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Equivariant Plateau problems

by

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Motivations.

I will present the results of my recent paper, "Equivariant Plateau problems" which may be found on ArXiV under the reference **math.DG/0602271**.

We wish to obtain conditions for the existence of geometric realisations of certain algebraic objects. Explicitely:

Let Σ be a compact surface of genus at least 2.

Let M be a compact manifold of negative sectional curvature.

Let $\theta : \pi_1(\Sigma) \rightarrow \pi_1(M)$ be a homomorphism.

Qn : Given prescribed curvature conditions (for example, minimality, constant mean curvature, constant Gaussian curvature, convexity, etc.), when does there exist an immersion $i : \Sigma \rightarrow M$ which satisfies these curvature conditions and realises θ, in other words, such that:

$$i_* = \theta?$$

A brief study of this problem reveals a relationship with the theory of complex projective structures:

Let $\tilde{\Sigma}$ and $\tilde{M}$ be the universal covers of Σ and M respectively.

The problem now becomes:

Qn : When does there exists an immersion $i : \tilde{\Sigma} \rightarrow \tilde{M}$ which satisfies these curvature conditions such that i is equivariant under the action of θ :

$$\forall \gamma \in \pi_1(\Sigma), \qquad i \circ \gamma = \theta(\gamma) \circ i?$$

$\tilde{M}$ is a Hadamard manifold. Let $\partial_\infty \tilde{M} \cong S^2$ be the ideal boundary.

We define the Gauss-Minkowski mapping over the unitary bundle of $\tilde{M}$ (see figure 1):

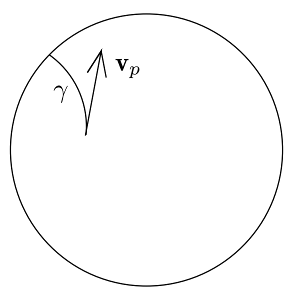

Figure 1

$$\overrightarrow{n} : U\tilde{M} \to \partial_\infty \tilde{M}.$$

For a given vector $\mathbf{v}_p$ in $U\tilde{M}$, we define $\overrightarrow{n}(\mathbf{v}_p)$ to be the point of arrival in the ideal boundary of $\tilde{M}$ of the unique geodesic in $\tilde{M}$ which leaves the point p in the direction of $\mathbf{v}_p$:

$$\overrightarrow{n}(\mathbf{v}_p) = \gamma(+\infty),$$

where γ is the unique geodesic such that $\partial_t\gamma(0) = \mathbf{v}_p$.

Let $\mathsf{N_i} : \tilde{\Sigma} \to \mathsf{U\tilde{M}}$ be the normal vector field over i (see figure 2).

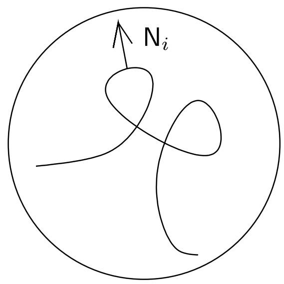

Figure 2

Since the group $\pi_1(M)$ acts by isometries over $\tilde{M}$, it acts by homeomorphisms over $\partial_\infty \tilde{M}$. We consider the following equivariant mapping:

$$\varphi = \overrightarrow{n} \circ \mathsf{N_i} : \tilde{\Sigma} \to \partial_\infty \tilde{\mathsf{M}},$$

$$\varphi \circ \gamma = \theta(\gamma) \circ \varphi \qquad \forall \gamma \in \pi_1(\Sigma).$$

When i is locally convex, φ is a local homeomorphism. Likewise, given a local homeomorphism φ which is equivariant under the action of θ, there exists a locally convex immersion i (which is not necessarily unique) such that:

$$\varphi = \overrightarrow{n} \circ \mathsf{N_i}.$$

Thus, in the case where the curvature condition is that of convexity, the initial problem becomes the following:

Qn: When does there exist a local homeomorphism $\varphi : \tilde{\Sigma} \to \partial_\infty \tilde{M}$ which is equivariant under the action of θ?

We call such a structure an "equivariant Plateau problem".

In principal, we could state the same thing for any curvature condition that we might chose to impose (and not just that of convexity). We thus consider the study of equivariant Plateau problems to be the most general version of our original problem.

Complex Projective Structures.

When M is of constant sectional curvature, we transform this problem into another well known problem:

$$\begin{array}{lll} \tilde{M} & \text{becomes} & \mathbb{H}^3, \\ \partial_\infty \tilde{M} & \text{becomes} & \partial_\infty \mathbb{H}^3 \cong \hat{\mathbb{C}}, \\ \pi_1(M) & \subseteq & \mathrm{Isom}(\mathbb{H}^3) = \mathbb{P}SL(2,\mathbb{C}), \\ \theta : \pi_1(\Sigma) & \to & \mathbb{P}SL(2,\mathbb{C}). \end{array}$$

Since φ is a local homeomorphism, we pull back the holomorphic structure over $\hat{\mathbb{C}}$ to obtain a holomorphic structure over $\tilde{\Sigma}$. We have:

$$\varphi : \tilde{\Sigma} \to \hat{\mathbb{C}} \text{ loc. conformal, equivariant.}$$

The (φ, θ) is thus a complex projective structure with holonomy:

$$(\varphi, \theta) \text{ is a } \mathbb{P}SL(2,\mathbb{C}) \text{ structure with holonomy.}$$

We now recall a result of Gallo, Kapovich and Marden:

Theorem (Gallo, Kapovich & Marden)

Let $\theta : \pi_1(\Sigma) \to \mathbb{P}SL(2,\mathbb{C})$ be a homomorphism. There exists φ such that (φ, θ) is a complex projective structure if and only if:

(i) θ is non-elementary and

(ii) $\mathrm{SW}_2(\theta) = 0$.

The meaning of these two conditions will be explained presently. The essential point is that this result provides necessary and sufficient *algebraic* conditions on θ for the existence of an equivariant Plateau problem. We now use an existence theorem of Labourie:

Theorem (Labourie)

Let (φ, θ) be a complex projective structure and suppose that Σ is compact. Then, there exists a unique immersion $i : \Sigma \to \mathbb{H}^3$ such that:

(i) i is complete,

(ii) i is of constant Gaussian curvature equal to k, and

(iii) $\overrightarrow{n} \circ \mathsf{N}_i = \varphi$.

Moreover, the uniqueness of i implies that:

(iv) i is equivariant under the action of θ.

By taking the quotient of this immersion, we obtain a constant Gaussian curvature immersion of Σ in M which realises θ. We have thus solved the problem for constant Gaussian curvature. In other words, we have obtained necessary and sufficient *algebraic* conditions over θ for the existence of a constant Gaussian curvature immersion of Σ in M with realises θ.

The Existence Theorem.

We will now study the definitions of these algebraic conditions in order to see how they may be generalised to the case where M is not necessarily of constant curvature. First:

θ is said to be non-elementary if and only if the subgroup $\theta(\pi_1(\Sigma))$ does not have any fixed points in $\partial_\infty\tilde{M} = \partial_\infty\mathbb{H}^3 = \hat{\mathbb{C}}$.

This condition may be easily generalised to the case where M is not necessarily of constant sectional curvature. In order to understand the condition concerning the second Steifel-Whitney class of θ, we must first do a bit of algebra. We have the following short exact sequence:

$$0 \to \mathbb{Z}_2 \to \mathrm{SL}(2,\mathbb{C}) \to \mathbb{P}SL(2,\mathbb{C}) \to 0.$$

Using Cech homology, θ generates a homological class:

$$\theta \to [\theta] \in H^1(M, \mathbb{P}SL(2,\mathbb{C})).$$

If we now apply the boundary mapping of the Meyer-Vietoris long exact sequence, we obtain the second Steifel-Whitney class of θ:

$$\delta([\theta]) = \mathrm{SW}_2(\theta) \in H^2(M, \mathbb{Z}_2).$$

In fact, it isn't necessary to understand the details of this algebraic construction: the second Steifel-Whitney class of θ vanishes if and only if there exists a lifting $\tilde{\theta}$ of θ in $\mathrm{SL}(2,\mathbb{C})$, the universal cover of $\mathbb{P}SL(2,\mathbb{C})$:

$$\mathrm{SW}_2(\theta) = 0 \Leftrightarrow \exists\tilde{\theta} : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{C}) \text{ s.t. } \pi \circ \tilde{\theta} = \theta.$$

We generalise SW_2 to the case where M is a manifold of strictly negative sectional curvature. Indeed, the isometry group of the universal cover of M acts on the ideal boundary of M by homeomorphisms. Thus:

$$\theta : \pi_1(\Sigma) \to \pi_1(M) \cong \mathrm{Isom}(\tilde{M}) \to \mathrm{Homeo}_0(\partial_\infty\tilde{M}) \cong \mathrm{Homeo}_0(S^2).$$

By Friberg's theorem, the fundamental group of the homeomorphism group of the sphere is isomorphic to $\mathbb{Z}_2$. The universal cover of this group is thus a two-fold covering, and we obtain the following short exact sequence:

$$0 \to \mathbb{Z}_2 \to \widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M}) \to \mathrm{Homeo}_0(\partial_\infty\tilde{M}) \to 0.$$

Consequently, using Cech homology and the boundary operator of the Meyer-Vietoris sequence, as before, we may define the following element:

$$\mathrm{SW}_2(\theta) \in H^2(\Sigma; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

This is the second Steifel-Whitney class of θ. It vanishes if and only if θ lifts to a homomorphism taking values in the universal cover of the group of homeomorphisms of the sphere:

$$\mathrm{SW}_2(\theta) = 0 \Leftrightarrow \exists \tilde{\theta} : \pi_1(\Sigma) \rightarrow \widetilde{\mathrm{Homeo}}_0(\partial_\infty \tilde{M}) \text{ s.t. } \pi \circ \tilde{\theta} = \theta.$$

We may now state the main result of this presentation:

Theorem

If θ is non-elementary and $\mathrm{SW}_2(\theta) = 0$, then there exists a local homeomorphism $\varphi : \tilde{\Sigma} \rightarrow \partial_\infty \tilde{M}$ which is equivariant under the action of θ :

$$\forall \gamma \in \pi_1(\Sigma), \qquad \varphi \circ \gamma = \theta(\gamma) \circ \varphi.$$

We thus obtain a complete locally convex immersion. Taking the quotient, this yields:

Corollary

If θ is non-elementary and $\mathrm{SW}_2(\theta) = 0$, then there exists a locally convex immersion $i : \Sigma \rightarrow M$ which realises θ :

$$i_* = \theta.$$

The proof follows a strategy analogous to that used by Gallo, Kapovich et Marden. Indeed, the first step is almost identical to their own. First, by using the fact that θ is non-elementary, we obtain a trouser decomposition of Σ such that the image under θ of the fundamental group of each trouser is a Schottky group. The method of proof is not too unlike playing with a Rubix cube.

Schottky Groups and Fundamental Domains.

let $C_a^{\pm}$,$C_b^{\pm}$ be Jordan curves in $\partial_\infty \tilde{M}$ oriented such that each one of these curves lies in the exterior of the three others (see figure 3).

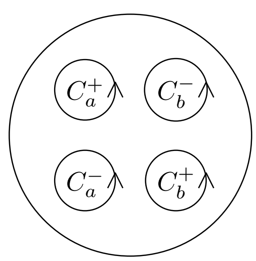

Figure 3

Let $\alpha, \beta \in \mathrm{Isom}(\tilde{M})$ be such that:

$$\begin{aligned} \mathrm{Ext}(C_a^-) \cdot \alpha &= \mathrm{Int}(C_a^+), \\ \mathrm{Ext}(C_b^+) \cdot \alpha &= \mathrm{Int}(C_b^+). \end{aligned}$$

Then the group $\langle \alpha, \beta \rangle$ generated by α and β is a Schottky group.

The result is now easy to prove for pairs of trousers where the image of the fundamental group under the action of θ is a Schottky group.

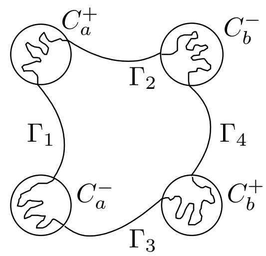

Figure 4

We join together the four Jordan curves by simple curves. The closure of the union of the images of these simple curves under the action of the Schottky group defines another Jordan curve which is invariant under the action of this group (see figure 4):

$$C = \overline{\bigcup_{\gamma \in \langle \alpha, \beta \rangle} (\Gamma_1 \cup ... \cup \Gamma_4) \cdot \gamma}.$$

The interior if this curve thus defines a Jordan domain which is also invariant under the action of the Schottky group:

$$\Omega = \mathrm{Int}(C).$$

We call this an invariant domain of the group $\langle \alpha, \beta \rangle$. The quotient of Ω under the action of this group is a pair of trousers:

$\Omega / \langle \alpha, \beta \rangle$ is a pair of trousers.

The fundamental group of this pair of trousers is canonically isomorphic to the Schottky group generated by α and β:

$$\pi_1(\Omega / \langle \alpha, \beta \rangle) \cong \langle \alpha, \beta \rangle.$$

We thus obtain the solution in this case.

Let P be a pair of trousers. Let $\theta : \pi_1(P) \to \pi_1(M)$ be a homomorphism whose image is a Schottky group:

$$\mathrm{Im}(\theta) = \langle \alpha, \beta \rangle.$$

Let Ω be a fundamental domain of $\mathrm{Im}(\theta)$.

Using elementary topology, we obtain a homeomorphism:

$$\varphi : P \to \Omega/\langle\alpha, \beta\rangle \text{ s.t. } \varphi_* = \theta.$$

The mapping φ lifts to a mapping of the universal cover of P into Ω :

$$\tilde{\varphi} : \tilde{P} \to \Omega.$$

The mapping $\tilde{\varphi}$ is a solution. Indeed:

(i) $\tilde{\varphi}$ is a local homeomorphism, and

(ii) $\tilde{\varphi} : \tilde{P} \to \partial_\infty \tilde{M}$ is equivariant under the action of θ.

The Proof of the Theorem.

It now suffices to establish under which conditions the solutions obtained for each pair of trousers may be joined together to obtain a solution over the entire surface.

We recall that Σ is decomposed into $2g-2$ pairs of trousers $(P_i)_{2g-2}$. For each i, we denote the boundary components of P_i by $(C_{i,j})_{j\in\{1,2,3\}}$ (see figure 5).

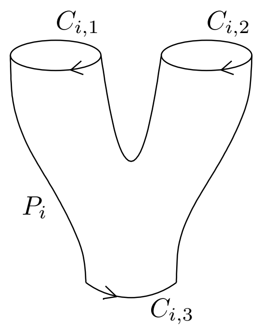

Figure 5

Each boundary component of P_i defines a conjugacy class in the fundamental group of the pair of trousers. Using the homomorphism θ and the combinatorics of the decomposition, we associate to each boundary component $C_{i,j}$ an element of $\pi_1(M)$ which we consider to be the unique image of $C_{i,j}$ under the action of θ:

$$C_{i,j} \to \alpha_{i,j} \in \pi_1(M) \ [= \theta(C_{i,j})].$$

Bearing in mind that these components are oriented such that P_i always lies to their left, each time these two components are joined together, the orientation of one of them must be reversed. Thus:

$$C_{i,j} \text{ joined to } C_{i',j'} \Rightarrow \alpha_{i',j'} = \alpha_{i,j}^{-1}.$$

Since θ lifts, we may equally well define the lifts of each of the $\alpha_{i,j}$:

$$C_{i,j} \rightarrow \tilde{\alpha}_{i,j} \in \widetilde{\text{Homeo}}_0(\partial_\infty \tilde{M}) \; [= \tilde{\theta}(C_{i,j})].$$

As before, we have:

$$C_{i,j} \text{ joined to } C_{i',j'} \Rightarrow \tilde{\alpha}_{i',j'} = \tilde{\alpha}_{i,j}^{-1}.$$

We recall that, since M is compact, every element of the fundamental group of M acts hyperbolically over $\tilde{M}$. In particular, the action of each element has exactly two fixed points in $\partial_\infty \tilde{M}$.

For all i,j let $\alpha_{i,j}^{\pm}$ be the two fixed points of $\alpha_{i,j}$ in $\partial_\infty \tilde{M}$.

We now define the torus:

$$\mathbb{T}_{i,j} = (\tilde{M} \setminus \{\alpha_{i,j}^{\pm}\})/\langle \alpha_{i,j} \rangle.$$

Trivially:

$$C_{i,j} \text{ joined to } C_{i',j'} \Rightarrow \mathbb{T}_{i,j} = \mathbb{T}_{i',j'}.$$

The solutions that we have constructed over each pair of trousers define classes in the first homology groups of these tori:

$$\varphi_i : \tilde{T}_i \rightarrow \partial_\infty \tilde{M} \Rightarrow \xi_{i,j} \in H_1(\mathbb{T}_{i,j}).$$

Explicitly:

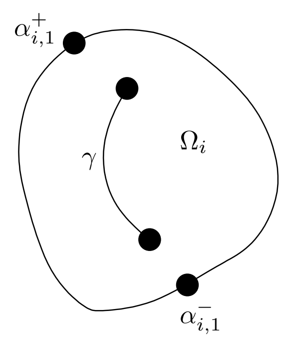

Figure 6

Let $\Omega_i = \text{Im}(\varphi_i)$ be a Jordan domain which is invariant under the action of $\theta(\pi_1(\mathbb{T}_{i,j}))$. For all j, $\alpha_{i,j}^{\pm} \in \partial\Omega_i$, let $\gamma : I \rightarrow \Omega_i$ be a curve such that $\gamma(0) \cdot \alpha_{i,j} = \gamma(1)$ (see figure 6).

The image of this γ under the canonical projection defines a closed loop in the torus $\mathbb{T}_{i,j}$, and thus defines a homology class $\xi_{i,j}$ in this torus:

$$\xi_{i,j} = [\pi \circ \gamma] \in H_1(\mathbb{T}_{i,j}).$$

Since Ω_i is simply connected, this element does not depend on the curve chosen. It thus only depends on Ω_i and is consequently only a function of φ_i. Later, we will denote it by:

$$\xi_{i,j}(\varphi_i).$$

Suppose now that $C_{i,j}$ is joined to $C_{i',j'}$:

$$C_{i,j} \text{ joined to } C_{i',j'}.$$

Gallo, Kapovich and Marden showed that φ_i and φ_j may be extended to a solution over the union of P_i and P_j when $\xi_{i,j} = -\xi_{i',j'}$ (see figure 7) :

$$\xi_{i,j} = -\xi_{i',j'} \Rightarrow \varphi_i \text{ and } \varphi_j \text{ generate } \varphi_{i,j} \text{ above } T_i \cup T_j.$$

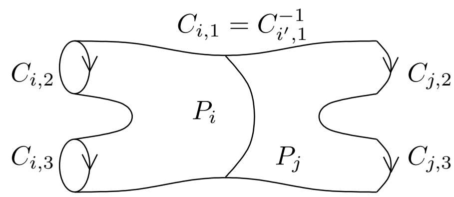

Figure 7

Consequently, in order to obtain the solution, it suffices to chose the fundamental domains (that is, the Ω_i) such that the homological classes that they generate (the $\xi_{i,j}$) coincide along the boundary components which are joined together.

The only remaining question is thus, “When is this possible?”.

The Algebraic Obstruction.

We return to the definition of the homological class generated by the immersion:

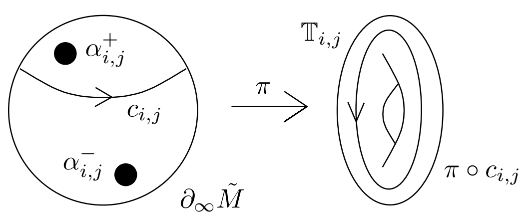

Figure 8

Let $c_{i,j}$ be a Jordan curve in $\partial_\infty \tilde{M}$ such that $\alpha^+_{i,j}$ lies in its interior and $\alpha^-_{i,j}$ in its exterior (see figure 8). This projects onto a simple closed curve in $\mathbb{T}_{i,j}$:

$$\gamma(0)\cdot\alpha_{i,j} = \gamma(1) \Rightarrow \pi\circ\gamma \text{ crosses } \pi\circ c \text{ once going from right to left.}$$

Using Poincaré duality, we thus obtain:

$$\langle \pi\circ\gamma, \pi\circ c_{i,j}\rangle = 1.$$

We define:

$$L_{i,j} = \{[\gamma]\in H_1(\mathbb{T}_{i,j}) \text{ s.t. } \langle[\gamma], \pi\circ c_{i,j}\rangle = 1\}.$$

We have:

$$\begin{aligned} C_{i,j} \text{ joined to } C_{i',j'} \quad &\Rightarrow \ c_{i,j} \text{ runs in the opposite direction to } c_{i',j'}.\\ &\Rightarrow L_{i,j} = -L_{i',j'}. \end{aligned}$$

Trivially:

$$\xi_{i,j}\in L_{i,j} \text{for all } i,j.$$

We now define the mapping $\mathrm{Lift}_{i,j}$ which sends $L_{i,j}$ into $\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$ as follows:

Let $[\gamma]\in L_{i,j}$. Let $\tilde{\gamma}$ be a lifting of γ in $\partial_\infty\tilde{M}\setminus\{\alpha^\pm_{i,j}\}$. We have $\gamma(0)\cdot\alpha_{i,j}=\gamma(1)$.

We view the points $\alpha^\pm_{i,j}$ as constant curves in $\partial_\infty\tilde{M}$, and we define a braid:

$$(\alpha^-_{i,j}, \gamma, \alpha^+_{i,j}) \text{ is a braid.}$$

Moreover, the mapping $\alpha_{i,j}$ sends the start point of this braid onto its end point:

$$(\alpha^-_{i,j}, \gamma(0), \alpha^+_{i,j})\cdot\alpha_{i,j} = (\alpha^-_{i,j}, \gamma(1), \alpha^+_{i,j}).$$

This braid thus defines a homotopy of curves in $\mathrm{Homeo}_0(\partial_\infty\tilde{M})$ going from Id to $\alpha_{i,j}$, and this defines a lifting of $\alpha_{i,j}$ in $\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$.

This lifting only depends on the homology class of $[\gamma]$ in $H_1(\mathbb{T}_{i,j})$. We thus obtain a mapping:

$$\mathrm{Lift}_{i,j} : L_{i,j}\to\widetilde{\mathrm{Homeo}}_o(\partial_\infty\tilde{M}).$$

In particular, we observe that:

$$\pi\circ\mathrm{Lift}_{i,j}([\gamma]) = \alpha_{i,j}.$$

Everything now depends on the following result:

Theorem

For $j\in\{1,2,3\}$, let $\eta_{i,j}$ be an element in $L_{i,j}$. There exists a homeomorphism $\varphi_i : \tilde{T}_i\to\partial_\infty\tilde{M}$ such that:

$$\xi_{i,j}(\varphi_i) = \eta_{i,j} \qquad \forall\ i,$$

if and only if:

$$\mathrm{Lift}(\eta_{i,1})\cdot\mathrm{Lift}(\eta_{i,2})\cdot\mathrm{Lift}(\eta_{i,3})^{-1} = \mathrm{Id}',$$

where Id' is the unique lifting of the identity in $\widetilde{\mathrm{Homeo}}_0(\partial_\infty\tilde{M})$ which is different to the identity.

The solution is now trivial:

(i) For all i,j, we choose $\xi_{i,j} \in L_{i,j}$ such that:

$$\begin{cases} C_{i,j} \text{ joined to } C_{i',j'} & \Rightarrow \xi_{i,j} = -\xi_{i',j'}, \\ \mathrm{Lift}_{i,j}(\xi_{i,j}) & = \mathrm{Id}' \cdot \tilde{\alpha}_{i,j}. \end{cases}$$

(a), (b) as above.

(ii) For all i, we have (see figure 9) :

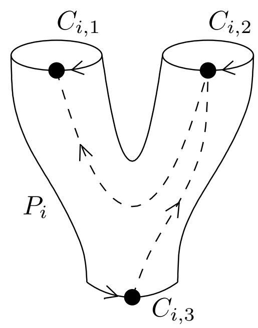

Figure 9

$$\begin{aligned}
&\mathrm{Lift}_{i,1}(\xi_{i,1}) \cdot \mathrm{Lift}_{i,2}(\xi_{i,2}) \cdot \mathrm{Lift}_{i,3}(\xi_{i,3})^{-1} \\
&\quad = \mathrm{Id}' \cdot \tilde{\alpha}_{i,1} \cdot \mathrm{Id}' \cdot \tilde{\alpha}_{i,2} \cdot \tilde{\alpha}_{i,3}^{-1} \cdot \mathrm{Id}' \\
&\quad = (\mathrm{Id}')^3 \cdot \tilde{\alpha}_{i,1} \cdot \tilde{\alpha}_{i,2} \cdot \tilde{\alpha}_{i,3}^{-1} \\
&\quad = (\mathrm{Id}')^3 \cdot \tilde{\theta}(C_{i,1}) \cdot \tilde{\theta}(C_{i,2}) \cdot \tilde{\theta}(C_{i,3}^{-1}) \\
&\quad = (\mathrm{Id}')^3 \cdot \tilde{\theta}(C_{i,1} \cdot C_{i,2} \cdot C_{i,3}^{-1}) \\
&\quad = (\mathrm{Id}')^3 \cdot \tilde{\theta}(\mathrm{Id}) \\
&\quad = \mathrm{Id}'.
\end{aligned}$$

There thus exists $\varphi_i : \tilde{T}_i \to \partial_\infty \tilde{M}$ such that:

$$\xi_{i,j}(\varphi_i) = \xi_{i,j}.$$

(iii) The solutions for each of these pairs of trousers may now be joined together, and the result now follows.