbb{Z})$$

$$\gamma_{d_k}(M)(x) = \int_{\mathcal{M}(P_k,g)} \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x)$$

HERE ARE A FEW OF THE THINGS WRONG WITH THIS "DEFINITION" :

- $\mathcal{M}(P_k, g)$ IS NOT COMPACT SO YOU CAN'T INTEGRATE OVER IT

- DEEP ANALYTICAL RESULTS OF UHLENBECK AND TAUBES LED DONALDSON TO A CANONICAL COMPACTIFICATION $\overline{\mathcal{M}}(P_k, g)$ OF $\mathcal{M}(P_k, g)$ AND AN EXTENSION $\bar{\mu} : H_2(M;\mathbb{Z}) \to H^2(\overline{\mathcal{M}}(P_k, g);\mathbb{Z})$ OF THE μ-MAP. HOWEVER,

- $\overline{\mathcal{M}}(P_k, g)$ IS NOT A MANIFOLD SO INTEGRATION OVER IT MUST BE REPLACED BY PAIRING WITH THE FUNDAMENTAL CLASS.

- REGRETTABLY, $\overline{\mathcal{M}}(P_k, g)$ ADMITS A FUNDAMENTAL CLASS ONLY FOR SUFFICIENTLY LARGE k, I.E., IN THE <u>STABLE</u> <u>RANGE</u>

$$k > \frac{3}{4}(1 + b_2^+(M))$$

OR EQUIVALENTLY

$$d_k > \frac{3}{2}(1 + b_2^+(M)).$$

IF d IS AN INTEGER SATISFYING

$$d \equiv -\frac{3}{2}(1 + b_2^+(M)) \bmod 4$$

AND

$$d > \frac{3}{2}(1 + b_2^+(M))$$

WE HAVE

$$\gamma_d(M) : H_2(M; \mathbb{Z}) \to \mathbb{Z}$$

- REMOVING THE STABLE RANGE CONDITION AND THE MOD 4 CONGRUENCE REQUIRES
  - ANOTHER μ-MAP $\mu : H_0(M; \mathbb{Z}) \to H^4(\mathcal{M}(P_k, g); \mathbb{Z})$
  - A DETOUR AROUND THE FACT THIS μ-MAP DOES NOT FULLY EXTEND TO $\overline{\mathcal{M}}(P_k, g)$
  - A BLOW-UP FORMULA RELATING $\gamma_d(M)$ AND $\gamma_d(M \# n\overline{\mathbb{CP}}^2)$

ADDENDUM 5 : DONALDSON POLYNOMIALS

THE END RESULT IS A SEQUENCE

$$\gamma_d(M)\ ,\quad d = 0, 1, 2, \ldots$$

OF ORIENTATION PRESERVING DIFFEOMORPHISM INVARIANTS OF M.

<u>DONALDSON</u> (FORMAL POWER) <u>SERIES</u>

$$\mathcal{D}_M(x) = \sum_{d=0}^{\infty} \frac{\gamma_d(M)(x)}{d!}$$

E.G., FOR $M = K3$

$$\mathcal{D}_{K3}(x) = \sum_{d=0}^{\infty} \frac{(Q_{K3}(x,x)/2)^d}{d!} = \exp(Q_{K3}(x,x)/2).$$

SPRING, 1994 : A BREAKTHROUGH

<u>KRONHEIMER</u> - <u>MROWKA</u> <u>STRUCTURE</u> <u>THEOREM</u> : IF M IS OF "D-SIMPLE TYPE", THEN THERE EXIST COHOMOLOGY CLASSES

$$K_1, \ldots, K_s \in H^2(M; \mathbb{Z})$$ (<u>D</u>-<u>BASIC</u> <u>CLASSES</u>)

AND RATIONAL NUMBERS

$$a_1, \ldots, a_s \in \mathbb{Q}$$ (<u>COEFFICIENTS</u>)

SUCH THAT

$$\mathcal{D}_M(x) = \exp(Q_M(x,x)/2) \sum_{r=1}^{s} a_r \exp(K_r(x))$$

MOREOVER, EACH K_r IS AN INTEGRAL LIFT OF THE SECOND STIEFEL-WHITNEY CLASS $w_2(M) \in H^2(M; \mathbb{Z}_2)$.

FALL, 1994 : RENDERED MOOT (DISCOVERY OF SEIBERG-WITTEN INVARIANTS)

EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN

WITTEN'S MAGICAL FORMULA

$$\mathcal{D}_M(x) = \exp(Q_M(x,x)/2) \sum_{\mathcal{L}\in\Lambda} 2^{m(M)} SW_0(M,\mathcal{L}) \exp(c_1(L^0(\mathcal{L}))(x))$$

DONALDSON INVARIANTS

SEIBERG-WITTEN INVARIANTS

AND THE PHYSICS IT CAME FROM

$$\Phi = (\omega, \phi, \lambda, \eta, \psi, \zeta)$$

$$S_{DW}[\Phi] = \int_M Tr \Big\{ \tfrac{1}{4} F_\omega \wedge {}^*F_\omega + \tfrac{1}{4} F_\omega \wedge F_\omega - \tfrac{1}{2}\psi \wedge [\phi,\psi] - i\, d^\omega \zeta \wedge \psi - 2i[\zeta, {}^*\zeta]\lambda + i\,\phi\, d^\omega {}^* d^\omega \lambda - \zeta \wedge {}^* d^\omega \eta \Big\}$$

$$Z_{DW} = \int \exp(-S_{DW}[\Phi]/e^2)\, \mathcal{D}\Phi$$

- MATHAI - QUILLEN (1986)

  " SUPERSYMMETRIC " FORMULAS FOR THE EULER CHARACTERISTIC OF FINITE - DIMENSIONAL VECTOR BUNDLES FROM EQUIVARIANT COHOMOLOGY.

- ATIYAH - JEFFREY (1990)

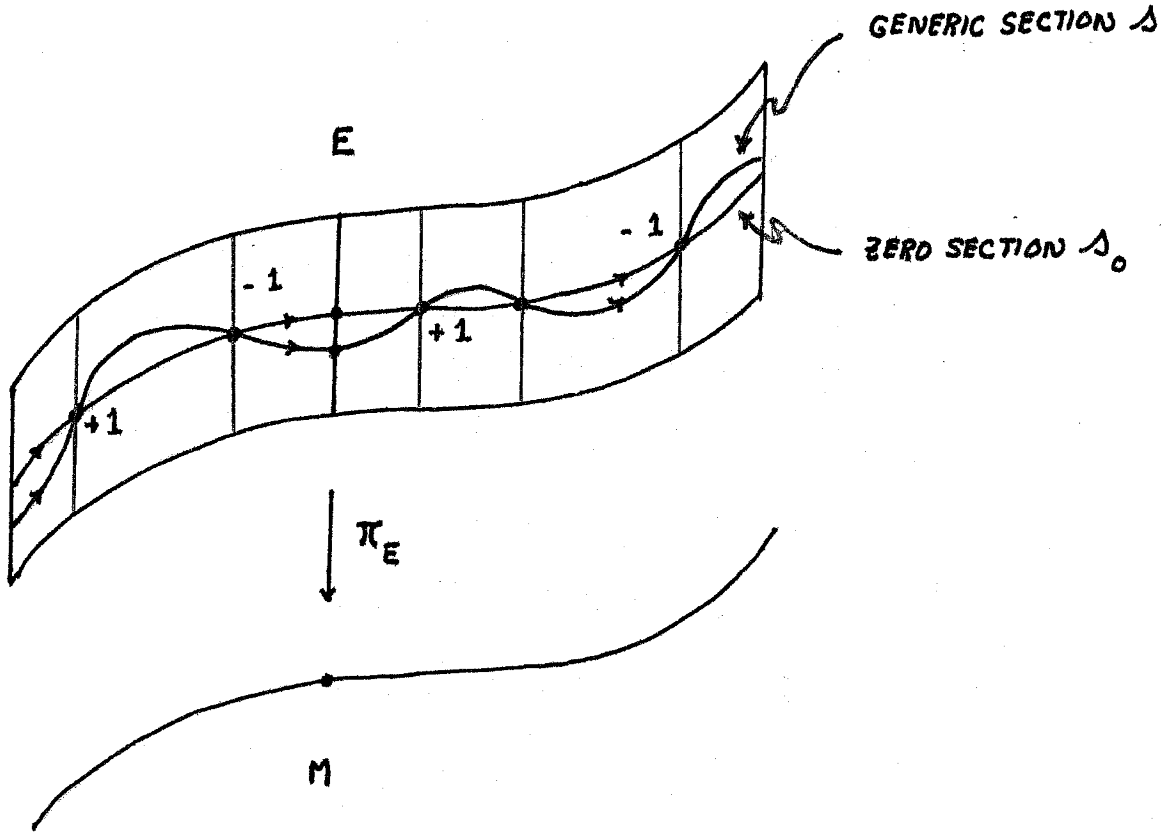
  FORMAL EXTENSION OF MATHAI - QUILLEN TO CERTAIN INFINITE - DIMENSIONAL VECTOR BUNDLE REPRODUCES $S_{DW}[\Phi]$ AND EXHIBITS Z_{DW} AS AN " EULER CHARACTERISTIC ".

- MATHAI - QUILLEN FORMALISM FOR TQFT OF " COHOMOLOGICAL TYPE ".

EULER CHARACTERISTIC OF AN ORIENTED, REAL VECTOR BUNDLE

$$E \xrightarrow{\pi_E} M$$

OF FIBER DIMENSION $2k$ OVER A COMPACT, ORIENTED MANIFOLD M OF DIMENSION $2k$ IS DEFINED AS FOLLOWS :



$\chi(E)$:= INTERSECTION NUMBER OF ANY GENERIC SECTION

INTEGRAL REPRESENTATION : $\chi(E) = \int_M e(E)$

$e(E)$ = EULER CLASS OF $E \xrightarrow{\pi_E} M$

TWO APPROACHES TO $e(E)$:

1. (CHERN-WEIL) FIBER METRIC ON $E \xrightarrow{\pi_E} M$ GIVES

$$SO(2k) \hookrightarrow F_{SO}(E) \xrightarrow{\pi_{SO}} M.$$

CHOOSE CONNECTION ω WITH CURVATURE Ω. CHOOSE THE ad - INVARIANT POLYNOMIAL

$$\text{PFAF} : so(2k) \longrightarrow \mathbb{R}$$

$$(2\pi)^{-k}\,\text{PFAF}(\Omega) = \frac{1}{2^{2k}\pi^k k!} \sum_{\sigma \in S_{2k}} (-1)^\sigma \Omega_{\sigma(1)\sigma(2)} \wedge \cdots \wedge \Omega_{\sigma(2k-1)\sigma(2k)}$$

IS A CLOSED FORM ON $F_{SO}(E)$ AND IS BASIC ($SO(2k)$ - INVARIANT AND HORIZONTAL) SO IT DESCENDS TO A CLOSED $2k$ - FORM ON M WHOSE COHOMOLOGY CLASS

$$e(E)$$

DOES NOT DEPEND ON THE CHOICE OF ω.

NOTE : $e(TS^2)$ IS COMPUTED IN ADDENDUM 6.

2. (<u>THOM</u> <u>CLASS</u>) THERE EXISTS A UNIQUE $U(E) \in H^{2k}_{cv}(E;\mathbb{R})$ WHOSE INTEGRAL OVER EACH FIBER $\pi_E^{-1}(p)$ IS 1. THEN

$$e(E) = s^*(U(E))$$

FOR ANY SECTION s OF $E \xrightarrow{\pi_E} M$.

MATHAI-QUILLEN WILL YIELD VARIOUS EXPLICIT REPRESENTATIVES OF THE THOM CLASS $U(E)$.

<u>ANOTHER VIEW OF THE PFAFFIAN</u> $\mathrm{PFAF}: so(2k) \longrightarrow \mathbb{R}$:

$$V = \text{TYPICAL FIBER OF } E \xrightarrow{\pi_E} M$$

$\{\psi^1, \dots, \psi^{2k}\}$ = ORIENTED, ORTHONORMAL BASIS FOR V

= ODD GENERATORS OF THE SUPERCOMMUTATIVE SUPERALGEBRA

$$\Lambda V = \bigoplus_{i=0}^{2k} \Lambda^i V = (\Lambda V)_0 \oplus (\Lambda V)_1$$

$$\mathrm{VOL} = \psi^1 \cdots \psi^{2k} \in \Lambda^{2k} V$$

$$A = (A_{ij}) \in so(2k) \longrightarrow \alpha_A = \frac{1}{2} A_{ij} \psi^i \psi^j \in \Lambda^2 V$$

$$= \frac{1}{2} \psi^T A \psi$$

$$= -\frac{1}{2} \sum_{l=1}^{2k} \psi^l A \psi^l$$

$$\frac{1}{k!}\alpha_A^{\,k} = \text{PFAF}(A)\ \text{VOL}$$

AS A <u>BEREZIN</u> (OR <u>FERMIONIC</u>) <u>INTEGRAL</u>:

ANY $f \in \Lambda V$ IS A POLYNOMIAL WITH REAL COEFFICIENTS IN THE ODD "VARIABLES" $\psi^1, \ldots, \psi^{2k}$. DEFINE

$$\int f\, \mathcal{D}\psi = f_{\text{VOL}} \in \mathbb{R}$$

$$= \text{COEFFICIENT OF VOL} = \psi^1 \cdots \psi^{2k}$$

THUS,

$$\text{PFAF}(A) = \int e^{\frac{1}{2}\psi^T A \psi}\, \mathcal{D}\psi = \int e^{-\frac{1}{2}\sum_{\ell} \psi^\ell A \psi^\ell}\, \mathcal{D}\psi$$

SUBSTITUTING FOR A THE CURVATURE Ω OF A CONNECTION ON $SO(2k) \hookrightarrow F_{SO}(E) \longrightarrow M$ GIVES A BEREZIN INTEGRAL REPRESENTATION OF A FORM WHICH DESCENDS TO A REPRESENTATIVE OF THE EULER CLASS ON M.

MATHAI-QUILLEN REPRESENTATIVES OF THE THOM CLASS ARE LIKEWISE GIVEN AS BEREZIN INTEGRALS.

EXTENDED BEREZIN INTEGRATION :

$$\mathcal{A} = \text{ANOTHER SUPERCOMMUTATIVE SUPERALGEBRA}$$

$$\text{E.G., } \Omega^*(V)$$

$\mathcal{A} \otimes \Lambda V$ = SUPER TENSOR PRODUCT

$$\left((a_1 \otimes f_1)(a_2 \otimes f_2) = (-1)^{\deg f_1 \deg a_2} (a_1 a_2) \otimes (f_1 f_2) \right)$$

= POLYNOMIALS F WITH COEFFICIENTS IN $\mathcal{A}$ IN THE ODD "VARIABLES" $\psi^1, \dots, \psi^{2k}$

$$\int F \, \mathcal{D}\psi = F_{\text{VOL}} \in \mathcal{A}$$

EXAMPLES :

1. $\mathcal{A} = \Omega^*(V)$ = COMPLEX - VALUED DIFFERENTIAL FORMS ON V

$$\Omega^*(V) \otimes \Lambda V$$

$\{ \mu_1, \dots, \mu_{2k} \}$ = BASIS FOR $V^* \subseteq \Omega^0(V)$ DUAL TO $\{ \psi^1, \dots, \psi^{2k} \}$

OMITING THE "$\otimes$" FOR ELEMENTS OF $\Omega^*(V) \otimes \Lambda V$ WE HAVE

$$-i \, d\mu_j \, \psi^j = i \, \psi^j d\mu_j = i \, \psi^T d\mu \in \Omega^*(V) \otimes \Lambda V$$

$$-\tfrac{1}{2} \|\mu\|^2 1 = -\tfrac{1}{2} \|\mu\|^2 = -\tfrac{1}{2} (\mu_1^2 + \dots + \mu_{2k}^2) \in \Omega^*(V) \otimes \Lambda V$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2 + i \psi^T d\mu} \in \Omega^*(V) \otimes \Lambda V$$

$$\int (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2 + i\psi^T du} \,\mathcal{D}\psi =$$

$$(2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2} e^{i\psi^T du} \,\mathcal{D}\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int e^{i\psi^1 du_1 + \cdots + i\psi^{2k} du_{2k}} \,\mathcal{D}\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int e^{i\psi^1 du_1} \cdots e^{i\psi^{2k} du_{2k}} \,\mathcal{D}\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int (1 + i\psi^1 du_1) \cdots (1 + i\psi^{2k} du_{2k}) \,\mathcal{D}\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int (i\psi^1 du_1) \cdots (i\psi^{2k} du_{2k}) \,\mathcal{D}\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} i^{2k} \int (-1)^{\frac{1}{2}(2k)(2k+1)} du_1 \cdots du_{2k} \,\psi^1 \cdots \psi^{2k} \,\mathcal{D}\psi =$$

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} du_1 \cdots du_{2k} \quad \in \Omega^*(V)$$

INTEGRATES TO 1 OVER V. THINK OF IT AS A "GAUSSIAN" REPRESENTATIVE OF THE THOM CLASS OF V, REGARDED AS A VECTOR BUNDLE OVER A POINT.

2. $\mathcal{A} = \mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V)$

$\{\xi_1, \ldots, \xi_n\}$ = BASIS FOR $\mathfrak{so}(V)$ $\quad (n = k(2k-1))$

$\{x^1, \ldots, x^n\}$ = DUAL BASIS FOR $\mathfrak{so}(V)^*$, REGARDED AS LINEAR FUNCTIONS ON $\mathfrak{so}(V)$

$\mathbb{C}[\mathfrak{so}(V)] = \mathbb{R}[x^1, \ldots, x^n] \otimes \mathbb{C}$

$\mathbb{C}[so(V)] \otimes \Omega^*(V)$ = SUMS OF TERMS

$$\alpha = \rho \otimes \varphi$$

WHICH ARE TO BE REGARDED AS $\Omega^*(V)$-VALUED POLYNOMIALS ON $so(V)$

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

<u>GRADING</u> : $\quad \deg(\rho \otimes \varphi) = 2 \deg \rho + \deg \varphi$

WE DESCRIBE AN ELEMENT ν OF $\mathbb{C}[so(V)] \otimes \Omega^*(V)$, CALLED THE <u>UNIVERSAL THOM FORM</u> OF V, AS THE BEREZIN INTEGRAL OF AN ELEMENT OF

$$\mathbb{C}[so(V)] \otimes \Omega^*(V) \otimes \Lambda V.$$

<u>NOTATION</u> : $\quad \xi \in so(V)$ GIVES A LINEAR TRANSFORMATION

$$M_\xi : V \to V$$

$$M_\xi(\psi) = \frac{d}{dt}(\exp(t\xi)(\psi))\Big|_{t=0}$$

IF $M_a = M_{\xi_a}$, THEN

$$M_\xi = x^a(\xi) M_a$$

$$\mathrm{PFAF}(M_\xi) = \int e^{-\frac{1}{2}\sum_{\ell} \psi^\ell x^a(\xi) M_a \psi^\ell} \mathcal{D}\psi$$

NOW NOTICE THAT

$$-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell} \in \mathbb{C}[so(V)]\otimes\Omega^{*}(V)\otimes\Lambda V$$

$$e^{-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell}} \in \mathbb{C}[so(V)]\otimes\Omega^{*}(V)\otimes\Lambda V$$

$$(2\pi)^{-k}\, e^{-\frac{1}{2}\|u\|^{2}+i\psi^{T}du-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell}} \in \mathbb{C}[so(V)]\otimes\Omega^{*}(V)\otimes\Lambda V$$

SO WE CAN DEFINE

$$\nu = (2\pi)^{-k}\int e^{-\frac{1}{2}\|u\|^{2}+i\psi^{T}du-\frac{1}{2}\sum_{\ell}\psi^{\ell}x^{a}M_{a}\psi^{\ell}}\,\mathcal{D}\psi \in \mathbb{C}[so(V)]\otimes\Omega^{*}(V$$

EXAMPLE : $V=\mathbb{R}^{2}$ (USUAL ORIENTATION AND INNER PRODUCT)

$\{\psi^{1},\psi^{2}\}$ = STANDARD BASIS

$\{u_{1},u_{2}\}$ = DUAL BASIS (COORDINATE FUNCTIONS)

$SO(V) \equiv SO(2)$

$\{\xi_{1}\} = \left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}$

$\{x^{1}\}$ = DUAL BASIS

$$\nu = (2\pi)^{-1}e^{-\frac{1}{2}(u_{1}^{2}+u_{2}^{2})}du_{1}du_{2} + (2\pi)^{-1}x^{1}e^{-\frac{1}{2}(u_{1}^{2}+u_{2}^{2})}$$

$\mathbb{C}^{0}[so(2)]\otimes\Omega^{2}(\mathbb{R}^{2})$

INTEGRATES TO 1 OVER $\mathbb{R}^{2}$

$\mathbb{C}^{1}[so(2)]\otimes\Omega^{0}(\mathbb{R}^{2})$

NOTE : THE PROOF IS IN ADDENDUM 7.

WE WILL SEE SHORTLY HOW THE UNIVERSAL THOM FORM OF V CAN BE USED TO PRODUCE (GAUSSIAN) REPRESENTATIVES OF THE THOM CLASS FOR ANY VECTOR BUNDLE WITH TYPICAL FIBER V.

FIRST, HOWEVER,

WHAT KIND OF "THING" IS ν ?

BRIEF ASIDE ON THE <u>CARTAN</u> <u>MODEL</u> OF <u>EQUIVARIANT</u> <u>COHOMOLOGY</u>:

M = SMOOTH MANIFOLD $\qquad$ G = COMPACT, CONNECTED LIE GROUP

LEFT ACTION : $\sigma : G \times M \to M$

$$\sigma(g,m) = g \cdot m = \sigma_g(m)$$

OBJECTIVE : COHOMOLOGY OF M/G

$\Omega^*(M)$ = GRADED ALGEBRA OF COMPLEX-VALUED DIFFERENTIAL FORMS ON M

$\mathbb{C}[\mathfrak{g}]$ = GRADED ALGEBRA OF COMPLEX-VALUED POLYNOMIALS ON THE LIE ALGEBRA $\mathfrak{g}$ OF G

$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ = SUMS OF $\alpha = \rho \otimes \varphi$: $\Omega^*(M)$-VALUED POLYNOMIALS ON $\mathfrak{g}$

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

GRADING : $\deg(\alpha) = \deg(\rho \otimes \varphi) = 2\deg(\rho) + \deg(\varphi)$

INDUCED ACTION OF G ON $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$:

$$(g\cdot\alpha)(\xi) = (g\cdot(p\otimes\varphi))(\xi) = p(g^{-1}\xi g)\,\sigma_{g^{-1}}^*(\varphi)$$

$$\Omega_G^*(M) = [\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)]^G$$

$= G$- INVARIANT ELEMENTS OF $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$

$= G$ - <u>EQUIVARIANT</u> <u>DIFFERENTIAL</u> <u>FORMS</u> ON M

G - <u>EQUIVARIANT</u> <u>EXTERIOR</u> <u>DERIVATIVE</u>:

$$d_G : \Omega_G^*(M) \to \Omega_G^*(M)$$

$$(d_G\alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi^*}(\alpha(\xi))$$

WHERE

$$\xi^*(m) = \frac{d}{dt}(\exp(-t\xi)\cdot m)\Big|_{t=0}.$$

$(\Omega_G^*(M), d_G)$ IS A COCHAIN COMPLEX AND ITS COHOMOLOGY

$$H_G^*(M)$$

IS THE <u>CARTAN</u> <u>MODEL</u> OF THE G- <u>EQUIVARIANT</u> <u>COHOMOLOGY</u> OF M.

<u>THEOREM</u> (CARTAN): IF THE ACTION OF G ON M IS FREE, THEN

$$H_G^*(M) \cong H^*_{deRham}(M/G; \mathbb{C}).$$

<u>NOTE</u>: MORE DETAILS AND $H_{S^1}^*(S^3)$ CAN BE FOUND IN ADDENDUM 8.

INTEGRATION OF EQUIVARIANT DIFFERENTIAL FORMS :

$$\int_M \; : \; \Omega_G^*(M) \longrightarrow \mathbb{C}[\mathfrak{g}]$$

$$\left(\int_M \alpha \right)(\xi) = \int_M \alpha(\xi) := \int_M \alpha(\xi)_{[2k]}$$

UNIVERSAL THOM FORM

$$\nu = (2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^T du - \frac{1}{2}\sum_\ell \psi^\ell x^a M_a \psi^\ell} \mathcal{D}\psi \in \Omega^*_{SO(V)}(V)$$

SATISFIES

$$d_{SO(V)}\nu = 0$$

AND

$$\int_V \nu = 1$$

NOTE : PROOFS FOR $V = \mathbb{R}^2$ IN ADDENDUM 9.

THOM FORMS FROM THE UNIVERSAL THOM FORM :

ANY ORIENTED REAL VECTOR BUNDLE WITH TYPICAL FIBER V CAN BE VIEWED AS THE VECTOR BUNDLE

$$P \times_\rho V$$

ASSOCIATED TO SOME PRINCIPAL G-BUNDLE

$$G \hookrightarrow P \longrightarrow M$$

BY SOME REPRESENTATION

$$\rho : G \to SO(V) \qquad (g\cdot\psi = \rho(g)(\psi))$$

$$(p\cdot g,\, g^{-1}\cdot\psi) \to [p,\psi] \qquad \begin{tikzcd} P\times V \arrow[r] \arrow[d] & P \arrow[d] \\ P\times_\rho V \arrow[r] & M \end{tikzcd}$$

$$\rho_* : \mathfrak{g} \to \mathfrak{so}(V)$$

$$\mathcal{P}\otimes\varphi \in [\mathbb{C}[\mathfrak{so}(V)]\otimes\Omega^*(V)]^{SO(V)}$$

$$\to (\mathcal{P}\circ\rho_*)\otimes\varphi \in [\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(V)]^{G}$$

E.G., $\nu \in \Omega^*_{SO(V)}(V) \to \nu_G \in \Omega^*_G(V)$

<u>CHERN-WEIL MAP</u>:

CHOOSE ω ON P

$$\mathbb{C}[\mathfrak{g}]^G \to \Omega^*(P)_{BASIC} \cong \Omega^*(M)$$

$$\mathcal{P} \to \mathcal{P}(\Omega)$$

<u>MATHAI-QUILLEN MAP</u>:

$$[\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(V)]^G \to \Omega^*(P\times V)_{BASIC} \cong \Omega^*(P\times_\rho V)$$

$$\mathcal{P}\otimes\varphi \to HOR_\omega(\mathcal{P}(\Omega)\wedge\varphi)$$

$$d_G\text{-CLOSED} \to d\text{-CLOSED}$$

<u>NOTE</u>: WE USE THE SAME SYMBOLS FOR FORMS ON P AND V AND THEIR PULLBACKS TO $P\times V$ BY A PROJECTION.

UNDER THE MATHAI-QUILLEN MAP

$$\nu_G = (2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^T du - \frac{1}{2}\sum_{\ell} \psi^{\ell}(X^{a}\circ\rho_*)M_a\psi^{\ell}} \mathcal{D}\psi$$

GOES TO

$\mathcal{V}$ = THE HORIZONTAL PROJECTION OF

$$(2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^T du - \frac{1}{2}\sum_{\ell} \psi^{\ell}(X^{a}(\rho_*\Omega))M_a\psi^{\ell}} \mathcal{D}\psi$$

$$= (2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^T du + \frac{1}{2}\psi^T(\rho_*\Omega)\psi} \mathcal{D}\psi$$

WHERE $(\rho_*\Omega)$ IS THE SKEW-SYMMETRIC MATRIX IMAGE OF THE G-CURVATURE UNDER ρ_*.

EXAMPLE : $V = \mathbb{R}^2$ AS IN THE EXAMPLE ON PAGE 10.

$$\nu = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(x' + du_1 du_2)$$

VECTOR BUNDLE $= TS^2$ (USUAL STRUCTURE ON S^2)

$G = SO(2)$

$SO(2) \hookrightarrow F_{SO}(TS^2) \longrightarrow S^2$

$\rho = id_{SO(2)}$ (SO $\rho_* = id_{so(2)}$)

$\omega = \omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\qquad$ $\Omega = \Omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathcal{V} = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(\Omega' + du_1 du_2 + \omega' \wedge (u_1 du_1 + u_2 du_2))$$

NOTE : THE PROOF IS IN ADDENDUM 10.

IN GENERAL,
$$U = HOR_\omega \left((2\pi)^{-k} \int e^{-\frac{1}{2}\|\mu\|^2 + i\psi^T d\mu + \frac{1}{2}\psi^T(\rho_*\Omega)\psi} \mathcal{D}\psi \right)$$

IS A BASIC FORM ON $P \times V$ WHICH DESCENDS TO A GAUSSIAN REPRESENTATIVE OF THE THOM CLASS ON $P \times_\rho V$.

PULLING BACK TO M BY A SECTION OF $P \times_\rho V \to M$ GIVES A REPRESENTATIVE OF THE EULER CLASS.

BUT EVERY SECTION OF $P \times_\rho V \to M$ IS OF THE FORM
$$m \longrightarrow (s(m), S(s(m))) \to [s(m), S(s(m))]$$

WHERE s IS A SECTION OF $G \hookrightarrow P \to M$ AND $S : P \to V$ IS EQUIVARIANT $\quad (S(p \cdot g) = \rho(g^{-1})(S(p)))$.

THUS, ONE CAN PULL U BACK DIRECTLY BY
$$(s, S \circ s)^* = ((1, S) \circ s)^* = s^* \circ (1, S)^*$$

I.E., PULL V-FACTORS BACK BY S TO GET A FORM ON P AND THEN PULL THIS BACK TO M BY s.

<u>EXAMPLE</u> : $\quad TS^2 = F_{SO}(TS^2) \times_{id_{SO(2)}} \mathbb{R}^2$ AS IN THE EXAMPLE ON PAGE 16.

$$\omega = \text{LEVI-CIVITA CONNECTION ON } F_{SO}(TS^2)$$

$$s = \text{SECTION OF } F_{SO}(TS^2)$$

$=$ ORIENTED, ORTHONORMAL FRAME FIELD ON S^2

$$\mathfrak{s}(\phi,\theta) = \left(\phi,\theta,\ \frac{\partial}{\partial\phi},\ \frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)$$

$S : F_{SO}(TS^2) \to \mathbb{R}^2$ AN EQUIVARIANT MAP

(I.E., A VECTOR FIELD ON S^2)

$$S(\mathfrak{s}(\phi,\theta)) = (\gamma\sin\theta,\ \gamma\cos\theta\cos\phi)$$

WHERE $\gamma \in \mathbb{R}$ IS ARBITRARY

THE RESULTING REPRESENTATIVE OF THE EULER CLASS IS

$$(2\pi)^{-1}\, e^{-\frac{1}{2}\gamma^2(\sin^2\theta+\cos^2\theta\cos^2\phi)}\,\sin\phi\,(1+\gamma^2\cos^2\theta\sin^2\phi)\,d\phi\wedge d\theta$$

<u>NOTE</u> : THE PROOF IS IN ADDENDUM II.

IN PARTICULAR, WE OBTAIN THAT, FOR ANY $\gamma \in \mathbb{R}$,

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} e^{-\frac{1}{2}\gamma^2(\sin^2\theta+\cos^2\theta\cos^2\phi)}\,\sin\phi\,(1+\gamma^2\cos^2\theta\sin^2\phi)\,d\phi\,d\theta$$

$$= \chi(S^2)$$

$$= 2$$

ATIYAH - JEFFREY TRANSFORMATION OF THE UNIVERSAL THOM FORM (SKETCH) :

NOTE : DETAILED CALCULATIONS CAN BE FOUND IN ADDENDUM 12.

$$(1) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \int \exp\left(i\,\psi^T d\mu + \tfrac{1}{2}\psi^T (\rho_* \Omega)\psi \right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

ADDITIONAL ASSUMPTIONS AND CHOICES :

1. $G \hookrightarrow P \xrightarrow{\pi_P} M$ IS ORIENTABLE (I.E., $\exists$ n-FORM Ψ ON P ($n = \dim G$) SUCH THAT, IF $m \in M$ AND $\iota_m : \pi_P^{-1}(m) \hookrightarrow P$, THEN $\iota_m^* \Psi$ IS AN ORIENTATION FOR $\pi_P^{-1}(m) \cong G$). FOLLOWS THAT P IS ORIENTED BY THE "LOCAL PRODUCT ORIENTATION" $\pi_P^*(\mathrm{vol}_M) \wedge \Psi$.

2. THE ACTION OF G ON P IS ORIENTATION PRESERVING ($\sigma_g : P \to P$, $\sigma_g(p) = p \cdot g$, IS AN ORIENTATION PRESERVING DIFFEOMORPHISM $\forall g \in G$).

3. A G-INVARIANT RIEMANNIAN METRIC $\langle\ ,\ \rangle$ HAS BEEN CHOSEN FOR P.

4. THE CONNECTION ω FOR P IS THE ONE WHOSE DISTRIBUTION OF HORIZONTAL SPACES IS THE FAMILY OF $\langle\, ,\, \rangle$- ORTHOGONAL COMPLEMENTS TO THE $G \hookrightarrow P \xrightarrow{\pi_p} M$ VERTICAL SPACES

5. AN ad-INVARIANT INNER PRODUCT $(\ ,\ )$ ON $\mathfrak{g}$ HAS BEEN CHOSEN AND NORMALIZED SO THAT THE VOLUME OF G (ARISING FROM THE CORRESPONDING BI-INVARIANT RIEMANNIAN METRIC ON G) IS 1.

NOW WE BEGIN TO MANIPULATE U, GIVEN BY (1) :

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega] \text{ AND}$$

$[\omega,\omega]$ VANISHES ON HORIZONTAL PARTS SO

$$(2) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\,\psi^T du + \tfrac{1}{2}\psi^T(\rho_*(d\omega))\psi \right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

<u>TOOLS</u> :

$$C_p : \mathfrak{g} \longrightarrow \mathrm{VERT}_p(P) \subseteq T_p(P)$$

$$C_p(\xi) = \xi^*(p) = \frac{d}{dt}(p \cdot \exp(t\xi))\Big|_{t=0}$$

$$C_p^* : T_p(P) \to \mathfrak{g}$$

$$\langle v_p, C_p(\eta)\rangle_p = (C_p^*(v_p), \eta) \quad \forall v_p \in T_p(P)\ \forall \eta \in \mathfrak{g}$$

$$R_p = C_p^* \circ C_p : \mathfrak{g} \to \mathfrak{g} \quad \text{(INVERTIBLE AND SELF-ADJOINT)}$$

$$R_p^{-1} : \mathfrak{g} \to \mathfrak{g}$$

THEN

$$\omega_p = R_p^{-1} \circ C_p^*$$

AS LIE ALGEBRA - VALUED 1-FORMS,

$$\omega = R^{-1} \circ C^*$$

SO

$$d\omega = dR^{-1} \wedge C^* + R^{-1} dC^*$$

C^* VANISHES ON HORIZONTAL VECTORS AND THEREFORE SO DOES $dR^{-1} \wedge C^*$ SO

$$(3) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^T du + \tfrac{1}{2}\psi^T(\rho_*(R^{-1}dC^*))\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

TO ELIMINATE THE EXPLICIT APPEARANCE OF THE INVERSE, USE THE FOURIER INVERSION FORMULA (ON $\mathfrak{g}$) AND A CHANGE OF VARIABLE.

$\phi = (\phi_1, \ldots, \phi_n)$ AND $\lambda = (\lambda_1, \ldots, \lambda_n)$ COORDINATES ON $\mathfrak{g}$

AND f A REAL-VALUED FUNCTION ON $\mathfrak{g}$

$$(\mathfrak{F}f)(\phi) = (2\pi)^{-n/2} \int_{\mathfrak{g}} e^{-i(\phi,\lambda)} f(\lambda)\, d\lambda$$

$$(\tilde{\mathfrak{F}}f)(\lambda) = (2\pi)^{-n/2} \int_{\mathfrak{g}} e^{i(\lambda,\phi)} f(\phi)\, d\phi$$

NOW, $f = \mathcal{F}(\tilde{\mathcal{F}} f)$ IMPLIES

$$f(\mu) = (2\pi)^{-n} \int_{\mathfrak{g}} \int_{\mathfrak{g}} e^{-i(\mu,\lambda)} e^{i(\lambda,\phi)} f(\phi)\, d\phi\, d\lambda$$

USE THIS TO EVALUATE $f(R^{-1}\mu)$ AND MAKE THE CHANGE OF VARIABLE $\lambda \to R\lambda$ TO OBTAIN

$$f(R^{-1}\mu) = (2\pi)^{-n} \int_{\mathfrak{g}} \int_{\mathfrak{g}} e^{-i(\mu,\lambda)} e^{i(\phi,R\lambda)} f(\phi) \det R \, d\lambda\, d\phi$$

APPLY THIS TO THE MATHAI-QUILLEN FORM WITH $\mu = dC^*$ TO OBTAIN

$$(4) \quad U = (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \iiint \exp\left(\tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T d\mu - i(dC^*,\lambda) + i(\phi,R\lambda)\right) \det R\, \mathcal{D}\psi\, d\phi\, d\lambda$$

(EVALUATED ON HORIZONTAL PARTS)

TO INCORPORATE "EVALUATED ON HORIZONTAL PARTS" DIRECTLY INTO THE INTEGRAL WE NEED:

A <u>NORMALIZED</u> <u>VERTICAL</u> <u>VOLUME</u> <u>FORM</u> FOR A PRINCIPAL G-BUNDLE $G \hookrightarrow Q \xrightarrow{\pi} X$ (WE HAVE IN MIND $G \hookrightarrow P \times V \longrightarrow P \times_\rho V$) IS AN n-FORM ($n = \dim G$) W ON Q SUCH THAT, IF $\iota_x : \pi^{-1}(x) \hookrightarrow Q$, THEN

$$\int_{\pi^{-1}(x)} \iota_x^* W = 1$$

$\forall x \in X$.

ONE CAN ALWAYS CONSTRUCT SUCH A THING AND, IN OUR CASE, IT CAN BE WRITTEN AS A BEREZIN INTEGRAL

$$W = (\det R)^{-1} \int e^{(c^*, \eta)} \mathcal{D}\eta$$

WHERE η IS A VARIABLE IN $\mathfrak{g}$ AND (c^*, η) IS THE 1-FORM ON $P \times V$ DEFINED BY

$$(c^*, \eta)(\xi) = (c^*(\xi), \eta) = \langle \xi, c(\eta) \rangle = \langle \xi, \eta^\# \rangle$$

FOR ANY VECTOR FIELD ξ.

WEDGING WITH W KILLS ANY VERTICAL PARTS AND LEAVES ONLY TERMS WITH HORIZONTAL PARTS AND A FACTOR OF W. THEN FIBER INTEGRATION (I.E., IGNORING W) LEAVES ONLY HORIZONTAL PARTS.

DOING THIS FOR U IN (4) ADDS (c^*, η) TO THE EXPONENT, CANCELS THE $\det R$ AND INTRODUCES ONE MORE BEREZIN INTEGRAL:

$$(5) \quad U = (2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \iiiint \exp\left(\tfrac{1}{2}\psi^T(\rho_+\phi)\psi + i\psi^T d\mu - i(dc^*, \lambda) + i(\phi, R\lambda) + (c^*, \eta)\right) \mathcal{D}\eta \mathcal{D}\psi \, d\phi \, d\lambda$$

PULLING BACK THE V-PARTS OF THIS FORM ON $P \times V$ BY AN EQUIVARIANT MAP

$$s : P \longrightarrow V$$

GIVES A FORM

$$(6) \quad (2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \iiiint \exp\Big(\tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T ds - i(dc^+_, \lambda) + i(\phi, R\lambda) + (c^+_, \eta)\Big) \mathcal{D}\eta\, \mathcal{D}\psi\, d\phi\, d\lambda$$

WHOSE INTEGRAL OVER P IS THE EULER NUMBER OF $P \times_\rho V$.

FINALLY, WE ADOPT THE PRACTICE IN (SUPERSYMMETRIC) PHYSICS OF WRITING THE INTEGRAL OF A FORM AS A BEREZIN INTEGRAL FOLLOWED BY A VOLUME INTEGRAL:

$$\int_P \alpha = \int_P \int \alpha(x^i, \xi_i)\, \mathcal{D}\xi\, d\omega$$

APPLY THIS TO (6) AND (FINALLY!) SUPPRESS ALL BUT ONE OF THE INTEGRAL SIGNS TO GET THE ATIYAH-JEFFREY FORMULA FOR THE EULER NUMBER OF $P \times_\rho V$:

$$(7) \quad (2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \int \exp\Big(\tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T ds - i(dc^+_, \lambda) + i(\phi, R\lambda) + (c^+_, \eta)\Big) \mathcal{D}\xi\, \mathcal{D}\eta\, \mathcal{D}\psi\, d\lambda\, d\phi\, d\omega$$

NOW WE WANT TO APPLY THIS (FORMALLY) TO AN INFINITE-DIMENSIONAL VECTOR BUNDLE THAT ARISES NATURALLY IN DONALDSON THEORY.

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD WITH $b_2^+(M) > 1$ AND ODD

g = GENERIC RIEMANNIAN METRIC ON M

$SU(2) \hookrightarrow P \xrightarrow{\pi} M$ WITH

$$8c_2(P) - 3(1 + b_2^+(M)) = 0$$

PRINCIPAL BUNDLE :

$$\hat{\mathcal{G}}(P) \hookrightarrow \hat{\mathcal{A}}(P) \longrightarrow \hat{\mathcal{B}}(P)$$

(NOTATION FROM ADDENDUM 2)

$\hat{\mathcal{G}}(P)$- ACTION ON $\Omega^2_+(M, ad\,P)$ GIVES AN ASSOCIATED VECTOR BUNDLE

$$\hat{\mathcal{A}}(P) \times_{\hat{\mathcal{G}}(P)} \Omega^2_+(M, ad\,P) \longrightarrow \hat{\mathcal{B}}(P)$$

s_0 = 0-SECTION

s = SELF-DUAL CURVATURE SECTION = F^+

$$s([\omega]) = [\omega, F^+_\omega]$$

$$\mathfrak{M}(P,g) = \mathcal{S}(\hat{\mathcal{B}}(P)) \cap \mathcal{S}_0(\hat{\mathcal{B}}(P))$$

$$\chi_0(M) = \text{"EULER NUMBER"}$$

ATIYAH - JEFFREY (SKETCH) :

<u>NOTE</u> : DETAILS ARE AVAILABLE IN ADDENDUM 13.

REQUIRES A $\hat{\mathcal{G}}$ - INVARIANT RIEMANNIAN METRIC FOR $\hat{\mathcal{A}}$ (I WILL DROP THE " (P) " EVERYWHERE FROM THIS POINT ON).

$\hat{\mathcal{A}}$ OPEN IN $\mathcal{A}$, WHICH IS AN AFFINE SPACE

$$T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, ad\,P)$$

<u>RECALL</u> : ON $\Omega^k(M, ad\,P)$,

$$\langle \mu, \nu \rangle_k = \int_M -2\,tr\,(\mu \wedge * \nu)$$

IS INVARIANT UNDER ADJOINT ACTION.

THUS, $\langle\ ,\ \rangle_1$ ON $T_\omega(\hat{\mathcal{A}})$ GIVES $\hat{\mathcal{G}}$ - INVARIANT RIEMANNIAN METRIC ON $\hat{\mathcal{A}}$.

THIS GIVES A CORRESPONDING CONNECTION ON $\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$:

$$T_\omega(\hat{\mathcal{A}}) \cong T_\omega(\omega\cdot\hat{\mathcal{G}}) \oplus \mathrm{KER}(\delta^\omega) \equiv \mathrm{IM}(d^\omega) \oplus \mathrm{KER}(\delta^\omega)$$

$$d^\omega : \Omega^0(M, \mathrm{ad}\,P) \to \Omega^1(M, \mathrm{ad}\,P)$$

$$\delta^\omega : \Omega^1(M, \mathrm{ad}\,P) \to \Omega^0(M, \mathrm{ad}\,P)$$

(SEE ADDENDUM 2)

$$\mathrm{HOR}_\omega(\hat{\mathcal{A}}) \cong \mathrm{KER}(\delta^\omega)$$

NOW EXAMINE EACH TERM IN THE ATIYAH-JEFFREY EXPONENT

$$-\tfrac{1}{2}\|s(\omega)\|^2 \quad + \quad \tfrac{1}{2}\psi^T(\rho_*\phi)\psi \quad + \; i\,\psi^T ds_\omega(\xi)$$

$$-\,i\,(dC_\omega^*(\xi), \lambda) + \quad i\,(\phi, R_\omega\lambda) \quad + (C_\omega^*(\xi), \eta)$$

SEPARATELY.

1. $$-\tfrac{1}{2}\|s(\omega)\|^2 = -\tfrac{1}{2}\|F_\omega^+\|^2 = \int_M \mathrm{tr}(F_\omega^+ \wedge {}^*F_\omega^+)$$

$$= \int_M \mathrm{tr}(F_\omega^+ \wedge F_\omega^+)$$

$$\boxed{-\tfrac{1}{2}\|s(\omega)\|^2 = \underbrace{\tfrac{1}{2}\int_M \mathrm{tr}(F_\omega \wedge {}^*F_\omega)}_{\text{YANG-MILLS TERM}} + \underbrace{\tfrac{1}{2}\int_M \mathrm{tr}(F_\omega \wedge F_\omega)}_{\text{TOPOLOGICAL TERM}}}$$

DONALDSON THEORY ANALOGUES OF C, C^* AND R :

$$\text{LIE ALGEBRA OF } \hat{\mathcal{G}} \cong \Omega^0(M, \operatorname{ad} P)$$

$$T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, \operatorname{ad} P)$$

SO $\forall\, \omega \in \hat{\mathcal{A}}$

$$C_\omega : \Omega^0(M, \operatorname{ad} P) \longrightarrow \Omega^1(M, \operatorname{ad} P)$$

$$C_\omega(\xi) = \frac{d}{dt}(\omega \cdot \exp(t\xi))\Big|_{t=0} = d^\omega \xi$$

$$C_\omega = d^\omega$$

THUS, RELATIVE TO $\langle\, ,\, \rangle_0$ AND $\langle\, ,\, \rangle_1$,

$$C_\omega^* : \Omega^1(M, \operatorname{ad} P) \rightarrow \Omega^0(M, \operatorname{ad} P)$$

$$C_\omega^* = \delta^\omega = -{*d^\omega *}$$

AND

$$R_\omega : \Omega^0(M, \operatorname{ad} P) \rightarrow \Omega^0(M, \operatorname{ad} P)$$

$$R_\omega = C_\omega^* \circ C_\omega = \delta^\omega \circ d^\omega = \Delta_0^\omega = -{*d^\omega * d^\omega}$$

= SCALAR LAPLACIAN OF ω

2. $i(\phi, R_\omega \lambda)$

BOTH ϕ AND λ ARE IN THE LIE ALGEBRA OF $\hat{\mathcal{G}}$ SO INTRODUCE TWO "BOSONIC" FIELDS

$$\phi, \lambda \in \Omega^0(M, \mathrm{ad}\, P)$$

AND INTERPRET $(\ , \ )$ AS $\langle \ , \ \rangle_0$. FOR EACH $\omega \in \hat{\mathcal{A}}$,

$$\boxed{i(\phi, R_\omega \lambda) = 2i \int_M \mathrm{tr}\, (\phi\, d^\omega * d^\omega \lambda)}$$

3. $(C_\omega^*(\xi), \eta)$

$C_\omega^* : \Omega^1(M, \mathrm{ad}\, P) \to \Omega^0(M, \mathrm{ad}\, P)$ SO INTRODUCE TWO "FERMIONIC" FIELDS

$$\xi \in \Omega^1(M, \mathrm{ad}\, P)$$

$$\eta \in \Omega^0(M, \mathrm{ad}\, P)$$

AND IDENTIFY $(\ , \ ) = \langle \ , \ \rangle_0$.

$$(C_\omega^*(\xi), \eta) = \langle C_\omega^*(\xi), \eta \rangle_0 = \langle \xi, C_\omega(\eta) \rangle_1$$

$$= \langle \xi, d^\omega \eta \rangle_1$$

$$\boxed{(C_\omega^*(\xi), \eta) = -2 \int_M \mathrm{tr}\, (\xi \wedge d^\omega \eta)}$$

4. $i\,\psi^T ds_\omega(\xi)$

$$s = F^+ : \hat{\mathcal{A}} \longrightarrow \Omega^2_+(M, \mathrm{ad}\,P)$$

$$ds_\omega = d^\omega_+ : \Omega^1(M, \mathrm{ad}\,P) \longrightarrow \Omega^2_+(M, \mathrm{ad}\,P)$$

$$ds_\omega(\xi) = d^\omega_+ \xi$$

INTRODUCE "FERMIONIC" FIELD

$$\psi \in \Omega^2_+(M, \mathrm{ad}\,P)$$

AND INTERPRET FINITE-DIMENSIONAL EXPRESSIONS

$$A^T B = (A^1 \cdots A^r) \begin{pmatrix} B^1 \\ \vdots \\ B^r \end{pmatrix}$$

AS THE APPROPRIATE INNER PRODUCT ON FIELDS.

$$i\,\psi^T ds_\omega(\xi) = i \langle \psi, d^\omega_+(\xi) \rangle_2$$

$$\boxed{i\,\psi^T ds_\omega(\xi) = -2i \int_M d^\omega \xi \wedge \psi}$$

5. $\frac{1}{2}\psi^T (\rho_* \phi)\, \psi$

ρ IS THE REPRESENTATION OF $\hat{\mathcal{G}}$ ON $\Omega^2_+(M, ad\,P)$ GIVING RISE TO THE ASSOCIATED VECTOR BUNDLE. REGARDING $\hat{\mathcal{G}}$ AS SECTIONS OF $Ad\,P$ THIS IS, POINTWISE, THE ADJOINT ACTION OF $SU(2)$ ON $su(2)$. INFINITESIMAL ACTION ρ_* IS BRACKET.

$$(\rho_* \phi)\psi = [\phi, \psi]$$

SO

$$\tfrac{1}{2}\psi^T(\rho_* \phi)\psi = \tfrac{1}{2}\langle [\phi,\psi], \psi\rangle_2$$

$$\boxed{\tfrac{1}{2}\psi^T(\rho_*\phi)\psi = -\int_M tr\,(\psi \wedge [\phi,\psi])}$$

6. $-i\,(dC^*_\omega(\zeta), \lambda)$

dC^* IS A 2-FORM ON $\hat{A}$ WITH VALUES IN $\Omega^0(M, ad\,P)$

COMPUTING dC^*_ω REQUIRES A LITTLE WORK (ADDENDUM 13), BUT THE RESULT IS

$$dC^*_\omega(\zeta_1, \zeta_2) = -2\,{}^*[\zeta_1, {}^*\zeta_2]$$

$$-i\,(dC^*_\omega(\zeta), \lambda) = 2i\,\langle {}^*[\zeta, {}^*\zeta], \lambda\rangle_0$$

$$\boxed{-i\,(dC_\omega^*(\xi),\lambda) = -4i\int_M tr\,([\xi,{}^*\xi]\lambda)}$$

ADDING TOGETHER ALL OF THE TERMS IN THE BOXES AND (IN DEFERENCE TO THE PHYSICS LITERATURE) SWITCHING TO

$$Tr = -2\,tr$$

GIVES

$$\int_M Tr\,\{-\tfrac{1}{4}F_\omega\wedge{}^*F_\omega - \tfrac{1}{4}F_\omega\wedge F_\omega + \tfrac{1}{2}\psi\wedge[\phi,\psi] + i\,d^\omega\xi\wedge\psi$$
$$+ 2i\,[\xi,{}^*\xi]\lambda - i\,\phi\, d^{\omega *}d^\omega\lambda + \xi\wedge{}^*d^\omega\eta\ \}$$

WRITING THE INTEGRAL IN (7) IN THE FORM

$$\int e^{-\int_M Tr\,\mathcal{L}_{DW}}\,\mathcal{D}\Phi$$

WE IDENTIFY THE <u>DONALDSON</u>-<u>WITTEN</u> <u>LAGRANGIAN</u>:

$$\mathcal{L}_{DW}[\Phi] = \tfrac{1}{4}F_\omega\wedge{}^*F_\omega + \tfrac{1}{4}F_\omega\wedge F_\omega - \tfrac{1}{2}\psi\wedge[\phi,\psi] - i\,d^\omega\xi\wedge\psi$$
$$-2i\,[\xi,{}^*\xi]\lambda + i\,\phi\, d^{\omega *}d^\omega\lambda - \xi\wedge{}^*d$$

<u>ACTION</u>: $\quad S_{DW}[\Phi] = \int_M Tr\,\mathcal{L}_{DW}$

<u>PARTITION</u> <u>FUNCTION</u>: $\quad Z_{DW} = \int e^{-S_{DW}[\Phi]/e^2}\,\mathcal{D}\Phi$

SEIBERG-WITTEN THEORY : PROLOGUE

WITTEN'S 1988 TOPOLOGICAL QUANTUM FIELD THEORY (TQFT) :

M AS BEFORE WITH RIEMANNIAN METRIC g. FIX SOME

$$SU(2) \hookrightarrow P \xrightarrow{\pi} M$$

FIELD CONTENT :

GAUGE FIELD (CONNECTION) ω WITH CURVATURE $F_\omega \in \Omega^2(M, \mathrm{ad}P)$

+ " MATTER FIELDS "

BOSONIC	FERMIONIC
$\phi \in \Omega^0(M, \mathrm{ad}P)$	$\eta \in \Omega^0(M, \mathrm{ad}P)$
$\lambda \in \Omega^0(M, \mathrm{ad}P)$	$\psi \in \Omega^1(M, \mathrm{ad}P)$
	$\chi \in \Omega^2_+(M, \mathrm{ad}P)$

$$\Phi = (\omega, \phi, \lambda, \eta, \psi, \chi)$$

+ " GHOST NUMBERS " + " SUPERSYMMETRY OPERATOR "

+ DONALDSON-WITTEN ACTION $S_{DW}[\Phi]$

$$S_{DW}[\Phi] = \int_M Tr\,\{ \tfrac{1}{4} F_\omega \wedge {}^*F_\omega + \tfrac{1}{4} F_\omega \wedge F_\omega - \tfrac{1}{2} \psi \wedge [\phi, \psi]$$
$$- i\, d^\omega \xi \wedge \psi - 2i\, [\xi, {}^*\xi]\,\lambda + i\phi\, d^{\omega *} d^\omega \lambda$$
$$- \xi \wedge {}^* d^\omega \eta \,\}$$

PARTITION FUNCTION: $\quad Z_{DW} = \int e^{-S_{DW}[\Phi]/e^2}\, \mathcal{D}\Phi$

EXPECTATION VALUES OF OBSERVABLES $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \int e^{-S_{DW}[\Phi]/e^2}\, \mathcal{O}[\Phi]\, \mathcal{D}\Phi$$

SYMMETRIES BUILT INTO S_{DW} + FORMAL PATH INTEGRAL MANIPULATIONS "PROVE" THAT THESE ARE INDEPENDENT OF THE CHOICES OF g AND e.

FORMALLY COMPUTING Z_{DW} IN THE "WEAK COUPLING LIMIT" $e \to 0$, THE STATIONARY PHASE APPROXIMATION HAPPENS TO BE EXACT AND THE INTEGRAL LOCALIZES TO THE ANTI-SELF-DUAL MODULI SPACE GIVING, WHEN $8k - 3(1 + b_2^+(M)) = 0$, THE 0-DIMENSIONAL DONALDSON INVARIANT.

$$Z_{DW} = \gamma_0(M)$$

WITTEN SIMILARLY OBTAINS THE REMAINING DONALDSON INVARIANTS (FORMALLY) AS EXPECTATION VALUES FOR A FAMILY OF OBSERVABLES PARAMETRIZED BY $x \in H_2(M;\mathbb{Z})$.

<u>DUALITY</u> <u>IN</u> <u>WITTEN'S</u> <u>TQFT</u> : 1988 - SPRING, 1994

$e \to 0$	$e \to \infty$
WEAK COUPLING	STRONG COUPLING
ULTRAVIOLET	INFRARED
NON-ABELIAN	ABELIAN
PERTURBATIVE	NONPERTURBATIVE
COMPUTABLE	INTRACTIBLE
$\boxed{\text{DONALDSON INVARIANTS}}$	$\boxed{?}$

SEIBERG-WITTEN (FALL, 1994) : EXACT SOLUTIONS IN THE INFRARED

$\boxed{\text{SEIBERG-WITTEN INVARIANTS}}$

THESE SEIBERG-WITTEN INVARIANTS ARISE IN MUCH THE SAME WAY AS THE DONALDSON INVARIANTS FROM MODULI SPACES OF SOLUTIONS TO THE "SEIBERG-WITTEN EQUATIONS".

ALGEBRAIC PRELIMINARIES (MORE DETAILS IN ADDENDUM 14)

MATRIX MODEL OF THE (REAL) ALGEBRA $\mathbb{H}$ OF QUATERNIONS :

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$q = q^1 \mathbb{1} + q^2 I + q^3 J + q^4 K$$

$$\bar{q} = q^1 \mathbb{1} - q^2 I - q^3 J - q^4 K$$

$=$ CONJUGATE TRANSPOSE OF THE MATRIX q

MATRIX MODEL OF $\mathbb{R}^4$: ALL 4×4 COMPLEX MATRICES OF THE FORM

$$x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$$

WITH

$$\|x\|^2 = \det x$$

$$\langle x, y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 \right).$$

ORTHONORMAL BASIS FOR $\mathbb{R}^4$:

$$E_1 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

SATISFIES

$$E_i E_j + E_j E_i = -2 \langle E_i, E_j \rangle \mathbb{1}, \quad i, j = 1, 2, 3, 4$$

REAL SUBALGEBRA OF $\mathbb{C}^{4 \times 4}$ GENERATED BY $\{E_1, E_2, E_3, E_4\}$ IS THE REAL CLIFFORD ALGEBRA OF $\mathbb{R}^4$ AND IS DENOTED $Cl(\mathbb{R}^4)$.

COMPLEX SUBALGEBRA OF $\mathbb{C}^{4 \times 4}$ GENERATED BY $\{E_1, E_2, E_3, E_4\}$ IS THE COMPLEX CLIFFORD ALGEBRA OF $\mathbb{R}^4$ AND IS DENOTED $Cl(\mathbb{R}^4) \otimes \mathbb{C}$.

BASIS :

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

$$E_1E_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad E_1E_3 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad E_1E_4 = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}$$

$$E_2E_3 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad E_2E_4 = \begin{pmatrix} -J & 0 \\ 0 & -J \end{pmatrix}, \quad E_3E_4 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$E_1E_2E_3 = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}, \quad E_1E_2E_4 = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}$$

$$E_1E_3E_4 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad E_2E_3E_4 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$E_1E_2E_3E_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbb{Z}_2$ - GRADINGS :

$$Cl(\mathbb{R}^4) \cong Cl_0(\mathbb{R}^4) \oplus Cl_1(\mathbb{R}^4)$$

DIAGONAL $\oplus$ ANTI-DIAGONAL

EVEN $\oplus$ ODD

AND SIMILARLY FOR $Cl(\mathbb{R}^4) \otimes \mathbb{C}$

CENTERS :

$$Z(Cl(\mathbb{R}^4)) = \mathrm{SPAN}_{\mathbb{R}}\{E_0\} \cong \mathbb{R}$$

$$Z(Cl(\mathbb{R}^4) \otimes \mathbb{C}) = \mathrm{SPAN}_{\mathbb{C}}\{E_0\} \cong \mathbb{C}$$

IDENTIFY $U(1)$ WITH THE SUBSET

$$U(1) = \{ e^{\theta i} E_0 : \theta \in \mathbb{R} \}$$

OF $Cl(\mathbb{R}^4) \otimes \mathbb{C}$ (AND GENERALLY WRITE $e^{\theta i}$ FOR $e^{\theta i} E_0 = e^{\theta i} 1$).

IDENTIFY $\mathbb{R}^4 = \text{SPAN}_{\mathbb{R}}\{E_1, E_2, E_3, E_4\} \subseteq Cl(\mathbb{R}^4) \subseteq Cl(\mathbb{R}^4) \otimes \mathbb{C}$.

$$xy + yx = -2\langle x, y\rangle \mathbb{1} \qquad \forall\, x, y \in \mathbb{R}^4$$

$Cl^{\times}(\mathbb{R}^4)$ = MULTIPLICATIVE GROUP OF UNITS IN $Cl(\mathbb{R}^4)$

$x \in \mathbb{R}^4$ WITH $\|x\| = 1 \Rightarrow x \in Cl^{\times}(\mathbb{R}^4)$ AND $x^{-1} = -x$

$\text{PIN}(\mathbb{R}^4)$ = SUBGROUP OF $Cl^{\times}(\mathbb{R}^4)$ GENERATED BY $\{x \in \mathbb{R}^4 : \|x\| = 1\} = S^3$

$$\text{SPIN}(\mathbb{R}^4) = \text{PIN}(\mathbb{R}^4) \cap Cl_0(\mathbb{R}^4)$$

(USUALLY WRITTEN SPIN(4))

<u>THEOREM</u> : $\text{SPIN}(4) = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : U_1, U_2 \in SU(2) \right\}$ IS THE UNIVERSAL DOUBLE COVER OF $SO(4)$.

$$\text{SPIN} : \text{SPIN}(4) \longrightarrow SO(\mathbb{R}^4) \cong SO(4)$$

$$(\text{SPIN}(\mu))(x) = \mu x \mu^{-1}$$

COMPLEX ANALOGUE OF SPIN(4) :

$\text{SPIN}^c(4)$ = SUBGROUP OF $Cl^{\times}(\mathbb{R}^4) \otimes \mathbb{C}$ GENERATED BY SPIN(4) AND $U(1)$.

$$= \left\{ e^{\theta i} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : \theta \in \mathbb{R},\ U_1, U_2 \in SU(2) \right\}$$

$$= \left\{ \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} : U_{\pm} \in U(2),\ \det U_+ = \det U_- \right\}$$

SPIN(4) AND $SPIN^c(4)$ ARE COMPACT LIE GROUPS.

$$SPIN(4) \times U(1) \longrightarrow SPIN^c(4)$$

$$(\mu, e^{\theta i}) \longrightarrow e^{\theta i}\mu$$

SURJECTIVE HOMOMORPHISM WITH KERNEL $\mathbb{Z}_2 = \pm(\mathbb{1}, \mathbb{1})$

$$SPIN^c(4) \cong SPIN(4) \times U(1)/\mathbb{Z}_2$$

<u>SOME MAPPINGS</u>:

1. $\boxed{\delta : SPIN^c(4) \rightarrow U(1)}$

$$\delta(\xi) = \delta\begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} = \delta\begin{pmatrix} e^{\theta i}U_1 & 0 \\ 0 & e^{\theta i}U_2 \end{pmatrix}$$

$$= \det U_+ = \det U_- = e^{2\theta i}$$

SURJECTIVE HOMOMORPHISM WITH KERNEL SPIN(4)

2. $\boxed{\pi : SPIN^c(4) \longrightarrow SO(4)}$

$$\pi(\xi) = \pi(e^{\theta i}\mu) = ad_\xi = ad_\mu \in SO(\mathbb{R}^4) \cong SO(4)$$

3. $\boxed{SPIN^c : SPIN^c(4) \longrightarrow SO(4) \times U(1)}$

$$SPIN^c(\xi) = SPIN^c(e^{\theta i}\mu) = (\pi(\xi), \delta(\xi))$$

$$= (SPIN(\mu), e^{2\theta i})$$

SURJECTIVE HOMOMORPHISM WITH KERNEL $\mathbb{Z}_2 = \pm\mathbb{1}$

LIE ALGEBRAS OF SPIN(4) AND SPIN^c(4) :

$$\mathfrak{spin}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \equiv \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1, A_2 \in \mathfrak{su}(2) \right\}$$

$$\mathfrak{spin}^c(4) \cong \mathfrak{spin}(4) \oplus \mathfrak{u}(1) \equiv \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + ti \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A_1, A_2 \in \mathfrak{su}(2),\ t \in \mathbb{R} \right\}$$

$Cl(\mathbb{R}^4) \otimes \mathbb{C} \subseteq \mathbb{C}^{4\times 4}$ AND $\dim_{\mathbb{C}} (Cl(\mathbb{R}^4) \otimes \mathbb{C}) = 16 \Rightarrow$

$$Cl(\mathbb{R}^4) \otimes \mathbb{C} = \mathbb{C}^{4\times 4} \equiv END_{\mathbb{C}} (S_{\mathbb{C}})$$

$$S_{\mathbb{C}} = \mathbb{C}^4 \quad \text{WITH} \quad \langle z, w \rangle = \bar{z}^1 w^1 + \bar{z}^2 w^2 + \bar{z}^3 w^3 + \bar{z}^4 w^4$$

THE ELEMENTS OF $Cl(\mathbb{R}^4) \otimes \mathbb{C}$ (IN PARTICULAR, THOSE IN $\mathbb{R}^4$, $Cl(\mathbb{R}^4)$, SPIN(4) AND SPIN^c(4)) ALL ACT ON $S_{\mathbb{C}}$.

THIS ACTION IS CALLED CLIFFORD MULTIPLICATION

AND IS DENOTED WITH A DOT $\cdot$

RESULTING REPRESENTATIONS :

1. $Cl(\mathbb{R}^4) \longrightarrow END_{\mathbb{C}} (S_{\mathbb{C}})$ (IRREDUCIBLE)

2. $\Delta_{\mathbb{C}} : SPIN(4) \longrightarrow AUT_{\mathbb{C}} (S_{\mathbb{C}})$

THIS IS NOT IRREDUCIBLE AS ONE SEES BY WRITING

$$S_{\mathbb{C}} \cong S_{\mathbb{C}}^{+} \oplus S_{\mathbb{C}}^{-}$$

$$\begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix} = \begin{pmatrix} z^1 \\ z^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z^3 \\ z^4 \end{pmatrix}$$

CLIFFORD MULTIPLICATION BY ELEMENTS OF $Cl_0(\mathbb{R}^4)$ PRESERVES $S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$, WHEREAS CLIFFORD MULTIPLICATION BY ELEMENTS OF $Cl_1(\mathbb{R}^4)$ REVERSES $S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$. $\quad SPIN(4) \subseteq Cl_0(\mathbb{R}^4) \Rightarrow$

$$\Delta_{\mathbb{C}} = \Delta_{\mathbb{C}}^{+} \oplus \Delta_{\mathbb{C}}^{-}$$

$$\Delta_{\mathbb{C}}^{\pm} : SPIN(4) \longrightarrow SU(S_{\mathbb{C}}^{\pm})$$

$\Delta_{\mathbb{C}}^{+}$ AND $\Delta_{\mathbb{C}}^{-}$ ARE INEQUIVALENT, IRREDUCIBLE REPRESENTATIONS OF SPIN(4) ON $\mathbb{C}^2$.

3. $$\hat{\Delta}_{\mathbb{C}} : SPIN^c(4) \longrightarrow AUT_{\mathbb{C}}(S_{\mathbb{C}})$$

$$\hat{\Delta}_{\mathbb{C}} = \hat{\Delta}_{\mathbb{C}}^{+} \oplus \hat{\Delta}_{\mathbb{C}}^{-}$$

$$\hat{\Delta}_{\mathbb{C}}^{\pm} : SPIN^c(4) \longrightarrow U(S_{\mathbb{C}}^{\pm})$$

A FEW MORE IMPORTANT ACTIONS :

4. $\mathbb{R}^4 \subseteq Cl_1(\mathbb{R}^4)$ ACTS BY CLIFFORD MULTIPLICATION ON $S_{\mathbb{C}}$, REVERSING $S_{\mathbb{C}}^+$ AND $S_{\mathbb{C}}^-$.

$$\mathbb{R}^4 \cdot (\,) : S_{\mathbb{C}}^{\pm} \longrightarrow S_{\mathbb{C}}^{\mp}$$

5. $SPIN^c(4)$ ACTS ON $\mathbb{C}$ VIA $\delta : SPIN^c(4) \to U(1)$ AND COMPLEX MULTIPLICATION.

$$\xi \cdot z = (e^{\theta i} u)\cdot z = \delta(\xi) z = e^{2\theta i} z$$

6. $SO(4)$ ACTS ON $Cl(\mathbb{R}^4)$:

SPIN(4) ACTS ON $Cl(\mathbb{R}^4)$ BY CONJUGATION

$$u \cdot p = u p u^{-1}$$

BUT $(-u)p(-u)^{-1} = upu^{-1}$ SO THIS DESCENDS TO AN ACTION OF $SO(4) \cong SPIN(4)/\mathbb{Z}_2$ ON $Cl(\mathbb{R}^4)$.

7. COMPLEX-VALUED 2-FORMS $\Omega^2(\mathbb{R}^4)$ ON $\mathbb{R}^4$ ACT ON $S_{\mathbb{C}}$ VIA THE LINEAR ISOMORPHISM

$$\rho : \Omega^2(\mathbb{R}^4) \hookrightarrow Cl_0(\mathbb{R}^4) \otimes \mathbb{C}$$

$$\rho(\eta) = \rho\Big(\sum_{i<j} \eta_{ij} E^i \wedge E^j\Big) = \sum_{i<j} \eta_{ij} E_i E_j =$$

$$\begin{pmatrix} (\eta_{12}+\eta_{34})I + (\eta_{13}-\eta_{24})J + (\eta_{14}+\eta_{23})K & 0 \\ 0 & (-\eta_{12}+\eta_{34})I + (-\eta_{13}-\eta_{24})J + (-\eta_{14}+\eta_{23})K \end{pmatrix}$$

η REAL-VALUED $\Rightarrow$ $\rho(\eta)$ SKEW-HERMITIAN

η Im $\mathbb{C}$-VALUED $\Rightarrow$ $\rho(\eta)$ HERMITIAN

$S^+_{\mathbb{C}}$ AND $S^-_{\mathbb{C}}$ INVARIANT UNDER ANY $\rho(\eta)$

$$\rho^{\pm}(\eta) = \rho(\eta) | S^{\pm}_{\mathbb{C}}$$

E.G., SUPPRESSING THE TWO ZERO ENTRIES

$$\rho^+(\eta) = (\eta_{12} + \eta_{34}) I + (\eta_{13} + \eta_{42}) J + (\eta_{14} + \eta_{23}) K$$

$$\rho^{\pm} : \Omega^2(\mathbb{R}^4) \longrightarrow \mathrm{END}_{\mathbb{C}}(S^{\pm}_{\mathbb{C}})$$

$$\Omega^2(\mathbb{R}^4) \cong \Omega^2_+(\mathbb{R}^4) \oplus \Omega^2_-(\mathbb{R}^4)$$

$$\rho^{\pm} | \Omega^2_{\pm}(\mathbb{R}^4) : \Omega^2_{\pm}(\mathbb{R}^4) \twoheadrightarrow \mathrm{END}_0(S^{\pm}_{\mathbb{C}})$$

(COMPLEX LINEAR ISOMORPHISM)

INVERSES :

$$\sigma^{\pm} : \mathrm{END}_0(S^{\pm}_{\mathbb{C}}) \longrightarrow \Omega^2_{\pm}(\mathbb{R}^4)$$

<u>EXAMPLE</u> : LET $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \in S^+_{\mathbb{C}}$. GIVES AN ENDOMORPHISM OF $S^+_{\mathbb{C}}$:

$$\psi \otimes \psi^* = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} (\bar{\psi}^1 \ \ \bar{\psi}^2) = \begin{pmatrix} |\psi^1|^2 & \psi^1 \bar{\psi}^2 \\ \bar{\psi}^1 \psi^2 & |\psi^2|^2 \end{pmatrix}$$

TRACE FREE PART IS

$$(\psi \otimes \psi^*)_0 = \psi \otimes \psi^* - \tfrac{1}{2}\,tr(\psi \otimes \psi^*)\,\mathbb{1}$$

$$= \begin{pmatrix} \tfrac{1}{2}(|\psi^1|^2 - |\psi^2|^2) & \psi^1\bar{\psi}^2 \\ \bar{\psi}^1\psi^2 & \tfrac{1}{2}(|\psi^2|^2 - |\psi^1|^2) \end{pmatrix}$$

ONE VERIFIES THAT

$$\sigma^+((\psi \otimes \psi^*)_0) = -\tfrac{1}{4}\left\{ (\psi^* I \psi)(E^1 \wedge E^2 + E^3 \wedge E^4) + (\psi^* J \psi)(E^1 \wedge E^3 + E^4 \wedge E^2) + (\psi^* K \psi)(E^1 \wedge E^4 + E^2 \wedge E^3) \right\}$$

NOW GLOBALIZE THESE ALGEBRAIC CONSTRUCTIONS TO BUNDLES OVER

M = COMPACT, SIMPLY CONNECTED, ORIENTED SMOOTH 4-MANIFOLD + RESTRICTIONS ON $b_2^+(M)$

ANY CHOICE OF RIEMANNIAN METRIC g FOR M GIVES

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M$$

WE FIX THE LEVI-CIVITA CONNECTION ω_{LC} ON THIS FRAME BUNDLE.

<u>NOTE</u> : FOR MORE DETAILS ON THE CONSTRUCTIONS WHICH FOLLOW, SEE <u>ADDENDUM</u> <u>15</u> .

A <u>SPIN^c STRUCTURE</u> $\mathcal{L}$ FOR M CONSISTS OF A $SPIN^c(4)$-BUNDLE

$$SPIN^c(4) \hookrightarrow S^c(M) \xrightarrow{\pi_{S^c}} M$$

OVER M AND A SMOOTH MAP $\Lambda : S^c(M) \longrightarrow F_{SO}(M)$ SATISFYING

$$\begin{tikzcd} S^c(M) \arrow[rr, "\Lambda"] \arrow[dr, "\pi_{S^c}"'] & & F_{SO}(M) \arrow[dl, "\pi_{SO}"] \\ & M & \end{tikzcd} \qquad \pi_{SO} \circ \Lambda = \pi_{S^c}$$

AND

$$\Lambda(p \cdot \xi) = \Lambda(p) \cdot \pi(\xi)$$

WHERE $\pi : SPIN^c(4) \longrightarrow SO(4)$ IS MAP #2 ON P. 7.

<u>NOTE</u>: ANY COMPACT, ORIENTED 4-MANIFOLD ADMITS A SPIN^c STRUCTURE FOR ANY CHOICE OF g (HIRZEBRUCH-HOPF) AND, IN THE SIMPLY CONNECTED CASE, EQUIVALENCE CLASSES OF SPIN^c STRUCTURES ON M ARE IN ONE-TO-ONE CORRESPONDENCE WITH THE ELEMENTS OF $H^2(M;\mathbb{Z})$ WHOSE MOD 2 REDUCTION IS THE SECOND STIEFEL-WHITNEY CLASS $w_2(M)$ OF M.

GIVEN A SPIN^c STRUCTURE WE HAVE THE FOLLOWING ASSOCIATED BUNDLES:

$$\mathcal{S}(\mathcal{L}) = S^c(M) \times_{\hat{\Delta}_{\mathbb{C}}} S_{\mathbb{C}}$$ (SPINOR BUNDLE)

$$\mathcal{S}^{\pm}(\mathcal{L}) = S^c(M) \times_{\hat{\Delta}^{\pm}_{\mathbb{C}}} S^{\pm}_{\mathbb{C}}$$ (POSITIVE AND NEGATIVE SPINOR BUNDLES)

$$L(\mathcal{L}) = S^c(M) \times_{\delta} \mathbb{C}$$ (DETERMINANT LINE BUNDLE)

SOME OTHER RELEVANT BUNDLES :

$$U(1) \hookrightarrow L^{\circ}(\mathcal{L}) \xrightarrow{\pi_{L^{\circ}}} M$$ (ORIENTED, ORTHONORMAL FRAME BUNDLE FOR SOME FIBER METRIC ON $L(\mathcal{L})$)

$$Cl(M) = F_{SO}(M) \times_{SO(4)} Cl(4)$$ (CLIFFORD BUNDLE)

$$Cl(M) \otimes \mathbb{C} = F_{SO}(M) \times_{SO(4)} (Cl(4) \otimes \mathbb{C})$$ (COMPLEXIFIED CLIFFORD BUNDLE)

NOTE : CAN SHOW THAT $w_2(M) = c_1(L^{\circ}(\mathcal{L}))$ MOD 2 AND, CONVERSELY, GIVEN A PRINCIPAL $U(1)$-BUNDLE L° OVER M WITH $w_2(M) = c_1(L^{\circ})$ MOD 2, THERE IS A SPIN^c STRUCTURE $\mathcal{L}$ FOR M WITH $L^{\circ}(\mathcal{L}) = L^{\circ}$.

GLOBAL VERSION OF $\sigma^+ : \mathrm{END}_0(S^+_{\mathbb{C}}) \to \Omega^2_+(\mathbb{R}^4)$:

$$\sigma^+ : \Gamma(\mathrm{END}_0(\mathcal{S}^+(\mathcal{L}))) \longrightarrow \Omega^2_+(M)$$

IDENTIFIES SECTIONS OF THE TRACE FREE ENDOMORPHISM BUNDLE OF THE POSITIVE SPINOR BUNDLE WITH SELF-DUAL 2-FORMS ON M.

FIELD CONTENT OF SEIBERG - WITTEN THEORY :

GAUGE FIELD : CONNECTION A ON $U(1) \hookrightarrow L^0(\mathcal{L}) \to M$

POSITIVE SPINOR FIELD : SECTION $\psi \in \Gamma(S^+(\mathcal{L}))$

CURVATURE OF A IS THE $\mathfrak{u}(1)$-VALUED 2-FORM dA ON $L^0(\mathcal{L})$, BUT $U(1)$ IS ABELIAN SO THIS IS UNIQUELY DETERMINED BY A $\mathfrak{u}(1)$-VALUED 2-FORM F_A ON M (GAUGE POTENTIAL)

$$\psi \to (\psi \otimes \psi^*)_0 \in \Gamma(\mathrm{END}_0(S^+(\mathcal{L}))) \to \sigma^+((\psi \otimes \psi^*)_0) \in \Omega^2_+(M)$$

1^{ST} **SEIBERG - WITTEN EQUATION** : $F_A^+ = \sigma^+((\psi \otimes \psi^*)_0)$

FOR THE 2^{ND} SEIBERG - WITTEN EQUATION WE INTRODUCE A **DIRAC OPERATOR** ON SPINOR FIELDS :

$\mathrm{SPIN}^c(4)$ DOUBLE COVERS $SO(4) \times U(1)$ BY THE MAP $\mathrm{SPIN}^c = (\pi, \delta)$ SO $S^c(M)$ DOUBLE COVERS THE FIBER PRODUCT $F_{SO}(M) \dot{\times} L^0(\mathcal{L})$. DENOTE THE MAP BY SPIN^c ALSO.

$$\begin{tikzcd}
\mathrm{SPIN}^c(4) \arrow[r, hook] & S^c(M) \arrow[r] \arrow[d, "\mathrm{SPIN}^c"] & M \\
SO(4) \times U(1) \arrow[r, hook] & F_{SO}(M) \dot{\times} L^0(\mathcal{L}) \arrow[r] & M
\end{tikzcd}$$

THIS GIVES

$$\begin{array}{ccccc} SPIN^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \big\downarrow {\scriptstyle Pr_F \circ SPIN^c} & & \\ SO(4) & \hookrightarrow & F_{SO}(M) & \longrightarrow & M \end{array} \qquad \begin{array}{ccccc} SPIN^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \big\downarrow {\scriptstyle Pr_{L^0} \circ SPIN^c} & & \\ U(1) & \hookrightarrow & L^0(\mathcal{L}) & \longrightarrow & M \end{array}$$

FIXED CONNECTION : ω_{LC} SW CONNECTION : A

<u>SPIN^c</u> <u>CONNECTION</u>

$$\omega_A = (SPIN^c)^*(Pr_F{}^*\omega_{LC} + Pr_{L^0}{}^*A)$$

ω_A GIVES COVARIANT DERIVATIVES ON SECTIONS OF ASSOCIATED BUNDLES, E.G.,

$$\nabla_A : \Gamma(S(\mathcal{L})) \longrightarrow \Omega^1(M) \otimes \Gamma(S(\mathcal{L}))$$

$$\Psi \in \Gamma(S(\mathcal{L})) \rightarrow \nabla_A \Psi \in \Omega^1(M) \otimes \Gamma(S(\mathcal{L}))$$

$$\nabla_A \Psi(V) \in \Gamma(S(\mathcal{L}))$$

FOR EVERY VECTOR FIELD V ON M

LET $\{E_1, E_2, E_3, E_4\}$ BE A LOCAL ORIENTED, ORTHONORMAL FRAME FIELD ON M, I.E., A SECTION OF $F_{SO}(M)$.

EACH E_i CAN BE REGARDED EITHER AS A VECTOR FIELD ON M (SO $\nabla_A \Psi(E_i)$ MAKES SENSE) OR AS A SECTION OF THE CLIFFORD BUNDLE $Cl(M)$ WHICH THEREFORE ACTS BY CLIFFORD MULTIPLICATION ON SECTIONS OF $S(\mathcal{L})$.

$$\tilde{\not{D}}_A \Psi = \sum_{i=1}^{4} E_i \cdot \nabla_A \Psi(E_i)$$

$$S(\mathcal{L}) = S^+(\mathcal{L}) \oplus S^-(\mathcal{L})$$

CLIFFORD MULTIPLICATION BY E_i INTERCHANGES $S^{\pm}(\mathcal{L})$. RESTRICTING $\tilde{\not{D}}_A$ TO $S^{\pm}(\mathcal{L})$ THEREFORE GIVES

$$\not{D}_A : \Gamma(S^+(\mathcal{L})) \longrightarrow \Gamma(S^-(\mathcal{L}))$$

$$\not{D}_A^{\,*} : \Gamma(S^-(\mathcal{L})) \longrightarrow \Gamma(S^+(\mathcal{L}))$$

$\not{D}_A$ IS OUR <u>DIRAC OPERATOR</u> (AND $\not{D}_A^{\,*}$ IS ITS ADJOINT RELATIVE TO THE NATURAL INNER PRODUCTS ON SPACES OF SECTIONS).

2^{ND} <u>SEIBERG-WITTEN EQUATION</u>: $\not{D}_A \psi = 0$

<u>SEIBERG-WITTEN</u>: GIVEN M, SELECT RIEMANNIAN METRIC g AND SPIN^c STRUCTURE $\mathcal{L}$ FOR THE CORRESPONDING $F_{SO}(M)$. <u>SW CONFIGURATION SPACE</u> IS

$$\mathcal{A}(\mathcal{L}) = \{ (A, \psi) : A \text{ IS A CONNECTION ON } U(1) \hookrightarrow L^0(\mathcal{L}) \to M \text{ AND } \psi \in \Gamma(S^+(\mathcal{L})) \text{ IS A POSITIVE SPINOR FIELD ON } M \}$$

$(A, \psi) \in \mathcal{A}(\mathcal{L})$ IS A <u>SW MONOPOLE</u> IF IT SATISFIES

(SW1) $$F_A^{\,+} = \sigma^+((\psi \otimes \psi^*)_0)\,, \qquad \text{(CURVATURE EQUATION)}$$

(SW2) $$\not{D}_A \psi = 0 \qquad \text{(DIRAC EQUATION)}$$

<u>NOTE</u>: TO SEE WHAT THESE EQUATIONS LOOK LIKE ON $\mathbb{R}^4$, SOME EXPLICIT SOLUTIONS, AND SOME BOUNDEDNESS RESULTS, CONSULT <u>ADDENDUM 16</u>.

SEIBERG-WITTEN GAUGE GROUP $\mathcal{G}(\mathcal{L})$:

1. ALL AUTOMORPHISMS σ OF $S^c(M)$ THAT COVER THE IDENTITY ON $F_{SO}(M)$

$$Pr_F \circ \mathrm{SPIN}^c \circ \sigma = Pr_F \circ \mathrm{SPIN}^c$$

$$\begin{array}{ccccc} \mathrm{SPIN}^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \downarrow & & \\ SO(4) & \hookrightarrow & F_{SO}(M) & \longrightarrow & M \end{array}$$

2. $C^\infty(M, U(1))$

$$\gamma \in C^\infty(M, U(1)) \longleftrightarrow \sigma_\gamma : S^c(M) \to S^c(M)$$

$$\sigma_\gamma(p) = p \cdot \gamma(\pi_{S^c}(p))$$

ACTION OF $\mathcal{G}(\mathcal{L})$ ON $(A, \psi) \in \mathcal{A}(\mathcal{L})$:

CONNECTION A : σ_γ INDUCES AN AUTOMORPHISM OF $L^o(\mathcal{L})$

$$\begin{array}{ccc} S^c(M) & \xrightarrow{\sigma_\gamma} & S^c(M) \\ \downarrow & & \downarrow \\ L^o(\mathcal{L}) & \underset{\sigma_\gamma'}{\dashrightarrow} & L^o(\mathcal{L}) \end{array}$$

$$A \cdot \gamma = (\sigma_\gamma')^* A$$

SPINOR FIELD ψ : IDENTIFY THE SECTION ψ OF $S^+(\mathcal{L})$ WITH AN EQUIVARIANT MAP $\psi : S^c(M) \to S^+_{\mathbb{C}}$

$$\psi \cdot \gamma = \sigma_\gamma^* \psi = \psi \circ \sigma_\gamma$$

$$(A, \psi) \cdot \gamma = (A \cdot \gamma, \psi \cdot \gamma) = ((\sigma_\gamma')^* A, \sigma_\gamma^* \psi)$$

<u>THEOREM</u>: THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON THE SW CONFIGURATION SPACE $\mathcal{A}(\mathcal{L})$ CARRIES SOLUTIONS TO (SW) ONTO OTHER SOLUTIONS TO (SW), I.E., IF $(A,\psi) \in \mathcal{A}(\mathcal{L})$ SATISFIES

$$\begin{cases} F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \\ \not{\partial}_A \psi = 0 \end{cases}$$

AND $\gamma \in C^\infty(M, U(1))$, THEN $(A,\psi)\cdot\gamma = (A\cdot\gamma, \psi\cdot\gamma)$ SATISFIES

$$\begin{cases} F_{A\cdot\gamma}^+ = \sigma^+(((\psi\cdot\gamma) \otimes (\psi\cdot\gamma)^*)_0) \\ \not{\partial}_{A\cdot\gamma}(\psi\cdot\gamma) = 0 \end{cases}$$

TWO MODULI SPACES:

$$\mathcal{B}(\mathcal{L}) = \mathcal{A}(\mathcal{L}) / \mathcal{G}(\mathcal{L})$$

$$\mathcal{M}(\mathcal{L}) = \left\{ (A,\psi) \in \mathcal{A}(\mathcal{L}) : F_A^+ = \sigma^+((\psi \otimes \psi^*)_0),\ \not{\partial}_A \psi = 0 \right\} / \mathcal{G}(\mathcal{L})$$

ASSUMING THAT EACH IS REPLACED BY AN "APPROPRIATE SOBOLEV COMPLETION" THE ANALYSIS IS VERY SIMILAR TO THAT OF THE DONALDSON MODULI SPACES.

WE WILL MENTION ONLY A FEW POINTS AT WHICH THERE ARE DIFFERENCES.

1. THE REDUCIBLE ELEMENTS OF $\mathcal{A}(\mathcal{L})$ ARE EASILY IDENTIFIED.

   LEMMA: AN ELEMENT (A, ψ) OF $\mathcal{A}(\mathcal{L})$ IS LEFT FIXED BY SOME NON-IDENTITY ELEMENT γ OF $\mathcal{G}(\mathcal{L})$ IF AND ONLY IF $\psi \equiv 0$ AND, IN THIS CASE, $\gamma : M \to U(1)$ MUST BE A CONSTANT MAP.

2. ASSOCIATED WITH ANY SOLUTION (A, ψ) TO (SW) IS A FUNDAMENTAL ELLIPTIC COMPLEX $\mathcal{E}(A, \psi)$ WITH FINITE DIMENSIONAL COHOMOLOGY GROUPS ADMITTING INTERPRETATIONS ANALOGOUS TO THOSE IN DONALDSON THEORY.

   $H^0(A, \psi) = 0 \Leftrightarrow (A, \psi)$ IRREDUCIBLE $\Leftrightarrow \psi \not\equiv 0$

   $H^1(A, \psi) =$ FORMAL TANGENT SPACE TO $\mathcal{M}(\mathcal{L})$ AT $[A, \psi]$

   $H^2(A, \psi) =$ OBSTRUCTION SPACE ($H^2(A, \psi) = 0 \Leftrightarrow$ IMPLICIT FUNCTION THEOREM GIVES A LOCAL MANIFOLD STRUCTURE FOR THE SET OF SOLUTIONS TO (SW) NEAR (A, ψ) OF DIMENSION $\dim H^1(A, \psi)$)

IF $H^0(A, \psi) = 0$ AND $H^2(A, \psi) = 0$ THEN THE ATIYAH-SINGER THEOREM GIVES

$$\frac{1}{4}\left(c_1(L^0(\mathcal{L}))^2 - 2\chi(B) - 3\sigma(B) \right)$$

AS THE DIMENSION OF $\mathcal{M}(\mathcal{L})$ NEAR $[A, \psi]$.

$H^0(A, \psi) = 0$ AND $H^2(A, \psi) = 0$ ARE THE GENERIC SITUATION, BUT NOW "GENERIC" MEANS SOMETHING DIFFERENT THAN IT DID IN DONALDSON THEORY.

(A) AS IN DONALDSON THEORY, IF $b_2^+(M) > 0$ THERE IS A DENSE G_δ SET $\mathrm{Gen}(\mathcal{R})$ OF RIEMANNIAN METRICS ON M SUCH THAT

FOR ANY $g \in \mathrm{Gen}(\mathcal{R})$ AND ANY CORRESPONDING SPIN^c STRUCTURE $\mathcal{L}$ ANY SOLUTION (A, ψ) TO (SW) HAS

$$H^0(A, \psi) = 0.$$

IF $b_2^+(M) > 1$ THIS IS TRUE FOR A GENERIC PATH OF RIEMANNIAN METRICS.

(B) THERE IS NO ANALOGOUS "GENERIC METRICS THEOREM" FOR $H^2(A, \psi)$. IN THIS CASE ONE MUST PERTURB NOT THE METRIC, BUT THE EQUATIONS THEMSELVES.

FIX g AND $\mathcal{L}$. FOR ANY $\eta \in \Omega_+^2(M, \mathrm{Im}\,\mathbb{C})$, THE η- <u>PERTURBED SW EQUATIONS</u> ARE

$$F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) + \eta$$

$$\not{D}_A \psi = 0$$

EVERYTHING WE HAVE SAID ABOUT (SW) IS TRUE OF (PSW). ALSO

THERE IS A DENSE G_δ SET $\mathrm{Gen}(\Omega_+^2)$ IN $\Omega_+^2(M, \mathrm{Im}\,\mathbb{C})$ SUCH THAT FOR ANY $\eta \in \mathrm{Gen}(\Omega_+^2)$ EVERY SOLUTION (A, ψ) TO (PSW) HAS

$$H^2(A, \psi) = 0.$$

THEOREM: IF $b_2^+(M) > 0$, $g \in \mathrm{Gen}(\mathcal{R})$, $\mathcal{L}$ IS ANY CORRESPONDING SPIN^c STRUCTURE AND $\eta \in \mathrm{Gen}(\Omega^2_+)$, THEN THE MODULI SPACE $\mathcal{M}(\mathcal{L}, \eta)$ OF SOLUTIONS TO THE η-PERTURBED SEIBERG-WITTEN EQUATIONS IS EITHER EMPTY OR A SMOOTH MANIFOLD OF DIMENSION

$$\frac{1}{4}\left(c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M)\right).$$

A CHOICE OF ORIENTATION FOR $H^2_+(M;\mathbb{R})$ CANONICALLY ORIENTS $\mathcal{M}(\mathcal{L}, \eta)$.

NOTE: IF $b_2^+(M) > 1$ THERE IS ALSO A "COBORDISM" RESULT FOR GENERIC PATHS OF METRICS AND PERTURBATIONS ANALOGOUS TO THAT IN DONALDSON THEORY (ONE CAN EFFECTIVELY "FIX" THE SPIN^c STRUCTURE ALONG A PATH OF METRICS BECAUSE THE FRAME BUNDLES ARE ALL NATURALLY ISOMORPHIC).

3. THE SEIBERG-WITTEN MODULI SPACES ARE ALWAYS COMPACT!

THIS IS THE MOST SIGNIFICANT DIFFERENCE BETWEEN DONALDSON AND SEIBERG-WITTEN THEORY.

ASD MODULI SPACES ARE NOT COMPACT BECAUSE THE ASD-EQUATIONS ARE CONFORMALLY INVARIANT IN DIMENSION 4.

BY CONTRAST, SEIBERG-WITTEN SOLUTIONS SATISFY A PRIORI UNIFORM BOUNDS, E.G., IF (A, ψ) SATISFIES (SW), THEN, FOR EVERY $x \in M$,

$$\|\psi(x)\|^2 \le K(B) = \max\left\{-\tfrac{1}{2}\mathcal{K}(x_0) : x_0 \in M\right\}.$$

NOW FIX A GENERIC METRIC g, GENERIC PERTURBATION η AND ORIENTATION FOR $H^2_+(M;\mathbb{R})$. SUPPOSE THERE IS A CORRESPONDING SPIN^c STRUCTURE $\mathcal{L}$ FOR WHICH

$$c_1(L^0(\mathcal{L}))^2 = 2\chi(M) + 3\sigma(M).$$

THE MODULI SPACE IS EITHER EMPTY OR A FINITE SET OF ISOLATED POINTS, EACH EQUIPPED WITH A SIGN ± 1.

DEFINE THE <u>0 - DIMENSIONAL</u> <u>SEIBERG-WITTEN</u> <u>INVARIANT</u>

$$SW_0(M,\mathcal{L})$$

OF M ASSOCIATED WITH $\mathcal{L}$ TO BE ZERO IN THE FIRST CASE AND THE SUM OF THE SIGNS IN THE SECOND.

> <u>NOTE</u> : WHEN $b_2^+(M) > 1$ A COBORDISM ARGUMENT SHOWS THAT $SW_0(M,\mathcal{L})$ IS INDEPENDENT OF g AND η AND IS, IN FACT, A DIFFERENTIAL TOPOLOGICAL INVARIANT OF M.

ANOTHER CONSEQUENCE OF THE <u>A PRIORI</u> BOUNDS :

> IF g AND η ARE FIXED, THEN THERE ARE ONLY FINITELY MANY (EQUIVALENCE CLASSES OF) SPIN^c STRUCTURES $\mathcal{L}$ ON B THAT SATISFY
>
> 1. $c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) \geq 0$, AND
> 2. $\mathcal{M}(\mathcal{L},\eta) \neq \phi$.

WHEN THE DIMENSION OF THE MODULI SPACE IS NOT ZERO, SEIBERG-WITTEN INVARIANTS ARE DEFINED BY INTEGRATING A CERTAIN COHOMOLOGY CLASS OVER IT:

FIX $p_0 \in M$. LET $\mathcal{G}_0(\mathcal{L})$ = SUBGROUP OF $\mathcal{G}(\mathcal{L})$ OF THOSE ELEMENTS THAT ACT TRIVIALLY ON FIBER OF $S^c(M)$ ABOVE p_0.

$\mathcal{M}_0(\mathcal{L},\eta)$ = SW-SOLUTIONS MODULO $\mathcal{G}_0(\mathcal{L})$

NATURAL PROJECTION : $\mathcal{M}_0(\mathcal{L},\eta) \to \mathcal{M}(\mathcal{L},\eta)$

$U(1)$-ACTION ON $\mathcal{M}_0(\mathcal{L},\eta)$: $[A,\psi]_0 \cdot e^{\theta i} = [A, e^{-\theta i}\psi]_0$

PRINCIPAL $U(1)$-BUNDLE :

$$U(1) \hookrightarrow \mathcal{M}_0(\mathcal{L},\eta) \to \mathcal{M}(\mathcal{L},\eta)$$

1^{ST} CHERN CLASS :

$$c_1(\mathcal{M}_0(\mathcal{L},\eta)) \in H^2(\mathcal{M}(\mathcal{L},\eta);\mathbb{Z})$$

$$SW(M,\mathcal{L}) = \int_{\mathcal{M}(\mathcal{L},\eta)} c_1(\mathcal{M}^0(\mathcal{L},\eta))^{d_{\mathcal{L}}}$$

WHERE $\quad 2d_{\mathcal{L}} = \frac{1}{4}\left(c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M)\right).$

EMPIRICAL EVIDENCE SUGGESTS THAT $SW(M,\mathcal{L}) \neq 0$ ONLY FOR THOSE $\mathcal{L}$ SATISFYING $\quad c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) = 0.$

THE ELEMENTS $c_1(L^0(\mathfrak{L})) \in H^2(M;\mathbb{Z})$ CORRESPONDING TO THOSE $\mathfrak{L}$ FOR WHICH $c_1(L^0(\mathfrak{L}))^2 - 2\chi(M) - 3\sigma(M) = 0$ ARE CALLED <u>SW - BASIC CLASSES</u>.

M IS SAID TO BE OF <u>SW SIMPLE TYPE</u> IF NONZERO SW-INVARIANTS ARISE ONLY FROM 0-DIMENSIONAL MODULI SPACES, I.E., IF

$$SW(M,\mathfrak{L}) \neq 0 \Rightarrow c_1(L^0(\mathfrak{L}))^2 - 2\chi(M) - 3\sigma(M) = 0.$$

THE WITTEN CONJECTURE

1. M IS OF D-SIMPLE TYPE $\Leftrightarrow$ M IS OF SW-SIMPLE TYPE
2. D-BASIC CLASSES COINCIDE WITH SW-BASIC CLASSES
3. $$\mathcal{D}_M(x) = \exp(Q_M(x,x)/2) \sum_{r=1}^{s} a_r \exp(K_r(x))$$
$$= \exp(Q_M(x,x)/2) \sum_{\mathfrak{L}\in\Lambda} 2^{m(M)} SW_0(M,\mathfrak{L}) \exp(c_1(L^0(\mathfrak{L}))(x))$$

WHERE Λ IS THE SET OF ALL (EQUIVALENCE CLASSES OF) SPIN^c- STRUCTURES FOR WHICH $c_1(L^0(\mathfrak{L}))^2 - 2\chi(M) - 3\sigma(M) = 0$ AND

$$m(M) = 2 + \frac{1}{4}(7\chi(M) + 11\sigma(M)).$$

<u>NOTE</u> : FOR A BRIEF TOUR OF HOW WITTEN ARRIVED AT THE CONJECTURE, SEE <u>ADDENDUM</u> <u>17</u> (BY MATILDE MARCOLLI).

ATTITUDES ONE MIGHT ADOPT TOWARD THIS CONJECTURE :

1. IT SHOULD BE RIGOROUSLY PROVED

   - PIDSTRIGATCH AND TYURIN

   - FEEHAN AND LENESS (PARTIAL RESULTS HAVE ALREADY PAID DIVIDENDS)

   - VAJIAC

2. RIGOROUSLY TRUE OR NOT, THE SW INVARIANTS PROVIDE A MUCH MORE TRACTABLE TOOL FOR THE STUDY OF 4 - MANIFOLDS, SO IT MAKES GOOD, PRACTICAL SENSE TO ABANDON THE ASD EQUATIONS IN FAVOR OF THE SW EQUATIONS.

   - ESSENTIALLY EVERYONE ELSE

3. IF PHYSICS IS TRULY CAPABLE OF CASTING SUCH A PENETRATING LIGHT UPON MATHEMATICS AT THE DEEPEST LEVELS, MATHEMATICIANS WILL WANT TO TAKE HEED AND TURN THEIR ATTENTION ONCE AGAIN TO THEIR HISTORICAL ROOTS IN PHYSICS.

   - ATIYAH

ADDENDUM 1 : CHERN-WEIL CHARACTERISTIC CLASSES

HERE WE WILL SIMPLY RECORD THE PERTINENT DEFINITIONS AND RESULTS AND, AS AN ILLUSTRATION, SKETCH ONE SIMPLE CALCULATION.

A SMOOTH PRINCIPAL BUNDLE CONSISTS OF A SMOOTH (C^∞) MANIFOLD P (BUNDLE SPACE), A SMOOTH MANIFOLD M (BASE SPACE), A SMOOTH MAP π (PROJECTION) OF P ONTO M, A LIE GROUP G (STRUCTURE GROUP) AND A SMOOTH RIGHT ACTION

$$\sigma : P \times G \longrightarrow P$$

$$\sigma(p,g) = p \cdot g = \sigma_p(g) = \sigma_g(p)$$

OF G ON P SUCH THAT THE FOLLOWING ARE SATISFIED :

1. σ PRESERVES FIBERS OF π : $\pi(p \cdot g) = \pi(p)$ $\forall p \in P \ \forall g \in G$

2. (LOCAL TRIVIALITY) FOR EACH $x \in M$ $\exists$ OPEN NEIGHBORHOOD U OF x IN M (TRIVIALIZING NBD) AND A DIFFEOMORPHISM $\Psi : \pi^{-1}(U) \longrightarrow U \times G$ (TRIVIALIZATION) OF THE FORM $\Psi(p) = (\pi(p), \psi(p))$, WHERE $\psi : \pi^{-1}(U) \longrightarrow G$ SATISFIES

$$\psi(p \cdot g) = \psi(p) \cdot g$$

$\forall p \in \pi^{-1}(U) \ \forall g \in G$ (ψ IS EQUIVARIANT WITH RESPECT TO σ AND THE NATURAL ACTION OF G ON $U \times G$).

$$G \hookrightarrow P \xrightarrow{\pi} M$$

(LOCAL) SECTION / GAUGE : $s : U \to \pi^{-1}(U)$, $\pi \circ s = id_U$

$$\text{SECTIONS} \longleftrightarrow \text{TRIVIALIZATIONS}$$

$$s(x)\cdot g \longleftrightarrow (x, g)$$

TRANSITION FUNCTIONS :

$$s_i : U_i \to \pi^{-1}(U_i)$$

$$s_j : U_j \to \pi^{-1}(U_j)$$

$$s_j(x) = s_i(x)\cdot g_{ij}(x)$$

ASSOCIATED FIBER BUNDLES : F SMOOTH MANIFOLD WITH LEFT ACTION OF G

$$(g, \xi) \to g\cdot\xi$$

RIGHT ACTION OF G ON $P \times F$: $(p, \xi)\cdot g = (p\cdot g, g^{-1}\cdot\xi)$

ORBIT SPACE :
$$\begin{array}{c} P \times_G F \\ \downarrow \\ M \end{array}$$

IF $F = V$ IS A VECTOR SPACE AND THE ACTION OF G COMES FROM A REPRESENTATION $\rho : G \to GL(V)$, THEN THE ASSOCIATED BUNDLE $P \times_\rho V$ IS A VECTOR BUNDLE, E.G., ADJOINT BUNDLE

$$\mathrm{ad}\, P = P \times_{\mathrm{ad}} \mathfrak{g}$$

THREE (EQUIVALENT) DEFINITIONS OF A <u>CONNECTION</u> ON $G \hookrightarrow P \xrightarrow{\pi} M$:

ASSUME G IS A MATRIX LIE GROUP WITH LIE ALGEBRA $\mathfrak{g}$ AND $\dim M = n$

1. n - <u>DIMENSIONAL</u> <u>DISTRIBUTION</u> $\quad p \in P \longrightarrow HOR_p(P) \subseteq T_p(P)$ S.T.

   (a) $T_p(P) = HOR_p(P) \oplus VERT_p(P)$, WHERE

   $VERT_p(P) = T_p(\pi^{-1}(\pi(p)))$

   (b) $HOR_{p \cdot g}(P) = (\sigma_g)_{*p}(HOR_p(P))$

2. $\mathfrak{g}$ - <u>VALUED</u> 1 - <u>FORM</u> ω ON P S.T.

   (a) $\omega_{p \cdot g}((\sigma_g)_{*p}(v_p)) = g^{-1}\omega_p(v_p)g$

   (b) $\omega_p(\xi^{\#}(p)) = \xi \quad \forall\, \xi \in \mathfrak{g}$, WHERE

   $$\xi^{\#}(p) = \frac{d}{dt}\left(p \cdot \exp(t\xi)\right)\Big|_{t=0}$$

3. A TRIVIALIZING COVER $\{(U_j, \Psi_j)\}$ OF M FOR $G \hookrightarrow P \xrightarrow{\pi} M$ AND, FOR EACH j, A $\mathfrak{g}$ - VALUED 1 - FORM $\mathcal{A}_j$ ON U_j S.T. WHENEVER $U_j \cap U_i \neq \emptyset$

$$\mathcal{A}_j = g_{ij}^{-1}\mathcal{A}_i\, g_{ij} + g_{ij}^{-1}dg_{ij}$$

<u>RELATIONS</u> : $\quad HOR_p(P) = \ker \omega_p \quad$ AND $\quad \mathcal{A}_j = s_j^{*}\omega$

CURVATURE OF A CONNECTION 1-FORM ω IS THE $\mathfrak{g}$-VALUED 2-FORM Ω ON P DEFINED BY

$$\Omega_p(v_p, w_p) = d\omega_p(v_p^{HOR}, w_p^{HOR})$$

WHERE v_p^{HOR} AND w_p^{HOR} ARE THE PROJECTIONS OF v_p AND w_p ONTO $HOR_p(P)$ IN $T_p(P) = HOR_p(P) \oplus VERT_p(P)$.

CARTAN STRUCTURE EQUATION: $\Omega = d\omega + \omega \wedge \omega$

LOCAL FIELD STRENGTHS: $\mathfrak{F}_j = s_j^* \Omega = d\mathcal{A}_j + \mathcal{A}_j \wedge \mathcal{A}_j$

$$\mathfrak{F}_j = g_{ij}^{-1} \mathfrak{F}_i \, g_{ij}$$

$\Rightarrow$ THE $\mathfrak{F}_j$ PIECE TOGETHER INTO A 2-FORM

$$F_\omega \in \Omega^2(M, ad\,P)$$

ON M WITH VALUES IN THE ADJOINT BUNDLE $ad\,P$. THIS IS ALSO CALLED THE CURVATURE OF ω.

CHERN-WEIL PROCEDURE:

ASSUME NOW THAT G IS A COMPACT MATRIX LIE GROUP WITH LIE ALGEBRA $\mathfrak{g}$ AND CONSIDER THE ALGEBRA

$$\mathbb{C}[\mathfrak{g}]^G$$

OF $\mathrm{ad}\,G$-INVARIANT, COMPLEX-VALUED POLYNOMIAL FUNCTIONS ON $\mathfrak{g}$.

<u>MORE DETAIL</u>: LET $\{\xi_1, \ldots, \xi_n\}$ BE A BASIS FOR $\mathfrak{g}$ AND $\{x^1, \ldots, x^n\}$ THE DUAL BASIS FOR $\mathfrak{g}^*$ ($x^a(\xi_b) = \delta^a_b$). THE SYMMETRIC ALGEBRA $S(\mathfrak{g}^*) = S[x^1, \ldots, x^n]$ IS THE ALGEBRA OF POLYNOMIALS WITH REAL COEFFICIENTS IN $x^1, \ldots, x^n$ (REAL-VALUED POLYNOMIAL FUNCTIONS ON $\mathfrak{g}$). THERE IS A LEFT ACTION OF G ON $S(\mathfrak{g}^*)$: $\rho \in S(\mathfrak{g}^*)$ AND $g \in G$ GIVE $g \cdot \rho \in S(\mathfrak{g}^*)$ DEFINED BY

$$(g \cdot \rho)(\xi) = \rho(\mathrm{ad}_{g^{-1}} \xi) = \rho(g^{-1} \xi g).$$

$$S(\mathfrak{g}^*)^G = \{\rho \in S(\mathfrak{g}^*) : g \cdot \rho = \rho \ \forall g \in G\}, \text{ I.E.,}$$

$$\rho(g^{-1} \xi g) = \rho(\xi).$$

$$\mathbb{C}[\mathfrak{g}] = S(\mathfrak{g}^*) \otimes \mathbb{C} \quad \text{AND} \quad \mathbb{C}[\mathfrak{g}]^G = S(\mathfrak{g}^*)^G \otimes \mathbb{C}.$$

EVERY $\rho \in \mathbb{C}^k[\mathfrak{g}]^G$ GIVES RISE, VIA POLARIZATION, TO A G-INVARIANT k-MULTILINEAR FUNCTION, ALSO DENOTED

$$\rho : \mathfrak{g} \times \overset{k}{\cdots} \times \mathfrak{g} \longrightarrow \mathbb{C}$$

CHERN-WEIL IS BASICALLY A MAP

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \Omega^*(P)_{\text{BASIC}}$$

FROM G-INVARIANT POLYNOMIALS ON $\mathfrak{g}$ TO DIFFERENTIAL FORMS ON P THAT ARE <u>BASIC</u>, I.E., <u>G-INVARIANT</u> ($\sigma_g^* \varphi = \varphi \ \forall g \in G$) AND <u>HORIZONTAL</u> ($\iota_V \varphi = 0 \ \forall$ VERTICAL VECTOR FIELD V ON P)

<u>NOTE</u>: BASIC FORMS ON THE PRINCIPAL BUNDLE SPACE P ARE PRECISELY THOSE $\varphi \in \Omega^*(P)$ WHICH DESCEND TO THE BASE MANIFOLD M, I.E., FOR WHICH $\exists\ \bar{\varphi} \in \Omega^*(M)$ WITH $\pi^*\bar{\varphi} = \varphi$.

GIVEN $\rho \in \mathbb{C}[\mathfrak{g}]^G$ (THOUGHT OF AS A k-MULTILINEAR FUNCTION) CHOOSE A CONNECTION ω ON $G \hookrightarrow P \xrightarrow{\pi} M$ AND LET Ω BE ITS CURVATURE. WRITE $\omega = \omega^a \xi_a$ AND $\Omega = \Omega^a \xi_a$. DEFINE $\rho(\Omega) \in \Omega^{2k}(P)$ BY

$$\begin{aligned}\rho(\Omega) &= \rho(\Omega^{a_1}\xi_{a_1}, \dots, \Omega^{a_k}\xi_{a_k}) \\ &= \rho(\xi_{a_1}, \dots, \xi_{a_k})\,\Omega^{a_1} \wedge \dots \wedge \Omega^{a_k}\end{aligned}$$

THEN $\rho(\Omega)$ IS A CLOSED, BASIC $2k$-FORM ON P SO IT DESCENDS TO A CLOSED $2k$-FORM $\bar{\rho}(\Omega)$ ON M:

$$\pi^*(\bar{\rho}(\Omega)) = \rho(\Omega)$$

ESSENTIALLY, $\bar{\rho}(\Omega)$ IS JUST THE RESTRICTION OF $\rho(\Omega)$ TO THE ω-HORIZONTAL SPACES IN TP.

THE COHOMOLOGY CLASS $[\bar{\rho}(\Omega)] \in H^{2k}(M; \mathbb{R})$ DOES NOT DEPEND ON THE CHOICE OF ω OR THE BASIS $\{\xi_1, \dots, \xi_n\}$ FOR $\mathfrak{g}$ AND IS CALLED A CHARACTERISTIC CLASS OF $G \hookrightarrow P \xrightarrow{\pi} M$.

AS AN EXAMPLE WE WILL CONSIDER THE 2^{ND} CHERN CLASS OF A PRINCIPAL $SU(2)$ - BUNDLE

$$SU(2) \hookrightarrow P \xrightarrow{\pi} M$$

OVER A COMPACT, CONNECTED, ORIENTED 4 - MANIFOLD M

<u>NOTE</u> : ANOTHER EXAMPLE (THE EULER CLASS) WILL BE DISCUSSED IN THE NEXT SECTION ON " EQUIVARIANT COHOMOLOGY AND THE WITTEN LAGRANGIAN " .

NEED TO CHOOSE

1. A BASIS $\{ \xi_1, \xi_2, \xi_3 \}$ FOR $su(2)$ (2×2 COMPLEX MATRICES THAT ARE SKEW-HERMITIAN AND TRACEFREE), E.G.,

$$\xi_1 = -\frac{i}{2}\sigma_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\xi_2 = -\frac{i}{2}\sigma_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\xi_3 = -\frac{i}{2}\sigma_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

2. A CONNECTION ω WITH CURVATURE Ω WHICH WE WRITE AS

$$\omega = \omega^a \xi_a = \frac{1}{2}\begin{pmatrix} -\omega^3 i & -\omega^2 - i\omega^1 \\ \omega^2 - i\omega^1 & \omega^3 i \end{pmatrix}$$

$$\Omega = \Omega^a \xi_a = \frac{1}{2}\begin{pmatrix} -\Omega^3 i & -\Omega^2 - i\Omega^1 \\ \Omega^2 - i\Omega^1 & \Omega^3 i \end{pmatrix}$$

3. AN INVARIANT POLYNOMIAL $\mathcal{P}$ ON $su(2)$. FOR THIS WE TAKE

$$\mathcal{P} : su(2) \longrightarrow \mathbb{C}$$

$$\mathcal{P}(A) = \frac{1}{8\pi^2} \, Tr\,(A^2)$$

THE CORRESPONDING BILINEAR FORM ON $su(2)$ IS CLEARLY

$$\mathcal{P} : su(2) \times su(2) \longrightarrow \mathbb{C}$$

$$\mathcal{P}(A,B) = \frac{1}{8\pi^2} \, Tr\,(AB)$$

THUS,

$$\begin{aligned} \mathcal{P}(\Omega) &= \mathcal{P}(\Omega^{a_1}\xi_{a_1}, \Omega^{a_2}\xi_{a_2}) \\ &= \mathcal{P}(\xi_{a_1}, \xi_{a_2})\,\Omega^{a_1} \wedge \Omega^{a_2} \\ &= \frac{1}{8\pi^2} \, Tr\,(\xi_{a_1}\xi_{a_2})\,\Omega^{a_1} \wedge \Omega^{a_2} \end{aligned}$$

FOR THE BASIS CHOSEN ABOVE

$$Tr\,(\xi_{a_1}\xi_{a_2}) = \begin{cases} 0 & , a_1 \neq a_2 \\ -\frac{1}{2} & , a_1 = a_2 \end{cases}$$

SO

$$\begin{aligned} \mathcal{P}(\Omega) &= \frac{1}{8\pi^2}\left(-\tfrac{1}{2}\right)(\Omega^1\wedge\Omega^1 + \Omega^2\wedge\Omega^2 + \Omega^3\wedge\Omega^3) \\ &= \frac{1}{8\pi^2} \, Tr\,(\Omega\wedge\Omega) \end{aligned}$$

WHERE $\Omega \wedge \Omega$ IS THE MATRIX PRODUCT WITH ENTRIES MULTIPLIED BY THE ORDINARY WEDGE PRODUCT.

WE KNOW THAT

$$\frac{1}{8\pi^2}\, Tr\,(\Omega \wedge \Omega)$$

IS A CLOSED, BASIC 4-FORM ON P AND SO DESCENDS TO A CLOSED 4-FORM ON M (I.E., IS π^* OF SOME CLOSED 4-FORM ON M).

THIS CLOSED 4-FORM ON M CAN BE DESCRIBED LOCALLY IN TERMS OF LOCAL FIELD STRENGTHS

$$\mathcal{F}_j = s_j^* \Omega$$

SINCE $\pi^* \mathcal{F}_j = \pi^*(s_j^* \Omega) = (s_j \circ \pi)^* \Omega = id^* \Omega = \Omega$,

$$\pi^* \left(\frac{1}{8\pi^2}\, Tr\,(\mathcal{F}_j \wedge \mathcal{F}_j) \right) = \frac{1}{8\pi^2}\, Tr\,(\Omega \wedge \Omega)$$

ON ITS DOMAIN. SINCE THE LOCAL FIELD STRENGTHS TRANSFORM UNDER THE ADJOINT REPRESENTATION ($\mathcal{F}_j = g_{ij}^{-1}\, \mathcal{F}_i\, g_{ij}$) THE SAME IS TRUE OF $\mathcal{F}_j \wedge \mathcal{F}_j$ AND Tr IS ad-INVARIANT SO THE

$$\frac{1}{8\pi^2}\, Tr\,(\mathcal{F}_j \wedge \mathcal{F}_j)$$

PIECE TOGETHER INTO THE GLOBALLY DEFINED 4-FORM ON M TO WHICH $\frac{1}{8\pi^2}\, Tr\,(\Omega \wedge \Omega)$ DESCENDS. WE WRITE THIS AS

$$\frac{1}{8\pi^2}\, Tr\,(F_\omega \wedge F_\omega)$$

AND CALL ITS COHOMOLOGY CLASS

$$c_2(P) \in H^4(M;\mathbb{R})$$

THE 2^{ND} <u>CHERN CLASS</u> OF $SU(2) \hookrightarrow P \xrightarrow{\pi} M$. ITS INTEGRAL OVER M

$$c_2(P)[M] = \int_M c_2(P) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(F_\omega \wedge F_\omega),$$

WHICH IS ACTUALLY AN INTEGER, IS CALLED THE 2^{ND} <u>CHERN NUMBER</u> OF $SU(2) \hookrightarrow P \xrightarrow{\pi} M$

<u>NOTE</u> : FOR THE QUATERNIONIC HOPF BUNDLE

$$SU(2) \hookrightarrow S^7 \xrightarrow{\pi} S^4$$

ONE CAN USE ANY OF THE BPST INSTANTON CONNECTIONS TO COMPUTE REPRESENTATIVES OF THE CHERN CLASS. MOREOVER, THE BUNDLE IS TRIVIAL OVER $S^4 - \{\text{POINT}\}$ WHICH STEREOGRAPHICALLY PROJECTS TO $\mathbb{R}^4$ SO THE CHERN NUMBER CAN BE COMPUTED AS AN INTEGRAL OVER $\mathbb{R}^4$. THE CALCULATION WAS SKETCHED IN THE LECTURE. THE RESULT IS 1.

FOR THE COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLDS M OF INTEREST TO US HERE, PRINCIPAL $SU(2)$-BUNDLES OVER M ARE CLASSIFIED UP TO EQUIVALENCE BY THEIR 2^{ND} CHERN CLASS / NUMBER.

ADDENDUM 2 : THE MODULI SPACES

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD

+ ASSUMPTIONS ON $b_2^+(M)$ AS WE PROCEED

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} M$$

IS THE PRINCIPAL $SU(2)$-BUNDLE OVER M WITH CHERN CLASS $k>0$.

$\mathcal{A}(P_k)$ = ALL SMOOTH CONNECTION 1-FORMS ON P_k

$\mathcal{G}(P_k)$ = GAUGE GROUP

$$= \{ f : P_k \to P_k \ : \ f \text{ IS A DIFFEOMORPHISM}, \ f(p\cdot g) = f(p)\cdot g \ \forall p \in P_k \ \forall g \in SU(2), \ \pi_k \circ f = \pi_k \}$$

RIGHT ACTION OF $\mathcal{G}(P_k)$ ON $\mathcal{A}(P_k)$: $\quad \omega \cdot f = f^*\omega$

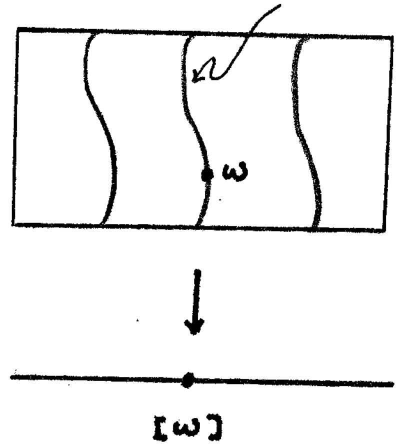
GAUGE EQUIVALENT CONNECTIONS : $\quad \omega_2 = \omega_1 \cdot f$

MODULI SPACE OF GAUGE EQUIVALENCE CLASSES OF CONNECTIONS :

$$\mathcal{B}(P_k) = \mathcal{A}(P_k)/\mathcal{G}(P_k)$$

$$= \{ [\omega] : \omega \in \mathcal{A}(P_k) \}$$

ORBIT $\omega \cdot \mathcal{G}(P_k)$



$$\mathcal{A}(P_k) \longrightarrow \mathcal{B}(P_k)$$

NOTE : FOR THE ANALYSIS THAT FOLLOWS ALL OF THESE C^∞ OBJECTS MUST BE REPLACED BY " APPROPRIATE SOBOLEV COMPLETIONS ". CONSIDER IT DONE !

$\mathcal{A}(P_k)$ IS AN AFFINE SPACE MODELED ON THE (INFINITE-DIMENSIONAL) VECTOR SPACE

$$\Omega^1_{ad}(P_k, su(2)) = \{ \varphi \in \Omega^1(P_k, su(2)) : \varphi \text{ IS HORIZONTAL AND } \sigma_g^* \varphi = g^{-1} \varphi g \}$$

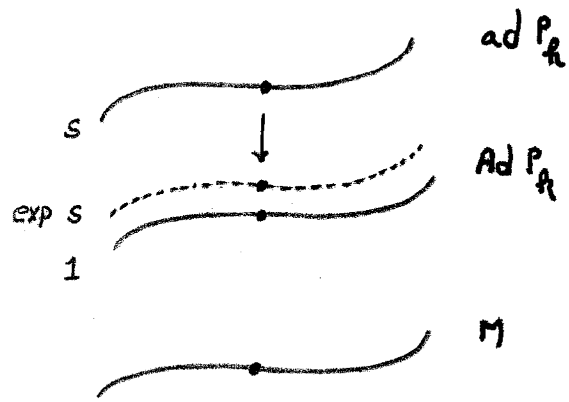
$$\cong \Omega^1(M, ad\, P_k)$$

THUS, $\mathcal{A}(P_k)$ IS AN INFINITE-DIMENSIONAL MANIFOLD WITH TANGENT SPACES

$$T_\omega(\mathcal{A}(P_k)) \cong \Omega^1_{ad}(P_k, su(2)) \cong \Omega^1(M, ad\, P_k)$$

$\mathcal{G}(P_k)$ CAN BE IDENTIFIED WITH THE SPACE OF SECTIONS OF THE NONLINEAR ADJOINT BUNDLE $\mathrm{Ad}\,P_k$ AND IS A HILBERT LIE GROUP WITH LIE ALGEBRA THAT CAN BE IDENTIFIED WITH THE SPACE OF SECTIONS OF THE (ORDINARY) ADJOINT BUNDLE $\mathrm{ad}\,P_k$.

POINTWISE EXPONENTIATE THE LATTER TO GET THE FORMER:



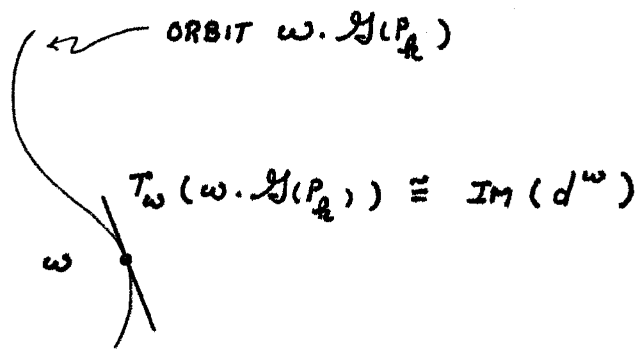
$$s \in \Omega^0(M, \mathrm{ad}\,P_k) \longrightarrow \exp s \in \Omega^0(M, \mathrm{Ad}\,P_k) = \mathcal{G}(P_k)$$

FOR A FIXED $\omega \in \mathcal{A}(P_k)$ THE MAP

$$f \to \omega . f \; : \; \mathcal{G}(P_k) \to \mathcal{A}(P_k)$$

HAS A DERIVATIVE AT $1 \in \mathcal{G}(P_k)$ THAT CAN BE IDENTIFIED WITH

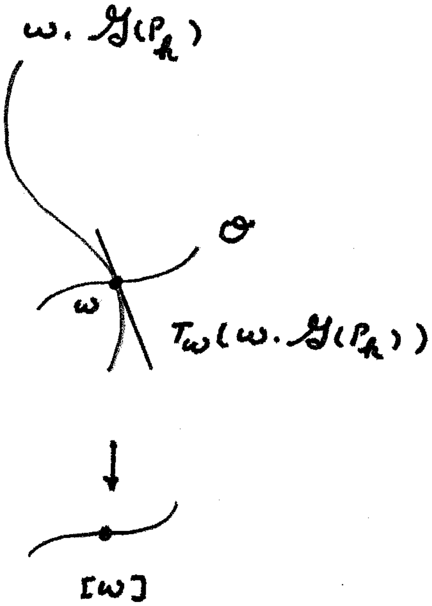
$$-d^{\omega} : \Omega^0(M, \mathrm{ad}\,P_k) \to \Omega^1(M, \mathrm{ad}\,P_k)$$



WOULD LIKE A "SLICE" OF THE $\mathcal{G}(P_k)$-ACTION ON $\mathcal{A}(P_k)$ NEAR ω, I.E., A SUBMANIFOLD $\mathcal{O}$ OF $\mathcal{A}(P_k)$ SUCH THAT

$$T_\omega(\mathcal{A}(P_k)) \cong T_\omega(\omega.\mathcal{G}(P_k)) \oplus T_\omega(\mathcal{O})$$

AND THE RESTRICTION TO $\mathcal{O}$ OF THE PROJECTION $\mathcal{A}(P_k) \to \mathcal{B}(P_k)$ IS INJECTIVE.



$$\boxed{\text{CHOOSE A RIEMANNIAN METRIC } g \text{ ON } M}$$

TOGETHER WITH AN ad-INVARIANT INNER PRODUCT ON $su(2)$ THIS GIVES NATURAL INNER PRODUCTS ON EACH $\Omega^i(M, \mathrm{ad}\, P_k)$ AND SO A FORMAL ADJOINT

$$\delta^\omega : \Omega^1(M, \mathrm{ad}\, P_k) \to \Omega^0(M, \mathrm{ad}\, P_k)$$

FOR

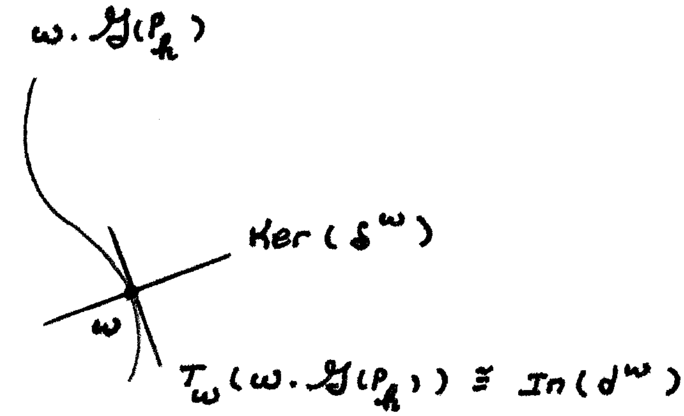
$$d^\omega : \Omega^0(M, \mathrm{ad}\, P_k) \to \Omega^1(M, \mathrm{ad}\, P_k).$$

$$\delta^{\omega} \circ d^{\omega} : \Omega^{0}(M, \mathrm{ad}\, P_k) \longrightarrow \Omega^{0}(M, \mathrm{ad}\, P_k)$$

IS (FORMALLY) SELF-ADJOINT AND ELLIPTIC.

ELLIPTIC THEORY (GENERALIZED HODGE DECOMPOSITION) GIVES THE ORTHOGONAL DECOMPOSITION

$$T_{\omega}(\mathcal{A}(P_k)) \cong \Omega^{1}(M, \mathrm{ad}\, P_k) \cong \mathrm{Im}(d^{\omega}) \oplus \mathrm{Ker}(\delta^{\omega})$$



$\mathrm{Ker}(\delta^{\omega})$ IS THE TANGENT SPACE TO THE AFFINE SUBSPACE

$$\omega + \mathrm{Ker}(\delta^{\omega}).$$

HOWEVER, PROJECTION OF $\omega + \mathrm{Ker}(\delta^{\omega})$ INTO $\mathcal{B}(P_k)$ NEED NOT BE INJECTIVE NEAR ω UNLESS ω IS "IRREDUCIBLE".

<u>STABILIZER</u> (OR <u>ISOTROPY SUBGROUP</u>) OF ω IS

$$\{ f \in \mathcal{G}(P_k) : f^{*}\omega = \omega \} \supseteq \mathbb{Z}_2 .$$

ω IS IRREDUCIBLE IF ITS STABILIZER IS PRECISELY $\mathbb{Z}_2$ AND REDUCIBLE OTHERWISE.

THE FOLLOWING ARE EQUIVALENT:

1. ω IS REDUCIBLE.
2. STABILIZER $(\omega)/\mathbb{Z}_2 \cong U(1)$
3. $d^\omega : \Omega^0(M, \mathrm{ad}\, P_k) \to \Omega^1(M, \mathrm{ad}\, P_k)$ HAS NONTRIVIAL KERNEL.

TO SEE THE RELEVANCE OF THIS: THE DERIVATIVE OF

$$\mathcal{G}(P_k) \times (\omega + \mathrm{Ker}(\delta^\omega)) \to \mathcal{A}(P_k)$$

$$(f, \omega') \to f^*\omega'$$

AT $(1, \omega)$ IS

$$\Omega^0(M, \mathrm{ad}\, P_k) \oplus \mathrm{Ker}(\delta^\omega) \to \Omega^1(M, \mathrm{ad}\, P_k) \cong \mathrm{Im}(d^\omega) \oplus \mathrm{Ker}(\delta^\omega)$$

$$d^\omega \oplus \mathrm{id}_{\mathrm{Ker}(\delta^\omega)}$$

WHICH IS ALWAYS SURJECTIVE AND INJECTIVE IF ω IS IRREDUCIBLE.

INVERSE FUNCTION THEOREM + BOOTSTRAPPING $\Rightarrow$

LOCALLY INJECTIVE NEAR ω IN $\omega + \mathrm{Ker}(\delta^\omega)$

ω IRREDUCIBLE $\Rightarrow$

A SUFFICIENTLY SMALL NEIGHBORHOOD OF ω IN $\omega + \mathrm{Ker}(S^{\omega})$ PROJECTS INJECTIVELY INTO THE MODULI SPACE $\mathcal{B}(P_k)$ AND SO PROVIDES A LOCAL (INFINITE DIMENSIONAL) MANIFOLD STRUCTURE NEAR $[\omega]$.

$$\hat{\mathcal{A}}(P_k) = \text{IRREDUCIBLE ELEMENTS OF } \mathcal{A}(P_k)$$

$$\hat{\mathcal{B}}(P_k) = \hat{\mathcal{A}}(P_k) / \mathcal{G}(P_k)$$

$$T_{[\omega]}(\hat{\mathcal{B}}(P_k)) \cong \mathrm{Ker}(S^{\omega})$$

NOW CONSIDER

$$Asd(P_k, g) = \text{ALL } \omega \in \mathcal{A}(P_k) \text{ WITH } {*F_\omega} = -F_\omega$$

$$\widehat{Asd}(P_k, g) = \text{IRREDUCIBLE ELEMENTS OF } Asd(P_k, g)$$

$$\mathcal{M}(P_k, g) = Asd(P_k, g) / \mathcal{G}(P_k)$$

$$\hat{\mathcal{M}}(P_k, g) = \widehat{Asd}(P_k, g) / \mathcal{G}(P_k)$$

TO STUDY THE LOCAL STRUCTURE OF THESE MODULI SPACES ASSOCIATE WITH EACH $\omega \in Asd(P_k, g)$ ITS <u>FUNDAMENTAL</u> <u>ELLIPTIC</u> <u>COMPLEX</u> $\mathcal{E}(\omega)$:

$$0 \to \Omega^0(M, ad P_k) \underset{\delta^\omega}{\overset{d^\omega}{\rightleftarrows}} \Omega^1(M, ad P_k) \underset{\delta^\omega_+}{\overset{d^\omega_+}{\rightleftarrows}} \Omega^2_+(M, ad P_k) \to 0$$

WHERE d^ω_+ IS d^ω FOLLOWED BY PROJECTION Pr_+ ONTO THE SELF-DUAL PART.

<u>NOTE</u> : $d^\omega \circ d^\omega = [F_\omega, \cdot\,] \Rightarrow d^\omega_+ \circ d^\omega = [F^+_\omega, \cdot\,]$
WHICH IS ZERO IF ω IS ASD.

ELLIPTIC THEORY $\Rightarrow$ COHOMOLOGY OF $\mathcal{E}(\omega)$ IS FINITE-DIMENSIONAL AND GIVEN BY

$$H^0(\omega) = Ker(d^\omega)$$

$$H^1(\omega) = Ker(d^\omega_+ \mid Ker(\delta^\omega))$$

$$H^2(\omega) = Im(d^\omega_+ \mid Ker(\delta^\omega))^\perp$$

HERE'S THE SIGNIFICANCE OF THIS :

$$F : \mathcal{A}(P_k) \longrightarrow \Omega^2(M, \operatorname{ad} P_k)$$

$$F(\omega) = F_\omega$$

LET $\mathcal{O}_{\omega,\varepsilon}$ = LOCAL SLICE NEAR ω

$$Pr_+ \circ F \mid \mathcal{O}_{\omega,\varepsilon} : \mathcal{O}_{\omega,\varepsilon} \longrightarrow \Omega^2_+(M, \operatorname{ad} P_k)$$

$$(Pr_+ \circ F \mid \mathcal{O}_{\omega,\varepsilon})^{-1}(0) = Asd(P_k, g) \cap \mathcal{O}_{\omega,\varepsilon}$$

THE DERIVATIVE OF $Pr_+ \circ F \mid \mathcal{O}_{\omega,\varepsilon}$ IS

$$d_+^\omega \mid Ker(\delta^\omega) : Ker(\delta^\omega) \rightarrow \Omega^2_+(M, \operatorname{ad} P_k)$$

WHICH IS <u>FREDHOLM</u> (BY FINITE DIMENSIONALITY OF $H^1(\omega)$ AND $H^2(\omega)$) AND IT IS <u>SURJECTIVE</u> IF AND ONLY IF

$$H^2(\omega) = 0.$$

IN THIS CASE THE (BANACH MANIFOLD) <u>IMPLICIT FUNCTION THEOREM</u> GIVES A LOCAL MANIFOLD STRUCTURE FOR $Asd(P_k, g) \cap \mathcal{O}_{\omega,\varepsilon}$ NEAR ω OF DIMENSION

$$\operatorname{DIM}\left(Ker(d_+^\omega \mid Ker(\delta^\omega))\right) = \operatorname{DIM} H^1(\omega).$$

IF, IN ADDITION,

$$H^0(\omega) = 0$$

(I.E., ω IS IRREDUCIBLE), THEN THE PROJECTION INTO THE MODULI SPACE IS INJECTIVE NEAR ω IN $Asd(P_k, g) \cap \mathcal{O}_{\omega,\varepsilon}$ SO $\hat{\mathcal{M}}(P_k, g)$ HAS A LOCAL MANIFOLD STRUCTURE NEAR $[\omega]$ OF DIMENSION

$$\begin{aligned} \dim H^1(\omega) &= -\dim H^0(\omega) + \dim H^1(\omega) - \dim H^2(\omega) \\ &= -\text{ INDEX OF THE COMPLEX } \mathcal{E}(\omega) \\ &= 8k - 3(1 + b_2^+(M)) \quad \text{BY THE ATIYAH-SINGER INDEX THEOREM} \end{aligned}$$

<u>NOTE</u>: INDEPENDENT OF ω

BUT "GENERICALLY", $H^0(\omega)$ AND $H^2(\omega)$ ARE ALWAYS TRIVIAL:

<u>GENERIC METRICS THEOREM</u>: LET $\mathcal{R}$ BE THE SPACE OF ALL RIEMANNIAN METRICS ON M. THEN

1. THERE IS A DENSE G_δ-SET IN $\mathcal{R}$ SUCH THAT, FOR EVERY g IN THIS SET, ANY g-ASD CONNECTION ω SATISFIES

$$H^2(\omega) = 0.$$

2. IF $b_2^+(M) > 0$, THEN THERE IS A DENSE G_δ-SET IN $\mathcal{R}$ SUCH THAT, FOR EVERY g IN THIS SET, ANY g-ASD CONNECTION ω SATISFIES ($H^2(\omega) = 0$) AND

$$H^0(\omega) = 0.$$

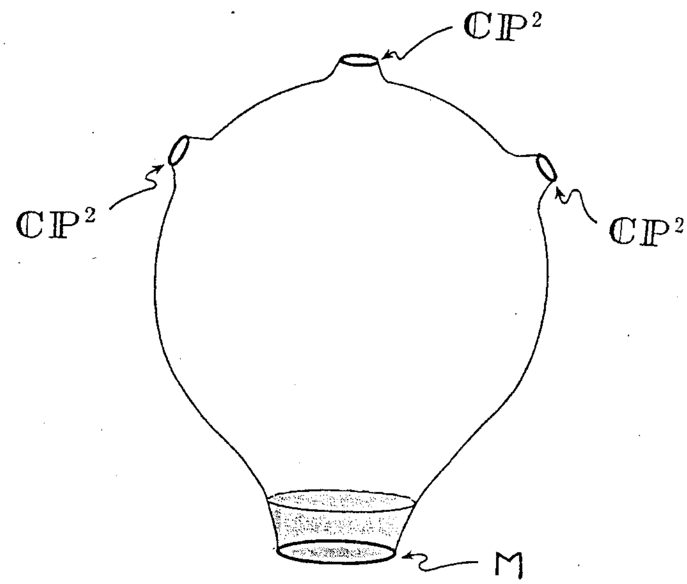
IN SHORT, FOR "GENERIC" g,

$$\mathcal{M}(P_k, g) = \hat{\mathcal{M}}(P_k, g)$$

IS (EITHER EMPTY OR) A SMOOTH MANIFOLD OF DIMENSION

$$8k - 3(1 + b_2^+(M)).$$

ADDENDUM 3 : "ETC."



HAVING DELETED THE (OPEN) TOP HALF OF EACH CONE AND THE (OPEN) BOTTOM HALF OF THE CYLINDER WE ARE LEFT WITH A MANIFOLD WITH BOUNDARY THAT IS COMPACT (BECAUSE K IS COMPACT) AND ORIENTED WITH BOUNDARY

$$M \sqcup p\,\mathbb{CP}^2 \sqcup q\,\overline{\mathbb{CP}}^2$$

WHERE

$$p + q = m .$$

THUS, M IS COBORDANT TO THE DISJOINT UNION $p\,\mathbb{CP}^2 \sqcup q\,\overline{\mathbb{CP}}^2$ SO

$$\sigma(M) = \sigma(p\,\mathbb{CP}^2 \sqcup q\,\overline{\mathbb{CP}}^2)$$

$$b_2^+(M) - b_2^-(M) = p - q$$

$$b_2^-(M) = q - p$$

$$b_2(M) = q - p \le q + p = m$$

WE SHOW NEXT THAT $b_2(M) \geq m$:

SELECT $x_1 \in H_2(M;\mathbb{Z})$ WITH $Q_M(x_1,x_1) = -1$ (THERE MUST BE AT LEAST ONE SUCH BECAUSE WE ASSUME THAT $b_2^+(M) = 0$, BUT $b_2(M) \neq 0$).

THEN THERE IS A Q_M - ORTHOGONAL DECOMPOSITION

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}x_1 \oplus G_1$$

NOW CONSIDER ANY $x_2 \in H_2(M;\mathbb{Z})$ WITH $Q_M(x_2,x_2) = -1$ AND $x_2 \neq \pm x_1$ (IF SUCH A THING HAPPENS TO EXIST). THE SCHWARTZ INEQUALITY GIVES

$$(Q_M(x_1,x_2))^2 < Q_M(x_1,x_1)\, Q_M(x_2,x_2) = 1 .$$

BUT $Q_M(x_1,x_2)$ IS AN INTEGER SO

$$Q_M(x_1,x_2) = 0$$

AND THEREFORE

$$x_2 \in G_1$$

NOW REPEAT THE ARGUMENT INSIDE G_1 AND CONTINUE INDUCTIVELY UNTIL YOU RUN OUT OF $x \in H_2(M;\mathbb{Z})$ FOR WHICH $Q_M(x,x) = -1$ (WHICH YOU WILL BECAUSE $H_2(M;\mathbb{Z})$ IS FINITELY GENERATED).

THE RESULT IS AN ORTHOGONAL DECOMPOSITION

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m \oplus G$$

WHERE G IS EITHER EMPTY OR THE ORTHOGONAL COMPLEMENT OF $\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m$. IN PARTICULAR,

$$m \leq b_2(M).$$

SINCE WE SHOWED EARLIER THAT $b_2(M) \leq m$ WE CONCLUDE THAT

$$b_2(M) = m = p+q .$$

THUS, IN FACT, G MUST BE EMPTY AND

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m$$

WHERE

$$Q_M(x_i) = -1 , \quad i = 1, \dots, m$$

AND THE MATRIX OF Q_M RELATIVE TO THE BASIS $\{x_1, \dots, x_m\}$ IS

$$-\,id_{m \times m} .$$

ADDENDUM 4 : THE μ-MAP

THE DEFINITION OF THE DONALDSON μ-MAP REQUIRES THE CONSTRUCTION OF A CERTAIN AUXILIARY SO(3)-BUNDLE WHICH WE BRIEFLY DESCRIBE AS FOLLOWS : TO ECONOMIZE ON NOTATION WE WILL LET

$$P_k = P, \ \hat{\mathcal{A}}(P_k) = \hat{\mathcal{A}}, \ \mathcal{G}(P_k) = \mathcal{G}, \ \hat{\mathcal{B}}(P_k) = \hat{\mathcal{B}}, \ \hat{\mathfrak{m}}(P_k, g) = \hat{\mathfrak{m}}$$

$\mathcal{G}$ ACTS ON $\hat{\mathcal{A}} \times P$ BY

$$(\omega, p) \cdot f = (f^* \omega, f^{-1}(p))$$

AND THE ACTION IS FREE (BECAUSE THE ELEMENTS OF $\hat{\mathcal{A}}$ ARE IRREDUCIBLE).

THE ORBIT SPACE

$$\hat{\mathcal{A}} \times_{\mathcal{G}} P$$

IS A HILBERT MANIFOLD AND

$$\mathcal{G} \hookrightarrow \hat{\mathcal{A}} \times P \longrightarrow \hat{\mathcal{A}} \times_{\mathcal{G}} P$$

$$(\omega, p) \longrightarrow [\omega, p]$$

IS A PRINCIPAL $\mathcal{G}$-BUNDLE.

THERE IS A NATURAL MAP

$$\hat{\mathcal{A}} \times_{\mathcal{G}} P \longrightarrow \hat{\mathcal{B}} \times M$$

$$[\omega, p] \longrightarrow ([\omega], \pi(p))$$

THE ACTION

$$(\omega, p)\cdot g = (\omega, p\cdot g)$$

OF $SU(2)$ ON $\hat{\mathcal{A}} \times P$ COMMUTES WITH THE ACTION OF $\mathcal{G}$ ON $\hat{\mathcal{A}} \times P$ SO IT DESCENDS TO AN ACTION OF $SU(2)$ ON $\hat{\mathcal{A}} \times_{\mathcal{G}} P$:

$$[\omega, p]\cdot g = [\omega, p\cdot g]$$

THIS ACTION IS NOT FREE, BUT BECAUSE THE ELEMENTS OF $\hat{\mathcal{A}}$ ARE IRREDUCIBLE ONE FINDS THAT

$$[\omega, p]\cdot g = [\omega, p] \iff g = \pm 1$$

THUS, THERE IS A FREE

$$SU(2)/\pm 1 \cong SO(3)$$

ACTION ON $\hat{\mathcal{A}} \times_{\mathcal{G}} P$ AND WE HAVE A PRINCIPAL $SO(3)$ - BUNDLE

$$\boxed{\begin{array}{ll} \mathcal{P}: & SO(3) \hookrightarrow \hat{\mathcal{A}} \times_{\mathcal{G}} P \\ & \qquad\qquad\quad \Big\downarrow \; [\omega, p] \to ([\omega], \pi(p)) \\ & \qquad\qquad\quad \hat{\mathcal{B}} \times M \end{array}}$$

AS DOES ANY $SO(3)$ - BUNDLE, THIS HAS A 1^{ST} PONTRYAGIN CLASS

$$p_1(\mathcal{P}) \in H^4(\hat{\mathcal{B}} \times M; \mathbb{Z})$$

ONE CAN SHOW THAT $p_1(\mathcal{P})$ IS DIVISIBLE BY 4 IN THE SENSE THAT

$$-\tfrac{1}{4}\, p_1(\mathcal{P})$$

IS STILL AN INTEGRAL CLASS.

> NOTE: THE REASON FOR INTRODUCING THE $-\frac{1}{4}$ IS AS FOLLOWS: WHEN THE PRINCIPAL $SO(3)$-BUNDLE $\mathcal{P}$ LIFTS TO A PRINCIPAL $SU(2)$-BUNDLE $\mathcal{P}'$ (AS IT ALWAYS DOES WHEN $k = c_2(P)$ IS ODD, FOR EXAMPLE),
>
> $$c_2(\mathcal{P}') = -\tfrac{1}{4}\, p_1(\mathcal{P})$$

NOW, THERE IS A GENERAL OPERATION IN ALGEBRAIC TOPOLOGY CALLED "SLANT PRODUCT". IT IS A MAP

$$H^{p+q}(X \times Y) \times H_p(Y) \longrightarrow H^q(X)$$

$$(\gamma, \alpha) \longrightarrow \gamma/\alpha$$

FIXING $\gamma \in H^{p+q}(X \times Y)$ THIS GIVES A MAP

$$H_p(Y) \longrightarrow H^q(X)$$

$$\alpha \longrightarrow \gamma/\alpha$$

APPLY THIS CONSTRUCTION TO

$$H^4(\hat{\mathcal{B}} \times M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \longrightarrow H^2(\hat{\mathcal{B}}; \mathbb{Z})$$

WITH γ FIXED AT $-\frac{1}{4} p_1(\mathcal{P}) \in H^4(\hat{\mathcal{B}} \times M; \mathbb{Z})$ TO OBTAIN A MAP

$$H_2(M; \mathbb{Z}) \longrightarrow H^2(\hat{\mathcal{B}}; \mathbb{Z})$$

$$x \longrightarrow -\tfrac{1}{4} p_1(\mathcal{P})/x$$

AND FOLLOW BY THE RESTRICTION TO $\hat{\mathcal{M}} \subseteq \hat{\mathcal{B}}$ TO OBTAIN

$$\mu : H_2(M; \mathbb{Z}) \to H^2(\hat{\mathcal{M}}; \mathbb{Z})$$

TO GET A MORE INTUITIVE FEEL FOR THE μ-MAP WE WILL NOW DESCRIBE IT MORE INFORMALLY IN THE LANGUAGE OF DE RHAM COHOMOLOGY :

KÜNNETH FORMULA GIVES

$$H^4(\hat{\mathcal{M}} \times M) \cong \bigoplus_{p+q=4} H^p(\hat{\mathcal{M}}) \otimes H^q(M)$$

SO $-\frac{1}{4} p_1(\mathcal{P} | \hat{\mathcal{M}} \times M)$ RESOLVES INTO A SUM OF $(p, 4-p)$-CLASSES. LET $[\alpha_1] \otimes [\alpha_2]$ BE THE "$(2,2)$-PART". EVERY ELEMENT x OF $H_2(M)$ CAN BE REPRESENTED BY A SMOOTHLY EMBEDDED, ORIENTED SURFACE Σ.

$$\mu(x) = \left(\int_\Sigma [\alpha_2] \right) [\alpha_1]$$

ADDENDUM 5 : DONALDSON POLYNOMIALS

HERE WE WILL ATTEMPT TO PROVIDE A VERY BRIEF MAP OF THE ROAD ONE MUST FOLLOW FROM THE " NAIVE DEFINITION " OF THE DONALDSON INVARIANTS

$$\gamma_{d_k}(M) : H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}]$$

$$x \in H_2(M;\mathbb{Z}) \longrightarrow \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x) \in H^{2d_k}(\mathcal{M}(P_k,g);\mathbb{Z})$$

$$\gamma_{d_k}(M)(x) = \int_{\mathcal{M}(P_k,g)} \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x)$$

TO AN HONEST DEFINITION.

FIRST STEP IS THE CONSTRUCTION OF A COMPACTIFICATION

$$\bar{\mathcal{M}}(P_k,g)$$

OF $\mathcal{M}(P_k,g)$.

RELIES ON DEEP ANALYTICAL RESULTS OF KAREN UHLENBECK AND IS ESSENTIALLY AN INTRICATE VARIATION ON THE PHENOMENON WE WITNESSED FOR THE BPST INSTANTONS ON S^4 :

SEQUENCE OF POINTS IN $\mathcal{M}(P_k, g)$ CAN FAIL TO HAVE A CONVERGENT SUBSEQUENCE ONLY IF THE CORRESPONDING CURVATURES HAVE POINTWISE NORMS THAT BECOME INCREASINGLY CONCENTRATED AT A POINT (OR FINITE SET OF POINTS) IN M.

INTUITIVELY, THESE CONVERGE TO "δ - CONNECTIONS" WHICH, BECAUSE OF THEIR SINGULAR NATURE, DO NOT APPEAR IN THE MODULI SPACE.

CONSTRUCTION OF $\bar{\mathcal{M}}(P_k, g)$ OCCURS IN LAYERS ("STRATA") BY ADDING ON THESE "VIRTUAL" CONNECTIONS, THEN LIMITS OF VIRTUAL CONNECTIONS, ...

NO RESTRICTION ON $b_2^+(M)$ REQUIRED. WHEN $b_2^+(M) = 0$ AND $k = 1$ THERE IS ONLY ONE STRATUM AND $\bar{\mathcal{M}}(P_1, g)$ IS OBTAINED FROM OUR EARLIER PICTURE OF $\mathcal{M}(P_1, g)$ BY ATTACHING THE "BOTTOM" COPY $M \times \{0\}$ OF M.

DONALDSON SHOWS THAT THE μ-MAP $\mu : H_2(M; \mathbb{Z}) \to H^2(\mathcal{M}(P_k, g); \mathbb{Z})$ EXTENDS TO

$$\bar{\mu} : H_2(M; \mathbb{Z}) \longrightarrow H^2(\bar{\mathcal{M}}(P_k, g); \mathbb{Z}).$$

0TH APPROXIMATION : $\gamma_{d_k}(x) = \int_{\mathcal{M}(P_k,g)} \overbrace{\mu(x)\wedge\cdots\wedge\mu(x)}^{d_k}$

PROBLEM : $\mathcal{M}(P_k, g)$ IS NOT COMPACT

1ST APPROXIMATION : $\gamma_{d_k}(x) = \int_{\bar{\mathcal{M}}(P_k,g)} \overbrace{\bar{\mu}(x)\wedge\cdots\wedge\bar{\mu}(x)}^{d_k}$

PROBLEM : $\bar{\mathcal{M}}(P_k, g)$ IS NOT A MANIFOLD

2ND APPROXIMATION : $\gamma_{d_k}(x) = \langle \overbrace{\bar{\mu}(x)\smile\cdots\smile\bar{\mu}(x)}^{d_k}, [\bar{\mathcal{M}}(P_k,g)] \rangle$

PROBLEM : $\bar{\mathcal{M}}(P_k, g)$ ADMITS A FUNDAMENTAL CLASS $[\bar{\mathcal{M}}(P_k,g)]$ ONLY IF k IS IN THE <u>STABLE RANGE</u>

$$k > \frac{3}{4}(1 + b_2^+(M))$$

OR, EQUIVALENTLY,

$$d_k > \frac{3}{2}(1 + b_2^+(M)).$$

AT LEAST FOR INTEGERS

$$d \equiv -\frac{3}{2}(1 + b_2^+(M)) \bmod 4$$

$$d > \frac{3}{2}(1 + b_2^+(M))$$

WE HAVE A

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \to \mathbb{Z}.$$

TO REMOVE THE RESTRICTIONS ON d :

EXTEND $\gamma_d(M) : H_2(M;\mathbb{Z}) \to \mathbb{Z}$ TO A d-MULTILINEAR MAP

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{d}{\cdots} \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$$

$$\gamma_d(M)(x_1,\dots,x_d) = \langle \bar{\mu}(x_1) \smile \cdots \smile \bar{\mu}(x_d),\ [\bar{\mathcal{M}}(P_k,g)] \rangle$$

EXTEND THE DEFINITION TO INCLUDE ARGUMENTS IN $H_*(M;\mathbb{Z})$ WITH ANOTHER μ-MAP

$$\mu : H_0(M;\mathbb{Z}) \longrightarrow H^4(\mathcal{M}(P_k,g);\mathbb{Z})$$

(ANALOGOUS DEFINITION).

$$H_0(M;\mathbb{Z}) \cong \mathbb{Z}$$

GENERATOR η

PROBLEM : $\mu(\eta)$ DOES NOT EXTEND TO THE ENTIRE UHLENBECK COMPACTIFICATION

$\mu(\eta)$ DOES, HOWEVER, ADMIT AN EXTENSION $\bar{\mu}(\eta)$ TO A LARGE ENOUGH SUBSET OF $\bar{\mathcal{M}}(P_k,g)$ THAT, UNDER CERTAIN ADDITIONAL RESTRICTIONS ON d, ONE CAN PRODUCE THE DESIRED EXTENSION OF $\gamma_d(M)$ TO INCLUDE THIS 4-DIMENSIONAL CLASS :

CONSIDER A $d \equiv -\frac{3}{2}(1+b_2^+(M)) \bmod 4$ AND

$$d = a + 2b$$

WHERE $b \geq 0$ AND

$$a > \frac{3}{2}(1+b_2^+(M)).$$

THEN ONE CAN DEFINE

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{a}{\cdots} \times H_2(M;\mathbb{Z}) \times H_0(M;\mathbb{Z}) \times \overset{b}{\cdots} \times H_0(M;\mathbb{Z}) \to \mathbb{Z}$$

BY

$$\gamma_d(M)(x_1,\dots,x_a, n_1\eta,\dots,n_b\eta) =$$

$$n_1 \cdots n_b \langle \bar{\mu}(x_1) \smile \cdots \smile \bar{\mu}(x_a) \smile \bar{\mu}(\eta)^b, [\bar{\mathcal{M}}(P_k, g)] \rangle$$

WHEN $\mathcal{M}(P_k, g) \neq \phi$ AND $\gamma_d(M) \equiv 0$ WHEN $\mathcal{M}(P_k, g) = \phi$ (HERE k IS THE INTEGER FOR WHICH $2d = 8k - 3(1+b_2^+(M))$).

FOR ANY INTEGER $k > 0$ SAY THAT A SEQUENCE

$$S = (x_1,\dots,x_a, n_1\eta,\dots,n_b\eta)$$

WITH $x_i \in H_2(M;\mathbb{Z})$, $i = 1,\dots,a$, AND $n_j\eta \in H_0(M;\mathbb{Z})$, $j = 1,\dots,b$, IS <u>k-STABLE</u> <u>FOR</u> M IF $b \geq 0$, $a > \frac{3}{2}(1+b_2^+(M))$, AND $a + 2b = d = 4k - \frac{3}{2}(1+b_2^+(M))$.

AT THIS POINT WE HAVE DEFINED $\gamma_d(M)(S)$ WHENEVER S IS k- STABLE FOR M.

NEXT WE NEED A "BLOW-UP FORMULA" OF DONALDSON.

RECALL :

- BLOW-UP OF M : $M \# \overline{\mathbb{CP}}^2$
- $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}) \cong \mathbb{Z}$, $Q_{\overline{\mathbb{CP}}^2} = (-1)$, $b_2^+(\overline{\mathbb{CP}}^2) = 0$
- $H_2(M \# \overline{\mathbb{CP}}^2; \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}}^2; \mathbb{Z})$
- $H_+^2(\overline{\mathbb{CP}}^2, \mathbb{R}) \cong 0$
- $H_+^2(M \# \overline{\mathbb{CP}}^2; \mathbb{R}) \cong H_+^2(M; \mathbb{R})$

THUS,

ORIENTING $H_+^2(M; \mathbb{R})$ ORIENTS $H_+^2(M \# \overline{\mathbb{CP}}^2; \mathbb{R})$

AND

STABLE RANGES OF M AND $M \# \overline{\mathbb{CP}}^2$ ARE THE SAME .

FIX A GENERATOR e FOR $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}) \hookrightarrow H_2(M \# \overline{\mathbb{CP}}^2; \mathbb{Z})$.

THEN

1. S k-STABLE FOR M (AND SO ALSO FOR $M \# \overline{\mathbb{CP}}^2$) $\Rightarrow$

$$\gamma_{d_k}(M \# \overline{\mathbb{CP}}^2)(S) = \gamma_{d_k}(M)(S).$$

2. SUPPOSE $i = 1, 2$, OR 3 AND S IS <u>NOT</u> k-STABLE FOR M, BUT $(S, e, \overset{i}{\ldots}, e)$ IS k-STABLE FOR $M \# \overline{\mathbb{CP}}^2$. THEN

$$\gamma_{d_k}(M \# \overline{\mathbb{CP}}^2)(S, e, \overset{i}{\ldots}, e) = 0.$$

3. S k-STABLE FOR M (AND SO ALSO FOR $M \# \overline{\mathbb{CP}}^2$) $\Rightarrow$ (S, e, e, e, e) $(k+1)$-STABLE FOR $M \# \overline{\mathbb{CP}}^2$ AND

$$\gamma_{d_k}(M)(S) = -\tfrac{1}{2}\,\gamma_{d_{k+1}}(M \# \overline{\mathbb{CP}}^2)(S, e, e, e, e).$$

FROM THESE :

$$M \# n\overline{\mathbb{CP}}^2 = M \# \overline{\mathbb{CP}}^2_1 \# \cdots \# \overline{\mathbb{CP}}^2_n$$

$$e_i \text{ GENERATOR FOR } H_2(\overline{\mathbb{CP}}^2_i ; \mathbb{Z})$$

S k-STABLE FOR M $\Rightarrow$ $(S, e_1, e_1, e_1, e_1, \ldots, e_n, e_n, e_n, e_n)$ $(k+n)$-STABLE FOR $M \# n\overline{\mathbb{CP}}^2$ AND

$$\boxed{\gamma_{d_k}(M)(S) = \left(-\tfrac{1}{2}\right)^n \gamma_{d_{n+k}}(M \# n\overline{\mathbb{CP}}^2)(S, e_1, e_1, e_1, e_1, \ldots, e_n, e_n, e_n, e_n)}$$

HERE'S THE POINT TO THIS <u>BLOW-UP</u> <u>FORMULA</u>:

EVEN WHEN $S = (x_1, \dots, x_a, n\eta, \dots, n_b\eta)$ SATISFIES

$$a + 2b = d_k$$

BUT <u>NOT</u>

$$a > \tfrac{3}{2}(1 + b_2^+(M))$$

SO THAT S IS NOT k-STABLE FOR M AND $\gamma_{d_k}(M)(S)$ HAS NOT YET BEEN DEFINED, THE RIGHT-HAND SIDE WILL BE DEFINED PROVIDED ONLY THAT n IS SUFFICIENTLY LARGE.

MOREOVER, THE RIGHT-HAND SIDE TAKES THE SAME VALUE FOR ALL SUFFICIENTLY LARGE n (BY #3 ABOVE) SO WE MAY USE IT TO <u>DEFINE</u> THE LEFT-HAND SIDE FOR ANY S WITH $a + 2b = d_k$.

$\gamma_{d_k}(M)$ NOW TAKES VALUES IN $\mathbb{Z}[\tfrac{1}{2}]$.

<u>STATUS</u> <u>REPORT</u>: LET d BE AN INTEGER SATISFYING

$$d \equiv -\tfrac{3}{2}(1 + b_2^+(M)) \bmod 4.$$

CHOOSE k SO THAT

$$2d = 8k - 3(1 + b_2^+(M))$$

AND CONSIDER

$$SU(2) \hookrightarrow P_k \longrightarrow M$$

AND

$$\mathcal{M}(P_k, g)$$

WHERE g IS GENERIC. THE FORMAL DIMENSION OF $\mathcal{M}(P_k, g)$ IS $2d$.

- $d < 0 \Rightarrow \gamma_d(M)$ IS TAKEN TO BE IDENTICALLY ZERO

- $d = 0 \Rightarrow$ WE HAVE DESCRIBED (PREVIOUS LECTURE) A NUMERICAL INVARIANT

$$\gamma_0(M) \in \mathbb{Z}.$$

- $d > 0 \Rightarrow$ WE HAVE, FOR ALL NON-NEGATIVE INTEGERS a AND b WITH $a + 2b = d$, A MULTILINEAR MAP

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{a}{\cdots} \times H_2(M;\mathbb{Z}) \times H_0(M;\mathbb{Z}) \times \overset{b}{\cdots} \times H_0(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$$

DEFINED, IF $a > \frac{3}{2}(1 + b_2^+(M))$, BY

$$\gamma_d(M)(x_1, \dots, x_a, n_1\eta, \dots, n_b\eta) =$$

$$n_1 \cdots n_b \langle \bar{\mu}(x_1) \smile \cdots \smile \bar{\mu}(x_a) \smile \bar{\mu}(\eta)^b, [\bar{\mathcal{M}}(P_k, g)] \rangle$$

AND BY THE BLOW-UP FORMULA WITH n SUFFICIENTLY LARGE OTHERWISE.

NEXT WE RELAX THE REQUIREMENT

$$d \equiv -\tfrac{3}{2}(1+b_2^+(M)) \bmod 4$$

TO

$$d \equiv -\tfrac{3}{2}(1+b_2^+(M)) \bmod 2 .$$

IF $d > 0$ SATISFIES THE MOD 2 CONGRUENCE, BUT NOT THE MOD 4 CONGRUENCE, THEN

$$d+2 \equiv -\tfrac{3}{2}(1+b_2^+(M)) \bmod 4$$

SO

$$\gamma_{d+2}(M) : H_2(M;\mathbb{Z}) \times \overset{d}{\cdots} \times H_2(M;\mathbb{Z}) \times H_0(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$$

IS DEFINED AND WE CAN DEFINE

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \overset{d}{\cdots} \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}]$$

BY

$$\gamma_d(M)(x_1,\dots,x_d) = \tfrac{1}{2}\gamma_{d+2}(M)(x_1,\dots,x_d,\eta) .$$

FINALLY, IF $d \not\equiv -\frac{3}{2}(1+b_2^+(M)) \bmod 2$, THEN WE TAKE

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \times \cdots \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}]$$

TO BE IDENTICALLY ZERO.

WRITING

$$\gamma_d(M)(x,\overset{d}{\dots},x) = \gamma_d(M)(x)$$

WE HAVE THE FULL CONTINGENT OF DONALDSON INVARIANTS

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \rightarrow \mathbb{Z}[\tfrac{1}{2}]$$

$$d = 1, 2, \ldots$$

(TOGETHER WITH $\gamma_0(M) \in \mathbb{Z}$).

FROM THESE KRONHEIMER AND MROWKA BUILT THE DONALDSON SERIES

$$\mathcal{D}_M(x) = \sum_{d=0}^{\infty} \frac{\gamma_d(M)(x)}{d!}$$

$$= \underbrace{\sum \frac{\gamma_d(M)(\overbrace{x,\ldots,x}^{d})}{d!}}_{\text{(MOD 4 CONGRUENCE)}} + \underbrace{\frac{1}{2}\sum \frac{\gamma_{d+2}(\overbrace{x,\ldots,x}^{d},\eta)}{d!}}_{\text{(MOD 2, BUT NOT MOD 4 CONGRUENCE)}}$$

M IS OF <u>D-SIMPLE TYPE</u> IF

$$\gamma_{d+4}(M)(x_1,\ldots,x_d,\eta,\eta) = 4\,\gamma_d(M)(x_1,\ldots,x_d)$$

FOR ALL $d > 0$ AND ALL $x_1,\ldots,x_d \in H_2(M;\mathbb{Z})$.

FOR THESE ONE CAN

- INDUCTIVELY EXTRACT ALL OF THE

$$\gamma_d(M)(x_1,\dots,x_a,\, n_1\eta,\dots,n_b\eta)$$

$(a+2b=d)$ FROM $\Theta_M(x)$.

- PROVE THE KRONHEIMER-MROWKA STRUCTURE THEOREM (STATED IN THE PREVIOUS LECTURE).

- FORMULATE WITTEN'S CONJECTURE (STILL TO COME).

ADDENDUM 6

THE EULER CLASS OF TS^2

CONSIDER THE 2-SPHERE S^2 WITH ITS USUAL ORIENTATION AND RIEMANNIAN METRIC AND LET $TS^2 \xrightarrow{\pi} S^2$ BE ITS TANGENT BUNDLE. THE CORRESPONDING ORIENTED, ORTHONORMAL FRAME BUNDLE IS

$$SO(2) \hookrightarrow F_{SO}(TS^2) \xrightarrow{\pi_{SO}} S^2.$$

A CONNECTION ω ON $F_{SO}(TS^2)$ IS AN $\mathfrak{so}(2)$-VALUED 1-FORM

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} 0 & \omega^1 \\ -\omega^1 & 0 \end{pmatrix} = \omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

ON $F_{SO}(TS^2)$. SINCE $SO(2)$ IS ABELIAN, THE CURVATURE IS

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = d\omega = d\omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

THE LEVI-CIVITA CONNECTION ON $F_{SO}(TS^2)$ IS CHARACTERIZED BY

$$d\Theta^i = -\omega_{ij} \wedge \Theta^j$$

WHERE $\{\Theta^1, \Theta^2\}$ IS THE BASIS OF 1-FORMS DUAL TO AN ORIENTED, ORTHONORMAL FRAME FIELD $\{e_1, e_2\}$ ON S^2 (I.E., A SECTION OF $F_{SO}(TS^2)$). INSERT IF ϕ AND Θ ARE THE USUAL SPHERICAL COORDINATES ON S^2, THEN WE MAY TAKE $\{e_1, e_2\} = \{\frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\Theta}\}$ AND $\{\Theta^1, \Theta^2\} = \{d\phi, \sin\phi\, d\Theta\}$. SINCE

INSERT: THE SAME EQUATIONS ARE CLEARLY ALSO SATISFIED BY THE PULLBACKS OF THESE FORMS TO S^2 BY A SECTION OF $F_{SO}(TS^2)$ AND WE WILL USE THE SAME SYMBOLS TO DENOTE THESE PULLBACKS.

$$d\Theta^1 = 0 = -\omega_{11} \wedge \Theta^1 - \omega_{12} \wedge \Theta^2 = -\omega_{12} \wedge \Theta^2 = -\omega_{12} \wedge (\sin\phi\, d\Theta)$$

$$d\Theta^2 = d(\sin\phi\, d\Theta) = \cos\phi\, d\phi \wedge d\Theta$$

$$= -\omega_{21} \wedge \Theta^1 - \omega_{22} \wedge \Theta^2$$

$$= -\omega_{21} \wedge \Theta^1$$

$$= \omega_{12} \wedge \Theta^1$$

$$= \omega_{12} \wedge d\phi$$

$$-\cos\phi\, d\Theta \wedge d\phi = \omega_{12} \wedge d\phi$$

WE HAVE

$$\omega_{12} = -\cos\phi \, d\theta$$

(WHICH ALSO SATISFIES THE FIRST CONDITION). THUS,

$$\omega = \begin{pmatrix} 0 & -\cos\phi\, d\theta \\ \cos\phi\, d\theta & 0 \end{pmatrix} = -\cos\phi\, d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

AND

$$\Omega = \sin\phi\, d\phi \wedge d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin\phi\, d\phi\wedge d\theta \\ -\sin\phi\, d\phi \wedge d\theta & 0 \end{pmatrix}$$

THUS,

$$(2\pi)^{-1}\, \mathrm{PFAF}\,(\Omega) = \frac{1}{4\pi} \sum_{\sigma \in S_2} (-1)^{\sigma}\, \Omega_{\sigma(1)\sigma(2)}$$

$$= \frac{1}{4\pi}\left[\Omega_{12} - \Omega_{21}\right] = \frac{1}{2\pi}\Omega_{12}$$

$$= \frac{1}{2\pi} \sin\phi \, d\phi \wedge d\theta$$

THEN, AS EXPECTED,

$$\int_{S^2} \frac{1}{2\pi} \sin\phi\, d\phi\wedge d\theta = 2 = \chi(S^2)$$

ADDENDUM 7

UNIVERSAL THOM FORM FOR $\mathbb{R}^2$

$V = \mathbb{R}^2$ (USUAL ORIENTATION AND INNER PRODUCT)

$\{\psi^1, \psi^2\}$ = STANDARD BASIS

$\{u_1, u_2\}$ = DUAL BASIS (COORDINATE FUNCTIONS)

$SO(V) \equiv SO(2)$

$\{\xi_1\} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$\{x^1\}$ = DUAL BASIS

WE COMPUTE

$$\nu = (2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^T du - \frac{1}{2}\sum_{\ell} \psi^\ell x^a M_a \psi^\ell} \, \mathcal{D}\psi .$$

$$-\tfrac{1}{2}\|u\|^2 = -\tfrac{1}{2}(u_1^2 + u_2^2)$$

$$i\psi^T du = i\psi^j du_j = i(\psi^1 du_1 + \psi^2 du_2)$$

$$x^a M_a = x^1 M_1 = x^1 M_{\xi_1} = x^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^1 \\ -x^1 & 0 \end{pmatrix}$$

$$\begin{aligned}
\sum_{\ell} \psi^\ell x^a M_a \psi^\ell &= \psi^1 x^1 M_1 \psi^1 + \psi^2 x^1 M_1 \psi^2 \\
&= \psi^1 \begin{pmatrix} 0 & x^1 \\ -x^1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi^2 \begin{pmatrix} 0 & x^1 \\ -x^1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \psi^1 \begin{pmatrix} 0 \\ -x^1 \end{pmatrix} + \psi^2 \begin{pmatrix} x^1 \\ 0 \end{pmatrix} \\
&= \psi^1 \left(-x^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + \psi^2 \left(x^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= -\psi^1 x^1 \psi^2 + \psi^2 x^1 \psi^1 \\
&= x^1(-\psi^1\psi^2 + \psi^2\psi^1) \\
&= -2x^1 \psi^1 \psi^2
\end{aligned}$$

$$-\frac{1}{2}\sum_{\ell}\psi^{\ell}\chi^{a}\Gamma_{a}\psi^{\ell} = \chi^{1}\psi^{1}\psi^{2}$$

THUS,

$$\nu = (2\pi)^{-1}\int e^{-\frac{1}{2}(u_1^2+u_2^2)+i(\psi^1 du_1+\psi^2 du_2)+\chi^1\psi^1\psi^2}\,\mathcal{D}\psi$$

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}\int e^{i(\psi^1 du_1+\psi^2 du_2)}\,e^{\chi^1\psi^1\psi^2}\,\mathcal{D}\psi$$

(EACH TERM IN THE EXPONENT IS EVEN SO THEY COMMUTE)

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [1+i(\psi^1 du_1+\psi^2 du_2)-\tfrac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2][1+\chi^1\psi^1\psi^2]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [1+i(\psi^1 du_1+\psi^2 du_2)-\tfrac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2+\chi^1\psi^1\psi^2+i(\psi^1 du_1+\psi^2 du_2)\chi^1\psi^1\psi^2-\tfrac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2\chi^1\psi^1\psi^2]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [-\tfrac{1}{2}(\psi^1 du_1+\psi^2 du_2)^2+\chi^1\psi^1\psi^2]\,\mathcal{D}\psi$$

(ALL OTHER TERMS ARE ZERO OR DO NOT CONTRIBUTE TO $\psi^1\psi^2$)

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [\chi^1\psi^1\psi^2-\tfrac{1}{2}(\psi^1 du_1\,\psi^2 du_2+\psi^2 du_2\,\psi^1 du_1)]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}\int [\chi^1\psi^1\psi^2-\tfrac{1}{2}(-du_1 du_2\,\psi^1\psi^2-du_1 du_2\,\psi^1\psi^2)]\,\mathcal{D}\psi$$

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}\int (\chi^1+du_1 du_2)\,\psi^1\psi^2\,\mathcal{D}\psi$$

$$= (2\pi)^{-1}e^{-\frac{1}{2}(u_1^2+u_2^2)}(\chi^1+du_1 du_2)$$

ADDENDUM 8

EQUIVARIANT COHOMOLOGY AND $H^*_{S^1}(S^3)$

HERE, VERY BRIEFLY, IS WHAT EQUIVARIANT COHOMOLOGY IS ALL ABOUT (IN THE SMOOTH CATEGORY): LET M BE A SMOOTH MANIFOLD AND G A COMPACT, CONNECTED LIE GROUP WITH LIE ALGEBRA $\mathfrak{g}$ AND SUPPOSE G ACTS SMOOTHLY ON M ON THE LEFT (BUT NOT NECESSARILY FREELY).

$$M/G = \text{ORBIT SPACE WITH THE QUOTIENT TOPOLOGY}$$

NOTE: IF THE ACTION HAPPENS TO BE FREE, THEN M/G ADMITS A NATURAL SMOOTH STRUCTURE AND, IN THIS CASE, THE "G-EQUIVARIANT COHOMOLOGY OF M" WILL TURN OUT TO BE JUST THE ORDINARY COHOMOLOGY OF M/G.

IN GENERAL, THE IDEA IS TO REPLACE M BY A SPACE M^G THAT IS (HOMOTOPICALLY) "VERY MUCH LIKE" M, BUT ON WHICH G ACTS FREELY AND TAKE THE G-EQUIVARIANT COHOMOLOGY OF M TO BE THE ORDINARY COHOMOLOGY OF M^G/G (AND THEN SHOW THAT THE RESULT DOESN'T DEPEND ON THE CHOICE OF M^G AND REDUCES TO THE ORDINARY COHOMOLOGY OF M/G WHEN THE ACTION OF G ON M IS FREE). THE CONSTRUCTION OF M^G IS BASED ON THE FOLLOWING:

> THEOREM: FOR EVERY COMPACT LIE GROUP G THERE EXISTS A CONTRACTIBLE TOPOLOGICAL SPACE EG ON WHICH G ACTS FREELY. THE QUOTIENT SPACE $BG = EG/G$ IS A UNIVERSAL CLASSIFYING SPACE FOR PRINCIPAL G-BUNDLES OVER COMPACT MANIFOLDS AND $G \hookrightarrow EG \longrightarrow BG$ IS THE UNIVERSAL BUNDLE FOR PRINCIPAL G-BUNDLES OVER COMPACT MANIFOLDS. EG IS UNIQUE UP TO EQUIVARIANT HOMOTOPY EQUIVALENCE.

THE OBVIOUS DIAGONAL ACTION OF G ON $EG \times M$ IS FREE (BECAUSE THE ACTION OF G ON EG IS FREE) AND $EG \times M$ IS EQUIVARIANTLY HOMOTOPY EQUIVALENT TO M (BECAUSE EG IS CONTRACTIBLE). WE TAKE $M^G = EG \times M$.

THUS, THE G-EQUIVARIANT COHOMOLOGY $H_G^*(M)$ OF M IS DEFINED TO BE THE ORDINARY COHOMOLOGY OF M^G/G.

ONE WOULD LIKE TO HAVE AN ALGEBRAIC SCHEME FOR CALCULATING $H_G^*(M)$ FROM A COCHAIN COMPLEX CONSTRUCTED FROM M (THUS, FOR EXAMPLE, IN THE CASE OF A FREE ACTION OBTAINING THE COHOMOLOGY OF THE QUOTIENT M/G FROM CALCULATIONS IN M, WHICH IS PRESUMABLY SIMPLER). THERE ARE A NUMBER OF SUCH ALGEBRAIC MODELS. WE WILL DESCRIBE THE CARTAN MODEL OF EQUIVARIANT COHOMOLOGY :

M = SMOOTH MANIFOLD

G = COMPACT CONNECTED LIE GROUP (LIE ALGEBRA $\mathfrak{g}$) WHICH ACTS SMOOTHLY ON M ON THE LEFT : $\sigma : G \times M \to M$

$$\sigma(g,p) = g \cdot p = \sigma_g(p) = \sigma_p(g)$$

VARIOUS OTHER ACTIONS :

1. LEFT ACTION OF G ON THE (GRADED) ALGEBRA $\Omega^*(M)$ OF COMPLEX-VALUED DIFFERENTIAL FORMS ON M : FOR $\varphi \in \Omega^*(M)$ AND $g \in G$ DEFINE
$$g \cdot \varphi = \sigma_{g^{-1}}^* \varphi$$
AN ELEMENT $\varphi \in \Omega^*(M)$ IS G-INVARIANT IF $g \cdot \varphi = \varphi$ AND THE SUBALGEBRA OF ALL SUCH IS DENOTED
$$\Omega^*(M)^G.$$

2. LEFT ACTION OF G ON $\mathfrak{g}^*$: FOR $\Theta \in \mathfrak{g}^*$ AND $g \in G$ DEFINE $g \cdot \Theta \in \mathfrak{g}^*$ BY
$$(g \cdot \Theta)(\xi) = \Theta(\mathrm{ad}_{g^{-1}} \xi) = \Theta(g^{-1} \xi g)$$
$\forall\, \xi \in \mathfrak{g}$.

3. LET $S(\mathfrak{g}^*)$ DENOTE THE SYMMETRIC ALGEBRA OF $\mathfrak{g}^*$. WE THINK OF THIS AS THE GRADED ALGEBRA OF COMMUTATIVE POLYNOMIAL FUNCTIONS ON $\mathfrak{g}$ IN THE FOLLOWING WAY: LET $\{\xi_1, \ldots, \xi_n\}$ BE A BASIS FOR $\mathfrak{g}$ AND $\{x^1, \ldots, x^n\}$ THE DUAL BASIS FOR $\mathfrak{g}^*$ $(x^a(\xi_b) = \delta^a_b)$. THEN $S(\mathfrak{g}^*) = \mathbb{R}[x^1, \ldots, x^n]$ IS THE ALGEBRA OF POLYNOMIALS WITH REAL COEFFICIENTS IN $x^1, \ldots, x^n$. THESE ACT ON $\xi \in \mathfrak{g}$ BY LETTING EACH x^a ACT ON ξ SO THE RESULT IS A POLYNOMIAL IN THE COMPONENTS $\xi^a = x^a(\xi)$ OF ξ RELATIVE TO $\{\xi_1, \ldots, \xi_n\}$.

THE ACTION OF G ON $\mathfrak{g}^*$ EXTENDS TO A LEFT ACTION OF G ON $S(\mathfrak{g}^*)$: FOR $\rho \in S(\mathfrak{g}^*)$ AND $g \in G$ DEFINE $g \cdot \rho$ BY

$$(g \cdot \rho)(\xi) = \rho(ad_{g^{-1}} \xi) = \rho(g^{-1} \xi g).$$

<u>NOTE</u>: AN ELEMENT ρ OF $S(\mathfrak{g}^*)$ IS <u>G-INVARIANT</u> IF $g \cdot \rho = \rho$ FOR EVERY $g \in G$, I.E., IF

$$\rho(g^{-1} \xi g) = \rho(\xi) \quad \forall g \in G \ \forall \xi \in \mathfrak{g}.$$

THE COLLECTION OF ALL SUCH IS A SUBALGEBRA WHICH WE DENOTE

$$S(\mathfrak{g}^*)^G.$$

TENSORING WITH $\mathbb{C}$ WE OBTAIN THE ALGEBRA $\mathbb{C}[\mathfrak{g}] = S(\mathfrak{g}^*) \otimes \mathbb{C}$ OF COMPLEX-VALUED POLYNOMIAL FUNCTIONS ON $\mathfrak{g}$ AND THE SUBALGEBRA $\mathbb{C}[\mathfrak{g}]^G$ OF G-INVARIANT ELEMENTS (THIS IS THE DOMAIN OF THE CLASSICAL "CHERN-WEIL MAP")

4. CONSIDER THE TENSOR PRODUCT $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ OF EXAMPLES #1 AND 3. WE THINK OF THIS AS THE ALGEBRA OF $\Omega^*(M)$-VALUED POLYNOMIAL FUNCTIONS ON $\mathfrak{g}$, E.G., IF $\alpha = \rho \otimes \varphi$ IS A HOMOGENEOUS ELEMENT, THEN

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

THE ACTIONS IN #1 AND 3 COMBINE TO GIVE A LEFT ACTION OF G ON $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ FOR HOMOGENEOUS ELEMENTS

$$\alpha = \rho \otimes \varphi$$

AND $g \in G$ DEFINE

$$\begin{aligned}(g\cdot\alpha)(\xi) &= (g\cdot(\rho\otimes\varphi))(\xi)\\ &= ((g\cdot\rho)\otimes(g\cdot\varphi))(\xi)\\ &= ((g\cdot\rho)(\xi))(g\cdot\varphi)\\ &= \rho(g^{-1}\xi g)\,\sigma^*_{g^{-1}}\varphi\end{aligned}$$

AN ELEMENT α OF $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ IS G-INVARIANT IF $g\cdot\alpha = \alpha$ FOR EACH $g \in G$ AND THE SUBALGEBRA OF ALL SUCH IS DENOTED

$$[\,\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)\,]^G$$

LEMMA : $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ IS G-INVARIANT IF AND ONLY IF

$$\alpha(g\cdot\xi) = g\cdot\alpha(\xi) \qquad [\alpha(g\xi g^{-1}) = \sigma^*_{g^{-1}}(\alpha(\xi))]$$

FOR EACH $g \in G$ AND $\xi \in \mathfrak{g}^*$.

PROOF : IT SUFFICES TO PROVE THIS FOR HOMOGENEOUS ELEMENTS $\alpha = \rho\otimes\varphi$. FIRST ASSUME THAT α IS G-INVARIANT. THEN

$$\begin{aligned}\alpha(g\cdot\xi) &= \alpha(g\xi g^{-1}) = (g\cdot\alpha)(g\xi g^{-1})\\ &= \rho(g^{-1}(g\xi g^{-1})g)\,\sigma^*_{g^{-1}}\varphi = \rho(\xi)\,\sigma^*_{g^{-1}}\varphi\\ &= \sigma^*_{g^{-1}}(\rho(\xi)\varphi) = \sigma^*_{g^{-1}}(\alpha(\xi))\\ &= g\cdot\alpha(\xi)\end{aligned}$$

CONVERSELY, IF $\alpha(g\cdot\xi) = g\cdot\alpha(\xi)\ \forall g,\xi$, THEN

$$\begin{aligned}(g\cdot\alpha)(\xi) &= \rho(g^{-1}\xi g)\,\sigma^*_{g^{-1}}\varphi = \sigma^*_{g^{-1}}(\rho(g^{-1}\xi g)\varphi) = \sigma^*_{g^{-1}}(\alpha(g^{-1}\xi g))\\ &= \sigma^*_{g^{-1}}(\alpha(g^{-1}\cdot\xi)) = \sigma^*_{g^{-1}}(g^{-1}\cdot\alpha(\xi)) = g\cdot(g^{-1}\cdot\alpha(\xi)) = \alpha(\xi). \qquad \square\end{aligned}$$

WE WILL CALL THE ELEMENTS OF $[\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)]^G$ __EQUIVARIANT DIFFERENTIAL FORMS__ ON M AND WRITE

$$\Omega_G^*(M) = [\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)]^G$$

__STRUCTURE OF__ $\Omega_G^*(M)$:

1. __GRADING__ : $\mathbb{C}[\mathfrak{g}] = \bigoplus_j \mathbb{C}^j[\mathfrak{g}]$ AND $\Omega^*(M) = \bigoplus_i \Omega^i(M)$ ARE NATURALLY $\mathbb{Z}$-GRADED, BUT RATHER THAN THE USUAL TENSOR PRODUCT GRADING ON $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ WE "DOUBLE THE DEGREES" IN $\mathbb{C}[\mathfrak{g}]$:

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M) = \bigoplus_{2j+i=k} \mathbb{C}^j[\mathfrak{g}] \otimes \Omega^i(M)$$

THUS, FOR HOMOGENEOUS ELEMENTS $\alpha = \rho \otimes \varphi$,

$$\deg \alpha = \deg(\rho \otimes \varphi) = 2\deg \rho + \deg \varphi .$$

$\Omega_G^*(M)$ IS THEN THE GRADED SUBALGEBRA AND EACH, WITH ITS NATURAL $\mathbb{Z}_2$-GRADING, IS A SUPERALGEBRA.

2. __EQUIVARIANT EXTERIOR DIFFERENTIAL__ d_G :

WE FIRST DEFINE AN OPERATOR d_G ON $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ AND THEN RESTRICT IT TO THE G-INVARIANT ELEMENTS. TO DO THIS WE RECALL A FEW BASIC CONSTRUCTIONS FROM DIFFERENTIAL GEOMETRY.

RECALL: GIVEN A $\xi \in \mathfrak{g}$ ONE HAS AN ASSOCIATED VECTOR FIELD $\xi^{\#}$ ON M DEFINED, AT EACH $p \in M$, BY

$$\xi^{\#}(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)\Big|_{t=0},$$

I.E., FOR EACH $\phi \in C^{\infty}(M)$,

$$(\xi^{\#}\phi)(p) = \frac{d}{dt}(\phi(\exp(-t\xi)\cdot p))\Big|_{t=0}$$

(NOTE THAT THE INFINITESIMAL GENERATOR OF THE 1-PARAMETER GROUP $t \to \exp(t\xi)$ USUALLY ASSOCIATED WITH ξ IS MINUS THIS $\xi^{\#}$). THE LIE DERIVATIVE

$$\mathcal{L}_{\xi^{\#}} : \Omega^{i}(M) \longrightarrow \Omega^{i}(M)$$

WITH RESPECT TO THIS VECTOR FIELD IS DEFINED BY

$$\mathcal{L}_{\xi^{\#}}\omega = \frac{d}{dt}(\sigma^{*}_{\exp(-t\xi)}\omega)\Big|_{t=0}$$

WHERE $\sigma_{\exp(-t\xi)}(p) = \exp(-t\xi)\cdot p$. INTERIOR MULTIPLICATION

$$\iota_{\xi^{\#}} : \Omega^{i}(M) \longrightarrow \Omega^{i-1}(M)$$

BY $\xi^{\#}$ IS JUST CONTRACTION WITH $\xi^{\#}$ IN THE FIRST VARIABLE. WE RECORD SOME STANDARD IDENTITIES:

1. $\mathcal{L}_{\xi^{\#}}(\omega \wedge \eta) = (\mathcal{L}_{\xi^{\#}}\omega)\wedge\eta + \omega\wedge(\mathcal{L}_{\xi^{\#}}\eta)$
2. $\iota_{\xi^{\#}}(\omega\wedge\eta) = (\iota_{\xi^{\#}}\omega)\wedge\eta + (-1)^{\deg\omega}\omega\wedge(\iota_{\xi^{\#}}\eta)$
3. $\mathcal{L}_{\xi^{\#}} = d\circ\iota_{\xi^{\#}} + \iota_{\xi^{\#}}\circ d$
4. $\iota_{\xi^{\#}}\circ\iota_{\zeta^{\#}} + \iota_{\zeta^{\#}}\circ\iota_{\xi^{\#}} = 0$
5. $\mathcal{L}_{\xi^{\#}}\circ\iota_{\zeta^{\#}} - \iota_{\zeta^{\#}}\circ\mathcal{L}_{\xi^{\#}} = \iota_{[\xi^{\#},\zeta^{\#}]}$
6. $\mathcal{L}_{\xi^{\#}}\circ\mathcal{L}_{\zeta^{\#}} - \mathcal{L}_{\zeta^{\#}}\circ\mathcal{L}_{\xi^{\#}} = \mathcal{L}_{[\xi^{\#},\zeta^{\#}]}$
7. $d\circ\iota_{\xi^{\#}} + \iota_{\xi^{\#}}\circ d = \mathcal{L}_{\xi^{\#}}$
8. $d\circ\mathcal{L}_{\xi^{\#}} - \mathcal{L}_{\xi^{\#}}\circ d = 0$
9. $\mathcal{L}_{(g\xi g^{-1})^{\#}} = \sigma^{*}_{g^{-1}}\circ\mathcal{L}_{\xi^{\#}}\circ\sigma^{*}_{g}$
10. $\iota_{(g\xi g^{-1})^{\#}} = \sigma^{*}_{g^{-1}}\circ\iota_{\xi^{\#}}\circ\sigma^{*}_{g}$

NOW DEFINE d_G ON $\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(M)$ AS FOLLOWS: FOR EACH $\alpha\in\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(M)$ AND EACH $\xi\in\mathfrak{g}$

$$\boxed{(d_G\alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi))}$$

ALTERNATIVE EXPRESSION: LET $\{\xi_1,\dots,\xi_n\}$ BE A BASIS FOR $\mathfrak{g}$ AND $\{x^1,\dots,x^n\}$ THE DUAL BASIS FOR $\mathfrak{g}^*$. WE WILL WRITE

$$\mathcal{L}_{\xi_a^\#} = \mathcal{L}_a \qquad \text{AND} \qquad \iota_{\xi_a^\#} = \iota_a$$

FOR EACH $a = 1,\dots,n$.

$\xi\in\mathfrak{g} \Rightarrow \xi = x^a(\xi)\xi_a \Rightarrow \xi^\# = x^a(\xi)\xi_a^\#$ AND $\iota_{\xi^\#} = x^a(\xi)\iota_a$. THUS, FOR HOMOGENEOUS ELEMENTS $\alpha = \rho\otimes\varphi$,

$$\iota_{\xi^\#}(\alpha(\xi)) = x^a(\xi)\,\iota_a(\alpha(\xi)) = x^a(\xi)\rho(\xi)\,\iota_a\varphi = (x^a\rho)(\xi)\,\iota_a\varphi$$

$$= ((x^a\otimes\iota_a)(\rho\otimes\varphi))(\xi) = ((x^a\otimes\iota_a)(\alpha))(\xi)$$

SINCE

$$d(\alpha(\xi)) = \rho(\xi)d\varphi = ((1\otimes d)(\rho\otimes\varphi))(\xi) = ((1\otimes d)(\alpha))(\xi)$$

WE FIND THAT

$$\boxed{d_G = 1\otimes d - x^a\otimes\iota_a}$$

WE CLAIM THAT

1. $d_G : \mathbb{C}[\mathfrak{g}]\otimes\Omega^*(M)\to\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(M)$ INCREASES THE ($\mathbb{Z}$-GRADED) DEGREE OF HOMOGENEOUS ELEMENTS BY 1.
2. d_G PRESERVES $\Omega_G^*(M) = [\mathbb{C}[\mathfrak{g}]\otimes\Omega^*(M)]^G$
3. $d_G\circ d_G = 0$ ON $\Omega_G^*(M)$ SO $(\Omega_G^*(M), d_G)$ IS A COCHAIN COMPLEX.

PROOFS :

1. LET $\alpha = \rho \otimes \varphi$. THEN $\deg(\alpha) = 2\deg\rho + \deg\varphi$. MOREOVER,

$$\deg((1\otimes d)(\alpha)) = 2\deg\rho + (\deg\varphi + 1)$$
$$\deg((x^a \otimes \iota_a)(\alpha)) = 2(\deg\rho + 1) + (\deg\varphi - 1)$$
$$= 2\deg\rho + (\deg\varphi + 1)$$

SO $\deg(d_G\alpha) = 2\deg\rho + (\deg\varphi + 1) = \deg\alpha + 1$.

2. LET $\alpha \in [\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)]^G$. THUS, $\alpha(g\cdot\xi) = g\cdot\alpha(\xi)$ FOR ALL $g \in G$ AND $\xi \in \mathfrak{g}$. THEN

$$(d_G\alpha)(g\cdot\xi) = d(\alpha(g\cdot\xi)) - \iota_{(g\cdot\xi)^\#}(\alpha(g\cdot\xi))$$

$$= d(g\cdot\alpha(\xi)) - \iota_{(g\xi g^{-1})^\#}(g\cdot\alpha(\xi))$$

$$= d(\sigma_{g^{-1}}^*\alpha(\xi)) - (\sigma_{g^{-1}}^* \circ \iota_{\xi^\#} \circ \sigma_g^*)(\sigma_{g^{-1}}^*\alpha(\xi))$$

$$= \sigma_{g^{-1}}^*(d(\alpha(\xi))) - \sigma_{g^{-1}}^*(\iota_{\xi^\#}(\alpha(\xi)))$$

$$= \sigma_{g^{-1}}^*(d(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi))) = g\cdot(d_G\alpha)$$

SO $d_G\alpha$ IS G-INVARIANT.

3. FOR ANY $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ (NOT NECESSARILY G-INVARIANT) WE HAVE

$$((d_G \circ d_G)(\alpha))(\xi) = (d_G(d_G\alpha))(\xi) = d((d_G\alpha)(\xi)) - \iota_{\xi^\#}((d_G\alpha)(\xi))$$
$$= d(d(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi))) - \iota_{\xi^\#}(d(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi)))$$
$$= -d(\iota_{\xi^\#}(\alpha(\xi))) - \iota_{\xi^\#}(d(\alpha(\xi)))$$
$$= -(d\circ\iota_{\xi^\#} + \iota_{\xi^\#}\circ d)(\alpha(\xi))$$
$$= -\mathcal{L}_{\xi^\#}(\alpha(\xi))$$

3. (CONTINUED) NOW SUPPOSE α IS G-INVARIANT ($\alpha(g\cdot\xi) = g\cdot\alpha(\xi)$ $\forall g \in G\ \forall \xi \in \mathfrak{g}$). THEN

$$\begin{aligned}
\mathcal{L}_{\xi^\#}(\alpha(\xi)) &= \frac{d}{dt}\left(\sigma^*_{\exp(-t\xi)}(\alpha(\xi))\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\exp(t\xi)\cdot(\alpha(\xi))\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\alpha(\exp(t\xi)\cdot\xi)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\alpha\left(\exp(t\xi)\,\xi\,\exp(-t\xi)\right)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\alpha(\xi)\right)\Big|_{t=0} \quad \text{SINCE } \xi \text{ COMMUTES WITH } \exp(t\xi) \\
&= 0
\end{aligned}$$

IN PARTICULAR, $d_G \circ d_G = 0$ ON $\Omega^*_G(M)$ AS REQUIRED. $\square$

SINCE $(\Omega^*_G(M), d_G)$ IS A COCHAIN COMPLEX ONE CAN COMPUTE ITS COHOMOLOGY. THE RESULT IS INTERESTING.

CARTAN'S THEOREM: IF THE ACTION OF G ON M IS LOCALLY FREE (ALL ISOTROPY SUBGROUPS ARE FINITE), THEN THE G-EQUIVARIANT COHOMOLOGY $H^*_G(M)$ OF M IS ISOMORPHIC TO THE COHOMOLOGY OF THE COCHAIN COMPLEX $(\Omega^*_G(M), d_G)$.

TO CONCLUDE WE WILL EXPLICITLY CALCULATE $H^*_{S^1}(S^3)$. MORE PRECISELY, WE CONSIDER

$$M = S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}$$

AND $G = S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ WITH THE LEFT ACTION GIVEN BY

$$e^{i\theta} \cdot (z^1, z^2) = (e^{i\theta} z^1, e^{i\theta} z^2)$$

THE ACTION IS CLEARLY FREE AND THE QUOTIENT SPACE IS, BY DEFINITION THE COMPLEX PROJECTIVE LINE $\mathbb{C}P^1$, WHICH IS DIFFEOMORPHIC TO S^2.

NOTE : THE RESULT OF THE CALCULATION SHOULD THEREFORE AGREE WITH THE USUAL (COMPLEX) DE RHAM COHOMOLOGY OF S^2 ($H^0(S^2;\mathbb{C}) = \mathbb{C}$, $H^1(S^2;\mathbb{C}) = 0$, $H^2(S^2;\mathbb{C}) = \mathbb{C}$, $H^k(S^2;\mathbb{C}) = 0$ FOR $k \geq 3$)

FOR THE CALCULATION WE NOTE THAT S^1 IS 1-DIMENSIONAL AND ABELIAN, DENOTE BY ξ_1 THE SINGLE GENERATOR OF THE LIE ALGEBRA $\mathfrak{g} = i\mathbb{R}$, AND BY $x^1 \in \mathfrak{g}^*$ THE DUAL TO ξ_1. WE ALSO HAVE THE CORRESPONDING OPERATORS

$$\mathcal{L}_1 = \mathcal{L}_{\xi_1^\#} \quad \text{AND} \quad \iota_1 = \iota_{\xi_1^\#}$$

ON $\Omega^*(S^3)$. SINCE S^1 IS ABELIAN, EVERY ELEMENT OF $\mathbb{C}[\mathfrak{g}]$ IS INVARIANT (CONJUGATING A VARIABLE DOES NOT CHANGE IT) SO

$$\Omega^*_{S^1}(S^3) = [\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(S^3)]^{S^1} = \mathbb{C}[x^1] \otimes \Omega^*(S^3)^{S^1}$$

THUS, REMEMBERING THAT WE DOUBLE THE DEGREES IN $\mathbb{C}[x^1]$,

$$\Omega^0_{S^1}(S^3) = \mathbb{C}^0[x^1] \otimes \Omega^0(S^3)^{S^1} \cong \Omega^0(S^3)^{S^1} = \text{SMOOTH FUNCTION ON } S^3 \text{ THAT ARE CONSTANT ON ORB}$$

$$\Omega^1_{S^1}(S^3) = \mathbb{C}^0[x^1] \otimes \Omega^1(S^3)^{S^1} \cong \Omega^1(S^3)^{S^1}$$

$$\Omega^2_{S^1}(S^3) = (\mathbb{C}^0[x'] \otimes \Omega^2(S^3)^{S^1}) \oplus (\mathbb{C}^1[x'] \otimes \Omega^0(S^3)^{S^1})$$

$$\cong \Omega^2(S^3)^{S^1} \oplus (\mathbb{C}\cdot x' \otimes \Omega^0(S^3)^{S^1})$$

$$\Omega^3_{S^1}(S^3) = (\mathbb{C}^0[x'] \otimes \Omega^3(S^3)^{S^1}) \oplus (\mathbb{C}^1[x'] \otimes \Omega^1(S^3)^{S^1})$$

$$\cong \Omega^3(S^3)^{S^1} \oplus (\mathbb{C}\cdot x' \otimes \Omega^1(S^3)^{S^1})$$

$$\Omega^4_{S^1}(S^3) = (\mathbb{C}^1[x'] \otimes \Omega^2(S^3)^{S^1}) \oplus (\mathbb{C}^2[x'] \otimes \Omega^0(S^3)^{S^1})$$

$$\cong (\mathbb{C}\cdot x' \otimes \Omega^2(S^3)^{S^1}) \oplus (\mathbb{C}^2[x'] \otimes \Omega^0(S^3)^{S^1})$$

AND SO ON.

THE EQUIVARIANT DIFFERENTIAL d_{S^1} IS DEFINED BY

$$d_{S^1} = 1 \otimes d - x' \otimes \iota_1$$

WHERE d IS THE ORDINARY EXTERIOR DERIVATIVE ON S^3

FOR $H^0_{S^1}(S^3)$ ONE CONSIDERS

$$0 \longrightarrow \Omega^0_{S^1}(S^3) \xrightarrow{d_{S^1}} \Omega^1_{S^1}(S^3)$$

AND $H^0_{S^1}(S^3) = \mathrm{KER}\,(d_{S^1})$ BUT, ON $\Omega^0_{S^1}(S^3) \cong \Omega^0(S^3)^{S^1}$, d_{S^1} COINCIDES WITH d AND, SINCE S^3 IS CONNECTED, $\mathrm{KER}\,(d)$ CONSISTS OF THE CONSTANT FUNCTIONS SO

$$H^0_{S^1}(S^3) = \mathbb{C}$$

BEFORE PROCEEDING WITH ANY FURTHER EXAMPLES WE MAKE A GENERAL OBSERVATION :

IF M IS A LEFT G-MANIFOLD, THEN ANY $\alpha \in \Omega^*(M)$ CAN BE "G-INVARIANTIZED", I.E., FOR EACH k THERE IS A CHAIN MAP

$$I : \Omega^k(M) \to \Omega^k(M)^G$$

WHICH REDUCES TO THE IDENTITY ON $\Omega^k(M)^G \subseteq \Omega^k(M)$.

ONE DEFINES I AS FOLLOWS : CHOOSE A BI-INVARIANT VOLUME FORM VOL_G ON G WITH UNIT VOLUME. FOR $v_1, \dots, v_k \in T_p(M)$ DEFINE

$$(I(\alpha))_p(v_1, \dots, v_k) = \int_G (\sigma_g^*\alpha)_p(v_1, \dots, v_k)\, \mathrm{VOL}_G$$

$$= \int_G \alpha_{g\cdot p}((\sigma_g)_{*p}(v_1), \dots, (\sigma_g)_{*p}(v_k))\, \mathrm{VOL}_G .$$

BEING A CHAIN MAP

$$d(I(\alpha)) = I(d\alpha)$$

AND WE CONCLUDE THAT IF AN INVARIANT FORM IS DE RHAM EXACT, THEN IT IS, IN FACT, COBOUNDED BY AN INVARIANT FORM, I.E.,

$$\beta \in \Omega^{k+1}(M)^G \text{ AND } \beta = d\alpha \text{ FOR SOME } \alpha \in \Omega^k(M) \Rightarrow$$

$$\beta = d(I(\alpha))$$

NOW, FOR $H^1_{S^1}(S^3)$ WE CONSIDER

$$\Omega^0_{S^1}(S^3) \xrightarrow{d^0_{S^1}} \Omega^1_{S^1}(S^3) \xrightarrow{d^1_{S^1}} \Omega^2_{S^1}(S^3)$$

AND $H^1_{S^1}(S^3) = \mathrm{KER}(d^1_{S^1})/\mathrm{IM}(d^0_{S^1})$. AN ELEMENT $\tilde{\eta} \in \Omega^1_{S^1}(S^3)$ CAN BE WRITTEN $1 \otimes \eta$ FOR SOME $\eta \in \Omega^1(S^3)^{S^1}$. THEN

$$d^1_{S^1}\tilde{\eta} = 1 \otimes d\eta - x^1 \otimes \iota_1 \eta$$

SO $d^1_{S^1}\tilde{\eta} = 0$ IMPLIES THAT $d\eta = 0$ AND $\iota_1\eta = 0$. THUS, η IS d-CLOSED AND, SINCE $H^1(S^3) = 0$, η IS d-EXACT, I.E., THERE IS AN $\alpha \in \Omega^0(S^3)$ WITH $d\alpha = \eta$. BY THE REMARKS ON THE PREVIOUS PAGE WE MAY ASSUME THAT α IS G-INVARIANT, I.E., $\alpha \in \Omega^0(S^3)^{S^1}$. THUS, $1 \otimes \alpha \in \Omega^0_{S^1}(S^3)$ AND

$$\begin{aligned} d^0_{S^1}(1 \otimes \alpha) &= 1 \otimes d\alpha - x^1 \otimes \iota_1\alpha = 1 \otimes d\alpha \\ &= 1 \otimes \eta \\ &= \tilde{\eta}. \end{aligned}$$

WE HAVE SHOWN THAT A $d^1_{S^1}$-CLOSED $\tilde{\eta} \in \Omega^1_{S^1}(S^3)$ IS $d^0_{S^1}$-EXACT, I.E.,

$$H^1_{S^1}(S^3) = 0.$$

NOW FOR $H^2_{S^1}(S^3)$ WE CONSIDER

$$\Omega^1_{S^1}(S^3) \xrightarrow{d^1_{S^1}} \Omega^2_{S^1}(S^3) \xrightarrow{d^2_{S^1}} \Omega^3_{S^1}(S^3)$$

AND $H^2_{S^1}(S^3) = \mathrm{KER}(d^2_{S^1})/\mathrm{IM}(d^1_{S^1})$. AN ELEMENT $\tilde{\omega} \in \Omega^2_{S^1}(S^3)$ CAN BE WRITTEN

$$\tilde{\omega} = 1 \otimes \omega + x^1 \otimes f$$

WHERE $\omega \in \Omega^2(S^3)^{S^1}$ AND $f \in \Omega^0(S^3)^{S^1}$. THUS,

$$
\begin{aligned}
d^2_{S^1}\tilde{\omega} &= (1\otimes d - x'\otimes\iota_1)(1\otimes\omega + x'\otimes f)\\
&= 1\otimes d\omega + x'\otimes df - x'\otimes\iota_1\omega - (x')^2\otimes\iota_1 f\\
&= 1\otimes d\omega + x'\otimes(df - \iota_1\omega)
\end{aligned}
$$

THUS, $d^2_{S^1}\tilde{\omega} = 0$ IMPLIES $d\omega = 0$ AND $df = \iota_1\omega$. WE CLAIM THAT THIS IMPLIES THAT $\tilde{\omega}$ IS d_{S^1} - COHOMOLOGOUS TO A COMPLEX MULTIPLE OF

$$x'\otimes 1$$

NOTE: $x'\otimes 1$ IS $d^2_{S^1}$ - CLOSED BECAUSE $d^2_{S^1}(x'\otimes 1) = x'\otimes d(1) - x'\otimes\iota_1(1)$. HOWEVER, IT IS NOT EXACT. TO SEE THIS SUPPOSE $1\otimes\eta\in\Omega^1_{S^1}(S^3)$ SATISFIES $d^1_{S^1}(1\otimes\eta) = x'\otimes 1$. THEN

$$1\otimes d\eta - x'\otimes\iota_1\eta = x'\otimes 1$$

SO

$$d\eta = 0 \quad \text{AND} \quad \iota_1\eta = -1.$$

BUT $d\eta = 0$ AND $H^1(S^3) = 0$ IMPLIES THAT $\eta = df$ FOR SOME $f\in\Omega^0(S^3)$ WHICH WE MAY ASSUME IS IN $\Omega^0(S^3)^{S^1}$ (PAGE 24). THUS,
$\iota_1\eta = \iota_1(df) = \mathcal{L}_1 f - d(\iota_1 f) = 0 - d(0) = 0$ SO WE CANNOT HAVE $\iota_1\eta = -1$.
THUS, $x'\otimes 1$ DETERMINES A COHOMOLOGY CLASS IN $H^2_{S^1}(S^3)$ AND, IF WE ESTABLISH THE CLAIM ABOVE, IT WILL FOLLOW THAT

$$H^2_{S^1}(S^3) = \mathbb{C}.$$

PROOF OF THE CLAIM: WE SHOW THAT $d\omega = 0$ AND $df = \iota_1\omega$ IMPLY THAT, FOR SOME $a\in\mathbb{C}$ AND SOME $\eta\in\Omega^1(S^3)^{S^1}$,

$$
\begin{aligned}
(1\otimes\omega + x'\otimes f) - a(x'\otimes 1) &= d^1_{S^1}(1\otimes\eta)\\
1\otimes\omega + x'\otimes(f - a) &= 1\otimes d\eta - x'\otimes\iota_1\eta
\end{aligned}
$$

IN ORDER FOR THIS TO BE TRUE WE MUST HAVE

$$d\eta = \omega$$

AND

$$\iota_1 \eta = a - f$$

WE SOLVE FOR η AND a.

$d\omega = 0$ AND $H^2(S^3) = 0$ IMPLY THAT THERE IS AN $\eta \in \Omega^1(S^3)$ SUCH THAT $\omega = d\eta$. AS USUAL WE CAN ASSUME $\eta \in \Omega^1(S^3)^{S^1}$ SO THE FIRST CONDITION ABOVE IS SATISFIED. FURTHERMORE, SINCE η IS S^1-INVARIANT

$$0 = \mathcal{L}_1 \eta = d(\iota_1 \eta) + \iota_1 (d\eta) = d(\iota_1 \eta) + \iota_1 \omega = d(\iota_1 \eta + f)$$

SINCE S^3 IS CONNECTED THIS IMPLIES THAT $\iota_1 \eta + f$ IS A CONSTANT FUNCTION a. FOR THIS a THE SECOND CONDITION ABOVE IS ALSO SATISFIED AND THE CLAIM IS PROVED.

ANALOGOUS ARGUMENTS SHOW THAT THE REMAINING S^1-EQUIVARIANT COHOMOLOGY GROUPS OF S^3 ARE TRIVIAL. ALTERNATIVELY ONE CAN APPEAL TO THE CARTAN THEOREM AND THE FACT THAT $H^k(S^2) = 0$ FOR $k > 2$. ONE CAN ALSO GENERALIZE AND CALCULATE THE S^1-EQUIVARIANT COHOMOLOGY OF $S^{2n-1} \subseteq \mathbb{C}^n$, I.E., THE ORDINARY COHOMOLOGY OF $\mathbb{CP}^{n-1}$.

NOTE: THERE ARE, OF COURSE, SLICKER WAYS TO DO ALL OF THIS, E.G., MAYER-VIETORIS, BUT THE OBJECTIVE HERE WAS TO GET SOME FEEL FOR THE DEFINITIONS.

ADDENDUM 9

UNIVERSAL THOM CLASS FOR $\mathbb{R}^2$

AS IN APPENDIX 2 : $V = \mathbb{R}^2$ (USUAL ORIENTATION AND INNER PRODUCT)

$\{\psi^1, \psi^2\}$ = STANDARD BASIS

$\{u_1, u_2\}$ = DUAL BASIS (COORDINATE FUNCTIONS)

$SO(V) = SO(2)$

$\{\xi_1\} = \{ \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \}$

$\{x^1\}$ = DUAL BASIS

WE DERIVED THE UNIVERSAL THOM FORM IN APPENDIX 2 :

$$\nu = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2 + u_2^2)} du_1 du_2 + (2\pi)^{-1} x^1 e^{-\frac{1}{2}(u_1^2+u_2^2)}$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\mathbb{C}^0[so(2)] \otimes \Omega^2(\mathbb{R}^2) \qquad\qquad \mathbb{C}^1[so(2)] \otimes \Omega^0(\mathbb{R}^2)$$

WE WILL VERIFY THE FOLLOWING GENERAL PROPERTIES OF THE UNIVERSAL THOM FORM :

1. ν IS A NONHOMOGENEOUS ELEMENT OF $\Omega^{2k}_{SO(V)}(V) = \Omega^2_{SO(2)}(\mathbb{R}^2)$ WITH

$$\int_V \nu = \int_{\mathbb{R}^2} \nu = 1$$

2. $d_{SO(V)} \nu = d_{SO(2)} \nu = 0$ SO ν DETERMINES A $[\nu] \in H^{2k}_{SO(V)}(V) = H^2_{SO(2)}(\mathbb{R}$

ν IS OBVIOUSLY NONHOMOGENEOUS AND (BECAUSE WE DOUBLED THE DEGREES IN $\mathbb{C}[so(V)]$) EACH TERM HAS DEGREE 2. TO SHOW THAT ν IS

SO(2)-INVARIANT WE EXAMINE EACH TERM SEPARATELY. RECALL THAT FOR HOMOGENEOUS ELEMENTS $\alpha = \rho \otimes \varphi$ OF $\mathbb{C}[so(2)] \otimes \Omega^*(\mathbb{R}^2)$, SO(2)-INVARIANCE MEANS

$$\rho(g^{-1}\xi g)\,\sigma_{g^{-1}}^{*}\varphi = \rho(\xi)\varphi$$

$\forall\, g \in SO(2)$ $\forall\, \xi \in so(2)$. FOR

$$(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1\, du_2 \in \mathbb{C}^0[so(2)] \otimes \Omega^2(\mathbb{R}^2)$$

THE ρ-PART (BEING CONSTANT) OBVIOUSLY SATISFIES $\rho(g^{-1}\xi g) = \rho(\xi)$ $\forall\, \xi$ $\forall\, g$. MOREOVER, THE φ-PART IS A ROTATIONALLY INVARIANT MULTIPLE OF THE VOLUME FORM ON $\mathbb{R}^2$ SO IT SATISFIES $\sigma_{g^{-1}}^{*}\varphi = \varphi$ $\forall\, g$. THUS, THIS FIRST PIECE IS OBVIOUSLY SO(2)-INVARIANT. FOR

$$(2\pi)^{-1} x^1 e^{-\frac{1}{2}(u_1^2+u_2^2)} \in \mathbb{C}^1[so(2)] \otimes \Omega^0(\mathbb{R}^2)$$

THE φ-PART $(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}$ IS AGAIN ROTATIONALLY INVARIANT SO $\sigma_{g^{-1}}^{*}\varphi = \varphi$. THE ρ PART IS x^1, BUT IT DOESN'T REALLY MATTER WHAT IT IS SINCE SO(2) IS ABELIAN SO $\rho(g^{-1}\xi g) = \rho(\xi)$ FOR ANY ρ.

RECALLING THE DEFINITION OF THE INTEGRAL

$$\int_V : \Omega^*_{SO(V)}(V) \rightarrow \mathbb{C}[so(V)]^{SO(V)}$$

$$\left(\int_V \alpha\right)(\xi) = \int_V \alpha(\xi) := \int_V \alpha(\xi)_{[2k]}$$

WE HAVE

$$\left(\int_{\mathbb{R}^2} \nu\right)(\xi) = \int_{\mathbb{R}^2} \nu(\xi)_{[2]}$$

$$= \int_{\mathbb{R}^2} (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2$$

$$= 1$$

$\forall\ \xi \in \mathfrak{so}(2)$ SO

$$\int_{\mathbb{R}^2} \nu = 1 \quad (\text{THE CONSTANT FUNCTION IN } \mathbb{C}[\mathfrak{so}(2)]^{SO(2)})$$

ALL THAT REMAINS IS TO SHOW

$$d_{SO(2)} \nu = 0.$$

RECALL THAT

$$(d_{SO(2)} \nu)(\xi) = d(\nu(\xi)) - \iota_{\xi^\#}(\nu(\xi))$$

$\forall\ \xi \in \mathfrak{so}(2)$, LET

$$\xi = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} = \lambda\, \xi_1$$

BE AN ARBITRARY ELEMENT OF $\mathfrak{so}(2)$. THEN $X'(\xi) = \lambda$ SO

$$\nu(\xi) = \underbrace{(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2}_{\Omega^2(\mathbb{R}^2)} + \underbrace{(2\pi)^{-1} \lambda e^{-\frac{1}{2}(u_1^2+u_2^2)}}_{\Omega^0(\mathbb{R}^2)}$$

THUS,

$$d(\nu(\xi)) = 0 + (2\pi)^{-1} \lambda\, d\left(e^{-\frac{1}{2}(u_1^2+u_2^2)}\right)$$

$$d(\nu(\xi)) = (2\pi)^{-1}\lambda\, e^{-\frac{1}{2}(u_1^2+u_2^2)}(-u_1 du_1 - u_2 du_2)$$

NEXT,

$$\iota_{\xi^\#}(\nu(\xi)) = \iota_{\xi^\#}\left((2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2 + (2\pi)^{-1}\lambda\, e^{-\frac{1}{2}(u_1^2+u_2^2)}\right)$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\,\iota_{\xi^\#}(du_1 du_2) + 0$$

(THE CONTRACTION OF ANY 0-FORM IS 0.)

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\left(\iota_{\xi^\#}(du_1)du_2 + (-1)^1 du_1\, \iota_{\xi^\#}(du_2)\right)$$

(APPENDIX 3, PAGE 10, # 2)

NOW WE CLAIM THAT

$$\iota_{\xi^\#}(du_1) = -\lambda u_2 \quad \text{AND} \quad \iota_{\xi^\#}(du_2) = \lambda u_1 .$$

TO SEE THIS NOTE THAT

$$\iota_{\xi^\#}(du_1) = u_1(\xi^\#) \quad \text{AND} \quad \iota_{\xi^\#}(du_2) = u_2(\xi^\#)$$

AND $\forall\, v \in \mathbb{R}^2$, $v = v_1\psi^1 + v_2\psi^2$,

$$\xi^\#(v) = \frac{d}{dt}\left(\exp(-t\xi)\cdot v\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\left(1 - t\xi + \tfrac{1}{2}t^2\xi^2 - \cdots\right)\cdot v\right)\Big|_{t=0}$$

$$= -\xi v = -\lambda \xi_1 v = -\lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda v_2 \\ \lambda v_1 \end{pmatrix}$$

THUS,

$$u_1(\xi^\#)(v) = -\lambda v_2 = -\lambda u_2(v)$$

AND

$$u_2(\xi^\#)(v) = \lambda v_1 = \lambda u_1(v)$$

$\forall\, v \in \mathbb{R}^2$ SO THE CLAIM FOLLOWS.

THUS,

$$\iota_{\xi^\#}(\nu(\xi)) = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (-\lambda u_2\, du_2 - du_1\, (\lambda u_1))$$

$$= (2\pi)^{-1} \lambda\, e^{-\frac{1}{2}(u_1^2+u_2^2)} (-u_1\, du_1 - u_2\, du_2)$$

$$= d(\nu(\xi))$$

SO

$$(d_{SO(2)} \nu)(\xi) = d(\nu(\xi)) - \iota_{\xi^\#}(\nu(\xi)) = 0$$

$\forall\, \xi \in so(2)$, I.E.,

$$d_{SO(2)} \nu = 0.$$

ADDENDUM 10

MATHAI - QUILLEN THOM FORM FOR TS^2

HERE WE USE THE UNIVERSAL THOM FORM

$$\nu = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (x' + du_1 du_2) \in \Omega^2_{SO(2)}(\mathbb{R}^2)$$

FOR $\mathbb{R}^2$ (SEE APPENDIX 2 AND APPENDIX 4) TO CONSTRUCT REPRESENTATIVES OF THE THOM CLASS FOR TS^2 AND THE EULER CLASS FOR S^2.

FOR THIS WE NEED TO REGARD TS^2 AS THE VECTOR BUNDLE ASSOCIATED TO SOME PRINCIPAL BUNDLE BY A REPRESENTATION OF ITS STRUCTURE GROUP.

$$\text{PRINCIPAL BUNDLE}: \quad SO(2) \hookrightarrow F_{SO}(TS^2) \xrightarrow{\pi_{SO}} S^2$$

$$\text{REPRESENTATION}: \quad \rho = id_{SO(2)} : SO(2) \to SO(2)$$

$$(\text{SO } \rho_* = id_{so(2)})$$

FOR ANY CONNECTION $\omega = \omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ON $F_{SO}(TS^2)$ (SEE APPENDIX 1),

$$\Omega = d\omega = d\omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Omega' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

SO

$$x'(\rho_*(\Omega)) = x'(\Omega) = \Omega'$$

(WE WILL MAKE A SPECIFIC CHOICE SHORTLY) SO

KEEP IN MIND THAT ALL FORMS ARE PULLED BACK TO $F_{SO}(TS^2) \times \mathbb{R}^2$ BY PROJECTIONS

$$(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (\Omega' + du_1 du_2)$$

INSERT: (FOR THE NEXT THREE PAGES I DROPPED THE NOTATIONAL CONVENTION OF SUPPRESSING THESE PROJECTIONS)

IS A FORM ON $P \times V = F_{SO}(TS^2) \times \mathbb{R}^2$ WHOSE HORIZONTAL PROJECTION (FOR THE CONNECTION $(PR_{F_{SO}(TS^2)})^* \omega$ ON $F_{SO}(TS^2) \times \mathbb{R}^2) \longrightarrow TS^2$) IS BASIC AND DESCENDS TO A REPRESENTATIVE OF THE THOM CLASS FOR TS^2.

WE COMPUTE THIS HORIZONTAL PROJECTION.

<u>NOTE</u>: FOR ANY PRINCIPAL BUNDLE $G \hookrightarrow P \xrightarrow{\pi} X$ WITH CONNECTION ω THERE IS AN EXPLICIT FORMULA FOR COMPUTING THE HORIZONTAL PROJECTION OF A FORM α (WHICH IS DEFINED TO BE THE FORM THAT EVALUATES α ON HORIZONTAL PARTS OF ITS ARGUMENTS). SINCE $\mathfrak{so}(2)$ HAS A SINGLE GENERATOR ξ_1 THIS IS PARTICULARLY SIMPLE: $HOR_\omega(\alpha) = \alpha - \omega^1 \wedge \iota_1 \alpha$, WHERE $\omega = \omega^1 \xi_1$ AND $\iota_1 = \iota_{\xi_1^\#}$ (SEE APPENDIX 3). IN OUR CASE, $P = F_{SO}(TS^2) \times \mathbb{R}^2$ AND ω IS $(PR_{F_{SO}(TS^2)})^* \omega$ WHERE ω IS A CONNECTION ON $SO(2) \hookrightarrow F_{SO}(TS^2) \xrightarrow{\pi_{SO}} S^2$.

LET

$$\alpha = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2 + u_2^2)} (\Omega^1 + du_1 du_2).$$

WE COMPUTE

$$\begin{aligned} \mho &= HOR_{(PR_{F_{SO}(TS^2)})^* \omega}(\alpha) \\ &= \alpha - (PR_{F_{SO}(TS^2)})^* \omega^1 \wedge \iota_{\xi_1^\#} \alpha \end{aligned}$$

WHERE $\xi_1^\#$ IS DEFINED FROM THE $SO(2)$-ACTION ON $F_{SO}(TS^2) \times \mathbb{R}^2$ GIVEN BY

$$(p, v) \cdot g = (p \cdot g, g^{-1} \cdot v)$$

WE COMPUTE $\iota_{\xi_1^\#}\alpha$. IDENTIFYING THE TANGENT BUNDLE OF THE PRODUCT MANIFOLD $F_{SO}(TS^2) \times \mathbb{R}^2$ WITH THE SUM OF THE TANGENT BUNDLES OF $F_{SO}(TS^2)$ AND $\mathbb{R}^2$ WE HAVE

$$\xi_1^{\#}(p,v) = \frac{d}{dt}\left((p,v)\cdot \exp(t\xi_1)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(p\cdot \exp(t\xi_1),\ \exp(-t\xi_1)\cdot v\right)\Big|_{t=0}$$

$$= \xi_{11}^{\#}(p) + \xi_{12}^{\#}(v),$$

WHERE WE USE THE EXTRA SUBSCRIPT TO DISTINGUISH THE VECTOR FIELDS ARISING FROM THE RIGHT ACTION OF $SO(2)$ ON $F_{SO}(TS^2)$ AND THE LEFT ACTION OF $SO(2)$ ON $\mathbb{R}^2$. SIMILARLY, FOR THE INTERIOR MULTIPLICATION $\iota_1 = \iota_{\xi_1^\#}$ WE WILL WRITE

$$\iota_1 = \iota_{11} + \iota_{12}$$

SINCE INSERTING $\xi_1^{\#}$ IN THE 1^{ST} SLOT IS THE SAME AS INSERTING $\xi_{11}^{\#}$ AND $\xi_{12}^{\#}$ SEPARATELY AND ADDING.

$$\iota_{\xi_1^\#}\alpha = \iota_{\xi_1^\#}\left((2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(\Omega' + du_1 du_2)\right)$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\ \iota_{\xi_1^\#}(\Omega' + du_1 du_2)$$

BECAUSE THE FORMS HAVE BEEN PULLED BACK BY PROJECTIONS.

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\left(\underbrace{\iota_{11}\Omega'}_{0 \text{ SINCE } \Omega' \text{ IS HORIZONTAL}} + \iota_{12}(du_1 du_2)\right)$$

$$= (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}\left(-u_1 du_2 - u_2 du_2\right)$$

(SEE ADDENDUM 9)

$$\iota_{\xi_1^{\#}}\alpha = -(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(u_1\,du_1 + u_2\,du_2)$$

THUS,

$$U = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(\Omega' + du_1\,du_2) - (PR_{F_{SO}(TS^2)})^{*}\omega' \wedge$$
$$(-(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(u_1\,du_1 + u_2\,du_2))$$

$$\boxed{U = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)}(\Omega' + du_1\,du_2 + (PR_{F_{SO}(TS^2)})^{*}\omega' \wedge (u_1\,du_1 + u_2\,du_2))}$$

<u>NOTE</u>: $(PR_{F_{SO}(TS^2)})^{*}\omega'$ OPERATES ON A TANGENT VECTOR V AT (p, v), WRITTEN AS $V = V_1 + V_2$ WITH V_1 TANGENT TO $F_{SO}(TS^2)$ AND V_2 TANGENT TO $\mathbb{R}^2$, TO GIVE $\omega'(V_1)$.

THUS,

$$HOR_{(PR_{F_{SO}(S^2)})^{*}\omega}(p, v) = HOR_{\omega}(p) \oplus T_{v}(\mathbb{R}^2)$$

ADDENDUM 11

MATHAI - QUILLEN EULER FORMS FOR TS^2

HERE WE CONTINUE WITH THE EXAMPLE IN APPENDIX 5 TO PRODUCE SOME REPRESENTATIVES OF THE EULER CLASS OF S^2. THUS, WE BEGIN WITH

$$U = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} (\Omega' + du_1 du_2 + \omega' \wedge (u_1 du_1 + u_2 du_2))$$

AND PULLBACK BY

$$(\sigma, s\circ\sigma)^* = ((1,s)\circ\sigma)^* = \sigma^* \circ (1,s)^*$$

WHERE σ IS A SECTION OF $F_{SO}(TS^2)$ AND $s : F_{SO}(TS^2) \to \mathbb{R}^2$ IS EQUIVARIANT FOR THE $SO(2)$-ACTIONS (WE MUST, OF COURSE, CHOOSE ω AND THESE IN ORDER TO GET CONCRETE REPRESENTATIVES OF THE EULER CLASS $e(TS^2)$).

WE PULL THE $\mathbb{R}^2$-PARTS OF U BACK BY s TO GET A FORM ON $F_{SO}(TS^2)$ WHICH IS THEN PULLED BACK TO S^2 BY σ.

AN EQUIVARIANT MAP $s : F_{SO}(TS^2) \to \mathbb{R}^2$ IS NOTHING OTHER THAN A SECTION OF TS^2, I.E., A VECTOR FIELD V ON S^2 (MORE PRECISEY, $x \to [p, s(p)]$, WHERE $p \in \pi_{SO}^{-1}(x)$, IS A SECTION OF $F_{SO}(TS^2) \times_{id} \mathbb{R}^2 = TS^2$).

A SECTION σ OF $F_{SO}(TS^2)$ IS NOTHING OTHER THAN AN ORIENTED, ORTHONORMAL FRAME FIELD ON S^2

AS OUR σ WE CHOOSE THE ORIENTED, ORTHONORMAL FRAME FIELD CORRESPONDING TO THE SPHERICAL COORDINATE CHART :

$$s(\phi,\theta) = (\phi, \theta, \frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta})$$

FOR S WE MAY CHOOSE ANY VECTOR FIELD ON S^2. WE HAVE SELECTED A CONSTANT (γ) MULTIPLE OF THE INFINITESIMAL GENERATOR FOR ROTATIONS ABOUT THE x-AXIS :

$$\gamma\left(\sin\theta \frac{\partial}{\partial\phi} + \cos\theta \cot\phi \frac{\partial}{\partial\theta}\right)$$

$$= \gamma \sin\theta \frac{\partial}{\partial\phi} + \gamma\cos\theta\cos\phi\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)$$

AS AN EQUIVARIANT MAP $S : F_{SO}(TS^2) \longrightarrow \mathbb{R}^2$ THIS IS DEFINED ON THE IMAGE OF s BY

$$(S\circ s)(\phi,\theta) = S(s(\phi,\theta)) = S(\phi,\theta, \frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta})$$

$$= (\gamma\sin\theta, \gamma\cos\theta\cos\phi)$$

AND ELSEWHERE BY EQUIVARIANCE ($S(p\cdot g) = g^{-1}\cdot S(p)$).

FOR THE CONNECTION ω ON $F_{SO}(TS^2)$ WE CHOOSE, AS IN APPENDIX 1, THE LEVI-CIVITA CONNECTION

$$\omega = -\cos\phi\, d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

WHOSE CURVATURE IS

$$\Omega = \sin\phi\, d\phi\wedge d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Omega^1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

NOW WE COMPUTE THE PULLBACK OF THE $\mathbb{R}^2$- PARTS OF $\mho$ BY S, I.E., SUBSTITUTE $\mu_1 = \gamma\sin\theta$ AND $\mu_2 = \gamma\cos\theta\cos\phi$:

- $$(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} = (2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta+\cos^2\theta\cos^2\phi)}$$

$$du_1 = \gamma\cos\theta\, d\theta$$
$$du_2 = -\gamma\sin\theta\cos\phi\, d\theta - \gamma\cos\theta\sin\phi\, d\phi$$

$$u_1 du_1 + u_2 du_2 = \gamma^2\sin\theta\cos\theta\, d\theta - \gamma^2\sin\theta\cos\theta\cos^2\phi\, d\theta - \gamma^2\cos^2\theta\sin\phi\cos\phi\, d\phi$$

$$= \gamma^2\sin\theta\cos\theta(1-\cos^2\phi)\, d\theta - \gamma^2\cos^2\theta\sin\phi\cos\phi\, d\phi$$

- $$u_1 du_1 + u_2 du_2 = \gamma^2\sin\theta\cos\theta\sin^2\phi\, d\theta - \gamma^2\cos^2\theta\sin\phi\cos\phi\, d\phi$$

$$\omega' \wedge (u_1 du_1 + u_2 du_2) = -\cos\phi\, d\theta \wedge (\gamma^2\sin\theta\cos\theta\sin^2\phi\, d\theta - \gamma^2\cos^2\theta\sin\phi\cos\phi\, d\phi)$$

$$= \gamma^2\cos^2\theta\sin\phi\cos^2\phi\, d\theta\wedge d\phi$$

- $$\omega' \wedge (u_1 du_1 + u_2 du_2) = -\gamma^2\cos^2\theta\sin\phi\cos^2\phi\, d\phi\wedge d\theta$$

$$du_1 du_2 = (\gamma\cos\theta\, d\theta)\wedge(-\gamma\sin\theta\cos\phi\, d\theta - \gamma\cos\theta\sin\phi\, d\phi)$$

$$= -\gamma^2\cos^2\theta\sin\phi\, d\theta\wedge d\phi$$

- $$du_1 du_2 = \gamma^2\cos^2\theta\sin\phi\, d\phi\wedge d\theta$$

SUBSTITUTING THESE AND $\Omega' = \sin\phi\, d\phi\wedge d\theta$ INTO THE EXPRESSION FOR $\mho$ GIVES

$$(2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta\cos^2\phi)} \left(\sin\phi \, d\phi \wedge d\theta + \gamma^2 \cos^2\theta \sin\phi \, d\phi \wedge d\theta \right.$$

$$\left. - \gamma^2 \cos^2\theta \sin\phi \cos^2\phi \, d\phi \wedge d\theta \right) =$$

$$(2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta\cos^2\phi)} \sin\phi \, (1 + \underbrace{\gamma^2\cos^2\theta - \gamma^2\cos^2\theta\cos^2\phi}_{\gamma^2\cos^2\theta\sin^2\phi}) \, d\phi \wedge d$$

$$(2\pi)^{-1} e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta\cos^2\phi)} \sin\phi \, (1 + \gamma^2\cos^2\theta\sin^2\phi) \, d\phi \wedge d\theta$$

ADDENDUM 12

ATIYAH-JEFFREY TRANSFORMATION OF THE UNIVERSAL THOM FORM

UNIVERSAL THOM FORM:

$$(1) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^T du + \tfrac{1}{2}\psi^T(\rho_*\Omega)\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

ADDITIONAL ASSUMPTIONS AND CHOICES:

1. $G \hookrightarrow P \xrightarrow{\pi_P} M$ IS ORIENTABLE

   THIS MEANS THAT $\exists$ n-FORM ($n = \dim G$) Ψ ON P S.T. IF $m \in M$ AND $\iota_m : \pi_P^{-1}(m) \hookrightarrow P$ IS THE INCLUSION MAP, THEN $\iota_m^* \Psi$ IS AN ORIENTATION FORM ON $\pi_P^{-1}(m) \cong G$. IT FOLLOWS THAT THE MANIFOLD P IS ORIENTED BY THE "LOCAL PRODUCT ORIENTATION" $\pi_P^*(\mathrm{vol}_M) \wedge \Psi$, WHERE vol_M IS A VOLUME FORM ON M.

2. THE ACTION OF G ON P IS ORIENTATION PRESERVING.

   THIS MEANS THAT EACH OF THE DIFFEOMORPHISMS $\sigma_g : P \to P$, $\sigma_g(p) = p \cdot g$, FOR $g \in G$ IS ORIENTATION PRESERVING FOR THE ORIENTATION DESCRIBED IN #1.

3. A RIEMANNIAN METRIC $\langle\ ,\ \rangle$ THAT IS G-INVARIANT HAS BEEN CHOSEN FOR P.

   THIS MEANS THAT THE DIFFEOMORPHISM $\sigma_g : P \to P$ ARE $\langle\ ,\ \rangle$-ISOMETRIES. SUCH A METRIC EXISTS WHENEVER G IS COMPACT.

4. THE CONNECTION ω FOR P IS THE ONE WHOSE DISTRIBUTION OF HORIZONTAL SUBSPACES IS THE FAMILY OF $\langle\ ,\ \rangle$ - ORTHOGONAL COMPLEMENTS OF THE $G \hookrightarrow P \xrightarrow{\pi_P} M$ VERTICAL SPACES (FIBERS).

THIS IS A CONNECTION BECAUSE $\langle\ ,\ \rangle$ IS G-INVARIANT.

5. AN ad-INVARIANT INNER PRODUCT $(\ ,\ )$ ON $\mathfrak{g}$ HAS BEEN CHOSEN AND NORMALIZED SO THAT THE VOLUME OF G (ARISING FROM THE CORRESPONDING BI-INVARIANT RIEMANNIAN METRIC ON G) IS 1.

NOW WE BEGIN WITH THE MANIPULATION OF $\mho$ IN (1).

FIRST NOTICE THAT

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega]$$

AND ω VANISHES ON HORIZONTAL PARTS SO THE SAME IS TRUE OF $[\omega,\omega]$.

NOTE : KEEP IN MIND THAT "ω" HERE MEANS THE PULLBACK BY THE PROJECTION $P \times V \to P$ OF THE CONNECTION WITH THE SAME NAME ON P. BY #4,

$$\mathrm{HOR}^{\omega}_{(p,\xi)}(P \times V) \cong T_p(p\cdot G)^{\perp} \oplus T_{\xi}(V).$$

SINCE THE EXPRESSION FOR $\mho$ IN (1) IS EVALUATED ON HORIZONTAL PARTS, $\frac{1}{2}[\omega,\omega]$ MIGHT AS WELL NOT BE THERE SO

$$(2) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^T du + \tfrac{1}{2}\psi^T(\rho_*(d\omega))\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

AS USUAL, THE RIGHT ACTION OF G ON P GIVES A LIE ALGEBRA HOMOMORPHISM

$$\mathfrak{g} \longrightarrow \Gamma(TP)$$

$$\xi \longrightarrow \xi^{\#}$$

$$\xi^{\#}(p) = \frac{d}{dt}(p \cdot \exp(t\xi))\Big|_{t=0}$$

(FOR A LEFT ACTION ONE REPLACES $t\xi$ BY $-t\xi$ SINCE OTHERWISE $\xi \to \xi^{\#}$ WOULD BE AN ANTI-HOMOMORPHISM). $\xi^{\#}(p)$ IS TANGENT TO THE G-ORBIT OF p FOR EACH p SO WE CAN DEFINE

$$C_p : \mathfrak{g} \longrightarrow \mathrm{VERT}_p(P) \subseteq T_p(P)$$

BY

$$C_p(\xi) = \xi^{\#}(p)$$

THE ACTION OF G ON P IS FREE SO C_p CARRIES $\mathfrak{g}$ ISOMORPHICALLY ONTO $\mathrm{VERT}_p(P)$, ALTHOUGH WE WILL WANT TO REGARD C_p AS A MAP INTO $T_p(P)$ AS WELL.

SINCE $\mathfrak{g}$ AND $T_p(P)$ HAVE INNER PRODUCTS $(\ ,\ )$ AND $\langle\ ,\ \rangle_p$, THE LINEAR MAP $C_p : \mathfrak{g} \to T_p(P)$ HAS AN ADJOINT

$$C_p^* : T_p(P) \longrightarrow \mathfrak{g}$$

$$\langle v_p, C_p(\eta)\rangle_p = (C_p^*(v_p), \eta) \qquad \forall\, v_p \in T_p(P) \ \ \forall\, \eta \in \mathfrak{g}$$

IN PARTICULAR,

$$\langle C_p(\xi), C_p(\eta)\rangle_p = (C_p^*(C_p(\xi)), \eta)$$
$$= (R_p(\xi), \eta)$$

WHERE

$$R_p = C_p^* \circ C_p : \mathfrak{g} \to \mathfrak{g}$$

<u>LEMMA</u>: $R_p : \mathfrak{g} \to \mathfrak{g}$ IS INVERTIBLE AND SELF-ADJOINT WITH RESPECT TO $(\, , \,)$.

<u>PROOF</u>: SELF-ADJOINTNESS FOLLOWS FROM

$$(R_p(\xi), \eta) = (C_p^*(C_p(\xi)), \eta)$$
$$= \langle C_p(\xi), C_p(\eta)\rangle_p$$
$$= \langle C_p(\eta), C_p(\xi)\rangle_p$$
$$= (C_p^*(C_p(\eta)), \xi)$$
$$= (R_p(\eta), \xi)$$
$$= (\xi, R_p(\eta)).$$

TO SEE THAT R_p IS INVERTIBLE, ASSUME $R_p(\xi) = 0$. THEN, $\forall\, \eta \in \mathfrak{g}$,

$$(R_p(\xi), \eta) = 0$$
$$(C_p^*(C_p(\xi)), \eta) = 0$$
$$\langle C_p(\xi), C_p(\eta)\rangle_p = 0$$

SINCE C_p CARRIES $\mathfrak{g}$ ISOMORPHICALLY ONTO $VERT_p(P)$, $C_p(\eta)$ CAN TAKE ON ANY VALUE IN $VERT_p(P)$. SINCE $\langle \, , \, \rangle_p$ IS NONDEGENERATE ON $VERT_p(P)$, $C_p(\xi) = 0$. BUT C_p IS ONE-TO-ONE SO $\xi = 0$. $\square$

NEXT NOTICE THAT, SINCE $C_p : \mathfrak{g} \to \mathrm{VERT}_p(P)$ IS AN ISOMORPHISM, WE HAVE AN INVERSE

$$C_p^{-1} : \mathrm{VERT}_p(P) \to \mathfrak{g}$$

LEMMA: $\forall\, w \in \mathrm{VERT}_p(P)$

$$C_p^{-1}(w) = \omega_p(w).$$

PROOF: $w \in \mathrm{VERT}_p(P) \Rightarrow w = C_p(\eta)$ FOR A UNIQUE $\eta \in \mathfrak{g}$
$= \eta^{\#}(p)$

BUT ω IS A CONNECTION FORM SO

$$\omega_p(\eta^{\#}(p)) = \eta = C_p^{-1}(w)$$

I.E.,

$$\omega_p(w) = C_p^{-1}(w). \qquad \square$$

LEMMA: $C_p^{*} : T_p(P) \to \mathfrak{g}$ VANISHES ON HORIZONTAL VECTORS.

PROOF: $w_p \in \mathrm{HOR}_p^{\omega}(P)$

$C_p^{*}(w_p)$ IS DEFINED BY

$$(C_p^{*}(w_p), \xi) = \langle w_p, C_p(\xi)\rangle_p \quad \forall\, \xi \in \mathfrak{g}$$

$= 0$ SINCE $C_p(\xi) \in \mathrm{VERT}_p(P)$ AND, BY DEFINITION OF ω, $\mathrm{HOR}_p^{\omega}(P)$ IS THE $\langle\,,\,\rangle_p$ - ORTHOGONAL COMPLEMENT OF $\mathrm{VERT}_p(P)$

NONDEGENERACY OF $(\,,\,) \Rightarrow C_p^{*}(w_p) = 0.$ $\square$

NOW WE WISH TO REGARD C^* AS A LIE ALGEBRA - VALUED 1-FORM ON P :

$$C^* \in \Omega^1(P, \mathfrak{g})$$
$$C^*(p) = C^*_p : T_p(P) \to \mathfrak{g}$$

WE SHOW NEXT THAT C^* DIFFERS FROM THE CONNECTION 1-FORM ω AT EACH POINT $p \in P$ ONLY BY THE SELF-ADJOINT LINEAR ISOMORPHISM R_p :

<u>LEMMA</u> : $C_p^* = R_p \circ \omega_p$

<u>PROOF</u> : ON $HOR_p^\omega(P)$ BOTH SIDES ARE ZERO. NEXT SUPPOSE $w \in VERT_p(P)$. THEN

$$R_p(\omega_p(w)) = (C_p^* \circ C_p)(C_p^{-1}(w)) \qquad (\text{PAGE } 5)$$
$$= C_p^*(w).$$

SINCE $T_p(P) \cong HOR_p^\omega(P) \oplus VERT_p(P)$, THIS COMPLETES THE PROOF. $\square$

SINCE $R_p : \mathfrak{g} \to \mathfrak{g}$ IS INVERTIBLE,

$$\omega_p = R_p^{-1} \circ C_p^*.$$

IF WE FIX A BASIS FOR $\mathfrak{g}$ AND REGARD THIS AS A MATRIX EQUATION

$$\omega_p = R_p^{-1} C_p^*$$

AT EACH $p \in P$ SO, WRITING R^{-1} FOR THE MAP $p \in P \to R_p^{-1} \in GL(n, \mathbb{R})$ $(n = \dim(\mathfrak{g}))$ AND dR^{-1} FOR ITS ENTRYWISE DIFFERENTIAL,

$$\omega = R^{-1} C^*$$
$$d\omega = dR^{-1} \wedge C^* + R^{-1} dC^*$$

HERE'S WHAT THIS REALLY MEANS : CHOOSE A BASIS $\{\xi_1, \ldots, \xi_n\}$ FOR $\mathfrak{g}$ AND WRITE $C^* = (C^*)^i \xi_i$. THEN,

BY DEFINITION, $dC^* = d(C^*)^i \xi_i$. WRITE THE MATRIX OF R^{-1} RELATIVE TO $\{\xi_i\}$ AS (R^{ij}), THEN (ILLUSTRATING FOR $n=2$)

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{pmatrix} \begin{pmatrix} (C^*)^1 \\ (C^*)^2 \end{pmatrix}$$

$$= \begin{pmatrix} R^{11}(C^*)^1 + R^{12}(C^*)^2 \\ R^{21}(C^*)^1 + R^{22}(C^*)^2 \end{pmatrix}$$

$$d\omega = \begin{pmatrix} d\omega^1 \\ d\omega^2 \end{pmatrix} = \begin{pmatrix} dR^{11}\wedge(C^*)^1 + R^{11}d(C^*)^1 + dR^{12}\wedge(C^*)^2 + R^{12}d(C^*)^2 \\ dR^{21}\wedge(C^*)^1 + R^{21}d(C^*)^1 + dR^{22}\wedge(C^*)^2 + R^{22}d(C^*)^2 \end{pmatrix}$$

$$= \begin{pmatrix} dR^{11}\wedge(C^*)^1 + dR^{12}\wedge(C^*)^2 \\ dR^{21}\wedge(C^*)^1 + dR^{22}\wedge(C^*)^2 \end{pmatrix} + \begin{pmatrix} R^{11}d(C^*)^1 + R^{12}d(C^*)^2 \\ R^{21}d(C^*)^1 + R^{22}d(C^*)^2 \end{pmatrix}$$

$$= \begin{pmatrix} dR^{11} & dR^{12} \\ dR^{21} & dR^{22} \end{pmatrix} \wedge \begin{pmatrix} (C^*)^1 \\ (C^*)^2 \end{pmatrix} + \begin{pmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{pmatrix} \begin{pmatrix} d(C^*)^1 \\ d(C^*)^2 \end{pmatrix}$$

$$d\omega = dR^{-1}\wedge C^* + R^{-1}dC^*$$

THE FIRST TERM VANISHES ON HORIZONTAL VECTORS (2^{ND} LEMMA, PAGE 5) SO IN (2) WE CAN REPLACE $d\omega$ WITH $R^{-1}dC^*$:

$$(3) \qquad U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^T du + \tfrac{1}{2}\psi^T(\rho_*(R^{-1}dC^*))\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

NOW WE WISH TO ELIMINATE THE EXPLICIT APPEARANCE OF THE INVERSE (R^{-1}) BY USING THE FOURIER INVERSION FORMULA.

BRIEF SYNOPSIS OF THE FOURIER TRANSFORM :

LET W BE AN ORIENTED, REAL VECTOR SPACE OF DIMENSION n WITH AN INNER PRODUCT $(\ ,\ )$. LET $x^1, \ldots, x^n$ DENOTE THE COORDINATE FUNCTIONS RELATIVE TO SOME ORIENTED, ORTHONORMAL BASIS FOR W. DENOTE BY dx THE CORRESPONDING VOLUME FORM $dx^1 \wedge \cdots \wedge dx^n$.

$\mathcal{S}$: SCHWARTZ SPACE OF RAPIDLY DECREASING FUNCTIONS IN $x^1, \ldots, x^n$

= ALL C^∞ COMPLEX-VALUED FUNCTIONS f OF $x^1, \ldots, x^n$ WHICH, TOGETHER WITH ITS DERIVATIVES, DECREASE FASTER AS $\|x\| \to \infty$ THAN THE RECIPROCOL OF ANY POLYNOMIAL, I.E.,

$\forall$ NON-NEGATIVE INTEGER k

$\forall$ MULTI-INDEX α

$\exists$ CONSTANT $C_{\alpha,k}$ S.T.

$$(1+\|x\|^2)^k \, |D^\alpha f(x)| \leq C_{\alpha,k}$$

$\forall\, x \in W$.

NOTE :
$C_0^\infty(W, \mathbb{C}) \subseteq \mathcal{S} \subseteq L^2(W)$

DEFINE TWO LINEAR TRANSFORMATIONS ON $\mathcal{S}$:

$$\mathcal{F} : \mathcal{S} \to \mathcal{S}$$

$$\mathcal{F} f = \hat{f}$$

$\forall y \in W$,

$$(\mathcal{F} f)(y) = \hat{f}(y) = (2\pi)^{-n/2} \int_W e^{-i\langle y, z\rangle} f(z)\, dz$$

$$\tilde{\mathcal{F}} : \mathcal{S} \to \mathcal{S}$$

$$(\tilde{\mathcal{F}} f)(z) = (2\pi)^{-n/2} \int_W e^{i\langle z, y\rangle} f(y)\, dy$$

PARSEVAL : ON $\mathcal{S}$
$\langle f, g\rangle_2 = \langle \hat{f}, \hat{g}\rangle_2$
$\therefore$ $\mathcal{F}$ HAS A UNIQUE EXTENSION TO A UNITARY OPERATOR ON $L^2(W)$.

(BOTH INTEGRALS ARE UNIFORMLY AND ABSOLUTELY CONVERGENT)

<u>FOURIER</u> <u>INVERSION</u> : $\mathcal{F}, \tilde{\mathcal{F}} : \mathcal{S} \to \mathcal{S}$ ARE INVERSE ISOMORPHISMS.

THUS, $\forall\ f \in \mathcal{S}$

$$f = \mathcal{F}(\tilde{\mathcal{F}} f)$$

$$f(x) = (2\pi)^{-n/2} \int_W e^{-i\langle x, z\rangle} (\tilde{\mathcal{F}} f)(z)\, dz$$

$$= (2\pi)^{-n/2} \int_W e^{-i\langle x, z\rangle} (2\pi)^{-n/2} \int_W e^{i\langle z, y\rangle} f(y)\, dy\, dz$$

$$f(x) = (2\pi)^{-n} \int_W \int_W e^{-i\langle x, z\rangle} e^{i\langle z, y\rangle} f(y)\, dy\, dz$$

NOW LET R BE A SELF-ADJOINT LINEAR TRANSFORMATION OF W WITH POSITIVE DETERMINANT AND USE THIS FORMULA TO COMPUTE $f(R^{-1}x)$:

$$f(R^{-1}x) = (2\pi)^{-n} \int_W \int_W e^{-i\langle R^{-1}x, z\rangle} e^{i\langle z, y\rangle} f(y)\, dy\, dz$$

MAKE THE LINEAR CHANGE OF VARIABLE

$$w = R^{-1} z \qquad (z = Rw)$$

$$dz = \det R\, dw$$

SO THAT

$$\langle R^{-1}x, Rw\rangle = \langle RR^{-1}x, w\rangle = \langle x, w\rangle$$

AND

$$f(R^{-1}x) = (2\pi)^{-n} \int_W \int_W e^{-i\langle x, w\rangle} e^{i\langle Rw, y\rangle} f(y) \det R\, dy\, dw$$

$$(4) \qquad f(R^{-1}x) = (2\pi)^{-n} \int_W \int_W e^{-i\langle x, w\rangle} e^{i\langle y, Rw\rangle} f(y) \det R\, dw\, dy$$

NOW WE WISH TO APPLY THIS TO THE LAST EXPRESSION (3) FOR $\mho$.

$$\mho = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp(i\psi^T du + \tfrac{1}{2}\psi^T(\rho_*(R^{-1}dC^*))\psi)\,\mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

TO SEE HOW TO DO THIS RECALL THAT $\mho$ WAS OBTAINED ORIGINALLY FROM

$$\nu_G = (2\pi)^{-\ell} e^{-\frac{1}{2}\|u\|^2} \int \exp(i\psi^T du - \tfrac{1}{2}\sum_\ell \psi^\ell (x^a \circ \rho_*) M_a \psi^\ell)\,\mathcal{D}\psi$$

$$\in [\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(V)]^G$$

BY EVALUATING $x^a \circ \rho_*$ ON THE CURVATURE Ω.

NOW THINK OF ν_G AGAIN AS AN $\Omega^*(V)$-VALUED FUNCTION OF $\phi \in \mathfrak{g}$

$$\nu_G(\phi) = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp(i\psi^T du - \tfrac{1}{2}\sum_\ell \psi^\ell (x^a(\rho_*\phi) M_a)\psi^\ell)\,\mathcal{D}$$

$$\in \Omega^*(V) \quad \forall\, \phi \in \mathfrak{g}$$

(E.G., FOR $V = \mathbb{R}^2$, $G = SO(2)$,

$$\nu_{SO(2)}(\phi) = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} x^1(\rho_*\phi) \; +$$

$$(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2$$

$$\in \Omega^0(V) \oplus \Omega^2(V) \;)$$

COMPONENTS OF $\nu_G(\phi)$ RELATIVE TO $du_1, \ldots, du_2$ ARE POLYNOMIALS IN THE COORDINATES OF $\phi = (\phi_1, \ldots, \phi_n)$. WANT TO APPLY THE INVERSION FORMULA (4), WITH $W = \mathfrak{g}$ AND $\langle\ ,\ \rangle = (\ ,\ )$, TO THESE COMPONENT FUNCTIONS

NOTE: BEING POLYNOMIALS, THESE COMPONENTS ARE NOT IN THE SCHWARTZ SPACE OF $\mathfrak{z}$ SO TO RIGOROUSLY APPLY THE INVERSION FORMULA (4) ONE WOULD HAVE TO, SAY, INSERT AN EXPONENTIAL DECAY FACTOR $e^{-\varepsilon(\phi,\phi)}$, $\varepsilon > 0$, AND TAKE THE LIMIT AS $\varepsilon \to 0$. SINCE OUR OBJECTIVES HERE ARE MORE FORMAL — WE WILL FOREGO THIS. WE WILL ALSO WRITE THE FORMULA DIRECTLY FOR $\nu_G(\phi)$ LEAVING IMPLICIT THE FACT THAT IT IS APPLIED TO EACH COMPONENT SEPARATELY. MOREOVER, WE APPLY FORMULA (4) WITH $x = dc^*$, AGAIN LEAVING IMPLICIT THE FACT THAT WE ARE DOING THIS ONCE FOR EVERY CHOICE OF A PAIR OF TANGENT VECTORS TO P. FINALLY, WE WILL USE $\phi = (\phi_1, \ldots, \phi_n)$ AND $\lambda = (\lambda_1, \ldots, \lambda_n)$ AS LIE ALGEBRA VARIABLES IN $\mathfrak{z}$ RATHER THAN y AND w.

$$\nu_G(R^{-1}dc^*) = (2\pi)^{-n} \int_{\mathfrak{z}} \int_{\mathfrak{z}} e^{-i(dc^*,\lambda)} e^{i(\phi, R\lambda)} \nu_G(\phi) \det R \, d\lambda \, d\phi =$$

$$(2\pi)^{-n} \int_{\mathfrak{z}} \int_{\mathfrak{z}} e^{-i(dc^*,\lambda)} e^{i(\phi,R\lambda)} (2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \int \exp\Big(i\psi^T d\mu - \frac{1}{2}\sum_{\ell} \psi^{\ell}(x^a(p_*\phi)\eta_a)\psi^{\ell}\Big) \mathcal{D}\psi \det R \, d\lambda \, d\phi =$$

$$(2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|\mu\|^2} \iiint \exp\Big(-i(dc^*,\lambda) + i(\phi,R\lambda) + i\psi^T d\mu - \frac{1}{2}\sum_{\ell}\psi^{\ell}(x^a(p_*\phi)\eta_a)\psi^{\ell}\Big) \det R \, \mathcal{D}\psi \, d\lambda \, d\phi$$

SO WE CAN WRITE

(5) $$\mho = (2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \iiint \exp\left(\tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T du - i(dc^*, \lambda) + i(\phi, R\lambda)\right) \det R \, \mathcal{D}\psi \, d\phi \, d\lambda$$

(EVALUATED ON HORIZONTAL PARTS)

<u>NOTE</u> : (2.9) OF ATIYAH-JEFFREY DIFFERS SLIGHTLY FROM THIS. SWITCHING THEIR NOTATION TO OURS, THEY HAVE

$$\mho = (2\pi)^{-n} \pi^{-k} e^{-\|u\|^2} \iiint \exp\left(\tfrac{1}{4}\psi^T(\rho_*\phi)\psi + i\, du^T\psi - i(dc^*, \lambda) + i(\phi, R\lambda)\right) \det R \, \mathcal{D}\psi \, d\phi \, d\lambda$$

ALL OF THESE ARE ACCOUNTED FOR BY THE DIFFERENT FORM IN WHICH THEY WRITE THE ORIGINAL MATHAI-QUILLEN THOM FORM

$$\pi^{-k} e^{-\|u\|^2} \int \exp\left(i\, du^T\psi + \tfrac{1}{4}\psi^T(\rho_*\Omega)\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

AS COMPARED WITH OURS (1)

$$(2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^T du + \tfrac{1}{2}\psi^T(\rho_*\Omega)\psi\right) \mathcal{D}\psi$$

(EVALUATED ON HORIZONTAL PARTS)

THESE ARE THE SAME : ALTHOUGH $du^T\psi = -\psi^T du$, ONLY THE SQUARE OF THIS IN $\exp(i\psi^T du)$ CONTRIBUTES TO THE BEREZIN INTEGRAL. THEN A CHANGE OF VARIABLE $u \to \frac{u}{\sqrt{2}}$ CONVERTS THE THE SECOND INTO THE FIRST. HERE'S A CHECK OF THIS FOR $V = \mathbb{R}^2$, $G = SO(2)$: APPENDIX 2 SHOWS THAT THE SECOND IS

$$\frac{1}{2\pi} e^{-\frac{1}{2}(u_1^2+u_2^2)}(x' + du_1 du_2) =$$

$$\frac{1}{2\pi} e^{-\left(\left(\frac{u_1}{\sqrt{2}}\right)^2+\left(\frac{u_2}{\sqrt{2}}\right)^2\right)}\left(x' + 2\,d\left(\frac{u_1}{\sqrt{2}}\right)d\left(\frac{u_2}{\sqrt{2}}\right)\right) =$$

$$\frac{1}{\pi} e^{-(w_1^2+w_2^2)}\left(\tfrac{1}{2}x' + dw_1 dw_2\right)$$

WHICH, FOLLOWING APPENDIX 2, IS WHAT THE FIRST WOULD GIVE.

THE NEXT OBJECTIVE IS TO INCLUDE THE PARETHETICAL REMARK "EVALUATED ON HORIZONTAL PARTS" IN (5) DIRECTLY INTO THE INTEGRAL. THE PROCEDURE WILL BE TO SHOW THAT COMPUTING THE HORIZONAL PROJECTION OF A FORM CAN BE ACCOMPLISHED BY MULTIPLYING BY A "NORMALIZED VERTICAL VOLUME FORM" FOLLOWED BY A FIBER INTEGRATION AND THEN THAT THIS NORMALIZED VERTICAL VOLUME FORM CAN BE WRITTEN AS A BEREZIN INTEGRAL AND SO INCLUDED IN (5).

A <u>NORMALIZED</u> <u>VERTICAL</u> <u>VOLUME</u> <u>FORM</u> FOR A PRINCIPAL G-BUNDLE $G \hookrightarrow Q \xrightarrow{\pi_Q} X$ (WE HAVE IN MIND $G \hookrightarrow P\times V \longrightarrow P\times_\rho V$) IS AN n-FORM ($n = \dim G$) W ON Q SUCH THAT, IF $\iota_x : \pi_Q^{-1}(x) \hookrightarrow Q$ IS THE INCLUSION MAP, THEN

$$\int_{\pi_Q^{-1}(x)} \iota_x^* W = 1$$

FOR EACH $x \in X$ (ANALOGUE OF THE THOM FORM FOR A VECTOR BUNDLE).

WE CONSTRUCT SUCH A THING AS FOLLOWS: $(\,,\,)$ IS AN ad-INVARIANT INNER PRODUCT ON $\mathfrak{g}$ FOR WHICH THE VOLUME OF G IN THE CORRESPONDING

BI-INVARIANT RIEMANNIAN METRIC IN 1. LET $\xi_1, \dots, \xi_n$ BE AN ORIENTED, ORTHONORMAL BASIS FOR $\mathfrak{g}$. THEN

$$e_i = \xi_i^{\#} \quad , i = 1, \dots, n,$$

GIVES AN ORIENTED, ORTHONORMAL FRAME FIELD ON G ($\xi_i^{\#}$ IS THE VECTOR FIELD ON G ARISING FROM THE RIGHT ACTION OF G ON G BY MULTIPLICATION) LET

$$e^1, \dots, e^n$$

BE THE 1-FORMS ON G DUAL TO

$$e_1, \dots, e_n$$

$(e^i(e_j) = \delta^i_j)$. THEN $e^1 \wedge \cdots \wedge e^n$ IS A NONZERO BI-INVARIANT n-FORM ON G AND, BY CONSTRUCTION

$$\int_G e^1 \wedge \cdots \wedge e^n = 1.$$

NOW WE WISH TO PRODUCE A COPY OF THIS NORMALIZED VOLUME FORM ON EACH FIBER OF $G \hookrightarrow Q \xrightarrow{\pi_Q} X$.

FOR THIS WE NEED A CONNECTION ω ON $G \hookrightarrow Q \xrightarrow{\pi_Q} X$. WRITE

$$\omega = \omega^1 \xi_1 + \cdots + \omega^n \xi_n$$

WHERE $\omega^i \in \Omega^1(Q)$, $i = 1, \dots, n$. WE CLAIM THAT

$$W = \omega^1 \wedge \cdots \wedge \omega^n$$

IS A NORMALIZED VERTICAL VOLUME FORM FOR $G \hookrightarrow Q \xrightarrow{\pi_Q} X$. TO SEE THIS, FIX $x \in X$ AND LET $\iota_x : \pi_Q^{-1}(x) \hookrightarrow Q$ BE THE INCLUSION. THEN

$$\iota_x^* W = (\iota_x^* \omega^1) \wedge \cdots \wedge (\iota_x^* \omega^n)$$

AND, BY OUR CONSTRUCTION ON PAGE 14, IT WILL SUFFICE TO SHOW THAT THE FORMS

$$\iota_x^* \omega^1 \; , \ldots , \; \iota_x^* \omega^n$$

ON $\pi_Q^{-1}(x) \cong G$ ARE DUAL TO THE VECTOR FIELDS $\xi_1^{\#}, \ldots, \xi_n^{\#}$ (THESE ARE THE VECTOR FIELDS ON Q DETERMINED BY THE RIGHT ACTION OF G ON Q WHICH, BY LOCAL TRIVIALITY, COINCIDES ON EACH FIBER WITH THE RIGHT ACTION OF G ON G BY MULTIPLICATION). BUT A DEFINING PROPERTY OF A CONNECTION FORM ω IS THAT $\omega(\xi^{\#}) = \xi \;\; \forall\, \xi \in \mathfrak{g}$ SO

$$\xi_i = \omega(\xi_i^{\#}) = (\omega^j \xi_j)(\xi_i^{\#}) = \omega^j(\xi_i^{\#})\, \xi_j$$

SO $\omega^j(\xi_i^{\#}) = \delta^j_i$ FOR $i, j = 1, \ldots, n$, AS REQUIRED.

NEXT WE SHOW THAT OUR VERTICAL VOLUME FORM $W = \omega^1 \wedge \cdots \wedge \omega^n$ CAN BE WRITTEN AS A BEREZIN INTEGRAL. DENOTE BY $\eta_1, \ldots, \eta_n$ AN ORTHONORMAL BASIS FOR $\mathfrak{g}$ (SAME BASIS ELEMENTS AS $\xi_1, \ldots, \xi_n$, BUT, AS WE WILL SEE BELOW, POSSIBLY RE-ORDERED). REGARD THESE AS ODD GENERATORS FOR $\Lambda(\mathfrak{g})$ AND CONSIDER THE FOLLOWING ELEMENT OF $\Omega^*(Q) \otimes \Lambda(\mathfrak{g})$ (THE "$\omega \otimes \eta$" IS JUST A CONVENIENT SHORTHAND FOR $\sum_{i=1}^n \omega_i \eta_i$):

$$e^{\omega \otimes \eta} = e^{(\omega_1 \eta_1 + \cdots + \omega_n \eta_n)} = e^{\omega_1 \eta_1} \cdots e^{\omega_n \eta_n} = (1 + \omega_1 \eta_1) \cdots (1 + \omega_n \eta_n)$$

THEN

$$\int e^{\omega \otimes \eta} \mathcal{D}\eta = \int (1+\omega_1\eta_1)\cdots(1+\omega_n\eta_n)\,\mathcal{D}\eta$$

$$= \int (\omega_1\eta_1)\cdots(\omega_n\eta_n)\,\mathcal{D}\eta$$

$$= (-1)^{1+2+\cdots+(n-1)} \int \omega_1\cdots\omega_n\,\eta_1\cdots\eta_n\,\mathcal{D}\eta$$

$$= (-1)^{n(n-1)/2} \int \omega_1\cdots\omega_n\,\eta_1\cdots\eta_n\,\mathcal{D}\eta$$

THUS, IF WE AGREE TO ORDER $\eta_1,\dots,\eta_n$ APPROPRIATELY (SAME AS $\xi_1,\dots,\xi_n$ IF $n(n-1)/2$ IS EVEN AND AN ODD PERMUTATION OF THESE IF $n(n-1)/2$ IS ODD), THIS WILL GIVE

$$\int e^{\omega\otimes\eta}\,\mathcal{D}\eta = \omega_1\wedge\cdots\wedge\omega_n = W.$$

THIS STILL IS NOT QUITE THE FORM USED BY ATIYAH AND JEFFRY, HOWEVER. TO OBTAIN THIS WE WILL UTILIZE THE MAPS C_p, C_p^*, AND R_p INTRODUCED EARLIER (NOW FOR THE BUNDLE $G \hookrightarrow Q \xrightarrow{\pi_Q} X$ FOR WHICH WE NOW MAKE ALL OF THE SAME ASSUMPTIONS WE MADE OF $G \hookrightarrow P \xrightarrow{\pi_P} M$ AT THE OUTSET). OUR OBJECTIVE IS TO EXPLAIN AND JUSTIFY THE EQUALITY

$$(6) \qquad \int e^{\langle C^*,\eta\rangle}\,\mathcal{D}\eta = \det(R)\int e^{\omega\otimes\eta}\,\mathcal{D}\eta = \det(R)\,W.$$

NOTE THAT, FOR EACH FIXED $\eta \in \mathfrak{g}$, $\langle\cdot, C\eta\rangle$ IS A 1-FORM ON Q WHOSE VALUE AT ANY VECTOR FIELD Ψ ON Q IS $\langle\Psi, C\eta\rangle$, I.E.,

$$\langle\Psi, C\eta\rangle(p) = \langle\Psi(p), C_p(\eta)\rangle_p$$

$\forall\, p\in Q$. BUT

$$\langle \psi, C\eta \rangle = (C^*\psi, \eta)$$

SO WE CAN REGARD

$$(C^*, \eta) = \langle \cdot \, , C\eta \rangle$$

AS A LINEAR FUNCTION OF η WHOSE VALUES ARE 1-FORMS ON Q, I.E.,

$$(C^*, \eta) \in \mathbb{C}[\mathfrak{z}] \otimes \Omega^*(Q).$$

THUS, $e^{(C^*, \eta)}$ AND $\int e^{(C^*, \eta)} \mathcal{D}\eta$ AT LEAST MAKE SENSE.

TO JUSTIFY (6) WE WRITE THIS ELEMENT OUT IN MORE DETAIL AS FOLLOWS.

LET $\{A_1, \ldots, A_n, A_{n+1}, \ldots, A_d\}$ BE LOCAL COORDINATES ON Q WITH $A_1, \ldots, A_n$ VERTICAL ($\frac{\partial}{\partial A_1}, \ldots, \frac{\partial}{\partial A_n}$ TANGENT TO THE FIBERS) AND $A_{n+1}, \ldots, A_{n+d}$ HORIZONTAL ($\frac{\partial}{\partial A_{n+1}}, \ldots, \frac{\partial}{\partial A_d}$ ORTHOGONAL TO THE FIBERS RELATIVE TO $\langle \, , \, \rangle$). THEN, SINCE $C\eta$ IS VERTICAL FOR EACH $\eta \in \mathfrak{z}$

$$(C^*, \eta) = \langle \cdot \, , C\eta \rangle = \sum_{i=1}^{d} \langle \frac{\partial}{\partial A_i}, C\eta \rangle \, dA_i$$

$$= \sum_{i=1}^{n} \langle \frac{\partial}{\partial A_i}, C\eta \rangle \, dA_i$$

$$= \sum_{i=1}^{n} (C^*(\frac{\partial}{\partial A_i}), \eta) \, dA_i$$

NOW WRITE $C^*(\frac{\partial}{\partial A_i}) = \sum_{j=1}^{n} a_i^j \eta_j$ SO THAT

$$(C^*(\tfrac{\partial}{\partial A_i}), \cdot) = \sum_{j=1}^{n} a_i^j (\eta_j, \cdot)$$

AND

$$(C^*, \eta) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^j (\eta_j, \eta) dA_i$$

$$= \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} a_i^j dA_i\Big)(\eta_j, \eta)$$

$$= \sum_{j=1}^{n} \beta_j \eta^j$$

WHERE

$$\beta_j = \sum_{i=1}^{n} a_i^j dA_i \in \Omega^1(Q)$$

AND

$$\{\eta^1, \dots, \eta^n\} = \{(\eta_1, \cdot), \dots, (\eta_n, \cdot)\}$$

IS THE BASIS FOR $\mathfrak{g}^*$ DUAL TO $\{\eta_1, \dots, \eta_n\}$.

IDENTIFYING $\mathfrak{g}^*$ AND $\mathfrak{g}$ VIA $(\ ,\ )$ WE CONSIDER THE ELEMENT $\sum_{j=1}^{n} \beta_j \eta_j$ OF $\Omega^*(Q) \otimes \Lambda(\mathfrak{g})$ AND COMPUTE, AS ON PAGE 16, THE BEREZIN INTEGRAL

$$\int e^{(C^*, \eta)} \mathcal{D}\eta = \int e^{(\beta_1\eta_1 + \dots + \beta_n\eta_n)} \mathcal{D}\eta$$

$$= (-1)^{n(n-1)/2} \beta_1 \wedge \dots \wedge \beta_n$$

$$= (-1)^{n(n-1)/2} (a_1^1 dA_1 + \dots + a_n^1 dA_n) \wedge \dots \wedge (a_1^n dA_1 + \dots + a_n^n dA_n)$$

$$= (-1)^{n(n-1)/2} \det(a_j^i)\, dA_1 \wedge \dots \wedge dA_n$$

NOW WE COMPUTE $\det(a_j^i)$ AS FOLLOWS. $C^*(\frac{\partial}{\partial A_i}) = \sum_{j=1}^{n} a_i^j \eta_j$ IMPLIES

$$a_i^j = (C^*(\tfrac{\partial}{\partial A_i}), \eta_j) = \langle \tfrac{\partial}{\partial A_i}, C(\eta_j) \rangle$$

$$= \langle C(\omega(\tfrac{\partial}{\partial A_i})), C(\eta_j) \rangle$$ BECAUSE $\omega(\frac{\partial}{\partial A_i}) = C^{-1}(\frac{\partial}{\partial A_i})$ (FIRST LEMMA, PAGE 5)

$$= (C^* \circ C(\omega(\tfrac{\partial}{\partial A_i})), \eta_j)$$

$$= (R(\omega(\tfrac{\partial}{\partial A_i})), \eta_j)$$

NOW WRITE

$$\omega = \tilde{\omega}^1 \eta_1 + \cdots + \tilde{\omega}^n \eta_n$$

(NOTE THAT $\{\tilde{\omega}^1, \ldots, \tilde{\omega}^n\}$ DIFFERS FROM $\{\omega^1, \ldots, \omega^n\}$ AT MOST IN THE ORDERING). THEN

$$\omega(\tfrac{\partial}{\partial A_i}) = \tilde{\omega}^1(\tfrac{\partial}{\partial A_i})\eta_1 + \cdots + \tilde{\omega}^n(\tfrac{\partial}{\partial A_i})\eta_n$$

$$R(\omega(\tfrac{\partial}{\partial A_i})) = \tilde{\omega}^1(\tfrac{\partial}{\partial A_i})R(\eta_1) + \cdots + \tilde{\omega}^n(\tfrac{\partial}{\partial A_i})R(\eta_n)$$

$$a_i^j = \tilde{\omega}^1(\tfrac{\partial}{\partial A_i})(R(\eta_1), \eta_j) + \cdots + \tilde{\omega}^n(\tfrac{\partial}{\partial A_i})(R(\eta_n), \eta_j)$$

$$(a_i^j) = \begin{pmatrix} (R(\eta_1), \eta_1) & \cdots & (R(\eta_n), \eta_1) \\ \vdots & & \vdots \\ (R(\eta_1), \eta_n) & \cdots & (R(\eta_n), \eta_n) \end{pmatrix} \begin{pmatrix} \tilde{\omega}^1(\tfrac{\partial}{\partial A_1}) & \cdots & \tilde{\omega}^1(\tfrac{\partial}{\partial A_n}) \\ \vdots & & \vdots \\ \tilde{\omega}^n(\tfrac{\partial}{\partial A_1}) & \cdots & \tilde{\omega}^n(\tfrac{\partial}{\partial A_n}) \end{pmatrix}$$

THUS,

$$\det(a_i^j) = \det(R)\det(\tilde{\omega}_i^j)$$

WHERE

$$\tilde{\omega}^j = \sum_{i=1}^n \tilde{\omega}_i^j \, dA_i = \sum_{i=1}^n \tilde{\omega}^j(\tfrac{\partial}{\partial A_i}) dA_i$$

CONSEQUENTLY,

$$\int e^{(c^*,\eta)}\,\mathcal{D}\eta = (-1)^{n(n-1)/2}\det(a_i^j)\,dA_1\wedge\cdots\wedge dA_n$$

$$= (-1)^{n(n-1)/2}\det(R)\det(\tilde{\omega}_i^j)\,dA_1\wedge\cdots\wedge dA_n$$

$$= (-1)^{n(n-1)/2}\det(R)\,\Big(\sum_{i=1}^{n}\tilde{\omega}_i^1 dA_i\Big)\wedge\cdots\wedge\Big(\sum_{i=1}^{n}\tilde{\omega}_i^n dA_i\Big)$$

$$= (-1)^{n(n-1)/2}\det(R)\;\tilde{\omega}^1\wedge\cdots\wedge\tilde{\omega}^n$$

$$= \det(R)\,\big((-1)^{n(n-1)/2}\,\tilde{\omega}^1\wedge\cdots\wedge\tilde{\omega}^n\big)$$

$$= \det(R)\;\omega^1\wedge\cdots\wedge\omega^n$$

$$= \det(R)\;W$$

AS PROMISED IN (6).

NOW WE HAVE AN EXPRESSION THAT WILL ALLOW US, IN (5), TO INCORPORATE THE PARENTHETICAL PHRASE "EVALUATED ON HORIZONTAL PARTS" DIRECTLY INTO THE INTEGRAL. THE IDEA IS THIS. THE NORMALIZED VERTICAL VOLUME FORM IS "VERTICAL" IN THE SENSE THAT, IN LOCAL COORDINATES $A_1,\dots,A_n, A_{n+1},\dots,A_d$ AS ON PAGE 17, IT IS A MULTIPLE OF $dA_1\wedge\cdots\wedge dA_n$ (THE CONNECTION COMPONENTS $\omega^1,\dots,\omega^n$ CANNOT CONTAIN $dA_{n+1},\dots,dA_d$ SINCE "HORIZONTAL" MEANS "$\langle\,,\,\rangle$-ORTHOGONAL TO THE FIBERS", AND THEREFORE NEITHER CAN $\omega^1\wedge\cdots\wedge\omega^n$). THUS, WEDGING ANY FORM ON Q WITH W KILLS ANY TERM WITH A VERTICAL PART AND LEAVES ALL REMAINING TERMS WITH HORIZONTAL COMPONENTS AND A FACTOR OF W. IGNORING THE FACTOR OF W (OR, MORE PRECISELY, PERFORMING

A FIBER INTEGRATION) LEAVES A HORIZONTAL FORM WHOSE VALUE AT ANY SET OF TANGENT VECTORS IS THE SAME AS THE ORIGINAL FORM "EVALUATED ON HORIZONTAL PARTS".

ALL OF THIS WE NOW APPLY TO THE BUNDLE $G \hookrightarrow P \times V \longrightarrow P \times_\rho V$ WITH THE ORIENTATION, METRIC AND CONNECTION PULLED BACK FROM $G \hookrightarrow P \to X$ BY THE PROJECTION $P \times V \to P$ (AS ALWAYS, WE USE THE SAME SYMBOLS FOR THESE PULLBACKS). THE CORRESPONDING NORMALIZED VERTICAL VOLUME FORM CAN BE WRITTEN

$$W = (\det R)^{-1} \int e^{(c^*, \eta)} \mathcal{D}\eta$$

SO WEDGING THE EXPRESSION FOR U IN (5) WITH W CANCELS THE $\det R$ AND ADDS THE TERM (c^*, η) TO THE EXPONENT.

$$(7) \quad U = (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \iiiint \exp\left(\tfrac{1}{2} \psi^T (\rho_* \phi) \psi + i \psi^T du - i (dc^*, \lambda) + i (\phi, R\lambda) + (c^*, \eta) \right) \mathcal{D}\eta\, \mathcal{D}\psi\, d\phi\, d\lambda$$

A FIBER INTEGRATION OVER THE FIBERS OF $G \hookrightarrow P \times V \longrightarrow P \times_\rho V$ IS NECESSARY TO OBTAIN A FORM ON $P \times_\rho V$, I.E., A THOM FORM, BUT WE WILL EITHER LEAVE THIS IMPLICIT, OR BYPASS THE NEED TO DESCEND TO $P \times_\rho V$ AS FOLLOWS :

OUR REAL OBJECTIVE IS TO FIND REPRESENTATIVES OF THE EULER CLASS OF $P \times_\rho V$ AND THESE ARE OBTAINED BY PULLING BACK A THOM FORM OF $P \times_\rho V$ BY A SECTION OF $P \times_\rho V$. BUT EVERY SECTION OF THE ASSOCIATED

VECTOR BUNDLE $P\times_\rho V$ IS OF THE FORM

$$m \in M \longrightarrow (s(m), S(s(m))) \xrightarrow{q} [s(m), S(s(m))]$$

WHERE s IS A SECTION OF $G \hookrightarrow P \longrightarrow M$ AND $S: P \longrightarrow V$ IS AN EQUIVARIANT MAP $(S(p\cdot g) = \rho(g^{-1})(S(p)))$ AND q IS THE QUOTIENT MAP $P\times V \longrightarrow P\times_\rho V$.

NOW, IF WE TEMPORARILY DENOTE BY U THE FORM ON $P\times V$ GIVEN BY (7) PRIOR TO THE FIBER INTEGRATION AND BY $\tilde{U}$ THE FORM ON $P\times_\rho V$ TO WHICH IT DESCENDS (SO $U = q^*\tilde{U}$), THEN

$$\begin{aligned}(q\circ(s, S\circ s))^* &= (s, S\circ s)^*\circ q^* \\ &= ((1,S)\circ s)^*\circ q^* \\ &= s^*\circ(1,S)^*\circ q^*\end{aligned}$$

SO

$$(q\circ(s, S\circ s))^*\tilde{U} = s^*((1,S)^*U)$$

SO OUR EULER FORM CAN BE GOTTEN DIRECTLY FROM U WITHOUT THE NEED FOR $\tilde{U}$.

EVEN MORE EXPLICITLY, ONE OBTAINS AN EULER FORM FOR $P\times_\rho V$ FROM U ON $P\times V$ BY

(i) PULLING BACK THE V PARTS OF U BY THE EQUIVARIANT MAP $S: P\to V$ $((1,S)^*U)$ AND THEN

(ii) PULLING THE RESULTING FORM ON P BACK TO M BY A SECTION s OF $G\hookrightarrow P \rightarrow M$.

<u>NOTE</u>: THERE IS AN EXAMPLE IN ADDENDUM II.

IN FACT, IF WE REFRAIN FROM PULLING BACK TO M BY Δ WE HAVE A FORM $(1,S)^*\mho$ ON P WHOSE INTEGRAL OVER P IS STILL THE EULER NUMBER OF $P \times_\rho V$. THE REASON IS AS FOLLOWS: FOR ANY (INTEGRABLE) TOP FORM α ON THE BASE X OF A PRINCIPAL BUNDLE $G \hookrightarrow Q \xrightarrow{\pi_Q} X$ WITH NORMALIZED VERTICAL VOLUME FORM W,

$$\int_X \alpha = \int_Q \pi_Q^* \alpha \wedge W .$$

IN OUR CASE, α IS THE DESCENT TO X OF A BASIC FORM ON Q WITH A FACTOR OF W IN EACH TERM SO $\pi_Q^*\alpha \wedge W$ IS JUST THE FORM ON Q WHICH DESCENDS TO α.

THE IS JUST ONE LAST BIT OF COSMETIC SURGERY WE NEED TO APPLY TO $\mho$ IN (7) TO ARRIVE AT THE ATIYAH-JEFFREY FORM. TECHNICALLY, WHAT WE DO IS VALID ONLY FOR FINITE-DIMENSIONAL VECTOR SPACES, BUT, AS A NOTATIONAL DEVICE, IT IS COMMONLY USED IN MUCH MORE GENERAL CONTEXTS IN (SUPERSYMMETRIC) PHYSICS.

LET P DENOTE A FINITE-DIMENSIONAL, ORIENTED, REAL VECTOR SPACE WITH A POSITIVE DEFINITE INNER PRODUCT (THOUGHT OF AS AN ORIENTED RIEMANNIAN MANIFOLD). LET $x^1, \dots, x^n$ BE THE COORDINATE FUNCTIONS FOR P CORRESPONDING TO SOME ORIENTED, ORTHONORMAL BASIS FOR P. LET $d\omega = dx^1 \wedge \dots \wedge dx^n$ BE THE VOLUME FORM FOR P. THE INTEGRAL OF ANY (PROPERLY DECAYING) FUNCTION φ ON P IS THEN DEFINED BY

$$\int_P e \, d\omega .$$

INTRODUCE ODD GENERATORS $\xi^1, \dots, \xi^n$ FOR ΛP.
IF α IS ANY (PROPERLY DECAYING) FORM ON P WRITTEN IN TERMS OF x^i AND dx^i ($\alpha(x^i, dx^i)$), THEN THE FORMAL SUBSTITUTION

$$dx^i \longrightarrow \xi^i$$

$i = 1, \dots, n$ GIVES AN ELEMENT OF $C^\infty(P) \otimes \Lambda P$ WHOSE FERMIONIC INTEGRAL

$$\int \alpha(x^i, \xi^i) \, \mathcal{D}\xi$$

IS PRECISELY THE FUNCTION ONE INTEGRATES (NEXT TO $d\omega$ AS ABOVE) TO GET $\int_P \alpha$.
THUS,

$$\int_P \alpha = \int_P \int \alpha(x^i, \xi^i) \, \mathcal{D}\xi \, d\omega$$

THE POINT IS THAT INTEGRALS OF FORMS CAN BE THOUGHT OF AS A BEREZIN INTEGRAL FOLLOWED BY A VOLUME INTEGRAL.

TO ARRIVE AT OUR FINAL EXPRESSION FOR THE <u>EULER NUMBER</u> FOR $P \times_\rho V$ WE THEREFORE PROCEED AS FOLLOWS: LET $s : P \to V$ BE AN EQUIVARIANT MAP ($s(p \cdot g) = \rho(g^{-1})(s(p))$). PULL THE V-PARTS OF $\mathcal{U}$ BACK BY s (I.E., SUBSTITUTE $u = s$). TO INTEGRATE THE RESULTING FORM OVER P, INTRODUCE AN ODD ("FERMIONIC") VARIABLE ξ, REGARD ALL OF THE TERMS IN (7) GIVING RISE TO FORMS ON P (I.E., $i\psi^T ds$, $i(dc^*, \lambda)$ AND (c^*, η)) AS FUNCTIONS OF ξ RATHER THAN FORMS, AND LET $d\omega$ BE THE VOLUME FORM ON P AND THEN DO THE

THE BEREZIN INTEGRAL WITH RESPECT TO $\mathcal{D}\xi$ FOLLOWED BY THE VOLUME INTEGRAL WITH RESPECT TO $d\omega$.

THUS, ONE OBTAINS FROM (7), FINALLY SUPPRESSING ALL BUT ONE OF THE INTEGRAL SIGNS,

$$(8) \quad (2\pi)^{-n}(2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \int \exp\Big(\tfrac{1}{2}\psi^T(\rho + \phi)\psi + i\psi^T ds - i(dc^*, \lambda) + i(\phi, R\lambda) + (c^*, \eta)\Big) \mathcal{D}\xi\, \mathcal{D}\eta\, \mathcal{D}\psi\, d\lambda\, d\phi\, d\omega$$

ADDENDUM 13

THE WITTEN LAGRANGIAN

OBJECTIVE IS TO FORMALLY APPLY THE ATIYAH - JEFFREY INTEGRAL REPRESENTATION FOR THE EULER NUMBER

$$(1) \quad (2\pi)^{-n}(2\pi)^{-k} \int \exp\{-\tfrac{1}{2}\|s\|^2 + \tfrac{1}{2}\psi^T(\rho_* \phi)\psi + i\psi^T ds$$

$$- i(dC^*, \lambda) + i(\phi, R\lambda) + (C^*, \eta)\}\, \mathcal{D}\xi\, \mathcal{D}\eta\, \mathcal{D}\psi\, d\lambda\, d\phi\, d\omega$$

(ADDENDUM 12) TO A CERTAIN INFINITE-DIMENSIONAL VECTOR BUNDLE THAT ARISES NATURALLY IN DONALDSON THEORY TO OBTAIN THE PARTITION FUNCTION

$$Z_{DW} = \int \exp(-S_{DW}[\Phi]/e^2)\, \mathcal{D}\Phi$$

AND THEREBY THE ACTION $S_{DW}[\Phi]$ AND SO THE LAGRANGIAN $\mathcal{L}_{DW}[\Phi]$ OF DONALDSON-WITTEN THEORY (WITTEN'S 1988 TQFT IN WHICH Z_{DW} WAS IDENTIFIED WITH THE 0-DIMENSIONAL DONALDSON INVARIANT AND HIGHER DONALDSON INVARIANTS APPEARED AS EXPECTATION VALUES OF CERTAIN OBSERVABLES).

NOTE: THIS DEPENDS ON MAKING A VERY SPECIFIC CHOICE FOR THE SECTION SO IN THIS REGARD IT IS QUITE UNLIKE THE REPRESENTATION (1) FOR THE EULER NUMBER OF A FINITE-DIMENSIONAL VECTOR BUNDLE.

FOR CLARITY WE BEGIN BY REWRITING (1), EXPLICITLY INDICATING ALL OF THE DEPENDENCIES ON THE THREE "BOSONIC" (λ, ϕ, ω) AND THREE

"FERMIONIC" (ζ, η, ψ) VARIABLES (KEEP IN MIND THAT $d\omega$ REPRESENTS A VOLUME FORM ON THE PRINCIPAL BUNDLE SPACE P SO ω REPRESENTS A POINT IN P AND ALL TERMS THAT GIVE RISE TO FORMS ON P ARE REGARDED, NOT AS FORMS, BUT AS FUNCTIONS OF THE FERMIONIC VARIABLE ζ).

$$(2)\quad (2\pi)^{-n}(2\pi)^{-k}\int \exp\{-\tfrac{1}{2}\|S(\omega)\|^2+\tfrac{1}{2}\psi^T(\rho_*\phi)\psi+i\,\psi^T ds_\omega(\zeta)$$

$$-i(dC^*_\omega(\zeta),\lambda)\ +i(\phi,R_\omega\lambda)+(C^*_\omega(\zeta),\eta)\}\,\mathcal{D}\zeta\,\mathcal{D}\eta\,\mathcal{D}\psi\, d\lambda\, d\phi\, d\omega$$

NOW WE DESCRIBE THE BUNDLE WE HAVE IN MIND.

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD WITH $b_2^+(M) > 1$ AND ODD

$SU(2) \hookrightarrow P \xrightarrow{\pi} M$ A FIXED PRINCIPAL $SU(2)$-BUNDLE OVER M

g = GENERIC RIEMANNIAN METRIC ON M

$\hat{\mathcal{A}}$ = SPACE OF IRREDUCIBLE CONNECTIONS ON P (OPEN IN THE AFFINE SPACE $\mathcal{A}$ OF ALL CONNECTIONS ON P)

$\hat{\mathcal{G}} = \mathcal{G}/\mathbb{Z}_2$ = GAUGE TRANSFORMATIONS MODULO CENTER

$\hat{\mathcal{B}} = \hat{\mathcal{A}}/\hat{\mathcal{G}} = \hat{\mathcal{A}}/\mathcal{G}$

$\mathcal{M} = \hat{\mathcal{M}}$ = MODULI SPACE OF g-ASD CONNECTIONS ON P

$\hat{\mathcal{G}}$ ACTS FREELY ON $\hat{\mathcal{A}}$ SO

$$\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$$

IS A SMOOTH (INFINITE-DIMENSIONAL) PRINCIPAL BUNDLE.

$\Omega^2_+(M, \mathrm{ad}\, P)$ = (INFINITE-DIMENSIONAL) VECTOR SPACE OF SELF-DUAL 2-FORMS WITH VALUES IN THE ADJOINT BUNDLE ad P.

$\hat{\mathcal{G}}$ ACTS ON $\Omega^2_+(M, \mathrm{ad}\, P)$ ON THE LEFT (THINK OF $\hat{\mathcal{G}}$ AS SECTIONS OF THE NONLINEAR ADJOINT BUNDLE Ad P AND CONJUGATE POINTWISE).

THUS, THERE IS A VECTOR BUNDLE

$$\hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega^2_+(M, \mathrm{ad}\, P)$$

ASSOCIATED TO $\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$.

A SECTION IS DETERMINED BY THE EQUIVARIANT MAP

$$s = F^+ : \hat{\mathcal{A}} \longrightarrow \Omega^2_+(M, \mathrm{ad}\, P)$$

$$s(\omega) = F_\omega^+ = \tfrac{1}{2}(F_\omega + {}^*F_\omega)$$

$\mathcal{M}$ = ZERO SET OF THIS SECTION
= INTERSECTION OF THIS SECTION WITH THE ZERO-SECTION OF $\hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega^2_+(M, \mathrm{ad}\, P)$

NOW WE ASSUME THAT WE ARE IN THE SITUATION IN WHICH $\mathcal{M}$ IS NONEMPTY AND $8k - 3(1 + b_2^+(M)) = 0$ SO $\mathcal{M}$ IS A FINITE SET OF SIGNED POINTS. THUS, THE 0-DIMENSIONAL DONALDSON INVARIANT $\gamma_0(M)$ CAN BE VIEWED AS AN "INTERSECTION NUMBER" FOR THE TWO SECTIONS OF $\hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega^2_+(M, \mathrm{ad}\, P)$, WHICH MOTIVATES LOOKING FOR AN "EULER

NUMBER" INTERPRETATION AND THIS IS WHERE THE MATHAI-QUILLEN FORM ENTERS THE PICTURE. THE ATTEMPT TO PRODUCE SUCH AN INTERPRETATION WHICH ALSO HAPPENS TO AGREE WITH WITTEN'S PARTITION FUNCTION IS THE MOTIVATION BEHIND (2).

WE APPLY (2) TO THE VECTOR BUNDLE

$$\hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega^2_+(M, \mathrm{ad}\,P)$$

ASSOCIATED TO $\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$.

REQUIRES A RIEMANNIAN METRIC ON $\hat{\mathcal{A}}$ FOR WHICH $\hat{\mathcal{G}}$ ACTS BY ISOMETRIES (ADDENDUM 12).

$\hat{\mathcal{A}}$ IS OPEN IN $\mathcal{A}$ WHICH IS AN AFFINE SPACE MODELED ON $\Omega^1(M, \mathrm{ad}\,P)$ SO

$$T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, \mathrm{ad}\,P)$$

$\forall\ \omega \in \hat{\mathcal{A}}$.

<u>RECALL</u>: GIVEN THE RIEMANNIAN METRIC g AND ORIENTATION ON M THERE IS A HODGE STAR $*$ AND SO A POINTWISE INNER PRODUCT $\langle\ ,\ \rangle_p$ ON ANY $\Omega^k(M, \mathbb{R})$ GIVEN AS FOLLOWS: $\alpha, \beta \in \Omega^k(M, \mathbb{R})$,

$$\alpha \wedge {}^*\beta = \langle \alpha, \beta \rangle_p \,\mathrm{vol}_g .$$

LOCALLY,

$$\langle \alpha, \beta \rangle_p = \frac{1}{k!}\, \alpha_{i_1 \cdots i_k} \beta^{i_1 \cdots i_k}$$

THEN A GLOBAL L^2- INNER PRODUCT $\langle\ ,\ \rangle_k$ ON $\Omega^k(M, \mathbb{R})$

IS OBTAINED BY INTEGRATION :

$$\langle \alpha, \beta \rangle_k = \int_M \alpha \wedge {}^*\beta = \int_M \langle \alpha, \beta \rangle_p \, \mathrm{vol}_g$$

NEXT SUPPOSE $\mu, \nu \in \Omega^k(M, su(2))$. WE SELECT THE ad-INVARIANT INNER PRODUCT $(\ ,\ )_{su(2)}$ ON $su(2)$ GIVEN BY

$$(A, B)_{su(2)} = -2 \, tr(AB).$$

THEN

$$T_1 = -\tfrac{1}{2} i \sigma_1 = -\tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$
$$T_2 = -\tfrac{1}{2} i \sigma_2 = -\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$T_3 = -\tfrac{1}{2} i \sigma_3 = -\tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

IS AN ORTHONORMAL BASIS FOR $su(2)$ AND THE STRUCTURE CONSTANTS ARE $[T_a, T_b] = \sum_{c=1}^{3} \varepsilon_{abc} T_c$.

NOW WRITE

$$\mu = \mu^a T_a \quad \text{AND} \quad \nu = \nu^b T_b$$

WITH $\mu^a, \nu^b \in \Omega^k(M, \mathbb{R})$ AND DEFINE

$$\langle \mu, \nu \rangle_p = \langle \mu^a, \nu^b \rangle_p \, (T_a, T_b)_{su(2)} = \langle \mu^a, \nu^b \rangle_p \, \delta_{ab}$$

THIS IS AN ad-INVARIANT POINTWISE INNER PRODUCT ON $\Omega^k(M, su(2))$ SO

$$\langle \mu, \nu \rangle_k = \int_M \langle \mu, \nu \rangle_p \, \mathrm{vol}_g$$

IS A GLOBAL ad-INVARIANT INNER PRODUCT ON $\Omega^k(M, su(2))$. IF $\mu \wedge {}^*\nu$ IS INTERPRETED AS THE MATRIX PRODUCT OF

μ AND $^*\nu$ (WITH * COMPUTED COMPONENTWISE, I.E., ENTRYWISE), THEN IT IS EASY TO SEE THAT THIS IS GIVEN BY

$$\langle \mu, \nu \rangle_k = \int_M -2 \operatorname{tr}(\mu \wedge {}^*\nu)$$

SINCE $ad\,P$-VALUED FORMS ARE LOCALLY $su(2)$-VALUED FORMS, THE SAME WORKS FOR $\Omega^k(M, ad P)$.

IN PARTICULAR, EACH $T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, ad P)$ HAS AN ad-INVARIANT INNER PRODUCT AND THIS DEFINES A METRIC ON $\hat{\mathcal{A}}$ WHICH IS INVARIANT UNDER THE ACTION OF $\hat{\mathcal{G}}$ (POINTWISE CONJUGATION BY SECTIONS OF THE NONLINEAR ADJOINT BUNDLE).

THIS METRIC THEN DEFINES A CONNECTION ON $\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ WHOSE HORIZONTAL SPACES ARE THE ORTHOGONAL COMPLEMENTS TO THE GAUGE ORBITS. INDEED, WE HAVE ALREADY SEEN (ADDENDUM 2) THIS ORTHOGONAL DECOMPOSITION:

$$T_\omega(\hat{\mathcal{A}}) \cong T_\omega(\omega \cdot \hat{\mathcal{G}}) \oplus \operatorname{KER}(\delta^\omega) \cong \operatorname{Im}(d^\omega) \oplus \operatorname{KER}(\delta^\omega),$$

WHERE $d^\omega : \Omega^0(M, ad P) \to \Omega^1(M, ad P)$ IS THE COVARIANT EXTERIOR DERIVATIVE OF ω AND $\delta^\omega : \Omega^1(M, ad P) \to \Omega^0(M, ad P)$ IS ITS FORMAL ADJOINT RELATIVE TO THE INNER PRODUCTS DEFINED ABOVE.

NOTE: WE ALSO SAW IN ADDENDUM 2 THAT THE LIE ALGEBRA OF OUR STRUCTURE GROUP CAN BE IDENTIFIED $\Omega^0(M, ad P)$

WE ARE NOW READY TO BEGIN THE FORMAL IDENTIFICATION OF THE TERMS IN (2) WITH APPROPRIATE OBJECTS IN $\hat{\mathcal{A}} \times_{\tilde{\mathcal{G}}} \Omega^2_+(M, \mathrm{ad}\,P)$. WE WOULD DO WELL, HOWEVER, TO STRESS THE WORD "FORMAL". DESPITE APPEARENCES TO THE CONTRARY WE DO NOT "COMPUTE" THESE TERMS. RATHER WE DETERMINE HOW THEY MIGHT BE REASONABLY INTEPRETED IN THIS NEW, INFINITE-DIMENSIONAL CONTEXT.

NOW WE WILL SIMPLY ATTACK ALL OF THE TERMS IN THE EXPONENT OF (2), DECIDING AS WE GO WHAT SHOULD BE THE APPROPRIATE FIELD THEORETIC ANALOGUES OF THE VARIABLES THAT APPEAR THERE.

THE FIRST, $-\frac{1}{2}\|s(\omega)\|^2$, IS SIMPLE ENOUGH. OUR s IS THE SELF-DUAL CURVATURE MAP $s = F^+ : \hat{\mathcal{A}} \to \Omega^2_+(M, \mathrm{ad}\,P)$,

$$s(\omega) = F^+_\omega = \tfrac{1}{2}(F_\omega + {}^*F_\omega),$$

AND THE NORM WILL BE COMPUTED FROM THE INNER PRODUCT $\langle\ ,\ \rangle_2$ ON $\Omega^2(M, \mathrm{ad}\,P)$ AT EACH $\omega \in \hat{\mathcal{A}}$. THUS,

$$-\tfrac{1}{2}\|s(\omega)\|^2 = -\tfrac{1}{2}\|F^+_\omega\|^2 = \int_M \mathrm{tr}(F^+_\omega \wedge {}^*F^+_\omega)$$

$$= \int_M \mathrm{tr}(F^+_\omega \wedge F^+_\omega)$$

NOW NOTICE THAT

$$\begin{aligned} \mathrm{tr}(F_\omega \wedge {}^*F_\omega + F_\omega \wedge F_\omega) &= \mathrm{tr}(F_\omega \wedge ({}^*F_\omega + F_\omega)) \\ &= \mathrm{tr}(F_\omega \wedge (2F^+_\omega)) \\ &= 2\,\mathrm{tr}(F_\omega \wedge F^+_\omega) \\ &= 2\,\mathrm{tr}((F^+_\omega + F^-_\omega) \wedge F^+_\omega) \end{aligned}$$

$$= 2\,tr\,(F_\omega^+ \wedge F_\omega^+) + 2\,tr\,(F_\omega^- \wedge F_\omega^+)$$
$$= 2\,tr\,(F_\omega^+ \wedge F_\omega^+) + 2\,tr\,(F_\omega^- \wedge {}^*F_\omega^+)$$

SO INTEGRATION GIVES

$$\int_M tr\,(F_\omega \wedge {}^*F_\omega + F_\omega \wedge F_\omega) = 2\int_M tr\,(F_\omega^+ \wedge F_\omega^+)$$

(THE LAST TERM BEING ZERO BECAUSE THE HODGE DECOMPOSITION $F = F^+ + F^-$ IS L^2-ORTHOGONAL). WE CONCLUDE THAT

$$(3) \qquad -\tfrac{1}{2}\|s(\omega)\|^2 = \tfrac{1}{2}\int_M tr\,(F_\omega \wedge {}^*F_\omega) + \tfrac{1}{2}\int_M tr\,(F_\omega \wedge F_\omega)$$

THE FIRST TERM IS OF THE TYPICAL YANG-MILLS VARIETY, WHEREAS THE SECOND WITTEN REFERS TO AS A "TOPOLOGICAL TERM" BECAUSE IT AGREES (UP TO A FACTOR OF $\frac{1}{4\pi^2}$) WITH THE 2^{ND} CHERN CLASS OF $SU(2) \hookrightarrow P \to M$ (SEE ADDENDUM 2, PAGE 10).

NOTE: WITTEN EMPLOYS THE NOTATION MORE COMMON IN PHYSICS WHEREBY EVERYTHING IS WRITTEN SO AS TO APPEAR LOCAL. THE TERMS IN HIS LAGRANGIAN CORRESPONDING TO THOSE IN (3) ARE WRITTEN

$$(4) \qquad \tfrac{1}{4}\int_M d^4x\,\sqrt{g}\;Tr\,(F_{\alpha\beta}F^{\alpha\beta}) + \tfrac{1}{4}\int_M d^4x\,\sqrt{g}\;Tr\,(F_{\alpha\beta}\tilde{F}^{\alpha\beta})$$

I WILL NOT ATTEMPT A COMPLETE COMPARISON OF OUR NOTATION WITH THAT OF WITTEN, BUT IN THIS CASE WE WILL WORK IT OUT. WITH $\{T_1, T_2, T_3\}$ THE ORTHONORMAL BASIS FOR $su(2)$ GIVEN ON PAGE 5, WE (LOCALLY) WRITE

$$F_\omega = \tfrac{1}{2}\,F_{\alpha\beta}\,dx^\alpha \wedge dx^\beta,$$

WHERE

$$F_{\alpha\beta} = F^a_{\alpha\beta} T_a$$

THEN

$$F_\omega = \frac{1}{2} F^a_{\alpha\beta} T_a \, dx^\alpha \wedge dx^\beta$$

$$= \underbrace{\left(\frac{1}{2} F^a_{\alpha\beta} dx^\alpha \wedge dx^\beta \right)}_{F^a \ (\mathbb{R}\text{-VALUED 2-FORMS})} T_a$$

$$= F^a T_a$$

$$= -\frac{1}{2} \begin{pmatrix} F^3 i & F^2 + F^1 i \\ -F^2 + F^1 i & -F^3 i \end{pmatrix}$$

THUS,

$$*F_\omega = *F^a T_a = -\frac{1}{2} \begin{pmatrix} *F^3 i & *F^2 + *F^1 i \\ -*F^2 + *F^1 i & -*F^3 i \end{pmatrix}$$

SO

$$\mathrm{tr}(F_\omega \wedge *F_\omega) = \frac{1}{4} \Big[\big((F^3 i) \wedge (*F^3 i) + (F^2 + F^1 i) \wedge (-*F^2 + *F^1 i) \big) + \big((-F^2 + F^1 i) \wedge (*F^2 + *F^1 i) + (-F^3 i) \wedge (-*F^3 i) \big) \Big]$$

$$= \frac{1}{4} \big[-F^3 \wedge *F^3 - F^2 \wedge *F^2 - F^1 \wedge *F^1 - F^2 \wedge *F^2 - F^1 \wedge *F^1 - F^3 \wedge *F^3 \big]$$

$$= -\frac{1}{2} \left(F^1 \wedge *F^1 + F^2 \wedge *F^2 + F^3 \wedge *F^3 \right)$$

$$= -\frac{1}{2} \left(\langle F^1, F^1 \rangle_p + \langle F^2, F^2 \rangle_p + \langle F^3, F^3 \rangle_p \right) \mathrm{VOL}_g$$

$$= -\frac{1}{2} \langle F^a, F^b \rangle_p (T_a, T_b)_{su(2)} \mathrm{VOL}_g$$

$$= -\frac{1}{2} \langle F_\omega, F_\omega \rangle_p \mathrm{VOL}_g$$

($-2\,\mathrm{tr}(\mu \wedge *\nu) = \langle \mu, \nu \rangle_p \mathrm{VOL}_g$ FOLLOWS IN THE SAME WAY). NOW RAISE INDICES WITH g TO OBTAIN

$$F^{\alpha\beta} = g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha'\beta'}$$

$$= g^{\alpha\alpha'} g^{\beta\beta'} (F^a_{\alpha'\beta'} T_a)$$

$$= F^{a\alpha\beta} \, T_a$$

$$= -\frac{1}{2} \begin{pmatrix} F^{3\alpha\beta} i & F^{2\alpha\beta} + F^{1\alpha\beta} i \\ -F^{2\alpha\beta} + F^{1\alpha\beta} i & -F^{3\alpha\beta} i \end{pmatrix}$$

THEN

$$tr\,(F_{\alpha\beta} F^{\alpha\beta}) = \frac{1}{4} \Big[((F^3_{\alpha\beta} i) (F^{3\alpha\beta} i) + (F^2_{\alpha\beta} + F^1_{\alpha\beta} i) (-F^{2\alpha\beta} + F^{1\alpha\beta} i)) \\ + ((-F^2_{\alpha\beta} + F^1_{\alpha\beta} i) (F^{2\alpha\beta} + F^{1\alpha\beta} i) + (-F^3_{\alpha\beta} i) (-F^{3\alpha\beta} i)) \Big]$$

$$= \frac{1}{4} \Big[-F^3_{\alpha\beta} F^{3\alpha\beta} - F^2_{\alpha\beta} F^{2\alpha\beta} - F^1_{\alpha\beta} F^{1\alpha\beta} \\ - F^2_{\alpha\beta} F^{2\alpha\beta} - F^1_{\alpha\beta} F^{1\alpha\beta} - F^3_{\alpha\beta} F^{3\alpha\beta} \Big]$$

$$= -\frac{1}{2} \Big[F^1_{\alpha\beta} F^{1\alpha\beta} + F^2_{\alpha\beta} F^{2\alpha\beta} + F^3_{\alpha\beta} F^{3\alpha\beta} \Big]$$

BUT (SEE PAGE 4),

$$\langle F^a, F^a \rangle_p = \frac{1}{2!} F^a_{\alpha\beta} F^{a\alpha\beta} \qquad , \; a = 1, 2, 3$$

SO

$$tr\,(F_{\alpha\beta} F^{\alpha\beta}) = - [\langle F^1, F^1 \rangle_p + \langle F^2, F^2 \rangle_p + \langle F^3, F^3 \rangle_p]$$

$$= - \langle F^a, F^b \rangle_p \, (T_a , T_b)_{su(2)}$$

$$= - \langle F_\omega, F_\omega \rangle_p$$

AND THEREFORE

$$tr\,(F_\omega \wedge {}^*F_\omega) = \frac{1}{2} tr\,(F_{\alpha\beta} F^{\alpha\beta}) \, vol_g \, .$$

THUS,

$$\frac{1}{2}\int_M tr\,(F_\omega \wedge {}^*F_\omega) = \frac{1}{4}\int_M tr\,(F_{\alpha\beta}F^{\alpha\beta})\,vol_g$$

$$= \frac{1}{4}\int_M d^4x\,\sqrt{g}\;tr\,(F_{\alpha\beta}F^{\alpha\beta})$$

WHERE, IN THE LAST INTEGRAL, WE USE THE NOTATION FAVORED BY PHYSICISTS FOR THE VOLUME INTEGRAL. THUS, THE FIRST TWO INTEGRALS IN (3) AND (4) AGREE.

FOR THE SECOND INTEGRALS IN (3) AND (4) WE USE THE FACT THAT, FOR 2-FORMS ON A RIEMANNIAN MANIFOLD, $** = id$. THUS,

$$-2\,tr\,(F_\omega \wedge F_\omega) = -2\,tr\,(F_\omega \wedge {}^*({}^*F_\omega))$$

$$= \langle F_\omega, {}^*F_\omega\rangle_p\,vol_g \qquad \text{(PAGE 9)}$$

BUT WITH

$$^*F^{\alpha\beta} = g^{\alpha\alpha'}g^{\beta\beta'}\,{}^*F_{\alpha'\beta'}$$

THE SAME CALCULATIONS AS ON PAGE 10 GIVE

$$tr\,(F_{\alpha\beta}\,{}^*F^{\alpha\beta}) = -\langle F_\omega, {}^*F_\omega\rangle_p$$

SO

$$\frac{1}{2}\int_M tr\,(F_\omega \wedge F_\omega) = -\frac{1}{4}\int_M \langle F_\omega, {}^*F_\omega\rangle\,vol_g$$

$$= \frac{1}{4}\int_M tr\,(F_{\alpha\beta}\,{}^*F^{\alpha\beta})\,vol_g$$

$$= \frac{1}{4}\int d^4x\,\sqrt{g}\;tr\,(F_{\alpha\beta}\tilde{F}^{\alpha\beta})$$

WHERE WE HAVE USED WITTEN'S $\sim$ NOTATION FOR THE DUAL,

MUST SORT OUT THE APPROPRIATE ANALOGUES, IN THE DONALDSON THEORY CONTEXT, OF THE MAPS C, C^* AND R.

RECALL (FROM ADDENDUM 12) THAT AT EACH POINT ω IN THE PRINCIPAL BUNDLE SPACE $\hat{\mathcal{A}}$, C_ω IS THE MAP FROM THE LIE ALGEBRA OF THE STRUCTURE GROUP $\hat{\mathcal{G}}$, WHICH WE HAVE SEEN CAN BE IDENTIFIED WITH $\Omega^0(M, ad\,P)$, TO THE TANGENT SPACE $T_\omega(\hat{\mathcal{A}}) \cong \Omega^1(M, ad\,P)$ DEFINED, FOR EACH $\xi \in \Omega^0(M, ad\,P)$ BY

$$C_\omega(\xi) = \frac{d}{dt}(\omega \cdot \exp(t\xi))\Big|_{t=0}.$$

WE CLAIM THAT, FOR EACH $\omega \in \hat{\mathcal{A}}$,

(5) $$C_\omega : \Omega^0(M, ad\,P) \longrightarrow \Omega^1(M, ad\,P)$$

IS GIVEN BY

(6) $$C_\omega(\xi) = d^\omega \xi$$

HERE IS A ROUGH LOCAL ARGUMENT FOR THIS. LET $\mathcal{A}$ BE A LOCAL GAUGE POTENTIAL FOR ω (I.E., A PULLBACK OF ω TO M BY SOME LOCAL SECTION OF P). WE THINK OF THE ELEMENTS OF $\mathcal{G}$ AS SECTIONS OF THE NONLINEAR ADJOINT BUNDLE SO LOCALLY THEY ARE JUST $SU(2)$-VALUED MAPS ON M. SIMILARLY, $\xi \in \Omega^0(M, ad\,P)$ IS LOCALLY AN $su(2)$-VALUED FUNCTION ON M THAT IS EXPONENTIATED POINTWISE TO GET AN ELEMENT OF $\mathcal{G}$. WRITING

$$g_t = \exp(t\xi),$$

$\omega \cdot \exp(t\xi)$ IS GIVEN LOCALLY BY

$$g_t^{-1} \mathcal{A}\, g_t + g_t^{-1} dg_t = g_t^{-1} \mathcal{A}\, g_t + t\, d\xi$$

BECAUSE $g_t^{-1} dg_t = \exp(-t\xi)\, d(\exp(t\xi)) = \exp(-t\xi)\exp(t\xi)\, d(t\xi) = t\, d\xi$. THEN, SINCE THE CONJUGATION IS FIBERWISE BY ELEMENTS OF $SU(2)$,

$$\frac{d}{dt}\left(\exp(-t\xi)\, \mathcal{A} \exp(t\xi) + t\, d\xi\right)\Big|_{t=0}$$

$$\frac{d}{dt}\left(\left(1 - t\xi + \tfrac{1}{2}t^2\xi^2 - \cdots\right)\mathcal{A}\left(1 + t\xi + \tfrac{1}{2}t^2\xi^2 + \cdots\right) + t\, d\xi\right)\Big|_{t=0} =$$

$$\frac{d}{dt}\left(\mathcal{A} + t\mathcal{A}\xi - t\xi\mathcal{A} + \cdots + t\, d\xi\right)\Big|_{t=0} = d\xi + [\mathcal{A}, \xi] = d^\omega \xi$$

WITH C_ω THUS IDENTIFIED WITH d^ω, THE ADJOINT

(7) $$C_\omega^* : \Omega^1(M, \mathrm{ad}P) \longrightarrow \Omega^0(M, \mathrm{ad}P)$$

RELATIVE TO OUR NATURAL INNER PRODUCTS ON THESE SPACES OF FORMS IS

(8) $$C_\omega^* = \delta^\omega = -{*}d^\omega{*}$$

(THE LAST EQUALITY IS NOT OBVIOUS. SEE THEOREM 4.2.9 OF BLEECKER, GAUGE THEORY AND VARIATIONAL PRINCIPLES). CONSEQUENTLY,

(9) $$R_\omega = C_\omega^* \circ C_\omega : \Omega^0(M, \mathrm{ad}P) \longrightarrow \Omega^0(M, \mathrm{ad}P)$$

IS THE SCALAR LAPLACIAN OF ω

(10) $$R_\omega = \delta^\omega \circ d^\omega = \Delta_0^\omega = -{*}d^\omega{*}d^\omega$$

NOW LET'S TACKLE THE FIFTH TERM IN THE EXPONENT OF (2)

$$i(\phi, R_\omega \lambda).$$

THIS REQUIRES THAT BOTH ϕ AND λ LIE IN THE LIE ALGEBRA $\Omega^0(M, \mathrm{ad}\,P)$ AND THAT $(\ ,\ )$ BE THE NATURAL INNER PRODUCT $\langle\ ,\ \rangle_0$.
BECAUSE THE ϕ AND λ INTEGRALS IN (2) ARE ORDINARY ("BOSONIC") AS OPPOSED TO BEREZIN ("FERMIONIC"), WE INTRODUCE TWO

$$\underline{\text{BOSONIC}}\ \underline{\text{FIELDS}}: \ \phi, \lambda \in \Omega^0(M, \mathrm{ad}\,P)$$

AND, FOR EACH $\omega \in \hat{\mathcal{A}}$,

$$i(\phi, R_\omega \lambda) = i\langle \phi, \Delta_0^\omega \lambda \rangle_0 = i\int_M -2\,\mathrm{tr}(\phi \wedge {}^*\Delta_0^\omega \lambda)$$

$$= -2i\int_M \mathrm{tr}(\phi\, {}^*\Delta_0^\omega \lambda) \quad \text{BECAUSE } \phi \text{ IS A 0-FORM}$$

(11) $$i(\phi, R_\omega \lambda) = 2i\int_M \mathrm{tr}(\phi\, d^\omega * d^\omega \lambda)$$

<u>NOTE</u>: TO CONFORM TO WITTEN'S NOTATION WE MUST SUBSTITUTE $\phi \to i\phi$ AND $\lambda \to \frac{1}{2}\lambda$ THROUGHOUT. THE EFFECT HERE IS

$$2i\int_M \mathrm{tr}(i\phi\, d^\omega * d^\omega(\tfrac{1}{2}\lambda)) = -\int_M \mathrm{tr}(\phi\, d^\omega * d^\omega \lambda)$$

$$= \int_M \mathrm{Tr}(\tfrac{1}{2}\phi\, d^\omega * d^\omega \lambda)$$

WHERE $\mathrm{Tr} = -2\,\mathrm{tr}$ (WITTEN'S NOTATION).

NEXT WE CONSIDER THE TERM $(C_\omega^*(\xi), \eta)$ IN (2).

RECALL (ADDENDUM 12, PAGE 17) THAT (C^*, η) IS TO BE REGARDED AS A LINEAR FUNCTION OF η (IN THE LIE ALGEBRA). AT ANY η IT IS A 1-FORM ON THE PRINCIPAL BUNDLE SPACE $(\hat{A})$ WHOSE VALUE AT A TANGENT VECTOR $\zeta \in T_\omega(\hat{A}) = \Omega^1(M, \operatorname{ad} P)$ IS GIVEN BY $(C^*_\omega(\zeta), \eta) = \langle \zeta, C_\omega \eta \rangle$.

AGAIN BECAUSE THE ζ- AND η-INTEGRALS IN (2) ARE BEREZIN, WE INTRODUCE TWO

$$\underline{\text{FERMIONIC}}\ \underline{\text{FIELDS}}: \quad \begin{aligned} &\eta \in \Omega^0(M, \operatorname{ad} P) \\ &\zeta \in \Omega^1(M, \operatorname{ad} P) \end{aligned}$$

AND, FOR EACH $\omega \in \hat{A}$,

$$\langle C^*_\omega(\zeta), \eta \rangle_0 = \langle \zeta, C_\omega(\eta) \rangle_1 = \langle \zeta, d^\omega \eta \rangle_1$$

$$(12) \qquad (C^*_\omega(\zeta), \eta) = -2 \int_M tr(\zeta \wedge {}^* d^\omega \eta)$$

NOW WE TURN TO

$$i \psi^T ds_\omega(\zeta)$$

IN (2).

THE FERMIONIC VARIABLE ψ IN THE MATHAI-QUILLEN FORMALISM ARISES FROM THE ODD GENERATORS OF THE EXTERIOR ALGEBRA OF THE FIBER VECTOR SPACE V. IN OUR CASE THIS VECTOR SPACE IS $\Omega^2_+(M, \operatorname{ad} P)$ SO WE INTRODUCE A

$$\underline{\text{FERMIONIC}}\ \underline{\text{FIELD}}: \quad \psi \in \Omega^2_+(M, \operatorname{ad} P)$$

NOW, S IS THE SELF-DUAL CURVATURE MAP $S = F^+ : \hat{A} \longrightarrow \Omega^2_+(M, \operatorname{ad} P)$ AND, AT EACH $\omega \in \hat{A}$, ds_ω IS A 1-FORM WITH VALUES IN $\Omega^2_+(M, \operatorname{ad} P)$

THAT CAN BE IDENTIFIED WITH

$$d_+^\omega = pr_+ \circ d^\omega : \Omega^1(M, \mathrm{ad}P) \to \Omega_+^2(M, \mathrm{ad}P)$$

SO, FOR EACH $\zeta \in \Omega^1(M, \mathrm{ad}P)$,

$$ds_\omega(\zeta) = d_+^\omega(\zeta)$$

WE WILL INTERPRET FINITE-DIMENSIONAL EXPRESSIONS SUCH AS

$$A^T B = (A^1 \cdots A^r)\begin{pmatrix} B^1 \\ \vdots \\ B^r \end{pmatrix}$$

$$= (A^1B^1 + \cdots + A^rB^r)$$

IN TERMS OF THE APPROPRIATE INNER PRODUCT ON FORMS SO

$$i\,\psi^T ds_\omega(\zeta) = i\langle \psi, d_+^\omega(\zeta)\rangle_2$$

$$= i\langle \psi, d^\omega \zeta\rangle_2 \quad \text{BECAUSE } \psi \text{ IS SELF-DUAL}$$

$$= i\langle d^\omega \zeta, \psi\rangle_2$$

$$= -2i\int_M d^\omega \zeta \wedge * \psi$$

(13) $$i\,\psi^T ds_\omega(\zeta) = -2i\int_M d^\omega \zeta \wedge \psi$$

NEXT CONSIDER THE TERM

$$\frac{1}{2}\psi^T(\rho_* \phi)\psi$$

IN (2).

WE KNOW ALREADY THAT $\phi \in \Omega^0(M, \mathrm{ad}\,P)$ AND $\psi \in \Omega^2_+(M, \mathrm{ad}\,P)$. IN THE MATHAI - QUILLEN FORM, ρ CORRESPONDS TO THE ACTION OF G ON V THAT GIVES RISE TO THE ASSOCIATED VECTOR BUNDLE. IN OUR CASE, $\hat{\mathcal{G}}$ (REGARDED AS SECTIONS OF THE NONLINEAR ADJOINT BUNDLE) ACTS ON $\Omega^2_+(M, \mathrm{ad}\,P)$ POINTWISE BY CONJUGATION. AT EACH POINT THIS IS JUST THE ORDINARY ADJOINT ACTION OF $SU(2)$ ON $su(2)$. THE CORRESPONDING INFINITESIMAL ACTION OF $su(2)$ ON $su(2)$ IS SIMPLY BRACKET. THUS, FOR EACH $\phi \in \Omega^0(M, \mathrm{ad}\,P)$, $\rho_*\phi$ ACTS ON $\psi \in \Omega^2_+(M, \mathrm{ad}\,P)$ BY

$$(\rho_*\phi)\psi = [\phi, \psi]$$

AND WE INTERPRET $\frac{1}{2}\psi^T(\rho_*\phi)\psi$ AS

$$\frac{1}{2}\langle \psi, [\phi,\psi]\rangle_2 = \frac{1}{2}\langle [\phi,\psi], \psi\rangle_2$$

$$= -\int_M \mathrm{tr}\,([\phi,\psi]\wedge *\psi)$$

$$= -\int_M \mathrm{tr}\,([\phi,\psi]\wedge \psi)$$

BECAUSE ψ IS SELF-DUAL

$$= -\int_M \mathrm{tr}\,(\psi \wedge [\phi,\psi])$$

BECAUSE $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

(14) $$\frac{1}{2}\psi^T(\rho_*\phi)\psi = -\int_M \mathrm{tr}\,(\psi\wedge[\phi,\psi])$$

FINALLY, WE CONSIDER THE TERM

$$-i\,(dC^*_\omega(\xi),\lambda)$$

IN (2). THIS ONE TAKES A BIT MORE WORK.

FIRST RECALL THAT C^* IS A 1-FORM ON $\hat{\mathcal{A}}$ WITH VALUES IN THE LIE ALGEBRA $\Omega^0(M, \mathrm{ad}\,P)$ OF $\hat{\mathcal{G}}$. CONSEQUENTLY, dC^* IS A 2-FORM ON $\hat{\mathcal{A}}$ WITH VALUES IN $\Omega^0(M, \mathrm{ad}\,P)$. FOR EACH $\omega \in \hat{\mathcal{A}}$ AND ALL TANGENT VECTORS $\xi_1, \xi_2 \in T_\omega(\hat{\mathcal{A}}) = \Omega^1(M, \mathrm{ad}\,P)$ WE MUST COMPUTE

$$dC^*_\omega(\xi_1,\xi_2).$$

BECAUSE $\hat{\mathcal{A}}$ IS AN OPEN SUBSET OF AN AFFINE SPACE WE CAN REGARD ξ_1 AND ξ_2 AS CONSTANT VECTOR FIELDS ON $\hat{\mathcal{A}}$. THEN

$$\begin{aligned} dC^*(\xi_1,\xi_2) &= \xi_1(C^*\xi_2) - \xi_2(C^*\xi_1) - C^*([\xi_1,\xi_2]) \\ &= \xi_1(C^*\xi_2) - \xi_2(C^*\xi_1) \end{aligned}$$

WHERE $C^*\xi_i$ STANDS FOR THE FUNCTION ON $\hat{\mathcal{A}}$ DEFINED BY

$$(C^*\xi_i)(\omega) = C^*_\omega\,\xi_i = \delta^{\omega}\xi_i.$$

NOW,

$$\begin{aligned} (\xi_1(C^*\xi_2))(\omega) &= \xi_1(\omega)\,[C^*\xi_2] \\ &= \xi_1\,[C^*\xi_2] \\ &= \frac{d}{dt}\,(C^*\xi_2)(\omega+t\xi_1)\Big|_{t=0} \\ &= \frac{d}{dt}\,C^*_{\omega+t\xi_1}(\xi_2)\Big|_{t=0} \\ &= \frac{d}{dt}\,\delta^{\omega+t\xi_1}(\xi_2)\Big|_{t=0} \end{aligned}$$

NOW COMPUTE $\delta^{\omega+t\xi_1}(\xi_2)$ AS FOLLOWS: FOR ANY $\lambda \in \Omega^0(M, \operatorname{ad} P)$,

$$
\begin{aligned}
d^{\omega+t\xi_1}(\lambda) &= d\lambda + [\omega + t\xi_1, \lambda] \\
&= d\lambda + [\omega, \lambda] + t[\xi_1, \lambda] \\
&= d^{\omega}\lambda + t[\xi_1, \lambda] \\
&= d^{\omega}\lambda + t\, b_{\xi_1}(\lambda)
\end{aligned}
$$

WHERE

$$b_{\xi_1} : \Omega^0(M, \operatorname{ad} P) \to \Omega^1(M, \operatorname{ad} P)$$

$$b_{\xi_1}(\lambda) = [\xi_1, \lambda].$$

THUS,

$$\delta^{\omega+t\xi_1}(\xi_2) = \delta^{\omega}\xi_2 + t\, b^*_{\xi_1}(\xi_2)$$

WHERE

$$b^*_{\xi_1} : \Omega^1(M, \operatorname{ad} P) \to \Omega^0(M, \operatorname{ad} P)$$

IS THE ADJOINT OF b_{ξ_1}. WE CLAIM THAT

$$b^*_{\xi_1}(\xi_2) = -*[\xi_1, *\xi_2].$$

INDEED, FOR ANY $\lambda \in \Omega^0(M, \operatorname{ad} P)$,

$$
\begin{aligned}
\langle b_{\xi_1}(\lambda), \xi_2 \rangle_1 &= \langle [\xi_1, \lambda], \xi_2 \rangle_1 \\
&= -2 \int_M \operatorname{tr}([\xi_1, \lambda] \wedge *\xi_2) \\
&= 2 \int_M \operatorname{tr}([\lambda, \xi_1] \wedge *\xi_2)
\end{aligned}
$$

$$= 2\int_M tr(\lambda \wedge [\xi_1, {}^*\xi_2])$$

$$= 2\int_M tr(\lambda \wedge {}^*({}^*[\xi_1, {}^*\xi_2]))$$

$$= -\langle \lambda, {}^*[\xi_1, {}^*\xi_2]\rangle_0$$

$$= \langle \lambda, -{}^*[\xi_1, {}^*\xi_2]\rangle_0$$

$$\langle b_{\xi_1}(\lambda), \xi_2\rangle_1 = \langle \lambda, -{}^*[\xi_1, {}^*\xi_2]\rangle_0$$

WHICH PROVES THE CLAIM. THUS,

$$\delta^{\omega + t\xi_1}(\xi_2) = \delta^\omega \xi_2 - t\,{}^*[\xi_1, {}^*\xi_2]$$

AND COMPUTING THE DERIVATIVE AT $t=0$ GIVES

$$(\xi_1(c^*\xi_2))(\omega) = -{}^*[\xi_1, {}^*\xi_2]$$

INTERCHANGING ξ_1 AND ξ_2 GIVES

$$(\xi_2(c^*\xi_1))(\omega) = -{}^*[\xi_2, {}^*\xi_1]$$

NOTICE THAT THESE ARE INDEPENDENT OF ω SO

$$\xi_1(c^*\xi_2) = -{}^*[\xi_1, {}^*\xi_2]$$
$$\xi_2(c^*\xi_1) = -{}^*[\xi_2, {}^*\xi_1]$$

AND

$$dc^*(\xi_1, \xi_2) = -{}^*[\xi_1, {}^*\xi_2] + {}^*[\xi_2, {}^*\xi_1].$$

NOTE: $\alpha, \beta \in \Omega^1(M, \text{ad}\,P) \Rightarrow {}^*[\beta, {}^*\alpha] = -{}^*[\alpha, {}^*\beta]$

TO SEE THIS WE CAN ARGUE LOCALLY SO THAT α AND β ARE $su(2)$-VALUED AND WE WRITE $\alpha = \alpha^a T_a$, $\beta = \beta^b T_b$, ${}^*\alpha = {}^*\alpha^a T_a$, ${}^*\beta = {}^*\beta^b T_b$. THEN

$$[\alpha, {}^*\beta] = [\alpha^a T_a, {}^*\beta^b T_b]$$
$$= (\alpha^a \wedge {}^*\beta^b)[T_a, T_b]$$
$$= (\langle \alpha^a, \beta^b \rangle_p \,\text{vol}_g)[T_a, T_b]$$
$${}^*[\alpha, {}^*\beta] = \langle \alpha^a, \beta^b \rangle_p [T_a, T_b]$$

ALSO,

$${}^*[\beta, {}^*\alpha] = \langle \beta^c, \alpha^d \rangle_p [T_c, T_d]$$
$$= -\langle \alpha^d, \beta^c \rangle_p [T_d, T_c]$$
$$= -{}^*[\alpha, {}^*\beta].$$

THUS,

$$dC^*(\eta_1, \eta_2) = -2\,{}^*[\eta_1, {}^*\eta_2]$$

AT ANY $\omega \in \hat{\mathcal{A}}$ AND

$$-i\langle dC^*_\omega(\eta_1, \eta_2), \lambda \rangle_0 = -i\langle \lambda, dC^*_\omega(\eta_1, \eta_2) \rangle_0$$
$$= -i\langle \lambda, -2\,{}^*[\eta_1, {}^*\eta_2] \rangle_0$$
$$= 2i\langle \lambda, {}^*[\eta_1, {}^*\eta_2] \rangle_0$$
$$= -4i \int_M \text{tr}(\lambda \wedge [\eta_1, {}^*\eta_2])$$
$$= -4i \int_M \text{tr}(\lambda [\eta_1, {}^*\eta_2])$$
$$= -4i \int_M \text{tr}([\eta_1, {}^*\eta_2]\lambda)$$

THUS, OUR INTERPRETATION OF $-i\,(dC^*_\omega(\zeta),\lambda)$ IS

$$(15) \qquad -i\,(dC^*_\omega(\zeta),\lambda) = -4i\int_M tr\,([\zeta,{}^*\zeta]\,\lambda)$$

<u>NOTE</u>: CONSIDERING THE MACHINATIONS REQUIRED TO ARRIVE AT THIS WE SHOULD ATTEMPT TO MAKE SOME CONTACT WITH WITTEN. FOR THIS WE RETURN TO THE NOTATION EMPLOYED IN THE NOTE ON PAGE 21 WHERE WE SHOWED THAT, FOR $\alpha, \beta \in \Omega^1(M, ad P)$,

$$*[\alpha, *\beta] = \langle \alpha^a, \beta^b\rangle_p\,[T_a, T_b]$$

NOW LET US WRITE

$$\begin{aligned} \alpha &= \alpha^a\, T_a & \beta &= \beta^b\, T_b \\ &= (\alpha^a_i\, dx^i)\, T_a & &= (\beta^b_j\, dx^j)\, T_b \\ &= (\alpha^a_i\, T_a)\, dx^i & &= (\beta^b_j\, T_b)\, dx^j \\ &= \alpha_i\, dx^i & &= \beta_j\, dx^j \end{aligned}$$

AND RAISE THE INDICES OF β WITH THE RIEMANNIAN METRIC g

$$\beta^j = g^{jk}\beta_k = g^{jk}\beta^b_k\, T_b\,.$$

THEN

$$\begin{aligned} [\alpha_i, \beta^i] &= [\alpha^a_i\, T_a,\, g^{ik}\beta^b_k\, T_b] \\ &= g^{ik}\alpha^a_i\,\beta^b_k\,[T_a, T_b] \\ &= \langle \alpha^a, \beta^b\rangle_p\,[T_a, T_b] \end{aligned}$$

SO WE HAVE

$$[\alpha_i, \beta^i] = *[\alpha, *\beta]$$

THUS,

$$*[\zeta, *\zeta] = [\zeta_i, \zeta^i]$$

SO

$$\langle \lambda, *[\zeta, *\zeta] \rangle_0 = \langle \lambda, [\zeta_i, \zeta^i] \rangle$$

WHICH CORRESPONDS TO THE WAY THE TERM IS WRITTEN BY WITTEN.

NOW PUT ALL OF THESE PIECES TOGETHER.

$$-\tfrac{1}{2}\|S(\omega)\|^2 + \tfrac{1}{2}\psi^T(\rho_*\phi)\psi + i\psi^T dS_\omega(\zeta) - i(dC^*_\omega(\zeta), \lambda)$$
$$+ i(\phi, R_\omega \lambda) + (C^*_\omega(\zeta), \eta) =$$

$$\int_M tr\left\{ \tfrac{1}{2}F_\omega \wedge {}^*F_\omega + \tfrac{1}{2}F_\omega \wedge F_\omega - \psi\wedge[\phi,\psi] - 2i\, d^\omega\zeta\wedge\psi \right.$$
$$\left. -4i\,[\zeta, *\zeta]\lambda + 2i\,\phi\, d^{\omega *} d^\omega \lambda - 2\,\zeta\wedge {}^*d^\omega \eta \right\} =$$

$$\int_M Tr\left\{ -\tfrac{1}{4}F_\omega\wedge {}^*F_\omega - \tfrac{1}{4}F_\omega\wedge F_\omega + \tfrac{1}{2}\psi\wedge[\phi,\psi] + i\,d^\omega\zeta\wedge\psi \right.$$
$$\left. + 2i\,[\zeta, *\zeta]\lambda - i\,\phi\, d^{\omega *} d^\omega\lambda + \zeta\wedge {}^*d^\omega\eta \right\}$$

WHERE $Tr = -2\,tr$.

NOW, WRITING THE INTEGRAL IN (1) IN THE FORM

$$\int e^{-\int_M Tr\, \mathcal{L}_{DW}} \mathcal{D}\Phi$$

WE IDENTIFY THE DONALDSON-WITTEN LAGRANGIAN $\mathcal{L}_{DW}$ WITH

$$\mathcal{L}_{DW} = \frac{1}{4} F_\omega \wedge {}^*F_\omega + \frac{1}{4} F_\omega \wedge F_\omega - \frac{1}{2} \psi \wedge [\phi, \psi] - i\, d^\omega \chi \wedge \psi$$

$$- 2i\, [\chi, {}^*\chi] \lambda + i\, \phi\, d^\omega {}^* d^\omega \lambda - \chi \wedge {}^* d^\omega \eta$$

ACTION

$$S_{DW} = \int_M Tr\, \mathcal{L}_{DW}$$

AND PARTITION FUNCTION

$$Z_{DW} = \int e^{-S_{DW}[\Phi]/e^2}\, \mathcal{D}\Phi$$

WHERE e IS A COUPLING CONSTANT WHICH WE HAVE JUST INSERTED MANUALLY HERE, BUT COULD HAVE BEEN INCLUDED AT THE OUTSET BY RENORMALIZING THE KILLING FORM Tr ON $su(2)$. WE HAVE ALSO DROPPED THE $(2\pi)^{-n}(2\pi)^{-k}$ IN (1) AND (2) BECAUSE IN OUR PRESENT CIRCUMSTANCES BOTH n AND k WOULD BE "INFINITE".

ADDENDUM 14 : ALGEBRAIC DETAILS

NOTE : IN GENERAL, ANY FINITE DIMENSIONAL VECTOR SPACE V WITH AN INNER PRODUCT $\langle\, , \rangle$ HAS A CLIFFORD ALGEBRA $Cl(V)$. IF $\{e_1, \ldots, e_n\}$ IS AN ORTHONORMAL BASIS, THEN $Cl(V)$ IS THE REAL ASSOCIATIVE ALGEBRA WITH UNIT 1 GENERATED BY $e_1, \ldots, e_n$ SUBJECT TO THE RELATIONS

$$e_i e_j + e_j e_i = -2\langle e_i, e_j\rangle 1,$$

$i, j = 1, \ldots, n$. IT CAN BE DEFINED ABSTRACTLY AS THE QUOTIENT OF THE TENSOR ALGEBRA $T(V)$ BY THE TWO-SIDED IDEAL $I(V)$ GENERATED BY $v \otimes v + \langle v, v\rangle$, $v \in V$, AND SO DOES NOT DEPEND ON THE CHOICE OF $\{e_1, \ldots, e_n\}$.

WE WILL CONSTRUCT $Cl(\mathbb{R}^4)$ EXPLICITLY BY IDENTIFYING $\mathbb{R}^4$ WITH A REAL LINEAR SUBSPACE OF A MATRIX ALGEBRA, FINDING A BASIS FOR THIS MATRIX MODEL OF $\mathbb{R}^4$ THAT SATISFIES THE DEFINING RELATIONS FOR MATRIX MULTIPLICATION, AND FORMING THE REAL SUBALGEBRA IT GENERATES.

OBJECTIVE :

$$\begin{array}{ccccccccc} \mathbb{R}^4 & \hookrightarrow & Cl(\mathbb{R}^4) & \hookrightarrow & Cl(\mathbb{R}^4)\otimes_{\mathbb{R}}\mathbb{C} & = & \mathbb{C}^{4\times 4} & = & END_{\mathbb{C}}(\mathbb{C}^4) \\ & & \uparrow & & \uparrow & & & & \\ & & SPIN(4) & & SPIN^c(4) & & & & \end{array}$$

WE WILL USE THE QUATERNION STRUCTURE OF $\mathbb{R}^4$.

TWO VIEWS OF THE QUATENIONS $\mathbb{H}$:

1. $$\begin{aligned} q &= q_1 1 + q_2 i + q_3 j + q_4 k \\ &= q_1 \quad + q_2 i + q_3 j + q_4 k \\ &= (q_1 + q_2 i) + (q_3 + q_4 i) j \\ &= \quad \alpha \quad + \quad \beta j \end{aligned}$$

$$\boxed{\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = -ji &= k \\ jk = -kj &= i \\ ki = -ik &= j \end{aligned}}$$

AN ASSOCIATIVE, REAL, DIVISION ALGEBRA WITH INVOLUTION ($q \to \bar{q} = q_1 - q_2 i - q_3 j - q_4 k$), NORM ($\|q\|^2 = q\bar{q} = \bar{q}q = (q^1)^2 + (q^2)^2 + (q^3)^2 + (q^4)^2$) AND, BY POLARIZATION, INNER PRODUCT $\langle\ ,\ \rangle$. FOR $q \neq 0$, $q^{-1} = \bar{q}/\|q\|^2$

2. DEFINE $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$ BY $\gamma(q) = \gamma(\alpha + \beta j) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$

REAL LINEAR, INJECTIVE, $\gamma(qr) = \gamma(q)\gamma(r)$, $\gamma(1) = \mathbb{1}$, $\gamma(\bar{q}) = \overline{\gamma(q)}^T$, DET $(\gamma(q)) = \|q\|^2$.

$$\gamma(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \qquad \gamma(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = I$$

$$\gamma(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J \qquad \gamma(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = K$$

CAN IDENTIFY $q = q_1 \mathbb{1} + q_2 I + q_3 J + q_4 K$, ETC.

1^{ST} MATRIX MODEL OF $\mathbb{R}^4$:

CONSIDER THE REAL ASSOCIATIVE ALGEBRA

$$\mathbb{H}^{2\times 2} = \left\{ \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} : q_{ij} \in \mathbb{H} \right\}$$

AND THE REAL LINEAR SUBSPACE CONSISTING OF ALL ELEMENTS

(1) $$x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \quad , q \in \mathbb{H}$$

THIS SUBSPACE IS OBVIOUSLY ISOMORPHIC TO $\mathbb{R}^4$. DEFINING A NORM (AND, BY POLARIZATION, AN INNER PRODUCT) ON THIS COPY OF $\mathbb{R}^4$ BY

$$\|x\|^2 = \mathrm{DET}(x) = \|q\|^2$$

IT IS, IN FACT, ISOMETRIC TO $\mathbb{R}^4$.

NOTE : $\mathbb{R}^4$ IS NOT A SUBALGEBRA OF $\mathbb{H}^{2\times 2}$.

ORTHONORMAL BASIS FOR THIS COPY OF $\mathbb{R}^4$:

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \; e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \; e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} , \; e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

MATRIX MULTIPLICATION SHOWS THAT $e_i^2 = -\mathbb{1}$, $i = 1,2,3,4$ AND $e_i e_j = -e_j e_i$, $i,j = 1,2,3,4$, $i \neq j$ SO

(2) $$e_i e_j + e_j e_i = -2\langle e_i, e_j \rangle \mathbb{1} , \quad i,j = 1,2,3,4.$$

FOLLOWS THAT

(3) $$xy + yx = -2\langle x, y\rangle \mathbb{1}, \quad x, y \in \mathbb{R}^4.$$

THE REAL SUBALGEBRA OF $\mathbb{H}^{2\times 2}$ GENERATED BY e_1, e_2, e_3, e_4 IS THE REAL CLIFFORD ALGEBRA $Cl(4) = Cl(\mathbb{R}^4)$ OF $\mathbb{R}^4$ (WITH ITS USUAL INNER PRODUCT). BY FORMING PRODUCTS OF THE BASIS ELEMENTS AND USING (2) TO ELIMINATE LINEAR DEPENDENCIES ONE OBTAINS THE FOLLOWING BASIS FOR $Cl(4)$:

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

$$e_1e_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad e_1e_3 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \quad e_1e_4 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$$

$$e_2e_3 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad e_2e_4 = \begin{pmatrix} -j & 0 \\ 0 & -j \end{pmatrix} \quad e_3e_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$$e_1e_2e_3 = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \quad e_1e_2e_4 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$e_1e_3e_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad e_2e_3e_4 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$e_1e_2e_3e_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

THE SPAN OF THESE IS, IN FACT, ALL OF $\mathbb{H}^{2\times 2}$ SO

(4) $$Cl(4) = \mathbb{H}^{2\times 2}$$

<u>NOTE</u>: USING THE MAP $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$ (PAGE 2) WE CAN, AND SOON WILL, REGARD $Cl(4)$ AS AN ALGEBRA OF 4×4 COMPLEX MATRICES.

THERE IS A NATURAL $\mathbb{Z}_2$ - GRADING OF $Cl(4)$:

(5) $$Cl(4) = Cl_0(4) \oplus Cl_1(4)$$

WHERE

$$Cl_0(4) = C_0(4) \oplus C_2(4) \oplus C_4(4)$$

$$Cl_1(4) = C_1(4) \oplus C_3(4)$$

AND

$$C_0(4) = \text{SPAN}\{e_0\} = \left\{ \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix} : x_0 \in \mathbb{R} \right\} \cong \mathbb{R}$$

$$C_1(4) = \text{SPAN}\{e_1, e_2, e_3, e_4\} = \left\{ \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} : q \in \mathbb{H} \right\} \cong \mathbb{R}^4$$

$$C_2(4) = \text{SPAN}\{e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4\}$$

$$= \left\{ \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} : Q_1, Q_2 \in \operatorname{Im} \mathbb{H} \right\} \cong sp(1) \times sp(1)$$

$$C_3(4) = \text{SPAN}\{e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4\}$$

$$= \left\{ \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} : q \in \mathbb{H} \right\} \cong \mathbb{R}$$

$$C_4(4) = \text{SPAN}\{e_1e_2e_3e_4\} = \left\{ \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\} \cong \mathbb{R}$$

THUS, (5) IS JUST

$$Cl(4) = Cl_0(4) \oplus Cl_1(4) = \left\{ \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix} \right\}$$

ELEMENTS OF $Cl_0(4)$ ARE <u>EVEN</u>. ELEMENTS OF $Cl_1(4)$ ARE <u>ODD</u>

REGARDING $\mathbb{Z}_2 = \{0,1\}$ WITH ADDITION MODULO 2,

$$(Cl_i(4))(Cl_j(4)) \subseteq Cl_{i+j}(4)$$

FOR ALL $i, j \in \mathbb{Z}_2$ SO $Cl(4)$ IS A $\mathbb{Z}_2$-GRADED ALGEBRA.

<u>A FURTHER DECOMPOSITION</u>: IF $\mathbb{R}^4$ IS GIVEN ITS USUAL ORIENTATION, THEN $\{1, i, j, k\}$ IS AN ORIENTED BASIS. ORIENT THE COPY OF $\mathbb{R}^4$ IN $Cl(4)$ BY DECLARING $\{e_1, e_2, e_3, e_4\}$ TO BE AN ORIENTED BASIS. DEFINE THE <u>VOLUME ELEMENT</u> ω OF $Cl(4)$ BY

$$\omega = -e_1e_2e_3e_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>NOTE</u>: SOME SOURCES CHOOSE $e_1e_2e_3e_4$. WE HAVE OPTED FOR THE MINUS SIGN BECAUSE WHEN WE COMPLEXIFY SHORTLY THE REQUIRED CHOICE IN DIMENSION 4 IS $i^{\frac{4}{2}} e_1e_2e_3e_4 = -e_1e_2e_3e_4$.

NOW, SINCE $\omega^2 = \mathbb{1}$, LEFT MULTIPLICATION BY ω ON $Cl_0(4)$ OR $Cl_1(4)$ HAS EIGENVALUES ± 1 SO WE HAVE DECOMPOSITIONS

$$Cl_0(4) = Cl_0^+(4) \oplus Cl_0^-(4)$$
$$Cl_1(4) = Cl_1^+(4) \oplus Cl_1^-(4)$$

INTO ± 1 EIGENSPACES (ω IS ACTUALLY IN THE CENTER OF $Cl_0(4)$, BUT ACTS ANTI-COMMUTATIVELY ON $Cl_1(4)$). THE EIGENSPACES ARE EASILY SEEN TO BE

$$Cl_0^+(4) = \left\{ \begin{pmatrix} q_{11} & 0 \\ 0 & 0 \end{pmatrix} \right\} \qquad Cl_0^-(4) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & q_{22} \end{pmatrix} \right\}$$

$$Cl_1^+(4) = \left\{ \begin{pmatrix} 0 & q_{12} \\ 0 & 0 \end{pmatrix} \right\} \qquad Cl_1^-(4) = \left\{ \begin{pmatrix} 0 & 0 \\ q_{21} & 0 \end{pmatrix} \right\}$$

THERE ARE OBVIOUS ALGEBRA ISOMORPHISMS

$$\mathbb{H} \longrightarrow Cl_0^{\pm}(4).$$

<u>THE</u> <u>PIN</u> <u>AND</u> <u>SPIN</u> <u>GROUPS</u> :

(6) $\quad Cl^{\times}(4)$ = MULTIPLICATIVE GROUP OF UNITS (INVERTIBLE ELEMENTS) OF $Cl(4)$

<u>NOTE</u> : ANY $x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \in \mathbb{R}^4 \subseteq Cl(4)$ WITH $\|x\| = 1$ IS IN $Cl^{\times}(4)$ AND

$$x^{-1} = -x$$

(BECAUSE $\langle x, x\rangle = 1$ AND $xx + xx = -2\langle x, x\rangle \mathbb{1}$ IMPLY $xx = -\mathbb{1}$)

(7) $\quad PIN(4)$ = MULTIPLICATIVE SUBGROUP OF $Cl^{\times}(4)$ GENERATED BY ALL $x \in \mathbb{R}^4$ WITH $\|x\| = 1$.

NOTE THAT $x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \in \mathbb{R}^4 \subseteq Cl(4)$ HAS $\|x\| = 1$ IF AND ONLY IF $\|q\| = 1$, I.E., q IS IN THE GROUP $Sp(1)$ OF UNIT QUATERNIONS. SINCE THE SET OF SUCH ELEMENTS IS CLOSED UNDER INVERSION ($x^{-1} = -x = \begin{pmatrix} 0 & -q \\ -(-\bar{q}) & 0 \end{pmatrix}$), <u>PIN (4) IS THE SET OF ALL PRODUCTS OF ELEMENTS OF THE SET</u>

$$\left\{ x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} : q \in Sp(1) \right\}$$

SINCE

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} ad & 0 \\ 0 & bc \end{pmatrix}$$
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma & 0 \\ 0 & \beta\delta \end{pmatrix}$$
$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 0 & a\beta \\ b\alpha & 0 \end{pmatrix}$$
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha a \\ \beta b & 0 \end{pmatrix}$$

THE ELEMENTS OF PIN (4) ARE EITHER DIAGONAL (I.E., EVEN) OR ANTI-DIAGONAL (I.E., ODD) AND THEIR NONZERO ELEMENTS ARE ALL IN $Sp(1)$.

$$(8) \qquad SPIN(4) = PIN(4) \cap Cl_0(4)$$
$$= \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} : u_1, u_2 \in Sp(1) \right\}$$
$$\cong Sp_1(1) \times Sp_2(1)$$

TOPOLOGY AND DIFFERENTIABLE STRUCTURE INHERITED FROM $Cl(4) = \mathbb{H}^{2\times 2} = \mathbb{H}^4$ ARE THE USUAL ONES. <u>COMPACT, SIMPLY CONNECTED LIE GROUP</u>. THE LIE

ALGEBRA IS

$$(9)\quad spin(4) = \left\{ \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} : Q_1, Q_2 \in \operatorname{Im} \mathbb{H} \right\} = C_2(4)$$

$$\cong sp_1(1) \times sp_2(1).$$

<u>THEOREM</u>: SPIN(4) IS THE UNIVERSAL DOUBLE COVER OF SO(4).

BECAUSE OF THE IMPORTANCE OF THIS RESULT WE WILL PAUSE TO GIVE A PROOF. THE ARGUMENT GENERALIZES TO GIVE THE UNIVERSAL DOUBLE COVER SPIN(n) OF SO(n).

<u>LEMMA</u>: THE CENTER $Z(Cl(4))$ OF $Cl(4)$ IS $C_0(4) = \{x_0 e_0 : x_0 \in \mathbb{R}\}$

<u>PROOF</u>: SINCE $e_0 = \mathbb{1}$, $C_0(4) \subseteq Z(Cl(4))$ IS CLEAR. TO COMPLETE THE PROOF IT WILL SUFFICE TO SHOW THAT NO $e_I = e_{i_1} \cdots e_{i_k}$, $1 \le k \le 4$, $1 \le i_1 < \cdots < i_k \le 4$ IS IN $Z(Cl(4))$. IF $k=1$ THIS IS CLEAR SINCE $e_i e_j = -e_j e_i$ FOR $i \ne j$. IF $k=4$, THEN $e_I = e_1 e_2 e_3 e_4$ AND WE HAVE $e_1 e_I = (e_1 e_1) e_2 e_3 e_4 = -e_2 e_3 e_4$, WHEREAS $e_I e_1 = (e_1 e_2 e_3 e_4)(e_1) = e_1 (e_2 e_3 e_4 e_1) = e_1(-e_1 e_2 e_3 e_4) = -(e_1 e_1)(e_2 e_3 e_4) = e_2 e_3 e_4$. NOW SUPPOSE $1 < k < 4$. THEN $e_I e_{i_1} = (-1)^{k-1} e_{i_1} e_I$ AND, IF e_ℓ IS NOT AMONG $e_{i_1}, \ldots, e_{i_k}$, $e_I e_\ell = (-1)^k e_\ell e_I$. THUS, e_I CANNOT COMMUTE WITH BOTH e_{i_1} AND e_ℓ. $\square$

PROOF OF THE THEOREM :

CONSIDER THE ADJOINT ACTION OF $Cl^{\times}(4)$ ON $Cl(4)$, I.E., FOR EACH $\mu \in Cl^{\times}(4)$ DEFINE

$$ad_{\mu} : Cl(4) \to Cl(4)$$

(10) $$ad_{\mu}(p) = \mu p \mu^{-1}$$

ALGEBRA ISOMORPHISM. PRESERVES GRADING $Cl(4) = Cl_0(4) \oplus Cl_1(4)$.

NOTE THAT IF $x \in \mathbb{R}^4 \subseteq Cl(4)$ HAS $\|x\| = 1$ (I.E., IS ONE OF THE GENERATORS OF $PIN(4)$), THEN, FOR EVERY $v \in \mathbb{R}^4 \subseteq Cl(4)$, $ad_x(v) = x v x^{-1} = x v(-x) = -x v x$ SO THE IDENTITY

$$v x + x v = -2\langle v, x\rangle \mathbb{1}$$

IMPLIES

$$x v x + x x v = -2\langle v, x\rangle x$$

$$x v x - v = -2\langle v, x\rangle x \qquad (x(-x) = \mathbb{1})$$

$$x v x = v - 2\langle v, x\rangle x$$

$$x v x = \underbrace{(v - \langle v, x\rangle x)}_{\text{PROJECTION OF } v \text{ INTO THE HYPERPLANE } x^{\perp} \text{ ORTHOGONAL TO } x} - \langle v, x\rangle x$$

$$x v x = \text{REFLECTION OF } v \text{ THROUGH } x^{\perp}$$

$$x v x = REFL_{x^{\perp}}(v).$$

THUS,

(11) $$ad_x(v) = -REFL_{x^{\perp}}(v) \qquad (x \in \mathbb{R}^4, \|x\| = 1)$$

IN PARTICULAR, $ad_x : \mathbb{R}^4 \to \mathbb{R}^4$ WHEN $x \in \mathbb{R}^4$ AND $\|x\| = 1$. BUT ANY ELEMENT OF $PIN(4)$ IS A PRODUCT OF SUCH ELEMENTS OF $\mathbb{R}^4$ SO

(12) $$ad_u : \mathbb{R}^4 \to \mathbb{R}^4 \qquad (u \in PIN(4)).$$

THIS IS TRUE, IN PARTICULAR FOR $u \in SPIN(4)$. SINCE ANY PRODUCT OF REFLECTIONS IS AN ORTHOGONAL TRANSFORMATION AND THE PRODUCT OF ANY EVEN NUMBER OF REFLECTIONS IS A ROTATION,

(13) $$ad_u \in SO(\mathbb{R}^4) \qquad (u \in SPIN(4)).$$

MOREOVER, ANY REFLECTION CAN BE WRITTEN AS ad_x FOR SOME $x \in \mathbb{R}^4$ WITH $\|x\| = 1$ AND ANY ROTATION CAN BE WRITTEN AS THE PRODUCT OF AN EVEN NUMBER OF REFLECTIONS SO WE HAVE A SURJECTIVE GROUP HOMOMORPHISM (THE "SPINOR MAP"):

$$SPIN : SPIN(4) \longrightarrow SO(\mathbb{R}^4) \cong SO(4)$$

(14) $$SPIN(u) = ad_u$$

TO SEE THAT THE KERNEL OF THIS MAP IS $\mathbb{Z}_2 = \{\pm \mathbb{1}\}$ NOTE THAT $ad_u = id_{\mathbb{R}^4}$ IF AND ONLY IF $u x u^{-1} = x$, I.E., $ux = xu$ FOR EVERY $x \in \mathbb{R}^4$. BUT THEN u MUST COMMUTE WITH EVERYTHING IN $Cl(4)$, I.E., $u \in \mathcal{Z}(Cl(4))$. BY THE LEMMA, $u = x_0 e_0 = x_0 \mathbb{1}$ FOR SOME $x_0 \in \mathbb{R}$. SINCE THE NONZERO ENTRIES IN AN ELEMENT OF $SPIN(4)$ MUST BE IN $Sp(1)$ (PAGE 8), $x_0^2 = 1$ SO $u = \pm \mathbb{1}$ AS REQUIRED.

SINCE WE HAVE ALREADY NOTED THAT $SPIN(4)$ IS A SIMPLY CONNECTED LIE GROUP, THE PROOF IS COMPLETE. $\square$

THUS FAR WE HAVE THIS MUCH

$$\begin{array}{ccc} \mathbb{R}^4 & \hookrightarrow & Cl(\mathbb{R}^4) \\ & & \uparrow \\ & & SPIN(4) \end{array}$$

OF OUR STATED OBJECTIVE (PAGE 1) AND NOW WE MUST <u>COMPLEXIFY EVERYTHING</u> :

FIRST USE THE MAP $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$ (PAGE 2) TO IDENTIFY $\mathbb{H}^{2\times 2}$ WITH A SUBSET OF $\mathbb{C}^{4\times 4}$, I.E., WE DEFINE

$$T : \mathbb{H}^{2\times 2} \longrightarrow \mathbb{C}^{4\times 4}$$

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma(q_{11}) & \gamma(q_{12}) \\ \gamma(q_{21}) & \gamma(q_{22}) \end{pmatrix} \quad (2\times 2 \text{ BLOCKS})$$

REAL LINEAR, INJECTIVE AND PRESERVES PRODUCTS SO, AS A REAL ALGEBRA, WE CAN IDENTIFY

$$Cl(4) = T(\mathbb{H}^{2\times 2})$$

RESTRICTION TO $\mathbb{R}^4 \subseteq Cl(4) = \mathbb{H}^{2\times 2}$ IS

$$(15) \qquad x = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} 0 & \gamma(q) \\ \gamma(-\bar{q}) & 0 \end{pmatrix}$$

SINCE $DET(T(x)) = DET(x) = \|x\|^2 = |q|^2$ WE CAN DEFINE AN INNER PRODUCT (VIA POLARIZATION) ON THIS COPY OF $\mathbb{R}^4$ BY $\|T(x)\|^2 = DET(T(x))$ AND $T|\mathbb{R}^4$ BECOMES AN ISOMETRY.

NEW IDENTIFICATIONS :

$$\mathbb{R}^4 : \quad E_1 = T(e_1) = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$E_2 = T(e_2) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$E_3 = T(e_3) = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_4 = T(e_3) = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$E_i E_j + E_j E_i = -2 \langle E_i, E_j \rangle \mathbb{1}, \qquad i, j = 1,2,3,4$$

$Cl(4)$: REAL SUBALGEBRA OF $\mathbb{C}^{4\times 4}$ GENERATED BY $\mathbb{R}^4$. A BASIS IS JUST AS DESCRIBED ON PAGE 4, BUT WITH EVERYTHING CAPITALIZED (AND 1 CHANGED INTO $\mathbb{1}$).

$$SPIN(4) : \quad \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : U_1, U_2 \in SU(2) \right\} \cong SU_1(2) \times SU_2(2)$$

$$spin(4) : \quad \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1, A_2 \in su(2) \right\} \cong su_1(2) \times su_2(2)$$

NOW WE REGARD $\mathbb{C}^{4\times4}$ AS A COMPLEX ALGEBRA. THE <u>COMPLEXIFIED</u> <u>CLIFFORD</u> <u>ALGEBRA</u>

$$Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$$

OF $\mathbb{R}^4$ IS THE COMPLEX SUBALGEBRA OF $\mathbb{C}^{4\times4}$ GENERATED BY $Cl(4)$. SAME BASIS AS $Cl(4)$, BUT WITH COMPLEX COEFFICIENTS. THUS,

$$(16) \qquad Cl(4) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{4\times4}$$

(SINCE BOTH HAVE DIMENSION 16 OVER $\mathbb{C}$).

<u>NOTE</u> : $Cl(4)$ (AND THEREFORE $\mathbb{R}^4$, SPIN(4) AND $\mathfrak{spin}(4)$) IS NATURALLY IDENTIFIED WITH A SUBSET OF $Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$.

NOW LET

$$S_{\mathbb{C}} = \mathbb{C}^4$$

BE THE COMPLEX VECTOR SPACE $\mathbb{C}^4$ WITH ITS USUAL HERMITIAN INNER PRODUCT $\left(\langle z, w \rangle = \bar{z}^1 w^1 + \bar{z}^2 w^2 + \bar{z}^3 w^3 + \bar{z}^4 w^4 \right)$ AND IDENTIFY

$$(17) \qquad Cl(4) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{END}_{\mathbb{C}}(S_{\mathbb{C}})$$

THUS, THE ELEMENTS OF $Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$ (AND THEREFORE ALSO $Cl(4)$, $\mathbb{R}^4$, AND SPIN(4)) ALL ACT AS ENDOMORPHISMS ON $S_{\mathbb{C}}$. THIS ACTION IS OFTEN CALLED <u>CLIFFORD</u> <u>MULTIPLICATION</u> AND WRITTEN WITH A DOT $\cdot$.

IN PARTICULAR, WE HAVE A REPRESENTATION OF (THE REAL ALGEBRA) $Cl(4)$ BY ENDOMORPHISMS ON $S_{\mathbb{C}}$

$$Cl(4) \longrightarrow END_{\mathbb{C}}(S_{\mathbb{C}})$$

("REPRESENTATIONS" OF ALGEBRAS ARE BY ENDOMORPHISMS RATHER THAN ISOMORPHISMS SINCE NOT ALL ELEMENTS OF AN ALGEBA ARE UNITS)

THIS REPRESENTATION IS EASILY SEEN TO BE <u>IRREDUCIBLE</u> (WRITE OUT THE REAL LINEAR COMBINATIONS OF THE BASIS $E_0, \ldots, E_1E_2E_3E_4$ FOR $Cl(4) \subseteq Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$).

RESTRICTING THIS REPRESENTATION TO $SPIN(4) \subseteq Cl(4)$ GIVES A (GROUP) REPRESENTATION OF $SPIN(4)$ BY AUTOMORPHISMS OF $S_{\mathbb{C}}$ DENOTED

(18) $$\Delta_{\mathbb{C}} : SPIN(4) \longrightarrow AUT_{\mathbb{C}}(S_{\mathbb{C}})$$

AND CALLED THE <u>COMPLEX SPIN REPRESENTATION</u>.

<u>NOTES</u> :

1. $SPIN(4)$ IS THE DOUBLE COVER OF $SO(4)$, BUT $\Delta_{\mathbb{C}}$ DOES <u>NOT</u> DESCEND TO A REPRESENTATION OF $SO(4)$ BECAUSE

$$\Delta_{\mathbb{C}}(-\mathbb{1}) = -\mathbb{1} \neq \mathbb{1} = \Delta_{\mathbb{C}}(\mathbb{1})$$

2. $\Delta_{\mathbb{C}}$ IS <u>NOT</u> IRREDUCIBLE. INDEED, IF WE SPLIT

(19) $$S_{\mathbb{C}} = S_{\mathbb{C}}^{+} \oplus S_{\mathbb{C}}^{-}$$

BY

$$\begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix} = \begin{pmatrix} z^1 \\ z^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z^3 \\ z^4 \end{pmatrix}$$

THEN CLIFFORD MULTIPLICATION BY ANY ELEMENT OF $Cl_0(4)$ (BECAUSE THEY ARE BLOCK DIAGONAL) PRESERVES $S_{\mathbb{C}}^{+}$ AND $S_{\mathbb{C}}^{-}$.

> <u>NOTE</u>: $S_{\mathbb{C}}^{\pm}$ ARE JUST THE SUBSPACES OF $S_{\mathbb{C}}$ ON WHICH THE VOLUME ELEMENT $\omega = -E_1E_2E_3E_4$ ACTS AS $\pm$ ID.

RESTRICTING THE ACTION OF $SPIN(4) \subseteq Cl_0(4)$ TO $S_{\mathbb{C}}^{\pm}$ THEREFORE GIVES

(20) $$\Delta_{\mathbb{C}} = \Delta_{\mathbb{C}}^{+} \oplus \Delta_{\mathbb{C}}^{-}$$

WHERE

(21) $$\Delta_{\mathbb{C}}^{\pm} : SPIN(4) \longrightarrow SU(S_{\mathbb{C}}^{\pm}).$$

(RECALL THAT $SPIN(4) = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : U_1, U_2 \in SU(2) \right\}$.)

$\Delta_{\mathbb{C}}^{\pm}$ ARE <u>INEQUIVALENT</u>, <u>IRREDUCIBLE</u> REPRESENTATIONS OF SPIN(4) OF DIMENSION 2.

NEXT NOTE THAT CLIFFORD MULTIPLICATION BY ELEMENTS OF $Cl_1(4)$ (BECAUSE THEY ARE ANTI-DIAGONAL) INTERCHANGES $S^+_{\mathbb{C}}$ AND $S^-_{\mathbb{C}}$.

IN PARTICULAR, THIS IS TRUE FOR ELEMENTS OF $\mathbb{R}^4 \subseteq Cl(4)$, I.E., FOR $x \in \mathbb{R}^4 \subseteq Cl(4)$,

$$Z \in S^{\pm}_{\mathbb{C}} \Rightarrow x \cdot Z \in S^{\mp}_{\mathbb{C}} \tag{22}$$

(THIS WILL BE CRUCIAL WHEN WE DEFINE THE "DIRAC OPERATOR" THAT APPEARS IN THE SEIBERG-WITTEN EQUATIONS.)

NEXT WE FIND THE <u>COMPLEX ANALOGUE OF THE SPIN GROUP</u>. RECALL THAT SPIN(4) IS THE SET OF ALL EVEN ELEMENTS IN THE SUBGROUP OF THE MULTIPLICATIVE GROUP OF UNITS IN $Cl(4)$ GENERATED BY THE UNIT SPHERE IN $\mathbb{R}^4 \subseteq Cl(4)$.

$SPIN^c(4)$ IS THE SUBGROUP OF THE MULTIPLICATIVE GROUP OF UNITS IN $Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$ GENERATED BY SPIN(4) AND THE UNIT CIRCLE

$$U(1) = \{ e^{\theta i} \mathbb{1} : \theta \in \mathbb{R} \}$$

IN $C_0(4) \otimes_{\mathbb{R}} \mathbb{C} \subseteq Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$.

SINCE $U(1)$ IS IN THE CENTER OF $Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$ WE HAVE

$$(23) \qquad \mathrm{SPIN}^c(4) = \{ e^{\theta i} u : \theta \in \mathbb{R},\ u \in \mathrm{SPIN}(4) \}$$

$$= \left\{ e^{\theta i} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} : \theta \in \mathbb{R},\ U_1, U_2 \in SU(2) \right\}$$

NOTE THAT $U_1, U_2 \in SU(2) \Rightarrow e^{\theta i} U_1,\ e^{\theta i} U_2 \in U(2)$ (AND EVERY ELEMENT OF $U(2)$ CAN BE WRITTEN IN THIS WAY) AND

$$(24) \qquad \mathrm{DET}(e^{\theta i} U_1) = \mathrm{DET}(e^{\theta i} U_2) = e^{2\theta i}$$

SO

$$(25) \qquad \mathrm{SPIN}^c(4) = \left\{ \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} : U_\pm \in U(2),\ \mathrm{DET}(U_+) = \mathrm{DET}(U_-) \right\}$$

SOME MAPPINGS:

1. $\mathrm{SPIN}(4) \times U(1) \longrightarrow \mathrm{SPIN}^c(4)$

   $(u, e^{\theta i} \mathbb{1}) \rightarrow e^{\theta i} u$

THIS IS A SURJECTIVE HOMOMORPHISM WITH KERNEL $\mathbb{Z}_2 = \langle (-\mathbb{1}, -\mathbb{1}) \rangle$ (THE KERNEL IS THE SET OF PAIRS (α, α^{-1}), WHERE $\alpha \in \mathrm{SPIN}(4) \cap U(1)$, BUT SPIN (4) INTERSECTS THE SCALARS ONLY IN $\pm \mathbb{1}$). THUS,

$$(26) \qquad \mathrm{SPIN}^c(4) \cong \mathrm{SPIN}(4) \times U(1) / \langle (-\mathbb{1}, -\mathbb{1}) \rangle$$

$$\cong SU_1(2) \times SU_2(2) \times U(1) / \langle (-\mathbb{1}, -\mathbb{1}, -\mathbb{1}) \rangle$$

2. $\delta : SPIN^c(4) \to U(1)$

FOR $\xi = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} = \begin{pmatrix} e^{\theta i} U_1 & 0 \\ 0 & e^{\theta i} U_2 \end{pmatrix} \in SPIN^c(4)$ DEFINE

(27) $$\delta(\xi) = DET(U_+) = DET(U_-) = e^{2\theta i}$$

THIS IS A SURJECTIVE HOMOMORPHISM WHOSE KERNEL IS

$$KER(\delta) = \left\{ \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} : U_\pm \in SU(2) \right\} = SPIN(4)$$

3. $\pi : SPIN^c(4) \to SO(4)$

THE ADJOINT ACTION OF $SPIN(4)$ ON $\mathbb{R}^4$ (PAGES 10-11) EXTENDS TO AN ADJOINT ACTION OF $SPIN^c(4)$ ON $\mathbb{R}^4$. INDEED, FOR $\xi = e^{\theta i}\mu \in SPIN^c(4)$

$$ad_\xi : \mathbb{R}^4 \to \mathbb{R}^4$$

IS GIVEN BY

$$\begin{aligned} ad_\xi(x) &= \xi x \xi^{-1} \\ &= (e^{\theta i}\mu) x (\mu^{-1} e^{-\theta i}) \\ &= \mu x \mu^{-1} \\ &= ad_\mu(x) \end{aligned}$$

SINCE $ad_\mu \in SO(\mathbb{R}^4)$ WE HAVE

(28) $$ad_\xi \in SO(\mathbb{R}^4) \cong SO(4)$$

AND WE TAKE

(29) $$\pi(\xi) = ad_\xi$$

4. $SPIN^c(4)$ IS A DOUBLE COVER OF $SO(4) \times U(1)$:

DEFINE $SPIN^c : SPIN^c(4) \to SO(4) \times U(1)$ BY

$$SPIN^c(\xi) = SPIN^c(e^{\theta i}\mu) = (SPIN(\mu), \delta(e^{\theta i}\mu))$$
$$= (SPIN(\mu), e^{2\theta i})$$

THIS IS A SURJECTIVE HOMOMORPHISM WHOSE KERNEL IS THE SET OF $e^{\theta i}\mu$ SUCH THAT $SPIN(\mu) = \mathbb{1}$ AND $e^{2\theta i} = 1$, I.E., $\mu = \pm \mathbb{1}$ AND $e^{\theta i} = \pm 1$ THUS, $e^{\theta i}\mu = \pm \mathbb{1}$ SO

$$KER(SPIN^c) = \mathbb{Z}_2 = \langle -\mathbb{1} \rangle$$

FOLLOWS THAT

$$spin^c(4) \cong so(4) \oplus u(1) \cong spin(4) \oplus \operatorname{Im} \mathbb{C}$$

CAN BE IDENTIFIED WITH THE SUBSET

$$\left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + ti \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} : A_1, A_2 \in su(2), t \in \mathbb{R} \right\}$$

OF $Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$.

<u>NOTE</u>: $SPIN^c(4)$ IS <u>NOT</u> THE UNIVERSAL COVER OF $SO(4) \times U(1)$ BECAUSE $\pi_1(SPIN^c(4)) = \mathbb{Z}$ (<u>REASON</u>: $SPIN(4)$ IS NATURALLY IDENTIFIED WITH A SUBSPACE OF $SPIN^c(4)$ AND IS SIMPLY CONNECTED. THE SEQUENCE

$$1 \to SPIN(4) \hookrightarrow SPIN^c(4) \xrightarrow{\delta} U(1) \to 1$$

GIVES

$$0 \to \pi_1(SPIN^c(4)) \xrightarrow{\delta_*} \mathbb{Z} \to 0$$

AND δ_* IS AN ISOMORPHISM.)

NOW, THE IDENTIFICATION $Cl(4) \otimes_{\mathbb{R}} \mathbb{C} = END_{\mathbb{C}}(S_{\mathbb{C}})$ AND THE FACT THAT THE ELEMENTS OF $SPIN^c(4)$ ARE ALL UNITS IMPLIES THAT THE COMPLEX SPIN REPRESENTATION $\Delta_{\mathbb{C}} : SPIN(4) \to AUT_{\mathbb{C}}(S_{\mathbb{C}})$ EXTENDS TO

(30) $$\hat{\Delta}_{\mathbb{C}} : SPIN^c(4) \to AUT_{\mathbb{C}}(S_{\mathbb{C}})$$

MOREOVER, SINCE THE ELEMENTS OF $SPIN^c(4)$ ARE ALL EVEN $\hat{\Delta}_{\mathbb{C}}$ ALSO SPLITS INTO

(31) $$\hat{\Delta}_{\mathbb{C}} = \hat{\Delta}_{\mathbb{C}}^{+} \oplus \hat{\Delta}_{\mathbb{C}}^{-}$$

WHERE

(32) $$\hat{\Delta}_{\mathbb{C}}^{\pm} : SPIN^c(4) \to U(S_{\mathbb{C}}^{\pm}) \cong U_{\pm}(2)$$

STILL IRREDUCIBLE AND AND INEQUIVALENT,

BEFORE WE GLOBALIZE ALL OF THIS MACHINERY THERE IS ONE MORE ALGEBRAIC PRELIMINARY REQUIRED.

NOTE THAT THERE IS A NATURAL LINEAR ISOMORPHISM FROM THE SPACE OF 2-FORMS $\Omega^2(\mathbb{R}^4)$ ON $\mathbb{R}^4$ INTO $C_2(4) \subseteq Cl(4)$ GIVEN BY

$$(33) \qquad \eta = \sum_{i<j} \eta_{ij} e^i \wedge e^j \longrightarrow \sum_{i<j} \eta_{ij} e_i e_j$$

<u>REMARKS</u>: HERE $\{e^1, e^2, e^3, e^4\}$ IS THE DUAL BASIS OF $\{e_1, e_2, e_3, e_4\}$. THIS MAP IS <u>NOT</u> MULTIPLICATIVE (E.G., $e^1 \wedge e^1 = 0$, BUT $e_1 e_1 = -\mathbb{1}$). THERE IS, OF COURSE, AN ANALOGOUS LINEAR ISOMORPHISM FOR ANY RANK.

COMPOSE WITH THE INJECTION $Cl(4) \hookrightarrow \mathrm{END}_{\mathbb{C}}(S_{\mathbb{C}})$ TO OBTAIN

$$(34) \qquad \rho : \Omega^2(\mathbb{R}^4) \longrightarrow \mathrm{END}_{\mathbb{C}}(S_{\mathbb{C}})$$

$$(35) \qquad \rho(\eta) = \rho\Big(\sum_{i<j} \eta_{ij} e^i \wedge e^j\Big) = \sum_{i<j} \eta_{ij} E_i E_j =$$

$$\begin{pmatrix} (\eta_{12}+\eta_{34})I + (\eta_{13}-\eta_{24})J + (\eta_{14}+\eta_{23})K & 0 \\ 0 & (-\eta_{12}+\eta_{34})I + (-\eta_{13}-\eta_{24})J + (-\eta_{14}+\eta_{23})K \end{pmatrix}$$

BEING EVEN (I.E., DIAGONAL) THE SUBSPACES $S^{\pm}_{\mathbb{C}}$ ARE INVARIANT UNDER $\rho(\eta)$ FOR EACH η SO WE MAY SET

(36) $$\rho^{\pm}(\eta) = \rho(\eta) \mid S_{\mathbb{C}}^{\pm}.$$

FOR EXAMPLE, SUPPRESSING THE TWO ZERO ENTRIES IN $S_{\mathbb{C}}^{+}$ (SEE (19))

(37) $$\rho^{+}(\eta) = (\eta_{12}+\eta_{34})I + (\eta_{13}+\eta_{42})J + (\eta_{14}+\eta_{23})K$$

COMPLEXIFY (I.E., USE COMPLEX SCALARS) AND ρ EXTENDS TO

(38) $$\rho : \Omega^2(\mathbb{R}^4)\otimes_{\mathbb{R}}\mathbb{C} \longrightarrow END_{\mathbb{C}}(S_{\mathbb{C}})$$

(USE SAME SYMBOL FOR THE EXTENSION).

$S_{\mathbb{C}}^{\pm}$ STILL INVARIANT SO WE HAVE

(39) $$\rho^{\pm} : \Omega^2(\mathbb{R}^4)\otimes_{\mathbb{R}}\mathbb{C} \longrightarrow END_{\mathbb{C}}(S_{\mathbb{C}}^{\pm})$$

PROPERTIES :

(a) $\eta \in \Omega^2(\mathbb{R}^4)\otimes_{\mathbb{R}}\mathbb{C}$ REAL-VALUED $\Rightarrow \rho(\eta)$ SKEW-HERMITIAN

PROOF : $$\overline{\rho(\eta)}^T = \sum_{i<j} \bar{\eta}_{ij}\, \overline{E_i E_j}^T = \sum_{i<j} \eta_{ij}\, \bar{E}_j^T \bar{E}_i^T$$
$$= \sum_{i<j} \eta_{ij}(-E_j)(-E_i) = \sum_{i<j} \eta_{ij} E_j E_i$$
$$= \sum_{i<j} \eta_{ij}(-E_i E_j) = -\rho(\eta).$$

(b) η IN $\mathbb{C}$-VALUED $\Rightarrow$ $\rho(\eta)$ HERMITIAN

(c) $\rho(e^1\wedge e^2 + e^3\wedge e^4) = 2\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$

$\rho(e^1\wedge e^3 + e^4\wedge e^2) = 2\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$

$\rho(e^1\wedge e^4 + e^2\wedge e^3) = 2\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$

$\rho(e^1\wedge e^2 - e^3\wedge e^4) = -2\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$

$\rho(e^1\wedge e^3 - e^4\wedge e^2) = -2\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$

$\rho(e^1\wedge e^4 - e^2\wedge e^3) = -2\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$

RECALL: LET $* : \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$ BE THE HODGE STAR OPERATOR ON 2-FORMS ON $\mathbb{R}^4$ (USUAL INNER PRODUCT AND ORIENTATION). SINCE $\{e_1, e_2, e_3, e_4\}$ IS AN ORIENTED, ORTHONORMAL BASIS FOR $\mathbb{R}^4$, THE SELF-DUAL (SD) AND ANTI-SELF-DUAL (ASD) 2-FORMS ARE GIVEN BY

$$\Omega^2_\pm(\mathbb{R}^4) = \text{SPAN}\{e^1\wedge e^2 \pm e^3\wedge e^4,\ e^1\wedge e^3 \pm e^4\wedge e^2,\ e^1\wedge e^4 \pm e^2\wedge e^3\}.$$

EVERY $\eta \in \Omega^2(\mathbb{R}^4)$ CAN BE UNIQUELY WRITTEN $\eta = \eta^+ + \eta^-$, WHERE $\eta^\pm \in \Omega^2_\pm(\mathbb{R}^4)$. INDEED, $\eta^\pm = \frac{1}{2}(\eta \pm *\eta)$.

FROM PROPERTY (c), $\rho^\pm$ CARRIES $\Omega^2_\pm(\mathbb{R}^4)$ TO $\text{END}_{\mathbb{C}}(S^\pm_{\mathbb{C}})$

IN FACT, WE CLAIM THAT

(40) $$\rho^{\pm} : \Omega^2_{\pm}(\mathbb{R}^4) \to su(S^{\pm}_{\mathbb{C}})$$

IS AN ISOMORPHISM OF $\Omega^2_{\pm}(\mathbb{R}^4)$ ONTO THE SKEW-HERMITIAN, TRACE-FREE ENDOMORPHISMS OF $S^{\pm}_{\mathbb{C}}$.

PROOF FOR ρ^+: $\eta \in \Omega^2_+(\mathbb{R}^4) \Rightarrow \eta$ REAL-VALUED $\Rightarrow$ $\rho(\eta)$ SKEW-HERMITIAN BY PROPERTY (a) $\Rightarrow \rho^+(\eta)$ SKEW-HERMITIAN. η SELF-DUAL AND I, J, K TRACE-FREE $\Rightarrow$ $\rho^+(\eta)$ TRACE-FREE BY PROPERTY (c). ρ^+ LINEAR AND INJECTIVE BECAUSE ρ IS LINEAR AND INJECTIVE. FINALLY, ρ^+ IS ONTO $su(S^+_{\mathbb{C}})$ BECAUSE EVERY 2×2 COMPLEX, SKEW-HERMITIAN, TRACE-FREE MATRIX IS A REAL LINEAR COMBINATION OF I, J AND K ($su(2) \cong \mathrm{Im}\,\mathbb{H}$).

COMPLEXIFY AND GET ISOMORPHISMS

(41) $$\rho^{\pm} : \Omega^2_{\pm}(\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{END}_0(S^{\pm}_{\mathbb{C}})$$

FROM THE COMPLEX-VALUED (A)SD 2-FORMS ON $\mathbb{R}^4$ TO THE TRACE-FREE ENDOMORPHISMS OF $S^{\pm}_{\mathbb{C}}$

NOTE: EVERY 2×2 COMPLEX, TRACE-FREE MATRIX IS A COMPLEX LINEAR COMBINATION OF I, J AND K.

THE FINAL ALGEBRAIC ITEM WE REQUIRE IS THE INVERSE

(42) $\sigma^+ = (\rho^+)^{-1} : \mathrm{END}_0(S^+_{\mathbb{C}}) \to \Omega^2_+(\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C}$.

THIS CAN BE EXPLICITLY DESCRIBED AS FOLLOWS : LET T BE A TRACE-FREE ENDOMORPHISM OF $S^+_{\mathbb{C}}$ (THOUGHT OF AS A 4×4 COMPLEX MATRIX WHICH HAS THE FORM $\begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix}$ IN TERMS OF 2×2 BLOCKS). THEN $\sigma^+(T)$ IS TO BE A COMPLEX-VALUED, SD 2-FORM ON $\mathbb{R}^4$. WE DESCRIBE ITS VALUE ON A PAIR (v, w) OF TANGENT VECTORS TO $\mathbb{R}^4$ BY LETTING $\begin{pmatrix} 0 & \gamma(v) \\ -\gamma(\bar{v}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & V \\ -\bar{V} & 0 \end{pmatrix}$ AND $\begin{pmatrix} 0 & \gamma(w) \\ -\gamma(\bar{w}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & W \\ -\bar{W} & 0 \end{pmatrix}$ BE THE IMAGES OF v AND w IN $\mathbb{R}^4 \subseteq Cl(4) \subseteq Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$. THEN

(43) $(\sigma^+(T))(v, w) = \frac{1}{4} \, \mathrm{TRACE} \begin{pmatrix} 0 & V \\ -\bar{V} & 0 \end{pmatrix} \begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & W \\ -\bar{W} & 0 \end{pmatrix}$

THE PROOF THAT $\sigma^+(\rho^+(\eta)) = \eta \;\; \forall \; \eta \in \Omega^2_+(\mathbb{R}^4) \otimes \mathbb{C}$ AND $\rho^+(\sigma^+(T)) = T \;\; \forall \; T \in \mathrm{END}_0(S^+_{\mathbb{C}})$ IS A (NOT PARTICULARLY ENLIGHTENING) CALCULATION WHICH WE SHALL OMIT IN FAVOR OF AN IMPORTANT

<u>EXAMPLE</u> :

LET $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ BE IN $S^+_{\mathbb{C}}$ (TEMPORARILY SUPPRESS THE TWO ZERO COMPONENTS IN $S^+_{\mathbb{C}}$). DEFINE AN ENDOMORPHISM ON $S^+_{\mathbb{C}}$ BY

$$\psi \otimes \psi^* = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} (\bar{\psi}^1 \;\; \bar{\psi}^2) = \begin{pmatrix} |\psi^1|^2 & \psi^1 \bar{\psi}^2 \\ \bar{\psi}^1 \psi^2 & |\psi^2|^2 \end{pmatrix}$$

THE TRACE-FREE PART OF THIS ENDOMORPHISM IS

(44) $$(\psi \otimes \psi^*)_0 = \psi \otimes \psi^* - \tfrac{1}{2}\,\text{TRACE}\,(\psi \otimes \psi^*)\,\mathbb{1}$$
$$= \begin{pmatrix} \tfrac{1}{2}(|\psi^1|^2 - |\psi^2|^2) & \psi^1\bar{\psi}^2 \\ \bar{\psi}^1\psi^2 & \tfrac{1}{2}(|\psi^2|^2 - |\psi^1|^2) \end{pmatrix}$$

TO COMPUTE THE IMAGE OF THIS TRACE-FREE ENDOMORPHISM OF $S_{\mathbb{C}}^+$ UNDER σ^+ WE REINSTATE THE SUPPRESSED ZERO COMPONENTS IN $S_{\mathbb{C}}^+$ AND COMPUTE $\frac{1}{4}$ THE TRACE OF

$$\begin{pmatrix} 0 & V \\ -\bar{V} & 0 \end{pmatrix}\begin{pmatrix} (\psi\otimes\psi^*)_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & W \\ -\bar{W} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\bar{V}(\psi\otimes\psi^*)_0 W \end{pmatrix}$$

THIS HAS THE SAME TRACE AS $-\bar{V}(\psi\otimes\psi^*)_0 W$. WRITING OUT THIS MATRIX PRODUCT AND TAKING ITS TRACE GIVES

$$\tfrac{1}{2}(|\psi^1|^2 - |\psi^2|^2)\big(2(-v_1w_2 + w_2v_1)i + 2(-v_3w_4 + v_4w_3)i\big)$$
$$+ (\bar{\psi}^1\psi^2)\big((v_3w_1 - v_1w_3) + (-v_4w_2 + v_2w_4) + ((v_3w_2 - v_2w_3) + (v_4w_1 - v_1w_4))i\big)$$
$$+ (\psi^1\bar{\psi}^2)\big((v_1w_3 - v_3w_1) + (-v_2w_4 + v_4w_2) + ((-v_1w_4 + v_4w_1) + (-v_2w_3 + v_3w_2))i\big)$$

$$= -i\,(|\psi^1|^2 - |\psi^2|^2)(e^1\wedge e^2 + e^3\wedge e^4)(v,w)$$
$$+ (\psi^1\bar{\psi}^2 - \psi^2\bar{\psi}^1)(e^1\wedge e^3 + e^4\wedge e^2)(v,w)$$
$$- i\,(\psi^1\bar{\psi}^2 + \psi^2\bar{\psi}^1)(e^1\wedge e^4 + e^2\wedge e^3)(v,w)$$

=

$$i\left[-(|\psi^1|^2-|\psi^2|^2)(e^1\wedge e^2+e^3\wedge e^4)+2\,\mathrm{Im}(\psi^1\bar{\psi}^2)(e^1\wedge e^3+e^4\wedge e^2)\right.$$
$$\left.-2\,\mathrm{Re}(\psi^1\bar{\psi}^2)(e^1\wedge e^4+e^2\wedge e^3)\right](v,w)$$

THUS,

$$(45)\qquad \sigma^+((\psi\otimes\psi^*)_0)=\tfrac{1}{4}i\left[-(|\psi^1|^2-|\psi^2|^2)(e^1\wedge e^2+e^3\wedge e^4)\right.$$
$$+\;2\,\mathrm{Im}(\psi^1\bar{\psi}^2)(e^1\wedge e^3+e^4\wedge e^2)$$
$$\left.-\;2\,\mathrm{Re}(\psi^1\bar{\psi}^2)(e^1\wedge e^4+e^2\wedge e^3)\right]$$

WHICH IS, INDEED, AN $\mathrm{Im}\,\mathbb{C}$-VALUED, SELF-DUAL 2-FORM.

AMUSING OBSERVATION: EXCEPT FOR THE FACTOR OF $-\frac{1}{4}i$ THE COMPONENTS OF $\sigma^+((\psi\otimes\psi^*)_0)$ COINCIDE WITH THE COORDINATE FUNCTIONS OF THE COMPLEX HOPF MAP $S^3\to S^2$.

ONE CAN CHECK THAT (45) CAN ALSO BE WRITTEN

$$(46)\qquad \sigma^+((\psi\otimes\psi^*)_0)=-\tfrac{1}{4}\left[(\psi^*I\psi)(e^1\wedge e^2+e^3\wedge e^4)+\right.$$
$$(\psi^*J\psi)(e^1\wedge e^3+e^4\wedge e^2)+$$
$$\left.(\psi^*K\psi)(e^1\wedge e^4+e^2\wedge e^3)\right]$$

ADDENDUM 15 : SEIBERG-WITTEN INVARIANTS (MORE DETAILS)

HERE WE COLLECT A FEW MISC. OBSERVATIONS AND ARGUMENTS FOR WHICH THERE WAS NO TIME IN LECTURE.

1. SPIN AND SPIN^c STRUCTURES VIA TRANSITION MAPS

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M$$

$\{U_\alpha\}_{\alpha \in \mathcal{A}}$ A TRIVIALIZING COVER OF M WITH TRIVIALIZATIONS

$$\varphi_\alpha : \pi_{SO}^{-1}(U_\alpha) \longrightarrow U_\alpha \times SO(4)$$

$$\varphi_\alpha(p) = (\pi_{SO}(p), g_\alpha(p))$$

CORRESPONDING TRANSITION MAPS

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(4)$$

$$g_\alpha(p) = g_{\alpha\beta}(\pi_{SO}(p))\, g_\beta(p)$$

NOW, A SPIN STRUCTURE ON M CONSISTS OF A PRINCIPAL SPIN(4)-BUNDLE

$$SPIN(4) \hookrightarrow S(M) \xrightarrow{\pi_S} M$$

OVER M TOGETHER WITH A SMOOTH MAP

$$\lambda : S(M) \longrightarrow F_{SO}(M)$$

SUCH THAT

$$\begin{tikzcd} S(M) \arrow[rr, "\lambda"] \arrow[dr, "\pi_S"'] & & F_{SO}(M) \arrow[dl, "\pi_{SO}"] \\ & M & \end{tikzcd}$$

COMMUTES AND

$$\lambda(p \cdot g) = \lambda(p) \cdot \text{SPIN}(g)$$

$\forall\ p \in S(M)\ \forall\ g \in \text{SPIN}(4)$, WHERE $\text{SPIN} : \text{SPIN}(4) \longrightarrow SO(4)$ IS THE DOUBLE COVER (ADDENDUM 14).

<u>THEOREM</u>: LET M BE AN ORIENTED, RIEMANNIAN 4-MANIFOLD WITH ORIENTED ORTHONORMAL FRAME BUNDLE

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M.$$

A SPIN STRUCTURE EXISTS IF AND ONLY IF $\exists$ TRIVIALIZING COVER $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ OF M FOR $F_{SO}(M)$ FOR WHICH THE TRANSITION MAPS $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(4)$ LIFT TO A FAMILY OF MAPS TO SPIN(4)

$$\begin{tikzcd} & \text{SPIN}(4) \arrow[d, "\text{SPIN}"] \\ U_\alpha \cap U_\beta \arrow[ur, dotted, "\tilde{g}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & SO(4) \end{tikzcd}$$

SATISFYING THE COCYCLE CONDITION $\tilde{g}_{\alpha\beta}\,\tilde{g}_{\beta\gamma}\,\tilde{g}_{\gamma\alpha} = id.$

<u>PROOF</u> : SUPPOSE FIRST THAT A SPIN STRUCTURE EXISTS. LET $\{U_\alpha\}_{\alpha\in A}$ BE A TRIVIALIZING COVER OF M FOR BOTH $S(M)$ AND $F_{SO}(M)$. SELECT LOCAL SECTIONS

$$\tilde{s}_\alpha : U_\alpha \longrightarrow \pi_S^{-1}(U_\alpha) = \lambda^{-1}(\pi_{SO}^{-1}(U_\alpha))$$

AND LET THE CORRESPONDING TRANSITION MAPS BE

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SPIN(4).$$

(THESE NECESSARILY SATISFY THE COCYCLE CONDITION SINCE THEY ARE TRANSITION MAPS).

DEFINE

$$s_\alpha : U_\alpha \to \pi_{SO}^{-1}(U_\alpha)$$

$$s_\alpha = \lambda \circ \tilde{s}_\alpha$$

$\forall \alpha \in A$. THESE ARE SECTIONS OF $F_{SO}(M)$ SINCE $\pi_{SO} \circ s_\alpha = (\pi_{SO} \circ \lambda) \circ \tilde{s}_\alpha = \pi_S \circ \tilde{s}_\alpha = id$. THUS, THEY DETERMINE TRIVIALIZATIONS OF $F_{SO}(M)$ AND WE CLAIM THAT THE CORRESPONDING TRANSITION MAPS $g_{\alpha\beta}$ ARE GIVEN BY $g_{\alpha\beta} = SPIN \circ \tilde{g}_{\alpha\beta}$. THIS FOLLOWS FROM

$$\begin{aligned} s_\beta(x) = \lambda(\tilde{s}_\beta(x)) &= \lambda(\tilde{s}_\alpha(x) \cdot \tilde{g}_{\alpha\beta}(x)) \\ &= \lambda(\tilde{s}_\alpha(x)) \cdot SPIN(\tilde{g}_{\alpha\beta}(x)) \\ &= s_\alpha(x) \cdot (SPIN \circ \tilde{g}_{\alpha\beta})(x). \end{aligned}$$

FOR THE CONVERSE, SUPPOSE $F_{SO}(M)$ HAS A TRIVIALIZING COVER $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ FOR WHICH THE TRANSITION FUNCTIONS $g_{\alpha\beta}$ LIFT

$$\begin{tikzcd} & SPIN(4) \arrow[d, "SPIN"] \\ U_\alpha \cap U_\beta \arrow[r, "g_{\alpha\beta}"'] \arrow[ur, dashed, "\tilde{g}_{\alpha\beta}"] & SO(4) \end{tikzcd}$$

TO MAPS $\tilde{g}_{\alpha\beta}$ WHICH SATISFY THE COCYCLE CONDITION. THIS CONDITION INSURES THAT $\exists$ UNIQUE (UP TO EQUIVALENCE) BUNDLE

$$SPIN(4) \hookrightarrow S(M) \xrightarrow{\pi_S} M$$

TRIVIALIZED OVER $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ WITH TRANSITION MAPS $\tilde{g}_{\alpha\beta}$.

LET $s_\alpha : U_\alpha \to \pi_{SO}^{-1}(U_\alpha)$ AND $\tilde{s}_\alpha : U_\alpha \to \pi_S^{-1}(U_\alpha)$ BE THE SECTIONS ASSOCIATED WITH THE TRIVIALIZATIONS OF $F_{SO}(M)$ AND $S(M)$, RESPECTIVELY. THUS, THE TRIVIALIZATIONS ARE GIVEN BY

$$\Psi_\alpha : \pi_{SO}^{-1}(U_\alpha) \to U_\alpha \times SO(4)$$

$$\Psi_\alpha(s_\alpha(x)\cdot h) = (x, h)$$

$$\tilde{\Psi}_\alpha : \pi_S^{-1}(U_\alpha) \to U_\alpha \times SPIN(4)$$

$$\tilde{\Psi}_\alpha(\tilde{s}_\alpha(x)\cdot g) = (x, g)$$

NOW DEFINE

$$\lambda_\alpha : \pi_S^{-1}(U_\alpha) \longrightarrow \pi_{SO}^{-1}(U_\alpha)$$

$$\lambda_\alpha = \psi_\alpha^{-1} \circ (id_{U_\alpha} \times SPIN) \circ \tilde{\psi}_\alpha$$

THEN

$$\begin{aligned} \lambda_\alpha(\tilde{s}_\alpha(x)\cdot g) &= \psi_\alpha^{-1} \circ (id_{U_\alpha} \times SPIN)(x, g) \\ &= \psi_\alpha^{-1}(x, SPIN(g)) \\ &= s_\alpha(x)\cdot SPIN(g) \end{aligned}$$

SO

$$\pi_{SO} \circ \lambda_\alpha = \pi_S$$

AND $\lambda_\alpha(p\cdot g) = p\cdot SPIN(g)$. MOREOVER, λ_α AND λ_β AGREE ON $U_\alpha \cap U_\beta$ WHENEVER THIS IS NONEMPTY SO THESE MAPS DETERMINE $\lambda : S(M) \to F_{SO}(M)$ WITH THE REQUIRED PROPERTIES. $\square$

TRANSLATED INTO THE LANGUAGE OF ČECH COHOMOLOGY WITH $\mathbb{Z}_2$ COEFFICIENTS THIS BECOMES

> AN ORIENTED RIEMANNIAN 4-MANIFOLD M ADMITS A SPIN STRUCTURE IF AND ONLY IF ITS 2ND STIEFEL-WHITNEY CLASS $w_2(M) \in \check{H}^2(M;\mathbb{Z}_2)$ IS TRIVIAL.

(SEE PAGES 388-404 OF REFERENCE [41] FOR THE DETAILS)

IN EXACTLY THE SAME WAY ONE SHOWS THAT M ADMITS A SPIN^c STRUCTURE

$$SPIN^c(4) \hookrightarrow S^c(M) \xrightarrow{\pi_{S^c}} M$$

$$\Lambda : S^c(M) \to F_{SO}(M)$$

IF AND ONLY IF THE TRANSITION MAPS $g_{\alpha\beta}$ FOR SOME TRIVIALIZING COVER FOR $F_{SO}(M)$ LIFT

$$\begin{tikzcd} & SPIN^c(4) \arrow[d, "\pi"] \\ U_\alpha \cap U_\beta \arrow[r, "g_{\alpha\beta}"'] \arrow[ur, dotted, "\tilde{\tilde{g}}_{\alpha\beta}"] & SO(4) \end{tikzcd}$$

TO MAPS SATISFYING THE COCYCLE CONDITION.

SUCH A LIFTING, TOGETHER WITH THE MAP

$$\delta : SPIN^c(4) \longrightarrow U(1)$$

$$\delta(\xi) = \delta\begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} = \delta\begin{pmatrix} e^{\theta i}U_1 & 0 \\ 0 & e^{\theta i}U_2 \end{pmatrix}$$
$$= \det U_+ = \det U_- = e^{2\theta i}$$

GIVES THE PRINCIPAL $U(1)$-BUNDLE, AND CORRESPONDING HERMITIAN COMPLEX LINE BUNDLE, WITH TRANSITION MAPS

$$\delta \circ \tilde{\tilde{g}}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$$

NOTE: IF THE SPIN^c STRUCTURE IS DENOTED $\mathcal{L}$, THEN THESE ARE $L^0(\mathcal{L})$ AND $L(\mathcal{L})$.

CAN SHOW THAT $w_2(M)$ IS THE MOD 2 REDUCTION OF THE 1ST CHERN CLASS OF THIS $U(1)$-BUNDLE. MOREOVER,

SPIN STRUCTURE + HERMITIAN LINE BUNDLE WHOSE 1ST CHERN CLASS REDUCES MOD 2 TO $w_2(M)$

$\Longrightarrow$ SPIN^c STRUCTURE

$$\begin{tikzcd} & \mathrm{SPIN}(4) \arrow[d, "\mathrm{SPIN}"] \\ U_\alpha \cap U_\beta \arrow[ur, dotted, "\tilde{g}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & SO(4) \end{tikzcd} \quad + \quad U_\alpha \cap U_\beta \xrightarrow[h_{\alpha\beta}]{} U(1)$$

$$\Longrightarrow \quad U_\alpha \cap U_\beta \xrightarrow[\tilde{\tilde{g}}_{\alpha\beta} = h_{\alpha\beta}\tilde{g}_{\alpha\beta}]{\hspace{3cm}} \mathrm{SPIN}^c(4)$$

NOW LET'S SEE WHAT HAPPENS IF

$$\begin{array}{c} \text{SPIN STRUCTURE} \\ + \\ \text{LINE BUNDLE} \end{array} \Longrightarrow \text{SPIN}^c \text{ STRUCTURE} \Longrightarrow \text{LINE BUNDLE}$$

$$\begin{array}{c}\tilde{g}_{\alpha\beta} \\ + \\ h_{\alpha\beta}\end{array} \Rightarrow \tilde{\tilde{g}}_{\alpha\beta} = h_{\alpha\beta}\,\tilde{g}_{\alpha\beta} \Rightarrow \delta \circ \tilde{\tilde{g}}_{\alpha\beta} = \delta \circ (h_{\alpha\beta}\,\tilde{g}_{\alpha\beta}) = h^2_{\alpha\beta}$$

WHICH ARE THE TRANSITION MAPS FOR THE (TENSOR) SQUARE OF THE ORIGINAL LINE BUNDLE.

IN THIS CASE THE DETERMINANT LINE BUNDLE L FOR THE RESULTING SPIN^c STRUCTURE HAS A (TENSOR) SQUARE ROOT (THE HERMITIAN LINE BUNDLE SELECTED TO SUPPLEMENT THE SPIN STRUCTURE) WHICH ONE CAN WRITE $L^{\frac{1}{2}}$.

<u>NOTE</u> : THE SPIN STRUCTURE $\mathrm{SPIN}(4) \hookrightarrow S(M) \xrightarrow{\pi_S} M$ AND THE REPRESENTATIONS $\Delta^{\pm}_{\mathbb{C}} : \mathrm{SPIN}(4) \longrightarrow SU(S^{\pm}_{\mathbb{C}})$ GIVE ASSOCIATED POSITIVE AND NEGATIVE (REAL) SPINOR BUNDLES WHICH ONE OFTEN SEES DENOTED $W^{\pm}$. THE CORRESPONDING (COMPLEX) SPINOR BUNDLES ASSOCIATED WITH THE SPIN^c STRUCTURE CONSTRUCTED FROM $L^{\frac{1}{2}}$ AND THE SPIN STRUCTURE ARE THEN

$$W^{\pm} \otimes L^{\frac{1}{2}}$$

REGRETTABLY, ONE OFTEN SEES $\mathcal{S}^{\pm}$ WRITTEN THIS WAY EVEN WHEN THE SPIN^c STRUCTURE DOES NOT COME FROM A SPIN STRUCTURE (SO THAT W^+, W^- AND $L^{\frac{1}{2}}$ DO NOT EXIST).

2. DIRAC OPERATOR INDEPENDENT OF ORIENTED, ORTHONORMAL FRAME FIELD

RECALL : ON THE SPIN^c-BUNDLE $SPIN^c(4) \hookrightarrow S^c(M) \to M$ WE HAVE A SPIN^c- CONNECTION ω_A WHICH GIVES A COVARIANT DERIVATIVE

$$\nabla = \nabla_A \ : \Gamma(S(\mathcal{L})) \to \Omega^1(M) \otimes \Gamma(S(\mathcal{L}))$$

ON SPINOR FIELDS. CHOOSE A LOCAL, ORIENTED, ORTHONORMAL FRAME FIELD $\{E_1, E_2, E_3, E_4\}$ ON M. DEFINE

$$\tilde{\not{D}}_A \Psi = \sum_{i=1}^{4} E_i \cdot \nabla_A \Psi(E_i)$$

NOW LET $\{\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4\}$ BE ANOTHER ORIENTED, ORTHONORMAL FRAME FIELD (W.L.O.G., ON THE SAME OPEN SET IN M AS $\{E_1, E_2, E_3, E_4\}$). POINTWISE,

$$\hat{E}_i = \sum_{j=1}^{4} B_{ij} E_j ,$$

WHERE $(B_{ij}) \in SO(4)$. THEN, FOR ANY SECTION $\Delta \in \Gamma(S(\mathcal{L}))$

$$\sum_{i=1}^{4} \hat{E}_i \cdot (\nabla_A \Psi(\hat{E}_i))(\Delta) = \sum_{i=1}^{4} \Big(\sum_{j=1} B_{ij} E_j \Big) \cdot \Big(\Big(\sum_{k=1} B_{ik} \nabla_A \Psi(E_k) \Big)(\Delta) \Big)$$

SINCE $\nabla_A \Psi(\cdot)$ IS LINEAR

$$= \sum_{i,j,k=1}^{4} B_{ij} B_{ik} \, E_j \cdot \nabla_A \Psi(E_k)(\Delta)$$

$$= \sum_{j,k=1}^{4} \delta_{jk} \, E_j \cdot \nabla_A \Psi(E_k)(\Delta)$$

$$= \sum_{j=1}^{4} E_j \cdot \nabla_A \Psi(E_j)(\Delta) \quad \text{AS REQUIRED.}$$

3. SEIBERG-WITTEN GAUGE GROUP

M = COMPACT, CONNECTED, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD

g = RIEMANNIAN METRIC ON M

$$SO(4) \hookrightarrow F_{SO}(M) \xrightarrow{\pi_{SO}} M$$

SPIN^c STRUCTURE $\mathcal{L}$:

$$\begin{tikzcd} SPIN^c(4) \arrow[r, hook] & S^c(M) \arrow[r, "\pi_{S^c}"] \arrow[d, "\Lambda"] & M \\ SO(4) \arrow[r, hook] & F_{SO}(M) \arrow[r, "\pi_{SO}"'] & M \end{tikzcd}$$

A : CONNECTION ON DETERMINANT LINE BUNDLE $U(1) \hookrightarrow L^0(\mathcal{L}) \to M$

$\psi \in \Gamma(S^+(\mathcal{L}))$ A POSITIVE SPINOR FIELD

<u>RECALL</u> : $S^c(M)$ DOUBLE COVERS THE FIBER PRODUCT $F_{SO}(M) \dot{\times} L^0(\mathcal{L})$ VIA THE MAP SPIN^c, GIVEN LOCALLY BY

$$(x,\xi) = (x, e^{\theta i}\mu) \to ((x,x), (\pi(\xi), \delta(\xi)))$$

$$= ((x,x), (SPIN(\mu), e^{2\theta i}))$$

LET Pr_F AND Pr_{L^0} BE THE PROJECTIONS OF $F_{SO}(M) \times L^0(\mathcal{L})$ ONTO $F_{SO}(M)$ AND $L^0(\mathcal{L})$, RESPECTIVELY.

THEN WE HAVE

$$\begin{array}{ccccc} Spin^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \big\downarrow {\scriptstyle Pr_F \circ spin^c} & & \\ SO(4) & \hookrightarrow & F_{SO}(M) & \longrightarrow & M \end{array} \qquad \begin{array}{ccccc} Spin^c(4) & \hookrightarrow & S^c(M) & \longrightarrow & M \\ & & \big\downarrow {\scriptstyle Pr_{L^0} \circ spin^c} & & \\ U(1) & \hookrightarrow & L^0(\mathcal{L}) & \longrightarrow & M \end{array}$$

AN AUTOMORPHISM $\sigma : S^c(M) \longrightarrow S^c(M)$ (DIFFEOMORPHISM SATISFYING $\pi_{S^c} \circ \sigma = \pi_{S^c}$ AND $\sigma(p \cdot \xi) = \sigma(p) \cdot \xi$) IS SAID TO <u>COVER THE IDENTITY ON</u> $F_{SO}(M)$ IF

$$Pr_F \circ spin^c \circ \sigma = Pr_F \circ spin^c$$

$\mathcal{G}(\mathcal{L})$ = GROUP (UNDER COMPOSITION) OF ALL SUCH.

<u>EXAMPLES</u>: FOR ANY SMOOTH MAP $\gamma \in C^\infty(M, U(1))$ OF M TO $U(1) \hookrightarrow Spin^c(4)$ DEFINE

$$\sigma_\gamma : S^c(M) \longrightarrow S^c(M)$$

$$\sigma_\gamma(p) = p \cdot \gamma(\pi_{S^c}(p))$$

σ_γ IS A DIFFEOMORPHISM AND (SINCE $U(1) \subseteq Z(Spin^c(4))$)

$$\begin{aligned} \sigma_\gamma(p \cdot \xi) &= (p \cdot \xi) \cdot \gamma(\pi_{S^c}(p \cdot \xi)) \\ &= (p \cdot \xi) \cdot \gamma(\pi_{S^c}(p)) \\ &= p \cdot (\xi \, \gamma(\pi_{S^c}(p))) \\ &= p \cdot (\gamma(\pi_{S^c}(p)) \, \xi) \\ &= (p \cdot \gamma(\pi_{S^c}(p))) \cdot \xi = \sigma_\gamma(p) \cdot \xi \end{aligned}$$

SO σ_γ IS AN AUTOMORPHISM. IT COVERS THE IDENTITY ON $F_{SO}(M)$ BECAUSE, LOCALLY, $Pr_F \circ SPIN^c$ IS GIVEN BY

$$p = (x, \xi) = (x, e^{\theta i}\mu) \longrightarrow (x, SPIN(\mu))$$

SO p AND $p \cdot e^{\phi i}$ ALWAYS HAVE THE SAME IMAGE.

IN FACT, EVERY ELEMENT OF $\mathcal{G}(\mathcal{L})$ IS σ_γ FOR SOME $\gamma \in C^\infty(M, U(1))$

PROOF: FIRST OBSERVE THAT TWO ELEMENTS OF $S^c(M)$ WITH THE SAME IMAGE UNDER $Pr_F \circ SPIN^c$ CAN DIFFER ONLY BY THE ACTION OF SOMETHING IN $U(1)$ (IF $p_1 = (x_1, \xi_1)$ AND $p_2 = (x_2, \xi_2)$ HAVE THE SAME IMAGE, THEN $x_1 = x_2$ AND $SPIN(\mu_1) = SPIN(\mu_2)$ SO $\mu_1 = \pm\mu_2$ AND

$$\begin{aligned} \xi_1 = e^{\theta_1 i}\mu_1 &= (\pm e^{\theta_1 i})\mu_2 \\ &= (\pm e^{(\theta_1 - \theta_2)i})e^{\theta_2 i}\mu_2 \\ &= \xi_2 (\pm e^{(\theta_1-\theta_2)i}) \end{aligned}$$

SO $p_1 = p_2 \cdot (e^{(\theta_1-\theta_2)i})$ OR $p_1 = p_2 \cdot (e^{(\theta_1 - \theta_2 + \pi)i})$).

THUS, IF AN AUTOMORPHISM $\sigma : S^c(M) \longrightarrow S^c(M)$ COVERS THE IDENTITY ON $F_{SO}(M)$, THEN

$$\sigma(p) = p \cdot (\text{SOMETHING IN } U(1))$$

$\forall p \in S^c(M)$. WE CLAIM THAT THIS "SOMETHING" MUST BE THE

SAME FOR ALL POINTS IN THE SAME FIBER OF π_{S^c}. INDEED, $\pi_{S^c}(p_1) = \pi_{S^c}(p_2)$ IMPLIES $p_2 = p_1 \cdot \xi$ FOR SOME $\xi \in SPIN^c(4)$ AND IF $\sigma(p_1) = p_1 \cdot e^{\phi i}$, THEN

$$\begin{aligned} \sigma(p_2) &= \sigma(p_1 \cdot \xi) = \sigma(p_1) \cdot \xi \\ &= (p_1 \cdot e^{\phi i}) \cdot \xi = (p_1 \cdot \xi) \cdot e^{\phi i} \\ &= p_2 \cdot e^{\phi i} \end{aligned}$$

AS WELL. THUS, $\sigma(p) = p \cdot \gamma(\pi_{S^c}(p))$ FOR SOME $\gamma \in C^\infty(M, U(1))$ SO $\sigma = \sigma_\gamma$. $\square$

THUS,

$$\boxed{\mathcal{G}(\mathcal{L}) \cong C^\infty(M, U(1))}$$

WHERE THE GROUP OPERATION ON $C^\infty(M, U(1))$ IS POINTWISE MULTIPLICATION IN $U(1)$.

WE WILL USE WHICHEVER VIEW OF THE GAUGE GROUP $\mathcal{G}(\mathcal{L})$ IS MOST CONVENIENT IN ANY GIVEN SITUATION.

NOW WE DESCRIBE THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON (A, ψ):

<u>THE SPINOR FIELD</u> $\psi \in \Gamma(S^+(\mathcal{L}))$:

FOR THIS WE IDENTIFY ψ WITH AN EQUIVARIANT $S^+_{\mathbb{C}}$-VALUED

MAP ON $S^c(M)$ $(\psi(p\cdot\xi) = \xi^{-1}\cdot\psi(p))$ AND DEFINE THE ACTION OF σ_γ ON ψ BY PULLBACK:

$$\psi\cdot\sigma_\gamma = \psi\cdot\gamma = \sigma_\gamma^{*}\psi = \psi\circ\sigma_\gamma .$$

THUS, AT EACH $p\in S^c(M)$,

$$(\psi\cdot\sigma_\gamma)(p) = \psi(\sigma_\gamma(p)) = \psi(p\cdot\gamma(\pi_{S^c}(p)))$$

$$= (\gamma(\pi_{S^c}(p)))^{-1}\cdot\psi(p)$$

THEN IF WE THINK OF ψ AGAIN AS A SECTION OF $\mathcal{S}^+(\mathcal{L})$,

$$\psi\cdot\gamma = \gamma^{-1}\psi$$

<u>THE</u> <u>CONNECTION</u> A ON $U(1)\hookrightarrow L^0(\mathcal{L})\longrightarrow M$:

NOTE THAT THE AUTOMORPHISM σ_γ OF $S^c(M)$ INDUCES AN AUTOMORPHISM σ_γ' OF $L^0(\mathcal{L})$:

$$\begin{array}{ccc} S^c(M) & \xrightarrow{\sigma_\gamma} & S^c(M) \\ {\scriptstyle Pr_{L^0}\circ SPIN^c}\downarrow & & \downarrow{\scriptstyle Pr_{L^0}\circ SPIN^c} \\ L^0(\mathcal{L}) & \underset{\sigma_\gamma'}{\dashrightarrow} & L^0(\mathcal{L}) \end{array}$$

$$\sigma_\gamma'\circ Pr_{L^0}\circ SPIN^c = Pr_{L^0}\circ SPIN^c\circ\sigma_\gamma$$

(WE WILL WRITE OUT AN EXPLICIT LOCAL EXPRESSION FOR σ_γ' IN A MOMENT)

NOW DEFINE THE ACTION OF σ_γ ON A BY

$$A \cdot \sigma_\gamma = A \cdot \gamma = (\sigma_\gamma')^* A$$

WE WILL NEED A LOCAL EXPRESSION FOR COMPUTING THIS ACTION SO LET s BE A LOCAL SECTION OF $L^\circ(\mathcal{L})$ AND WRITE

$$\mathcal{A} = s^* A$$

FOR THE CORRESPONDING LOCAL GAUGE POTENTIAL. ALSO DEFINE

$$\mathcal{A} \cdot \gamma = s^*(A \cdot \gamma) = s^*((\sigma_\gamma')^* A) = (\sigma_\gamma' \circ s)^* A$$

SINCE $\Pr_{L^\circ} \circ \mathrm{SPIN}^c$ IS LOCALLY GIVEN BY

$$(x, \xi) = (x, e^{\theta i} u) \longrightarrow (x, \delta(\xi)) = (x, e^{2\theta i})$$

IT SATISFIES

$$(\Pr_{L^\circ} \circ \mathrm{SPIN}^c)(p \cdot \xi_0) = ((\Pr_{L^\circ} \circ \mathrm{SPIN}^c)(p)) \cdot \delta(\xi_0)$$

SO

$$\begin{aligned} \sigma_\gamma'((\Pr_{L^\circ} \circ \mathrm{SPIN}^c)(p)) &= (\Pr_{L^\circ} \circ \mathrm{SPIN}^c)(\sigma_\gamma(p)) \\ &= (\Pr_{L^\circ} \circ \mathrm{SPIN}^c)(p \cdot \gamma(\pi_{S^c}(p))) \\ &= ((\Pr_{L^\circ} \circ \mathrm{SPIN}^c)(p)) \cdot \delta(\gamma(\pi_{S^c}(p))) \end{aligned}$$

SINCE $Pr_{L^0} \circ SPIN^c$ MAPS ONTO $L^0(\mathcal{L})$ WE CAN WRITE THIS AS

$$\sigma_\gamma'(x) = x \cdot \delta(\gamma(\pi_{L^0}(x))) = x \cdot (\gamma(\pi_{L^0}(x)))^2$$

$\forall\ x \in L^0(\mathcal{L})$.

IN PARTICULAR,

$$\sigma_\gamma' \circ s = s \cdot \gamma^2$$

AT EVERY POINT OF M.

<u>NOTE</u> : s IS A SECTION OF $L^0(\mathcal{L})$ AND SO IS $\sigma_\gamma' \circ s$. THE LAST EQUALITY IDENTIFIES THE TRANSITION MAP THAT RELATES THE TWO SECTIONS AS γ^2.

SINCE $$a = s^* A \Rightarrow a \cdot \gamma = (\sigma_\gamma' \circ s)^* A$$

WE CONCLUDE THAT

$$a \cdot \gamma = (\gamma^2)^{-1} a (\gamma^2) + (\gamma^2)^{-1} d(\gamma^2)$$

$$= a + (\gamma^2)^{-1}(2\gamma d\gamma)$$

$$a \cdot \gamma = a + 2\gamma^{-1} d\gamma$$

WHICH IS OUR LOCAL EXPRESSION FOR THE ACTION OF THE GAUGE GROUP ON THE CONNECTION.

APPLYING $\pi_{L^0}^*$ TO BOTH SIDES GIVES

$$A\cdot\gamma = A + \pi_{L^0}^*(2\gamma^{-1}d\gamma)$$

NOW WE HAVE THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON SEIBERG-WITTEN CONFIGURATIONS:

$$(A,\psi)\cdot\sigma_\gamma = (A,\psi)\cdot\gamma = ((\sigma_\gamma')^*A,\ \sigma_\gamma^*\psi)$$

OR, LOCALLY ON M,

$$(\mathcal{A},\psi)\cdot\gamma = (\mathcal{A}+2\gamma^{-1}d\gamma,\ \gamma^{-1}\psi)$$

BEFORE PROVING OUR MAJOR RESULT ON SOLUTIONS (A,ψ) TO (SW) (THAT THE ACTION OF $\mathcal{G}(\mathcal{L})$ CARRIES SOLUTIONS TO SOLUTIONS) WE NEED TO OBSERVE THAT THE SPIN^c CONNECTION CORRESPONDING TO $A\cdot\gamma$ IS THE PULLBACK BY σ_γ OF THAT CORRESPONDING TO A:

$$\omega_{A\cdot\gamma} = \sigma_\gamma^*\,\omega_A$$

HERE'S THE PROOF: BY DEFINITION,

$$\sigma_\gamma^*\,\omega_A = (\mathrm{SPIN}^c \circ \sigma_\gamma)^*(Pr_F^*\,\omega_{LC} + Pr_{L^0}^*A).$$

MOREOVER,

$$\omega_{A\cdot\gamma} = (\mathrm{SPIN}^c)^*(Pr_F^*\,\omega_{LC} + Pr_{L^0}^*((\sigma_\gamma')^*A))$$

$$= (Pr_F \circ \mathrm{SPIN}^c)^*\omega_{LC} + (\sigma_\gamma' \circ Pr_{L^0} \circ \mathrm{SPIN}^c)^*A$$

$$= \underbrace{(Pr_F \circ \mathrm{SPIN}^c \circ \sigma_\gamma)^*\omega_{LC}}_{\substack{\sigma_\gamma \text{ COVERS} \\ \text{THE IDENTITY} \\ \text{ON } F_{SO}(M)}} + \underbrace{(Pr_{L^0} \circ \mathrm{SPIN}^c \circ \sigma_\gamma)^*A}_{\substack{\text{DEFINITION OF} \\ \sigma_\gamma'}}$$

$$= (\mathrm{SPIN}^c \circ \sigma_\gamma)^*(Pr_F^*\,\omega_{LC} + Pr_{L^0}^*A)$$

$$= \sigma_\gamma^*\,\omega_A$$

AS REQUIRED.

ANOTHER COMPUTATION OF $\omega_{A\cdot\gamma}$ (USING $A\cdot\gamma = A + \pi_{L^0}^*(2\gamma^{-1}d\gamma)$) GIVES

$$\omega_{A\cdot\gamma} = (\mathrm{SPIN}^c)^*(Pr_F^*\,\omega_{LC} + Pr_{L^0}^*(A + \pi_{L^0}^*(2\gamma^{-1}d\gamma)))$$

$$= \omega_A + (\mathrm{SPIN}^c)^*(Pr_{L^0}^*(\pi_{L^0}^*(2\gamma^{-1}d\gamma)))$$

$$= \omega_A + (\pi_{L^0} \circ Pr_{L^0} \circ \mathrm{SPIN}^c)^*(2\gamma^{-1}d\gamma)$$

$$= \omega_A + \pi_{S^c}^*(2\gamma^{-1}d\gamma)$$

NOW FOR OUR MAJOR RESULT :

<u>THEOREM</u>: THE ACTION OF $\mathcal{G}(\mathcal{L})$ ON (A, ψ) CARRIES SOLUTIONS TO (SW) ONTO OTHER SOLUTIONS TO (SW). MORE PRECISELY, IF (A, ψ) SATISFIES

$$\begin{cases} \not{D}_A \psi = 0 \\ F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \end{cases}$$

THEN, FOR ANY $\sigma_\gamma \in \mathcal{G}(\mathcal{L})$, $(A, \psi)\cdot\gamma = (A\cdot\gamma, \psi\cdot\gamma)$ SATISFIES

$$\begin{cases} \not{D}_{A\cdot\gamma}(\psi\cdot\gamma) = 0 \\ F_{A\cdot\gamma}^+ = \sigma^+\big(((\psi\cdot\gamma)\otimes(\psi\cdot\gamma)^*)_0\big) \end{cases}$$

<u>PROOF</u>: THE CURVATURE EQUATION IS EASY. WE MAY PROVE THE EQUALITY LOCALLY SO LET s BE A SECTION OF $L^0(\mathcal{L})$ AND $\mathcal{A} = s^* A$. WE HAVE SEEN THAT $\mathcal{A}\cdot\gamma = (\sigma_\gamma' \circ s)^* A$ SO $\mathcal{A}$ AND $\mathcal{A}\cdot\gamma$ ARE GAUGE POTENTIALS FOR THE SAME CONNECTION ON $L^0(\mathcal{L})$. WE HAVE ALSO SEEN THAT THE TRANSITION MAP RELATING THE TWO SECTIONS IS γ^2. CONSEQUENTLY, THE (SELF-DUAL PARTS OF THE) CURVATURES ARE RELATED BY

$$F_{\mathcal{A}\cdot\gamma}^+ = \gamma^2 F_{\mathcal{A}}^+ (\gamma^2)^{-1} = F_{\mathcal{A}}^+$$

(BECAUSE $U(1)$ IS ABELIAN). SIMILARLY, COMMUTATIVITY OF $U(1)$ GIVES

$$(\psi\cdot\gamma)\otimes(\psi\cdot\gamma)^{*} = (\gamma^{-1}\psi)\otimes(\gamma^{-1}\psi)^{*}$$
$$= (\gamma^{-1}\psi)\otimes(\gamma\psi^{*})$$
$$= (\gamma^{-1}\gamma)\,\psi\otimes\psi^{*}$$
$$= \psi\otimes\psi^{*}.$$

THUS, $F_A^+ = \sigma^+((\psi\otimes\psi^*)_0)$ IMPLIES $F_{A\cdot\gamma}^+ = \sigma^+(((\psi\cdot\gamma)\otimes(\psi\cdot\gamma)^*)_0)$.

TO VERIFY THE ANALOGOUS STATEMENT FOR THE DIRAC EQUATION IT WILL SURELY BE ENOUGH TO SHOW THAT

$$\not{D}_{A\cdot\gamma}(\psi\cdot\gamma) = (\not{D}_A\psi)\cdot\gamma\ .$$

FOR THIS WE IDENTIFY ψ WITH AN EQUIVARIANT $S_{\mathbb{C}}^+$ - VALUED MAP ON $S^c(M)$ AND COMPARE THE COVARIANT EXTERIOR DERIVATIVES $d_A\psi$ AND $d_{A\cdot\gamma}(\psi\cdot\gamma)$. STANDARD FORMULAS FOR SUCH DERIVATIVES (E.G., THEOREM 3.1.5 OF REFERENCE [9]) GIVE

$$d_A\psi = d\psi + \tfrac{1}{2}\omega_A\psi$$

AND

$$d_{A\cdot\gamma}(\psi\cdot\gamma) = d(\psi\cdot\gamma) + \tfrac{1}{2}\omega_{A\cdot\gamma}(\psi\cdot\gamma)$$

WHERE, E.G., ω_A TAKES VALUES IN $\dot{spin}^c(4)$, IDENTIFIED WITH A SUBSET OF $Cl(4)\otimes\mathbb{C}$, AND SO $\omega_A\psi$ IS A MATRIX PRODUCT.

NOW WE COMPUTE

$$
\begin{aligned}
d_{A\cdot\gamma}(\psi\cdot\gamma) &= d_{A\cdot\gamma}\left((\gamma\circ\pi_{sc})^{-1}\psi\right) \\
&= d\left((\gamma\circ\pi_{sc})^{-1}\psi\right) + \tfrac{1}{2}\,\omega_{A\cdot\gamma}\left((\gamma\circ\pi_{sc})^{-1}\psi\right) \\
&= (\gamma\circ\pi_{sc})^{-1}d\psi - (\gamma\circ\pi_{sc})^{-2}d(\gamma\circ\pi_{sc})\psi \\
&\quad + \tfrac{1}{2}\left(\omega_A + 2(\gamma\circ\pi_{sc})^{-1}d(\gamma\circ\pi_{sc})\right)\left((\gamma\circ\pi_{sc})^{-1}\psi\right) \\
&= (\gamma\circ\pi_{sc})^{-1}\left(d\psi + \tfrac{1}{2}\,\omega_A\psi\right) - (\gamma\circ\pi_{sc})^{-2}d(\gamma\circ\pi_{sc})\psi \\
&\qquad + (\gamma\circ\pi_{sc})^{-2}d(\gamma\circ\pi_{sc})\psi \\
&= (\gamma\circ\pi_{sc})^{-1}\left(d_A\psi\right)
\end{aligned}
$$

THUS,

$$d_{A\cdot\gamma}(\psi\cdot\gamma) = (\gamma\circ\pi_{sc})^{-1}(d_A\psi)$$

AND SO

$$\nabla_{A\cdot\gamma}(\psi\cdot\gamma) = (\gamma\circ\pi_{sc})^{-1}\nabla_A\psi \; .$$

NOW, FOR THE DIRAC OPERATORS WE HAVE

$$
\begin{aligned}
\not{\partial}_{A\cdot\gamma}(\psi\cdot\gamma) &= \sum_{i=1}^{4} E_i \cdot \nabla_{A\cdot\gamma}\psi(E_i) \\
&= \sum_{i=1}^{4} E_i \cdot (\gamma\circ\pi_{sc})^{-1}\nabla_A\psi(E_i) \\
&= (\gamma\circ\pi_{sc})^{-1}\not{\partial}_A\psi = (\not{\partial}_A\psi)\cdot\gamma
\end{aligned}
$$

AS REQUIRED. ∎

BEFORE LEAVING THE SUBJECT OF THE GAUGE ACTION ON (A, ψ) LET US DETERMINE THE REDUCIBLES :

<u>THEOREM</u> : (A, ψ) IS LEFT FIXED BY SOME NON-IDENTITY ELEMENT σ_γ OF $\mathcal{G}(\mathcal{L})$ IF AND ONLY IF $\psi \equiv 0$ AND, IN THIS CASE, $\gamma : M \to U(1)$ MUST BE A CONSTANT MAP.

<u>PROOF</u> : SUPPOSE $(A, \psi) \cdot \sigma_\gamma = (A, \psi)$. THEN,

$$(A + 2\gamma^{-1} d\gamma \, , \, \gamma^{-1} \psi) = (A, \psi)$$

SO

$$2\gamma^{-1} d\gamma = 0 \quad \text{AND} \quad \gamma^{-1}\psi = \psi$$

SINCE $\gamma \neq 1$, THE SECOND OF THESE IMPLIES $\psi \equiv 0$. THE FIRST GIVES $d\gamma = 0$ AND, SINCE M IS CONNECTED, γ IS CONSTANT. CONVERSELY, IF $\psi \equiv 0$, THEN (A, ψ) IS LEFT FIXED BY ANY σ_γ WITH γ CONSTANT. $\square$

4. A PRIORI BOUNDS (AND COMPACTNESS)

HERE WE WILL SKETCH THE PROOF THAT ANY SOLUTION (A, ψ) TO

$$(SW) \qquad \begin{aligned} \not{\partial}_A \psi &= 0 \\ F_A^+ &= \sigma^+((\psi \otimes \psi^*)_0) \end{aligned}$$

EITHER HAS $\psi \equiv 0$ ((A, ψ) REDUCIBLE) OR

$$\|\psi(x)\|^2 \leq \mathcal{K}(M) := \max\left\{-\tfrac{1}{2}\mathcal{K}(x) : x \in M\right\}$$

WHERE $\mathcal{K}(x)$ IS THE SCALAR CURVATURE OF M AT x.

> NOTE : IN PARTICULAR, IF THE METRIC HAS POSITIVE SCALAR CURVATURE, THEN ALL SOLUTIONS (A, ψ) HAVE $\psi \equiv 0$.

WE WRITE THE SECOND (SW) EQUATION AS

$$\rho^+(F_A) = (\psi \otimes \psi^*)_0$$

(ADDENDUM 14, PAGE 25).

WE WILL ALSO APPEAL TO THE FAMOUS WEITZENBÖCK FORMULA FROM DIFFERENTIAL GEOMETRY WHICH, IN OUR PRESENT CIRCUMSTANCES, ASSERTS THAT

(1) $\quad \not{D}_A^* \circ \not{D}_A \psi = \nabla_A^* \circ \nabla_A \psi + \frac{1}{4} \kappa \psi + \rho^+(F_A)\psi$

WHERE THE * INDICATES FORMAL ADJOINT AND κ IS THE SCALAR CURVATURE OF M.

BECAUSE (A, ψ) SATISFIES (SW) THIS REDUCES TO

(2) $\quad 0 = \nabla_A^* \circ \nabla_A \psi + \frac{1}{4} \kappa \psi + (\psi \otimes \psi^*)\psi .$

TAKE THE POINTWISE INNER PRODUCT OF BOTH SIDES OF (2) WITH ψ TO GET

(3) $\quad 0 = \langle \nabla_A^* \circ \nabla_A \psi(x), \psi(x) \rangle + \frac{1}{4} \kappa(x) \|\psi(x)\|^2 + \frac{1}{2} \|\psi(x)\|^4$

(FOR THE LAST TERM USE (44) OF ADDENDUM 14 TO COMPUTE $(\psi \otimes \psi^*)_0 \psi$).

NOW, $\|\psi\|^2$ IS A CONTINUOUS FUNCTION ON THE COMPACT SPACE M SO THERE IS AN $x_0 \in M$ AT WHICH IT ACHIEVES AN ABSOLUTE MAXIMUM VALUE.
WE CLAIM THAT, AT THIS POINT

(4) $\quad \langle \nabla_A^* \circ \nabla_A \psi(x_0), \psi(x_0) \rangle \geq 0$

(NOTE THAT (3) IMPLIES THAT $\langle \nabla_A^* \circ \nabla_A \psi(x), \psi(x) \rangle$ IS REAL FOR ALL x).

THE PROOF OF (4) DEPENDS ON THE FOLLOWING IDENTITY:

(5) $$\Delta_g \|\psi\|^2 = -2\|\nabla_A \psi\|^2 + 2\langle \nabla_A^* \circ \nabla_A \psi, \psi \rangle$$

WHERE $\Delta_g = \delta \circ d$ IS THE SCALAR HODGE LAPLACIAN OF g. THIS CAN BE VERIFIED BY WRITING OUT THE LAPLACIAN IN A LOCAL ORTHONORMAL FRAME FIELD E_1, E_2, E_3, E_4 ON M :

$$\Delta_g \|\psi\|^2 = -\sum_{i=1}^{4} (\partial_i \partial_i \|\psi\|^2 + \operatorname{div}(E_i)\partial_i \|\psi\|^2$$

RECALL : THE (COVARIANT) DIVERGENCE $\operatorname{div} V$ OF A VECTOR FIELD V IS DEFINED TO BE THE TRACE OF THE ENDOMORPHISM $X \to \nabla_X V : TM \to TM$ AND, FOR A LOCAL ORTHONORMAL FRAME

$$\operatorname{div}(E_i) = -\sum_j T^i_{jj}$$

WHERE $T^k_{ij} = g(\nabla_{E_i} E_j, E_k)$.

$$= -2\sum_i (\partial_i \operatorname{Re}\langle \psi, \nabla_i \psi \rangle + \operatorname{div}(E_i)\operatorname{Re}\langle \psi, \nabla_i \psi \rangle)$$

$$= -2\sum_i \|\nabla_i \psi\|^2 - 2\sum_i \operatorname{Re}\langle \psi, \nabla_i \nabla_i \psi + \operatorname{div}(E_i)\nabla_i \psi \rangle$$

$$= -2\|\nabla_A \psi\|^2 + 2\operatorname{Re}\langle \psi, \nabla_A^* \circ \nabla_A \psi \rangle$$

BECAUSE $\nabla_i^* = -\nabla_i - \operatorname{div}(E_i)$

$$= -2\|\nabla_A \psi\|^2 + 2\langle \psi, \nabla_A^* \circ \nabla_A \psi \rangle$$

FOR SOLUTIONS TO (SW) BY (3).

NOW, TO PROVE (4) WE APPLY (5) AT X_0 TO OBTAIN

$$2 \langle \nabla_A^* \circ \nabla_A \psi(x_0) \rangle = \Delta_g \|\psi\|^2(x_0) + 2\|\nabla_A \psi\|^2(x_0).$$

THE SECOND TERM IS OBVIOUSLY NONNEGATIVE. SINCE $\|\psi\|^2$ ACHIEVES A MAXIMUM AT X_0 (AND SINCE THE HODGE LAPLACIAN HAS THAT ANNOYING EXTRA MINUS SIGN) THE SAME IS TRUE OF THE FIRST TERM SO (4) IS PROVED.

NOW EVALUATE (3) AT X_0 AND USE (4) TO CONCLUDE THAT

$$\frac{1}{4} \mathcal{K}(x_0) \|\psi(x_0)\|^2 + \frac{1}{2} \|\psi(x_0)\|^4 \leq 0,$$

THUS,

$$\|\psi(x_0)\|^4 \leq -\frac{1}{2} \mathcal{K}(x_0) \|\psi(x_0)\|^2 .$$

THERE ARE TWO POSSIBILITIES. EITHER $\|\psi(x_0)\| = 0$, IN WHICH CASE $\psi \equiv 0$, OR

$$\|\psi(x_0)\|^2 \leq -\frac{1}{2} \mathcal{K}(x_0)$$

AND CONSEQUENTLY

(6) $$\|\psi(x)\|^2 \leq -\frac{1}{2} \mathcal{K}(x_0)$$

$\forall\ x \in M$.

NOTE: WE ARE LOOKING FOR A UNIFORM BOUND ON $\|\psi(x)\|^2$ FOR ALL SOLUTIONS (A,ψ) TO (SW) AND (6) WILL NOT DO SINCE x_0 GENERALLY DEPENDS ON ψ.

NOW, $-\frac{1}{2}\mathcal{K}(x)$ IS A CONTINUOUS FUNCTION ON THE COMPACT SPACE M SO WE MAY LET

$$\mathcal{K}(M) = \max\left\{-\tfrac{1}{2}\mathcal{K}(x) : x \in M\right\}$$

AND CONCLUDE THAT

$$\|\psi(x)\|^2 \leq \mathcal{K}(M) \qquad \forall x \in M.$$

THUS, FOR ANY METRIC AND ANY SPIN^c STRUCTURE AND ANY SOLUTION (A,ψ) TO (SW), THE SPINOR PART ψ IS UNIFORMLY BOUNDED BY (0 IF THE SOLUTION IS REDUCIBLE AND, OTHERWISE) THE GEOMETRICAL CONSTANT $\mathcal{K}(M)$.

THE SECOND (SW) EQUATION NOW GIVES A UNIFORM BOUND ON THE SELF-DUAL PART OF THE CURVATURE FOR ANY SOLUTION. A BIT MORE WORK THE GIVES A BOUND ON THE ANTI-SELF-DUAL PART OF THE CURVATURE, AND THUS ON THE CURVATURE ITSELF.

THESE UNIFORM BOUNDS ON Ψ AND F_A ARE, BY THEMSELVES, NOT SUFFICIENT TO PROVE THE COMPACTNESS OF THE MODULI SPACE.

> <u>NOTE</u>: THEY ARE, HOWEVER, SUFFICIENT TO PROVE THAT, FOR A GIVEN g, THERE ARE AT MOST FINITELY MANY EQUIVALENCE CLASSES OF SPIN^c STRUCTURES FOR WHICH THE MODULI SPACE IS NONEMPTY AND HAS NONNEGATIVE FORMAL DIMENSION.

FOR THIS ONE MUST ALSO BOUND THE CONNECTION PART A OF A SOLUTION (A, Ψ) "UP TO GAUGE".

AT THIS POINT, HOWEVER, THE "SMOOTH" ARGUMENTS WE HAVE RELIED UPON THUS FAR FAIL US AND ONE MUST PROCEED BY WAY OF A "BOOTSTRAPING" ARGUMENT THROUGH SOBOLEV SPACES OF VARYING INDEX. SINCE THIS USES THE ENTIRE ARSENAL OF SOBOLEV EMBEDDING AND MULTIPLICATION THEOREMS, THE RELLICH THEOREM, ETC., I WILL NOT ATTEMPT A QUICK SYNOPSIS. A REASONABLY CONCISE DISCUSSION OF THE ARGUMENT IS AVAILABLE ON PAGES 80 - 85 OF REFERENCE [36].

ADDENDUM 16 : FINITE ENERGY SEIBERG-WITTEN SOLUTIONS

(SW1) $$\not{D}_A \psi = 0$$

(SW2) $$F_A^+ = \sigma^+((\psi \otimes \psi^*)_0)$$

WE WILL WRITE THEM OUT EXPLICITLY ON $\mathbb{R}^4$ AND PROVE A THEOREM OF WITTEN WHICH ASSERTS THAT ANY SOLUTION (A, ψ) WITH $\psi \in L^2(\mathbb{R}^4)$ MUST ACTUALLY HAVE $\psi \equiv 0$.

THUS, WE LET $M = \mathbb{R}^4$ WITH ITS USUAL RIEMANNIAN METRIC AND ORIENTATION. SINCE $\mathbb{R}^4$ IS CONTRACTIBLE ALL OF THE RELEVANT BUNDLES ARE TRIVIAL AND WE WILL WORK WITH EXPLICIT TRIVIALIZATIONS. THUS, THE ORIENTED, ORTHONORMAL FRAME BUNDLE IS

$$SO(4) \hookrightarrow \mathbb{R}^4 \times SO(4) \longrightarrow \mathbb{R}^4$$

AND THERE IS AN ESSENTIALLY UNIQUE SPIN^c STRUCTURE $\mathcal{L}$

$$\begin{tikzcd} \mathbb{R}^4 \times SPIN^c(4) \arrow[rr, "\Lambda"] \arrow[dr] & & \mathbb{R}^4 \times SO(4) \arrow[dl] \\ & \mathbb{R}^4 & \end{tikzcd}$$

WHERE $\Lambda(x, \xi) = (x, \pi(\xi))$. THE SPINOR BUNDLES ARE THEREFORE ALSO TRIVIAL SO THEIR SECTIONS (I.E., THE SPINOR FIELDS) CAN BE IDENTIFIED WITH GLOBALLY DEFINED FUNCTIONS ON $\mathbb{R}^4$:

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} : \mathbb{R}^4 \longrightarrow S_{\mathbb{C}} = \mathbb{C}^4$$

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ 0 \\ 0 \end{pmatrix} : \mathbb{R}^4 \longrightarrow S_{\mathbb{C}}^{+} \cong \mathbb{C}^2$$

$$\phi = \begin{pmatrix} 0 \\ 0 \\ \psi^3 \\ \psi^4 \end{pmatrix} : \mathbb{R}^4 \longrightarrow S_{\mathbb{C}}^{-} \cong \mathbb{C}^2$$

FOR CONVENIENCE WE WILL OFTEN ABUSE THE NOTATION AND WRITE

(1) $$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

WE WILL USE x^1, x^2, x^3, x^4 FOR THE STANDARD COORDINATES ON $\mathbb{R}^4$ AND WRITE

$$\partial_i = \frac{\partial}{\partial x^i} \quad , \quad i = 1, 2, 3, 4$$

(THESE BEING APPLIED COMPONENTWISE TO SPINOR FIELDS).

THE DETERMINANT LINE BUNDLE OF $\mathcal{L}$ IS LIKEWISE TRIVIAL AS IS THE CORRESPONDING PRINCIPAL $U(1)$-BUNDLE $U(1) \hookrightarrow L^0 \longrightarrow B$:

$$U(1) \hookrightarrow \mathbb{R}^4 \times U(1) \longrightarrow \mathbb{R}^4$$

A $U(1)$-CONNECTION A ON THIS BUNDLE IS THEN UNIQUELY DETERMINED BY A GLOBALLY DEFINED $\mathfrak{u}(1) = \mathrm{Im}\, \mathbb{C}$-VALUED 1-FORM ON $\mathbb{R}^4$:

$$A = A_i \, dx^i$$

$$A_i : \mathbb{R}^4 \longrightarrow \mathrm{Im}\, \mathbb{C} \quad , \quad i = 1, 2, 3, 4.$$

NOW, IN ORTHONORMAL COORDINATES THE COVARIANT DERIVATIVE INDUCED BY THE LEVI-CIVITA CONNECTION IS JUST ORDINARY (COMPONENTWISE) EXTERIOR DIFFERENTIATION (CHRISTOFFEL SYMBOLS ARE ZERO) SO THE COVARIANT DERIVATIVE $\nabla = \nabla_A$ INDUCED BY IT AND THE $U(1)$-CONNECTION A TAKES THE FORM $\nabla = d + A$, I.E.,

$$\nabla = \nabla_i \, dx^i = (\partial_i + A_i) dx^i ,$$

SO THAT

$$\nabla \Psi = (\partial_i \Psi + A_i \Psi) dx^i = \begin{pmatrix} (\partial_i \psi^1 + A_i \psi^1) dx^i \\ (\partial_i \psi^2 + A_i \psi^2) dx^i \\ (\partial_i \psi^3 + A_i \psi^3) dx^i \\ (\partial_i \psi^4 + A_i \psi^4) dx^i \end{pmatrix}$$

WITH $\{e_i\} = \{\partial_i\}$ THE STANDARD ORIENTED, ORTHONORMAL FRAME FIELD ON $\mathbb{R}^4$ WE THEREFORE HAVE $\nabla(\Psi)(e_i) = \partial_i \Psi + A_i \Psi$ AND FOR CONVENIENCE WE WILL WRITE THIS

$$\nabla_i \Psi = (\partial_i + A_i) \Psi = \begin{pmatrix} \partial_i \psi^1 + A_i \psi^1 \\ \partial_i \psi^2 + A_i \psi^2 \\ \partial_i \psi^3 + A_i \psi^2 \\ \partial_i \psi^4 + A_i \psi^4 \end{pmatrix}$$

THE DIRAC OPERATOR $\tilde{\not{D}}_A \Psi = \sum_{i=1}^{4} e_i \cdot \nabla_i \Psi$ REQUIRES THAT WE CLIFFORD MULTIPLY BY THE BASIS ELEMENTS e_i, I.E., MATRIX MULTIPLY BY $E_i = T(e_i) \in Cl(4) \otimes_{\mathbb{R}} \mathbb{C}$. FOR THIS WE WILL WRITE $\Psi = \binom{\psi}{\phi}$ AS IN (1) SO THAT

$$\tilde{\not{D}}_A \Psi = \sum_{i=1}^{4} e_i \cdot \nabla_i \Psi = \sum_{i=1}^{4} T(e_i) \nabla_i \Psi$$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \nabla_1 \psi \\ \nabla_1 \phi \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \nabla_2 \psi \\ \nabla_2 \phi \end{pmatrix} +$$

$$\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \begin{pmatrix} \nabla_3 \psi \\ \nabla_3 \phi \end{pmatrix} + \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} \nabla_4 \psi \\ \nabla_4 \phi \end{pmatrix}$$

$$= \begin{pmatrix} \nabla_1 \phi + I \nabla_2 \phi + J \nabla_3 \phi + K \nabla_4 \phi \\ -\nabla_1 \psi + I \nabla_2 \psi + J \nabla_3 \psi + K \nabla_4 \psi \end{pmatrix}$$

NOTE THAT, AS EXPECTED, $\not{D}_A$ SENDS POSITIVE SPINORS $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ TO NEGATIVE SPINORS AND NEGATIVE SPINORS $\Psi = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$ TO POSITIVE SPINORS.

SINCE (SW1) INVOLVES ONLY THE RESTRICTION OF $\tilde{\not{D}}_A$ TO POSITIVE SPINOR FIELDS IT WILL BE CONVENIET TO DROP THE EXTRANEOUS ZERO AND WRITE

(2) $$\not{D}_A \psi = -\nabla_1 \psi + I \nabla_2 \psi + J \nabla_3 \psi + K \nabla_4 \psi .$$

(BUT KEEP IN MIND THAT THIS IS A NEGATIVE SPINOR FIELD).

<u>REMARK</u>: IN THE MATHEMATICS (AS OPPOSED TO THE PHYSICS) LITERATURE IT IS COMMON TO REFER TO THIS RESTRICTED OPERATOR AS THE "DIRAC OPERATOR".

TO WRITE OUT (SW1) ON $\mathbb{R}^4$ WE NEED ONLY SET THE EXPRESSION IN (2) EQUAL TO ZERO. FOR (SW2) WE WILL USE EXPRESSION (46), ADDENDUM 14, FOR

$\sigma^+((\psi\otimes\psi^*)_0)$ AND THE FOLLOWING LOCAL DESCRIPTION OF F_A^+ :

WRITE $A = A_i dx^i$. THEN $F_A = dA = \sum_{i<j} F_{ij}\, dx^i \wedge dx^j$, WHERE $F_{ij} = \partial_i A_j - \partial_j A_i$, $i, j = 1,2,3,4$. A BASIS FOR THE SELF-DUAL 2-FORMS IS

$$\{dx^1\wedge dx^2 + dx^3\wedge dx^4,\ dx^1\wedge dx^3 + dx^4\wedge dx^2,\ dx^1\wedge dx^4 + dx^2\wedge dx^3\}$$

AND $F_A^+ = \frac{1}{2}(F_A + {}^*F_A)$ IS GIVEN BY

$$F_A^+ = \tfrac{1}{2}(F_{12}+F_{34})(dx^1\wedge dx^2 + dx^3\wedge dx^4) + \tfrac{1}{2}(F_{13}+F_{42})(dx^1\wedge dx^3 + dx^4\wedge dx^2) + \tfrac{1}{2}(F_{14}+F_{23})(dx^1\wedge dx^4 + dx^2\wedge dx^3)$$

THUS, WE OBTAIN

$$\text{(SW1)} \qquad \nabla_1\psi = \nabla_2\psi + \nabla_3\psi + \nabla_4\psi$$

$$\text{(SW2)} \qquad \begin{aligned} F_{12}+F_{34} &= -\tfrac{1}{2}\,\psi^* I\,\psi \\ F_{13}+F_{42} &= -\tfrac{1}{2}\,\psi^* J\,\psi \\ F_{14}+F_{23} &= -\tfrac{1}{2}\,\psi^* K\,\psi \end{aligned}$$

TO CONVEY SOME SENSE OF WHAT KIND OF EQUATIONS THESE ACTUALLY ARE WE WILL WRITE THEM OUT EXPLICITLY IN TERMS OF THE UNKNOWNS A_1, A_2, A_3, A_4 AND ψ^1, ψ^2 :

$$\begin{pmatrix} -(\partial_1+A_1)+i(\partial_2+A_2) & (\partial_3+A_3)+i(\partial_4+A_4) \\ -(\partial_3+A_3)+i(\partial_4+A_4) & -(\partial_1+A_1)-i(\partial_2+A_2) \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(\partial_1 A_2 - \partial_2 A_1) + (\partial_3 A_4 - \partial_4 A_3) = -\tfrac{1}{2} i \left(|\psi^1|^2 - |\psi^2|^2 \right)$$
$$(\partial_1 A_3 - \partial_3 A_1) + (\partial_4 A_2 - \partial_2 A_4) = -\, i \, Im(\bar{\psi}^1 \psi^2)$$
$$(\partial_1 A_4 - \partial_4 A_1) + (\partial_2 A_3 - \partial_3 A_2) = -\, i \, Re(\bar{\psi}^1 \psi^2)$$

NOTICE THAT THE EQUATIONS ARE ONLY RATHER MILDLY NONLINEAR (ON THE RIGHT-HAND SIDE OF (SW2)).

REMARK: IT IS PERHAPS WORTH OBSERVING THAT THESE EQUATIONS DO HAVE NONTRIVIAL SOLUTIONS: LET US TAKE $\psi \equiv 0$ (FOR REASONS WE WILL DISCUSS LATER, A SOLUTION (A, ψ) TO (SW) WITH $\psi \equiv 0$ IS SAID TO BE REDUCIBLE). THEN (SW1) IS, OF COURSE, IDENTICALLY SATISFIED AND (SW2) REDUCES TO THE STATEMENT THAT A IS AN ANTI-SELF-DUAL $U(1)$-CONNECTION. ONE CAN CONSTRUCT SUCH THINGS AS FOLLOWS: LET $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ BE A SMOOTH VECTOR FIELD ON $\mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$. LET $\Delta = -\sum_{i=1}^{3} \frac{\partial^2}{(\partial x^i)^2}$ BE THE USUAL LAPLACIAN ON $\mathbb{R}^3$ AND WRITE $\Delta \vec{F}$ FOR $(\Delta F_1, \Delta F_2, \Delta F_3)$. VECTOR IDENTITIES GIVE

$$\text{CURL}\,(\text{CURL}\,\vec{F}) = -\Delta \vec{F} + \text{GRAD}\,(\text{DIV}\,\vec{F}).$$

NOW CHOOSE $\vec{F}$ TO BE (COMPONENTWISE) HARMONIC WITH NONCONSTANT $\text{GRAD}\,(\text{DIV}\,\vec{F})$ (E.G., $\vec{F}(x^1, x^2, x^3) = (e^{x^1}\cos(x^2), 0, 0)$) AND SET

$$\vec{G} = (G_1, G_2, G_3) = \text{CURL}\,(\text{CURL}\,\vec{F}) = \text{GRAD}\,(\text{DIV}\,\vec{F}).$$

THEN $\vec{G}: \mathbb{R}^3 \to \mathbb{R}^3$ IS NONCONSTANT AND SATISFIES

(3) $$\text{DIV}\,\vec{G} = 0 \quad \text{AND} \quad \text{CURL}\,\vec{G} = \vec{0}$$

NOW THINK OF $\vec{G}$ AS A FUNCTION ON $\mathbb{R}^4$ THAT IS INDEPENDENT OF x^4 AND

DEFINE AN ANTI-SELF-DUAL 2-FORM ω ON $\mathbb{R}^4$ BY

$$\omega = G_1(dx^1\wedge dx^2 - dx^3\wedge dx^4) + G_2(dx^1\wedge dx^3 - dx^4\wedge dx^2) + G_3(dx^1\wedge dx^4 - dx^2\wedge dx^3)$$

IT FOLLOWS FROM (3) THAT $d\omega = 0$ SO, BY THE POINCARÉ LEMMA, THERE IS A 1-FORM η ON $\mathbb{R}^4$ WITH $d\eta = \omega$. THUS, IF WE LET $A = i\eta$ WE HAVE A $U(1)$-CONNECTION FORM WHICH IS ANTI-SELF-DUAL BECAUSE $F_A = dA = i\,d\eta = i\omega$.

OUR OBJECTIVE NOW IS TO PROVE THAT ANY SOLUTION (A, Ψ) TO (SW) ON $\mathbb{R}^4$ FOR WHICH THE SPINOR FIELD Ψ IS SQUARE INTEGRABLE ON $\mathbb{R}^4$ MUST, IN FACT, BE REDUCIBLE.

<u>THEOREM</u> (WITTEN): SUPPOSE A IS IN $\Omega^1(\mathbb{R}^4, \mathrm{Im}\,\mathbb{C})$ AND $\Psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}$ IS IN $C^\infty(\mathbb{R}^4, \mathbb{C}^2)$ AND THAT THE PAIR (A, Ψ) SATISFIES

(4) $$\nabla_1\Psi = I\nabla_2\Psi + J\nabla_3\Psi + K\nabla_4\Psi \qquad (\not{\partial}_A\Psi = 0)$$

(5a) $$F_{12} + F_{34} = -\tfrac{1}{2}\Psi^* I\Psi$$

(5b) $$F_{13} + F_{42} = -\tfrac{1}{2}\Psi^* J\Psi$$

(5c) $$F_{14} + F_{23} = -\tfrac{1}{2}\Psi^* K\Psi$$

THEN $\Psi \in L^2(\mathbb{R}^4)$ IMPLIES $\Psi \equiv 0$.

<u>NOTE</u>: $\Psi \in L^2(\mathbb{R}^4)$ MEANS $\int_{\mathbb{R}^4} \|\Psi(x)\|^2\, d\mathrm{VOL} < \infty$, WHERE $\|\Psi(x)\|^2$ IS THE SQUARED NORM OF $\Psi(x)$ DETERMINED BY THE USUAL HERMITIAN INNER

PRODUCT ON $\mathbb{C}^2$, I.E., $\|\psi(x)\|^2 = |\psi^1(x)|^2 + |\psi^2(x)|^2$, AND $d\mathrm{vol}$ IS THE USUAL VOLUME FORM ON $\mathbb{R}^4$.

PROOF: LET $\Delta = -\sum_{i=1}^{4} \frac{\partial^2}{(\partial x^i)^2}$ BE THE USUAL LAPLACIAN ON $\mathbb{R}^4$. THEN

$$(6) \qquad \Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \frac{\partial}{\partial x^i} \mathrm{Re} \langle \psi, \nabla_i \psi \rangle .$$

TO SEE THIS COMPUTE

$$\frac{\partial}{\partial x^i} \|\psi\|^2 = \frac{\partial}{\partial x^i} (\bar{\psi}^1 \psi^1 + \bar{\psi}^2 \psi^2)$$
$$= \bar{\psi}_1 \partial_i \psi^1 + \psi_1 \partial_i \bar{\psi}_1 + \bar{\psi}_2 \partial_i \psi^2 + \psi^2 \partial_i \bar{\psi}^2$$

AND

$$\langle \psi, \nabla_i \psi \rangle = \langle \psi, \partial_i \psi + A_i \psi \rangle$$
$$= \bar{\psi}^1 (\partial_i \psi^1 + A_i \psi^1) + \bar{\psi}^2 (\partial_i \psi^2 + A_i \psi^2)$$
$$= \bar{\psi}^1 \partial_i \psi^1 + \bar{\psi}^2 \partial_i \psi^2 + A_i (|\psi^1|^2 + |\psi^2|^2) .$$

THEN

$$2 \mathrm{Re} \langle \psi, \nabla_i \psi \rangle = \langle \psi, \nabla_i \psi \rangle + \overline{\langle \psi, \nabla_i \psi \rangle}$$
$$= \bar{\psi}^1 \partial_i \psi^1 + \psi^1 \partial_i \bar{\psi}^1 + \bar{\psi}^2 \partial_i \psi^2 + \psi^2 \partial_i \bar{\psi}^2$$

BECAUSE $A_i (|\psi^1|^2 + |\psi^2|^2) \in \mathrm{Im}\, \mathbb{C}$. THUS,

$$\frac{\partial}{\partial x^i} \|\psi\|^2 = 2 \mathrm{Re} \langle \psi, \nabla_i \psi \rangle$$

SO

$$-\frac{\partial^2}{(\partial x^i)^2} \|\psi\|^2 = -2 \frac{\partial}{\partial x^i} \mathrm{Re} \langle \psi, \nabla_i \psi \rangle$$

AND SUMMING OVER $i = 1,2,3,4$ GIVES (6). A SIMILAR (BUT SOMEWHAT LONGER) CALCULATION GIVES

(7) $$\frac{\partial}{\partial x^i} \mathrm{RE} \langle \psi, \nabla_i \psi \rangle = \| \nabla_i \psi \|^2 + \mathrm{RE} \langle \psi, \nabla_i \nabla_i \psi \rangle .$$

AT THIS POINT WE REQUIRE A SPECIAL CASE OF THE FAMOUS WEITZENBÖCK FORMULA WHICH, IN OUR PRESENT CIRCUMSTANCES, READS

(8) $$\not{D}_A^* \not{D}_A \psi + \sum_{i=1}^{4} \nabla_i \nabla_i \psi = (\rho^+(F_A))\psi$$

AND CAN BE PROVED BY DIRECT CALCULATION. NOW, USING THE FACT THAT (A, ψ) SATISFIES $\not{D}_A \psi = 0$, THIS BECOMES

$$\sum_{i=1}^{4} \nabla_i \nabla_i \psi = (\rho^+(F_A))\psi$$

AND, WRITING $\rho^+(F_A)$ AS IN (37), ADDENDUM 14, WE OBTAIN

(9) $$\sum_{i=1}^{4} \nabla_i \nabla_i \psi = (F_{12}+F_{34}) I \psi + (F_{13}+F_{42}) J \psi + (F_{14}+F_{23}) K \psi .$$

THUS, SUBSTITUTING (7) INTO (6) AND USING (9) GIVES

(10) $$\Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - 2\,\mathrm{RE} \langle \psi, (F_{12}+F_{34}) I \psi \rangle$$

$$- 2\,\mathrm{RE} \langle \psi, (F_{13}+F_{42}) J \psi \rangle - 2\,\mathrm{RE} \langle \psi, (F_{14}+F_{23}) K \psi \rangle$$

NOW WE EXAMINE THE LAST THREE TERMS IN (10). BY (5a),

$$\langle \psi, (F_{12}+F_{34}) I \psi \rangle = \langle \psi, -\tfrac{1}{2}(\psi^* I \psi) I \psi \rangle$$

AND, BY (45) AND (46), ADDENDUM 14, $-\frac{1}{2}(\psi^* I \psi) = -\frac{1}{2} i \left(|\psi^1|^2 - |\psi^2|^2\right)$ SO

$$\langle \psi, (F_{12}+F_{34}) I\psi \rangle = \langle \psi, -\tfrac{1}{2} i (|\psi^1|^2 - |\psi^2|^2) I\psi \rangle$$

$$= -\tfrac{1}{2} i (|\psi^1|^2 - |\psi^2|^2) \langle \psi, I\psi \rangle$$

$$= -\tfrac{1}{2} i (|\psi^1|^2 - |\psi^2|^2) \left\langle \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \begin{pmatrix} i\psi^1 \\ -i\psi^2 \end{pmatrix} \right\rangle$$

$$= \tfrac{1}{2} (|\psi^1|^2 - |\psi^2|^2)^2$$

$$= \tfrac{1}{2} |\psi^* I \psi|^2$$

SIMILARLY, (5b) AND (5c) GIVE

$$\langle \psi, (F_{13}+F_{42}) J\psi \rangle = \tfrac{1}{2} |\psi^* J \psi|^2$$

AND

$$\langle \psi, (F_{14}+F_{23}) K\psi \rangle = \tfrac{1}{2} |\psi^* K \psi|^2 .$$

OUR FINAL EXPRESSION FOR $\Delta \|\psi\|^2$ FOLLOWS FROM THESE AND (10).

$$(11) \qquad \Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - |\psi^* I \psi|^2 - |\psi^* J \psi|^2 - |\psi^* K \psi|^2$$

IN PARTICULAR, $\Delta \|\psi\|^2 \leq 0$ ON ALL OF $\mathbb{R}^4$ SO THE FUNCTION

$$x \longrightarrow \|\psi(x)\|^2 \quad : \mathbb{R}^4 \longrightarrow \mathbb{R}$$

IS SUBHARMONIC ON ALL OF $\mathbb{R}^4$.

<u>REMARK</u>: FOR BASIC PROPERTIES OF SUBHARMONIC FUNCTIONS SEE <u>FOUNDATIONS OF MODERN POTENTIAL THEORY</u>, N.S. LANDKOF, SPRINGER-VERLAG, NEW YORK BERLIN 1972

CONSEQUENTLY, THIS FUNCTION SATISFIES A MEAN VALUE PROPERTY ON $\mathbb{R}^4$. SPECIFICALLY, FOR ANY $r > 0$ AND ANY $x \in \mathbb{R}^4$,

$$(12) \qquad \|\psi(x)\|^2 \leq \frac{2}{\pi^2 r^4} \int_{B_r(x)} \|\psi(x)\|^2 d\,\mathrm{vol}$$

WHERE $B_r(x)$ IS THE CLOSED BALL OF RADIUS r ABOUT x.

NOW, ASSUMING $\psi \in L^2(\mathbb{R}^4)$, $\int_{\mathbb{R}^4} \|\psi(x)\|^2 d\,\mathrm{vol} < \infty$. DENOTING THE VALUE OF THIS INTEGRAL BY K WE FIND FROM (12) THAT, FOR ANY $x \in \mathbb{R}^4$,

$$\|\psi(x)\|^2 \leq \frac{2K}{\pi^2 r^4}$$

FOR ANY $r > 0$. THUS, $\|\psi(x)\| = 0$ FOR ANY $x \in \mathbb{R}^4$, I.E., $\psi \equiv 0$ AS REQUIRED. $\square$

<u>REMARKS</u> : WE BRIEFLY DESCRIBE A FEW EXTENSIONS AND GENERALIZATIONS FOR COMPACT M. NOTE THAT THE SOLUTIONS (A, ψ) TO THE SEIBERG-WITTEN EQUATIONS OBVIOUSLY GIVE THE ABSOLUTE MINIMA OF THE FUNCTIONAL

$$(13) \qquad \int_M \left(\|\not{D}_A \psi\|^2 + 2\,\left| F_A^+ - \sigma^+((\psi \otimes \psi^*)_0) \right|^2 \right) d\,\mathrm{vol}_g$$

(THE 2 IS FOR CONVENIENCE). USING THE GENERAL WEITZENBÖCK FORMULA ONE CAN SHOW THAT THIS FUNCTIONAL CAN BE WRITTEN IN TERMS OF THE <u>SEIBERG-WITTEN ENERGY FUNCTIONAL</u>

$$(14) \qquad E(A,\Psi) = \int_M (|\nabla_A \Psi|^2 + \tfrac{s}{4}|\Psi|^2 + \tfrac{1}{4}|\Psi|^4 + |F_A^+|^2)\, d\mathrm{vol}_g$$

(s = SCALAR CURVATURE OF (M,g)) AS FOLLOWS :

$$E(A,\Psi) = \int_M (\|\not{\partial}_A \Psi\|^2 + 2|F_A^+ - \sigma^+((\Psi \otimes \Psi^*)_0)|^2)\, d\mathrm{vol}_g - \pi^2 \langle c_1(L_0)^2, [M] \rangle .$$

SINCE $\pi^2 \langle c_1(L^0)^2, [M] \rangle = \int_M (2|F_A^+|^2 - |F_A|^2)\, d\mathrm{vol}_g$ WE FIND THAT, IF (A,Ψ) SATISFIES (SW), THEN

$$\int_M (|\nabla_A \Psi|^2 + \tfrac{s}{4}|\Psi|^2 + \tfrac{1}{4}|\Psi|^4 + 2|F_A^+|^2)\, d\mathrm{vol}_g = 0 .$$

THUS, IF $s \geq 0$ WE MUST HAVE $\Psi \equiv 0$ AND $F_A^+ \equiv 0$. WE CONCLUDE THAT IF M ADMITS A RIEMANNIAN METRIC WITH NON-NEGATIVE SCALAR CURVATURE, THEN FOR ANY SPIN^c- STRUCTURE ON THE CORRESPONDING ORIENTED, ORTHONORMAL FRAME BUNDLE ANY SOLUTION (A,Ψ) TO (SW) SATISFIES $\Psi \equiv 0$ AND $F_A^+ \equiv 0$.

NOW, THE INTEGRALS IN (13) AND (14) ARE GENERALLY NOT MEANINGFUL (FINITE) ON THE NONCOMPACT MANIFOLD $\mathbb{R}^4$. INDEED, IF WE DEFINE THE ENERGY $E(A,\Psi)$ OF A PAIR (A,Ψ) SATISFYING (4) AND (5a) - (5c) ON $\mathbb{R}^4$ BY (14) (WITH $s = 0$) ONE CAN SHOW THAT FINITE ENERGY IMPLIES NOT ONLY THAT $\Psi \equiv 0$, BUT ALSO THAT A IS FLAT (I.E., $F_A \equiv 0$).

THEOREM (WITTEN) : SUPPOSE A IS IN $\Omega^1(\mathbb{R}^4, \mathrm{Im}\,\mathbb{C})$ AND $\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ IS IN $C^\infty(\mathbb{R}^4, \mathbb{C}^2)$ AND THAT THE PAIR (A,Ψ) SATISFIES (SW) ON $\mathbb{R}^4$, I.E., (4) AND (5a) - (5c). LET

$$(15) \qquad E(A,\Psi) = \int_{\mathbb{R}^4} \left(\sum_{i=1}^4 \|\nabla_i \Psi\|^2 + \tfrac{1}{4}\|\Psi\|^4 + \sum_{i<j} |F_{ij}|^2 \right) d\mathrm{vol} .$$

THEN $E(A,\Psi) < \infty$ IMPLIES $\Psi \equiv 0$ AND $F_A \equiv 0$.

PROOF : WE USE THE SAME NOTATION AS IN THE PROOF OF THE PREVIOUS THEOREM. SINCE THE ASSUMPTION THAT $\psi \in L^2(\mathbb{R}^4)$ WAS NOT USED UNTIL THE LAST PARAGRAPH OF THAT PROOF WE MAY USE THE IDENTITY (11) DERIVED PRIOR TO THIS.

A SIMPLE CALCULATION SHOWS THAT

$$|\psi^* I \psi|^2 + |\psi^* J \psi|^2 + |\psi^* K \psi|^2 = \|\psi\|^4$$

SO (11) BECOMES

$$\Delta \|\psi\|^2 = -2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - \|\psi\|^4. \tag{16}$$

NOW WE APPEAL TO AN IDENTITY FROM VECTOR CALCULUS ($\Delta(fg) = f\Delta g - 2\nabla f \cdot \nabla g + g\Delta f$) TO OBTAIN

$$\begin{aligned} \Delta \|\psi\|^4 &= \Delta(\|\psi\|^2 \|\psi\|^2) \\ &= -2(\nabla \|\psi\|^2) \cdot (\nabla \|\psi\|^2) + 2\|\psi\|^2 \Delta \|\psi\|^2 \end{aligned}$$

WHICH, WITH (16), GIVES

$$\Delta \|\psi\|^4 = -2(\nabla \|\psi\|^2) \cdot (\nabla \|\psi\|^2) - 4\|\psi\|^2 \sum_{i=1}^{4} \|\nabla_i \psi\|^2 - 2\|\psi\|^6. \tag{17}$$

IN PARTICULAR, $\|\psi\|^4$ IS SUBHARMONIC ON $\mathbb{R}^4$. SINCE $E(A,\psi) < \infty$ IMPLIES $\int_{\mathbb{R}^4} \|\psi\|^4 \, d\mathrm{vol} < \infty$, THE SAME ARGUMENT WE USED FOR $\|\psi\|^2$ IN THE PREVIOUS PROOF SHOWS ONCE AGAIN THAT

$$\psi \equiv 0. \tag{18}$$

TO SHOW THAT $F_A \equiv 0$ AS WELL WE OBSERVE THAT $\psi \equiv 0$ AND $F_A^+ = \sigma^+((\psi \otimes \psi^*)_0)$ TOGETHER IMPLY THAT $F_A^+ = 0$, I.E., $F = F_A$ IS ANTI-SELF-DUAL. THUS,

(19) $$F = dA = -*dA.$$

WE CLAIM THAT IT FOLLOWS FROM THIS THAT F IS A HARMONIC 2-FORM, I.E.,

(20) $$\Delta_2 F = 0,$$

WHERE Δ_2 IS THE HODGE LAPLACIAN ON 2-FORMS.

<u>REMARK</u>: FOR p-FORMS ON AN ORIENTED, RIEMANNIAN n-MANIFOLD ONE DEFINES $\delta = (-1)^{n(p+1)+1} * d *$ AND THEN THE LAPLACIAN ON FORMS IS DEFINED BY

$$\Delta_p = d \circ \delta + \delta \circ d.$$

FOR 2-FORMS ON $\mathbb{R}^4$, $\delta = -*d*$ SO $\Delta_2 = -(d*d* + *d*d)$.
RELATIVE TO STANDARD COORDINATES, $F = \sum_{i<j} F_{ij}\, dx^i \wedge dx^j$ IMPLIES

(21) $$\Delta_2 F = \sum_{i<j} (\Delta F_{ij})\, dx^i \wedge dx^j$$

TO PROVE THAT (19) IMPLIES (20) WE JUST COMPUTE

$$\begin{aligned}
\Delta_2 F &= -(d*d* + *d*d)F = -(d*d* + *d*d)(dA) \\
&= -d*d*(dA) - *d*d(dA) = -d*d*(-*dA) + *d*(d(dA)) \\
&= d*d(**dA) + *d*(d^2A) = d*(d^2A) + *d*(d^2A) \\
&= 0 \quad \text{SINCE } d^2 = 0.
\end{aligned}$$

FROM (21) WE THEN CONCLUDE THAT EACH F_{ij} IS HARMONIC:

$$\Delta F_{ij} = 0 \;, \quad i,j = 1,2,3,4 \;, \quad i<j \;,$$

NOW, SINCE $F_{ij} : \mathbb{R}^4 \to \operatorname{Im}\mathbb{C}$, $|F_{ij}|^2 = -F_{ij}^{\;2}$ SO

$$\Delta \Big(\sum_{i<j} |F_{ij}|^2 \Big) = - \sum_{i<j} \Delta (F_{ij})^2$$

$$= - \sum_{i<j} \big[-2\nabla F_{ij} \cdot \nabla F_{ij} + 2 F_{ij}\, \Delta F_{ij} \big]$$

$$= 2 \sum_{i<j} \nabla F_{ij} \cdot \nabla F_{ij} = 2 \sum_{i<j} \Big(\sum_{k=1}^{4} \Big(\frac{\partial F_{ij}}{\partial x^k} \Big)^2 \Big)$$

$$= -2 \sum_{i<j} \Big(\sum_{k=1}^{4} \Big(\frac{\partial F_{ij}}{\partial x^k} \Big) \Big(- \frac{\partial F_{ij}}{\partial x^k} \Big) \Big)$$

$$= -2 \sum_{i<j} \Big(\sum_{k=1}^{4} \Big| \frac{\partial F_{ij}}{\partial x^k} \Big|^2 \Big)$$

$$\leq 0 \,.$$

THUS, $\sum_{i<j} |F_{ij}|^2$ IS SUBHARMONIC ON $\mathbb{R}^4$. BUT $E(A,\psi) < \infty$ IMPLIES

$$\int_{\mathbb{R}^4} \Big(\sum_{i<j} |F_{ij}|^2 \Big) \, dvol < \infty$$

SO ONCE AGAIN THE MEAN VALUE PROPERTY IMPLIES $\sum_{i<j} |F_{ij}|^2 \equiv 0$. FROM THIS, EACH $F_{ij} \equiv 0$ SO $F \equiv 0$. $\square$

ADDENDUM 17 : PHYSICS BEHIND WITTEN'S CONJECTURE

TAKEN FROM

SEIBERG-WITTEN GAUGE THEORY

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5.2.1 The Physics way: S-duality

The physicists' approach to the equivalence of Seiberg–Witten and Donaldson theory is based on Witten's interpretation of Donaldson's theory as a twisted supersymmetric Quantum Field Theory [64] and on the concept of electro-magnetic duality. We attempt here a very rough overview of some of these topics. From the mathematician's point of view this concept of "duality" is rather mysterious; however, we'll try to present the basic ideas, mainly based on [2], [63], and on the exposition [8]. We especially recommend the very nice introduction to S-duality given in [18].

Maxwell equations

The first appearance of electromagnetic duality is in Maxwell equations. It is well known that the Maxwell equations in vacuum can be written as

$$dF = 0 \quad d^*F = 0,$$

where $F = dA$ is an imaginary 2-form, the curvature of a $U(1)$ bundle with connection A. In Physics notation one would write

$$E_k = -iF_{k4}$$

for the electric field, and

$$B^k = -i\frac{1}{2}\epsilon^{kpq}F_{pq}$$

for the magnetic field. The symbol ϵ^{kpq} is ± 1 according to the sign of the permutation $\{k, p, q\}$ of $\{1, 2, 3\}$ and zero if any two indices are equal.

It is clear that there is a symmetry given by the Hodge $*$-operator

$$F \mapsto *F$$

that preserves the equations and interchanges electric and magnetic fields.

The Maxwell equations are no longer invariant under the $*$-operator if one considers the presence of electric charges and electromagnetic currents, unless one postulates the existence of isolated magnetic charges, namely magnetic monopoles.

Magnetic monopoles satisfy a quantisation condition which states that the magnetic and electric charges are related by

$$m = \frac{2\pi}{e}.$$

There is an elegant topological motivation for this quantisation condition which is beautifully explained by Raoul Bott in [11].

There is an analogue of electromagnetic duality for monopoles in non-abelian field theory, where again one can see that electric and magnetic charges live in dual lattices and the magnetic charge can be given a topological meaning.

The electric charge enters the Lagrangian as a coupling constant (as we are going to discuss in a moment). Thus, one can see how electromagnetic duality interchanges weak and strong coupling (a small with a large coupling constant). Interchanging a weak with a strong coupling means to exchange the range in which perturbative theory can be applied with one in which it cannot. This will be discussed in the following.

Modular forms

In the abelian context, that is, with structure group $U(1)$, we can write the Lagrangian density on a four-manifold X as

$$\mathcal{L} = \frac{1}{8\pi} \int_X \left(\frac{4\pi}{e^2} F \wedge *F + \frac{i\theta}{2\pi} F \wedge F\right).$$

The second part of the Lagrangian density is a topological term,

$$\frac{i\theta}{2\pi} c_1(L)^2,$$

where L is the chosen line bundle on which the Maxwell equations are considered. The angle θ is the $U(1)$-symmetry of the vacuum state.

3.

Upon setting

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2},$$

one can rewrite $\mathcal{L}$ in terms of τ,

$$\mathcal{L} = \frac{1}{8\pi}\int_X (\bar{\tau}(F^+)^2 - \tau(F^-)^2)dv.$$

The partition function, formally written as an infinite dimensional integral

$$Z \sim \int e^{-\mathcal{L}}\mathcal{D}A,$$

is invariant under the transformation

$$\tau \mapsto \tau + 2.$$

Under the transformation

$$\tau \mapsto \tau + 1$$

we have

$$Z \mapsto Z \cdot e^{\pi i c_1(L)^2}.$$

In the case of a *Spin*-manifold $c_1(L)^2$ is an even integer, hence there is an invariance under $\tau \mapsto \tau + 1$. A more complicated computation with the formal rules of infinite dimensional integrals "shows" that there is also an invariance under

$$\tau \mapsto -\frac{1}{\tau}.$$

This can be viewed as a consequence of a Poisson summation formula applied formally to the infinite dimensional integrals, which leads to the result

$$Z(-\frac{1}{\tau}) = \tau^{\frac{1}{4}(\chi(X)-\sigma(X))}\bar{\tau}^{\frac{1}{4}(\chi(X)+\sigma(X))}Z(\tau).$$

This implies that $Z(\tau)$ behaves like a modular form under the action of $SL(2,\mathbb{Z})$. This fact is an appearance of the phenomenon known as Montonen-Olive duality. It is related to electromagnetic duality, since the transformation

$$\tau \mapsto -\frac{1}{\tau}$$

corresponds to

$$F \mapsto *F,$$

in the sense that all the expectation values are preserved under the combined action of the transformations together. Thus, the modularity can

4.

be thought of as a refined version of the Hodge duality which manifests itself at the quantum level.

We should remark, however, that the picture presented here is quite incomplete. In fact it ignores the essential role of supersymmetry.

In the case of non-abelian monopoles the analogous phenomenon happens if one considers the Lagrangian density

$$\mathcal{L} = \frac{1}{g^2}\int_X Tr(F \wedge *F) + \frac{i\theta}{8\pi^2}\int_X Tr(F \wedge F).$$

The presence of the coefficient $\frac{1}{g^2}$ depends on the fact that the Killing form on the compact Lie group G is only defined up to a scalar multiple which is usually set equal to one in the mathematical literature, while it appears in Physics as a coupling constant. The second term represents the second Chern class of the vector bundle E on X on which the connection and curvature $F = dA + A \wedge A$ are considered. The fact that this topological term appears explicitly in the Lagrangian is already an effect of the presence of $N = 1$ supersymmetry. In fact also other terms appear in the partition function that contain the "auxiliary fields" introduced by the supersymmetry. These are the analogue of the elements of the algebra $\Lambda[w]$ in our definition of the fermionic integral in relation to the Mathai-Quillen formalism. The fact that the vacuum state (that is, the minimum of the classical potential) has a $U(1)$-symmetry which explains the presence of the angle θ is also an effect of the presence of the "unbroken" supersymmetry.

Thus the partition function can be formally written as

$$\int e^{-\mathcal{L}}\mathcal{D}A = \sum_{r=c_2(E)} e^{ir\theta}\int e^{-\mathcal{L}_r}\mathcal{D}A,$$

where $\mathcal{L}_r = \frac{1}{g^2}\int_X Tr(F \wedge *F)$ on the fixed bundle E.

Again one can introduce the variable

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

The modularity in this context can be formulated in a different way, which leads to an interesting conjecture [61].

Conjecture 5.2.5 *consider the expression*

$$Z_G(\tau) = q^c \sum_{r=0}^{\infty} \chi_r q^r,$$

where $q = e^{2\pi i\tau}$ and χ_r is some suitable regularised Euler characteristic of the moduli space of G instantons on the four-manifold X with instanton number $r = c_2(E)$. There is an action of $SL(2,\mathbb{Z})$ and Z_G transforms like

$$Z_G(-\frac{1}{\tau}) \sim Z_{\tilde{G}}(\tau),$$

where $\tilde{G}$ is the Langlands dual of G.

We do not discuss this statement any further, but just mention that the Langlands dual interchanges the torus lattice with its dual. It is thus related to electromagnetic duality for non-abelian monopoles.

Weak and strong coupling

As we have seen in the example of Maxwell theory, the interchanging of electric and magnetic charges due to Hodge duality also interchanges weak and strong coupling in the action. When the coupling constant is small, one can formally compute the infinite dimensional integral by means of a stationary phase approximation [11], [63]. The model is the finite dimensional situation in which one has a function $F(x)$ with an isolated minimum at $x = 0$. We can write $F(x) = F(0) + \frac{1}{2}Q(x) + \cdots$ and for a small coupling constant we can approximate the integral

$$Z = \int e^{-\frac{1}{\lambda}F(x)} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}} \sim Z_0 = \int e^{-\frac{1}{\lambda}(F(0)+\frac{1}{2}Q(x))} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}}.$$

The latter can be computed exactly and it gives

$$Z_0 = e^{-\frac{1}{\lambda}F(0)} \frac{\lambda^{n/2}}{\det(Q)^{1/2}}.$$

The finite dimensional computation can be easily related to the computation of the Pfaffian that we presented in relation to the Mathai-Quillen formalism, with the only difference that the matrix Q is symmetric instead of antisymmetric. This explains why one gets $\det(Q)^{-1/2}$ instead of $\det(Q)^{1/2} = Pf(Q)$.

In order to generalise this argument to the infinite dimensional context, the problem is reformulated in terms of a functional F with non-degenerate minima. The approximation of the partition function in this case can be taken to be the well defined mathematical object $\det(Q)^{-1/2}$, where Q is a positive elliptic operator (the Hessian of the functional F at a minimum) and the determinant is the Ray-Singer determinant [55], [56]. If the coupling constant is large this approximation method no longer works and the partition function is in general no longer computable.

The u-plane

In the case of $N = 2$ supersymmetry, the auxiliary fields that are introduced can be described as two independent variables of the type of the $\Lambda[w]$ used in the definition of the fermionic integral, and a field ϕ which is a section of the adjoint bundle of E. The classical potential can be written as a function $V(\phi)$ and as mentioned before the supersymmetry imposes that in the vacuum state $V(\phi) = 0$. This allows for certain symmetries of the vacuum. This means that the vacuum state is not an isolated point but there is some parametrisation of a certain manifold of possible vacuum states. In our case the parameter that classifies inequivalent vacua is $Tr(\phi^2)$.

This is better said by introducing a variable $u = \langle Tr(\phi^2)\rangle$ which is the expectation value (with respect to the partition function Z) of $Tr(\phi^2)$. The expectation value of the field ϕ is proportional to a variable a, $\langle\phi\rangle \sim a$. In the classical limit, that is, when the coupling is weak, one has the relation $u \sim \frac{1}{2}a^2$. In the strong coupling range the relation is more complicated.

In terms of the parameter a one has the corresponding modulus

$$\tau(a) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)}.$$

The symmetry of the action under the transformation $\tau \mapsto \frac{-1}{\tau}$ can be formally described in terms of a Legendre transformation over a potential (called *prepotential* in the Physics literature) $\mathcal{F}$. In fact, a dual variable a_D is introduced by the relation

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}, \tag{5.13}$$

and a dual field ϕ_D is defined by $\langle\phi_D\rangle \sim a_D$. Here "dual" is intended in analogy to coordinates and moments in classical mechanics that are related by a Legendre transform similar to (5.13). The transformation $\tau \mapsto \frac{-1}{\tau}$ exchanges the action Z with a dual action Z_D where the field ϕ is replaced with ϕ_D and a with a_D. This exchanges weak and strong coupling.

The reason why this can be still thought of as electromagnetic duality is that one thinks of the purely electric or purely magnetic charge as quantities $q_e = n_e a$ and $q_m = n_m a_D$, for a pair of integers (n_e, n_m). One can also consider states (which are called *dyons* in the literature) that have both electric and magnetic charge $q = n_e a + n_m a_D$. The group

7.

$SL(2;\mathbb{Z})$ acts by mixing the electric and the magnetic charge

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}.$$

If one wants to express the variables a and a_D as functions of the parameter that determines the vacuum state, $a(u)$ and $a_D(u)$, one gets two multivalued functions, defined for $u \in \mathbb{C}$ with branch cuts. In particular one can compute the monodromy at the branch points [8]. One point is certainly the one at infinity, where the weak coupling range is attained. In this case the prepotential takes the form $\mathcal{F}(a) \sim \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}$ and as $u \mapsto e^{2\pi i} u$ one has $a \mapsto -a$ and $a_D \mapsto -a_D + 2a$. Thus the monodromy at $u = \infty$ is

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

An argument depending on the factorisation of the matrix

$$M_\infty \in SL(2;\mathbb{Z})$$

shows that there are other two branching points. Up to the choice of a normalising constant these can be taken to be $u = \pm 1$. As $u \to \pm 1$ the strong coupling range is attained. The corresponding monodromies [8] are

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

and

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

The physical interpretation of the eigenvalues of the monodromy matrices leads to interpreting these branch points as the vacua at which a magnetic monopole (when $u = 1$) or a $(1,-1)$-dyon (when $u = -1$) become massless.

Elliptic curves

Given the data obtained above by physical arguments, namely the punctured sphere $\mathbb{C}P^1 - \{\infty, 1, -1\}$ (or u-plane) and the prescribed monodromies at the punctures, it is possible to proceed with a rigorous construction. The monodromies obtained above span a subgroup $\Gamma(2)$ in $SL(2;\mathbb{Z})$. The u-plane is equivalent to the quotient of the upper half plane with respect to the group $\Gamma(2)$. This gives the moduli of the family of elliptic curves

$$y^2 = (x^2 - 1)(x - u)$$

that becomes singular at the points $u = \pm 1$.

The functions $a(u)$ and $a_D(u)$ can be interpreted within this geometric picture as the periods

$$a = \int_{\gamma_1} \lambda \qquad a_D = \int_{\gamma_2} \lambda \ ,$$

with

$$\lambda = \frac{\sqrt{2(x-u)}}{2\pi\sqrt{x^2-1}}dx.$$

The relation of all this with the Witten conjecture comes when one reads the weak coupling limit of Z as Donaldson theory (that is twisted $N = 2$ supersymmetric Yang-Mills theory) and the strong coupling limit of Z as the Seiberg–Witten theory. Then the idea that leads to the equivalence of the two theories is that the geometric data encoded in this family of elliptic curves should provide "the gluing instructions" of how to interpolate for all values of u knowing the asymptotic behaviour at the singular points. The relation obtained would then be in the form given by Kronheimer and Mrowka, as in theorem 5.2.2.

More recently, the conjecture 5.2.3 has been extended by Moore and Witten [50] to the case of manifolds with $b_2^+(X) = 1$. In this case a correction term to the relation 5.2.3 comes from integration over the u-plane.