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Noncommutative Quantum Field Theory

by

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Chapter 1

Introduction

1.1 Motivation

Improving QFT. At the very beginning of the twentiest century, Max Planck gave his tremendously often cited talk about black body radiation which ushered in a new era of physics, namely that of quantum mechanics. Physicists investigating this new formulation of ”short scale” physics, like Bohr, Schrödinger, Heisenberg, and many more scheduled big success in having elaborated a predictive theory concerning atomic structures. It was just a matter of time when a symbiosis of special relativity and this new way of contemplating structures and processes of nature was desired. Until this quantum field theory was well established, it took quite a time, but the result was astonishing. Celebrated scientists like Dirac, Pauli, Feynman, Tomonaga, Schwinger and Dyson (just to name a few) delivered new insights into the world and made possible the most precise predictions for quantities like the anomalous magnetic moment of the electron g for decades. Bit by bit, the strong and weak nuclear forces could be included in what is now called the standard model.
Despite the impressive success of quantum field theory, there are problems that remain until today: even the best studied conceivable interaction theories, like ϕ^4 theory, the Yukawa model or even quantum electro dynamics (=: QED) are *not analytical solvable* and produce both infra-red (IR; describing regions governed by small momenta) and ultra-violet (UV; regions of large momenta) *divergences* perturbatively. These inconsistencies have to be cleared out by the scheme of renormalization to obtain a theory which is physical meaningful. It was Arthur Wightman who first formulated an axiomatic approach for quantum field theory, which should restrict the term ”quantum field” to a few basic specifications. Nevertheless, up until now,

there exists *no nontrivial example* of a theory satisfying all of these axioms in 4 dimensions.
Furthermore, relevant theories that have shown to be renormalizable are still lacking a proof of summability. This means the following: since these theories cannot be analytically solved, one has to treat them perturbatively and to prove their renormalizability order by order. But in fact most of these perturbation series diverge or even cannot be proven of being *Borel summable*, but are instead asymptotic series. Since there exist arbitrarily many different functions having the same asymptotic expansion, one cannot speak of predictability here. The phenomenon of the *Landau ghost* connected to this problem is considered later in this text.

Space-time structure. There might be a number of people who regard these problems of being mostly of mathematical interest, but there is another success to be aimed at: the implementation of the last remaining fundamental interaction, gravity, into the standard model. A lot of physicists and mathematicians have worked in this field of "grand unification", but with no striking success so far.

"Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

This astonishing piece of insight by Bernhard Riemann [5] was published a little more than 150 years ago, and should prove to be unexpectedly accurate. Moreover, there are many other examples of statements responding to the case (for example Heisenberg's letter to Ehrenfest (1930), Schrödinger (34), Heisenberg (1938), Peierls, Oppenheimer, ...). Snyder [11] was actually the first who wrote down a commutator relation for position operators:

$$[\hat{x}^\mu, \hat{x}^\nu] = iL^{\mu\nu} \quad , \quad [\hat{x}^\mu, L^{\sigma\tau}] = i(\delta^{\mu\sigma}\hat{x}^\tau - \delta^{\mu\tau}\hat{x}^\sigma)$$

The great success of QED seems to be responsible why the early ideas on noncommutative space-time had been forgotten.

Localization Due to an argument by John Archibald Wheeler there is a natural limit where measurements are made impossible by the space time structure. It goes as follows: to be able to resolve structures very close to one another (distance D), one has to invest a big amount of energy E. This value can become large enough that a signal (even a ray of light) is trapped within the *Schwartzschild radius* R_{ss} and therefore a measurement cannot take place.

$$R_{ss} = 2\frac{G}{c^4}E \geq \frac{G}{c^4}\frac{2\pi\hbar c}{D}$$

Requiring $D \geq R_{ss}$ gives the Planck length as a lower bound to a localization length:

$$\Rightarrow D \geq \sqrt{\frac{\hbar G}{c^3}}$$

So localization loses its operational meaning at very small distances.

Scales. From today's point of view, the arena of physics is situated within 61 orders of magnitude: from 10^{-35}m to 10^{-20} m is the *terra incognito*, i.e. somewhere in between changes have to take place while the largest scale structures are known to be of dimension 10^{26}m.
To resolve these problems, a quantum theory of gravitation is required. Currently there are three established candidates for this matter:

- String Theory
- Loop Quantum Gravity
- Noncommutative Quantum Field Theory

Each of them has on the one side some particular nice features but on the other side a set of painstaking problems. The approach of noncommutative quantum field theory (=: NCQFT) is to take a noncommutative manifold as the quantum space or, in other words, endow the space-time coordinates with a noncommutative structure: As will be seen later on, considering NCQFT connects short with large distance scales. We will also use the ideas of the *renormalization group* (RG), those are to integrate out degrees of freedom with the purpose that nonrenormalizable interactions will die out.

The following table collects some mile stones of recent NCQFT history:

commutative	noncommutative
measure theory	von Neumann algebras ($C^\star$-algebras)
manifolds: use algebra of functions over M	deformed algebra
differential calculus	keep differential structure
fields	projective modules
integral	trace

86	Alain Connes (mathematical concept) + Mark Rieffel: nc tori
90	application to classical Standard Model H.G. et al.: Schwingerterm, cyclic cohomology
92	H.G. J. Madore: Fuzzy S^2; regularize QFT
94	Doplicher, Fredenhagen, Roberts: Uncertainty relations, formulations of nc Minkowski free fields, [6]
95	T. Filk: Feynman rules [12]
99	Schomerus: Obtained nc models from string field theory limit [8]
2000	Grosse, Schweda, Wulkenhaar,..nc gauge models, expanded in Θ, QED turns out to be nonrenormalizable
2002	Sibold et al, Denk, Schweda,...Feynman rules [9]
2004	Grosse, Wulkenhaar, proof of a ren. nc scalar field theory
2006	Rivasseau et al., vanishing of the β function of this model to all orders ...

1.2 Formulation

In this section, the different approaches for gaining a physical theory are outlined in note form with emphasis on the relations between the observables and space-time.

Dynamical systems. Start from a manifold, go over to phase space - Define an algebra of observables - States are probability measures - Choose a Hamiltonian which gives the time evolution, TIME is a parameter, SPACE is part of phase space.

Quantum mechanics Start from a Hilbert space $\mathcal{H}$, define an operator algebra - Define observables - $[\hat{x}, \hat{p}] = i\hbar\,\mathbf{1}$ on a dense domain $\mathcal{D}$, states are linear continuous functionals like

$$a \mapsto \omega(a) = \frac{\mathrm{Tr}(\rho\, a)}{\mathrm{Tr}(\rho)}$$

The Hamiltonian determines the time evolution, TIME is still commutative!
"Space" $\rightarrow$ $\hat{x}$ operators in Hilbert space

QFT: Axioms The Wightman axioms for short demand the following from the quantum fields: they are described in terms of smeared field operators acting in a Hilbert space. The spectrum of the energy momentum operator is contained within the closed forward light cone, there exists a cyclic vacuum state and they obey to the principles of covariance and causality. Vacuum expectation values of product of fields like these are called *Wightman functions.*
Schwinger and Symanzik analytically continued these and ensured the properties: covariance, Osterwalder-Schrader positivity, symmetry and the cluster property. The analytically continued Wighman functions are called *Schwinger functions.*

Models: The case of $D = 1$ is quantum mechanics, for $D = 2, 3$ models have been constructed, some models in $D = 2$ are solvable and lead to integrable structures. For 4 dimensions only renormalized perturbation theory is available.

1.3 Ideas NCG $\rightarrow$ NCQFT

"Space" Works by Gelfand, Naimark (1947) and J. von Neumann showed that it is possible to encode the structure of manifolds into algebras of functions over the manifold. More precisely we have the

Theorem 1.1. *Gelfand-Naimark*
Any commutative $C^\star$-algebra A is isometrically *-isomorphic to the commutative $C^\star$-algebra of continuous complex functions $C(X)$ on the spectrum of A. Moreover, this spectrum is a compact, topological Hausdorff space.(For the proof see for example: [1])

Given such a space X one takes $(C(X), \| \|_\infty)$ as a C^* algebra; given the algebra, one takes the set of characters and uses Gelfands transform.

Deformed algebra (deformation quantization due to Fedosov - Kontsevich, ...) Starting with a formal power series $\mathcal{A}[[\hbar]] \ni f_0 + \hbar f_1 + \ldots$ we are able to define an associative star product:

$$f \star g = f \cdot g + \sum_{j=1}^{\infty} \hbar^j \Pi_j(f, g) \tag{1.1}$$

where $\Pi_1(f, g) - \Pi_1(g, f)$ equals the Poisson bracket of f and g. The example we will use all the time in this lecture is the **Moyal-Weyl product**

$$f \star_\theta g\,(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)\Big|_{x=y} \tag{1.2}$$

which is related to the *Weyl algebra*

$$W(f) = \int \mathrm{d}p\, e^{ip\hat{x}} \tilde{f}(p), \tag{1.3}$$

$$W(f)W(g) = W(f \star g)\,, \tag{1.4}$$

$\tilde{f}$ denoting the Fourier transform of f. There exist further examples of possible deformations like Lie algebra or quantum group deformations, which are not going to be treated here.

Chapter 2

Differential Calculus

2.1 Commutative

Getting started, the reader is reminded of some (more or less) popular manifolds considered in differential geometry: $\mathbb{R}^n$, S^n, $\mathbf{C}P^n$, $\mathbb{T}^n$, moduli spaces, Riemannian manifolds, Calabi-Yau spaces, ...
The ones of substantial interest are the compact, oriented and real manifolds. These can obtain boundaries and often the task is to embed them into $\mathbb{R}^M$. Once a manifold is picked, one usually considers a corresponding algebra of the form $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^M)/\text{Rel}$, goes over to tangent- resp. cotangent-spaces $T(M)$ resp. $T^\star(M)$ and works out the differential calculus in terms of the exterior derivative d.

Remarks. *Tangent vectors* are derivations on the algebra and therefore the *Leibniz rule* holds:

$$X(fg) = X(f)g + fX(g). \tag{2.1}$$

The set of vector fields over a manifold defines a left-module (since vector fields in reference to a holonomic basis ∂_i can be written as $v^i\partial_i$, they cannot form a right-module).

Differential forms are considered as p-linear maps with the defining property

$$(df)(X) := X(f) \tag{2.2}$$

and the space of all such differentials Ω^p forms a bimodule over $\mathcal{A}$.
Next we write down the action of the differential on 1-forms:

$$\omega \in \Omega^1 : \qquad (d\omega)(X_0, X_1) = \frac{1}{2}\left(X_0\omega(X_1) - X_1\omega(X_0) + \omega([X_0, X_1])\right) \tag{2.3}$$

while in general we have:

$$
\begin{aligned}
(d\omega_p)(X_0,\ldots,X_p) &= \frac{1}{p+1}\left(\sum_{j=0}^{p} X_j\omega_p(X_0\ldots \not{X}_j\ldots X_p)\right. \\
&\quad \left. + \sum_{0<j,i\leq p}(-1)^{i+j}\omega_p([X_j,X_i]\ldots\not{X}_j\ldots\not{X}_i\ldots X_p)\right)
\end{aligned} \tag{2.4}
$$

At the end of this subsection we remind the reader of some

<u>Further concepts:</u>

- Lie derivative
$$\mathcal{L}_X = d\circ i_X + i_X \circ d \tag{2.5}$$
- Covariant derivative
$$\nabla = d + A,\ \ A\in\Omega^1(M), \tag{2.6}$$
  with gauge transformation
$$
\begin{aligned}
\nabla &\to g^\dagger \nabla g,\ \ g^\dagger g = gg^\dagger = \mathbf{1}, \\
A &\to g^\dagger dg + g^\dagger A g
\end{aligned} \tag{2.7}
$$
- Curvature
$$F = \nabla^2 \to g^\dagger F g \tag{2.8}$$
- A gauge invariant action is given by
$$S[A] = -\frac{1}{4}\mathrm{Tr}\int F^2 \tag{2.9}$$

2.2 Matrix geometry

This term refers to the differential calculus established (say) on the space of N times N matrices with complex entries $A_N = \mathrm{Mat}(N,\mathbb{C}) = \{\mathbf{1}, \lambda_j\Big|_{j=1}^{N^2-1}\}$ with vector fields e_i - defined as inner derivations on A_N [3]:

$$e_i(\mathbf{1}) = 0,\ \ e_i(\lambda_j) = [\lambda_i,\lambda_j] = f_{ij}{}^k\lambda_k \tag{2.10}$$

In contrast to the commutative case, they do not define a left module. But exactly like in there (2.2), we define the differential forms by duality:

$$\begin{aligned} (d\lambda^j)(e_k) &:= e_k(\lambda^j) & (2.11)\\ d\mathbf{1} &= 0 & (2.12)\\ d^2 &= 0 & (2.13) \end{aligned}$$

and the left modules containing these can be written as follows:

$$\Omega^0(A_N) = A_N,\ \Omega^1(A_N) = \{fdg|f, g \in A_N\} \tag{2.14}$$

Again, these structures permit the differential complex $(\Omega^\star, d)$.
There is the possibility to choose a better basis for Ω^1: that is
$\theta^i : \theta^i(e_j) = \delta^i_j \mathbf{1}$.
d is a graded derivation and the main advantage of the new choice of basis is that the θ^k commute with all the elements of A_N [4].

As an <u>example</u>, we regard the differential calculus over 2×2 matrices:

$$A_2 = \{\mathbf{1}, \vec{\sigma}\}_4 = \Omega^0,\ \Omega^1 = \{\theta^k, \sigma_m\theta^k\}_{12},\ \Omega^2 = \{\theta^k \wedge \theta^l, \sigma_m\theta^k \wedge \theta^l\}_{12},$$
$$\Omega^3 = \{\sigma_k\theta^1 \wedge \theta^2 \wedge \theta^3, \theta^1 \wedge \theta^2 \wedge \theta^3\}_4, ...,\ \mathbf{\Delta 1} = 0,\ \Delta\sigma_k = 8\sigma_k, \Delta\theta^l 4\theta^l, ...$$

where the index numbers describe the number of linearly independent elements. Using the latter basis θ^k, we are now able to see the existence of an exceptional one form on A_N, $\Theta = -\lambda_k\theta^k$, the **Maurer-Cartan form**.

Proposition 2.1. *Basic properties of the Maurer-Cartan form*

α) $d\Theta + \Theta^2 = 0$

β) $\mathcal{L}_X\Theta = 0$

γ) $df = [f, \Theta], \forall f \in A_N$

Proof. • ad α)

$$\begin{aligned} d(\lambda_l\theta^l) &= (d\lambda_l)\theta^l + \lambda_l d\theta^l = f_{nl}{}^m\lambda_m\theta^n - \frac{1}{2}f^l{}_{mn}\theta^m \wedge \theta^n\\ \Theta^2 &= \lambda_l\theta^l\lambda_m\theta^m = \frac{1}{2}f_{lm}{}^n\lambda_n\theta^l\theta^m \end{aligned}$$

- ad β)

$$\mathcal{L}_X\Theta = i_X \circ d\Theta + d \circ i_X\theta = -(i_X\Theta)\Theta + \Theta i_X\theta - df = 0,$$

  if $i_X\Theta = -f$ and $X = \operatorname{ad} f$.

- ad γ) Let $f = \alpha_i\lambda^i$, $\alpha_i \in \mathbb{C}$, then

$$df = \alpha_i d\lambda^i = \alpha_i(e_j\lambda^i)\theta^j = \alpha_i[\lambda_j, \lambda^i]\theta^j = [f, \Theta]$$

□

2.3 Gauge model on A_N

We can now go a step further and use the differential calculus on matrices to construct a gauge model. In order to do so, we will need a matrix acting as the

Gauge potential: $A \in \Omega^1(A_N)$ with $A^\dagger = -A$. Similarly to (2.6) & (2.8), we define the

- covariant derivative $\nabla = d + A$
  and the
- curvature

$$\begin{aligned} F = \nabla^2 &= (d + A)(d + A) = d^2 + dA + Ad + A^2 = \\ &= (dA) - Ad + Ad + A^2 = (dA) + A^2 \end{aligned}$$

We impose the following

Gauge transformation: $g \in U_N$, i.e. $gg^\dagger = g^\dagger g = \mathbf{1}$ on A:

$$\begin{aligned} \nabla &\to g^\dagger \nabla g, \\ A^g &= g^\dagger A g + g^\dagger dg. \end{aligned}$$

Now another remarkable property of the Maurer Cartan form becomes evident: gauge invariance

$$\Theta^g = g^\dagger\Theta g + gdg \underbrace{=}_{\gamma)} g^\dagger\Theta g + g^\dagger(g\Theta - \Theta g) = \Theta \tag{2.15}$$

To gain something nontrivial[1] we choose Θ as origin and add a small perturbation

$$A = \Theta + \Phi, \tag{2.16}$$

with $\Phi = \phi_l \theta^l$.

$$\Rightarrow F = dA + A^2 = \frac{1}{2} F_{ij} \theta^i \wedge \theta^j, \tag{2.17}$$

where $F_{ij} = [\phi_i, \phi_j] - f_{ij}{}^k \phi_k$. Then the action

$$S[\phi] = -\frac{1}{4} Tr F_{ij} F^{ij}$$

gives the Mexican hat potential for the Higgs field. Two remarks on this point should not be left out; firstly, if we think of the graph of a ϕ^4 potential, we see two zeros: one occurs when $\phi = 0$ and the other when ϕ fulfils the Lie-algebra property. Secondly, this Mexican hat potential appears without putting anything in but the natural evolving exceptional 1-form. This allows us to put new light to the Higgs effect.

Concluding this chapter, we just tell that on a more general level, one can use the schemes explained here, too. This **universal differential calculus** starts with an associative unital algebra A and uses p-chains composed of tensor products of copies of A as a bimodule over A. The differential forms are built by "words" out of a and δa, $a \in A$ and δ being the differential which acts as a juxtaposition on these words, satisfying $\delta(1) = 0$. The term "universal" comes from the universal property of the graded differential calculus defined by $(\Omega(A), \delta)$ which guarantees the existence of a homomorphism towards another differential calculus on A.

[1] In the sense of Maurer-Cartan property α)

Chapter 3

Fuzzy Physics

As we have already remarked in the introduction, in the conventional formulation of quantum field theory, UV divergences arise when one attemps to measure the amplitude of field oscillations at a precise given point in spacetime. One way of circumventing this problem would be to modify the microscopic structure of space-time such that the concept of a space time point loses its meaning. This effect is achieved when from a sufficiently small length scale on, smooth functions are substituted by noncommuting operators which cannot be diagonalized simultaneously.
Consequently, as we saw that it is possible to have a differential calculus on finite dimensional algebras (matrix algebras) and that the differential geometry of a manifold can be described in terms of an algebra of functions defined on it, we proceed by deforming the algebra to obtain what are called *fuzzy* spaces. They can be seen as manifolds which intrinsically have got a lattice structure avoiding all possible UV divergences.
The process of taking the algebra of smooth functions defined on a manifold and deforming it in this way is called *truncation* for obvious reasons.

3.1 Fuzzy $S^2 \hookrightarrow \mathbb{R}^3$

The algebra of functions on the sphere S^2 can be described by the smooth functions modulo the constraint that they should live on the sphere, i.e.:

$$A_\infty = \frac{\{f(x^i)\}}{\{f(x^i) = 0 \text{ for } \vec{x}^2 = R^2\}} \tag{3.1}$$

This requirement in the scalar product is expressed in terms of a delta function:

$$\langle f|g\rangle = \int d^3x\, \delta(\vec{x}^2 - R^2)\, f^\star g\,; \tag{3.2}$$

while the generators of rotation are $L_i = \epsilon_{ijk} x^j \frac{\partial}{i\partial x_k}$
We can decompose the algebra of smooth functions A_∞ into irreducible spin representations:

$$A_\infty = \underbrace{[0] \oplus [1] \oplus \cdots \oplus [j]}_{A_j} \oplus \dots \tag{3.3}$$

Now the truncation comes about. If we discontinue the decomposition we obtain an algebra j which has a direct interpretation in terms of matrices: the representation of two spin states $\frac{j}{2}$ can be represented in terms of $(j+1) \times (j+1)$ matrices, i.e.

$$\mathcal{L}(\left[\frac{j}{2}\right], \left[\frac{j}{2}\right]) = [0] \oplus \cdots \oplus [j] \in \mathrm{Mat}(j+1) \tag{3.4}$$

since for $[j]$ there are $2j+1$ possibilities and $\sum_{j=0}^{N}(2j+1) = (N+1)^2$.
The truncated spaces are to be embedded into the next respectively

$$A_j \hookrightarrow A_{j+1} \hookrightarrow A_{j+2} \dots \hookrightarrow A_\infty \tag{3.5}$$

and the fuzzy analogon to the "Schrödinger" equation reads as follows:

$$[x_l^N, [x_l^N, \psi_{lm}^N]] = l(l+1)\psi_{lm}^N, \tag{3.6}$$

$$\psi_{lm}^N \psi_{l'm'}^N = \sum_{L,M} C_{mm'M}^{l\ l'\ L} \left\{ \begin{array}{ccc} l & l' & L \\ N & N & N \end{array} \right\} \psi_{LM}^N \tag{3.7}$$

In the first line, the commutator with the generator x_l^N acts like a derivative (which basic properties can be easily comprehended) and in the second one, inside the { } brackets we have the $6j$-symbol, while the C denotes the Clebsch-Gordan coefficients. Moreover, as we have mentioned, the trace over the matrix indices replaces an integral here.
With these methods, the treatment of a field theory on the fuzzy sphere is provided and, in the case of scalar fields, takes the form:
$\phi \in A_N$, $L_i \phi = [x_i^N, \phi]$

$$S_j[\phi] = \frac{1}{j+1} \, \mathrm{Tr}_{j+1}(L_i\phi \, L^i\phi + \mathrm{Pol}(\phi)) \tag{3.8}$$

$$\langle \mathcal{F} \rangle_j = \frac{1}{Z_j} \int [\mathrm{d}\phi]_j \, e^{-S_j[\phi]} \mathcal{F}(\phi) \tag{3.9}$$

3.2 Topological configurations: Projective modules

A module P over a nonzero unit ring is called projective $\rightleftharpoons$ there exists a module Q such that the direct sum $P \oplus Q$ is a free module.
This is the formal definition thus far. Actually, there is an elaborated mathematical theory behind the considerations of fuzzy structures in physics, but as we want to head for field theory in managable time, it can only be sketched here.
Our first example, the Hopf fibration, named after Heinz Hopf who studied it in a 1931 paper, was a landmark discovery in topology and is of fundamental importance in the theory of Lie groups. So let us start with the

Classical Hopf fibration It is defined by the following relations:

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \mathbb{C}^2; \qquad |\chi_1|^2 + |\chi_2|^2 = R^2,\ x_i = \chi^\dagger \sigma_i \chi \tag{3.10}$$

It can be easily seen by the explicit figure of the x_i that there exist transformations

$$\chi \quad \rightarrow \quad e^{-\frac{i}{2}\varphi}\chi$$

which leave them invariant:

$$x_i \quad \rightarrow \quad x_i$$

This intelligibly shows that the underlying bundle is nothing else than a $U(1)$ bundle over S^2.
Considering the functions on $R = \chi^\dagger\chi$ one has the
Definition:

$$\mathcal{A}_k = \{ f(\chi^\dagger\chi) | \sum_{\underline{m},\underline{n}} c_{\underline{m},\underline{n}} (\xi_1^*)^{m_1} (\chi_2^*)^{m_2} \chi_1^{n_1} \chi_2^{n_2} \Big|_{2k=m_1+m_2-n_1-n_2} \} \tag{3.11}$$

$f \rightarrow e^{ik\varphi} f$, generators of rotations: $L_i = i\chi^* \sigma^i \partial_\chi + h.c. \ldots$

Quantized: To get to a quantized version of the fibration, we contemplate the Jordan-Schwinger representation of $su(2)$, which is defined like this: the functions χ, $\chi^\dagger$ are displaced by creation and annihilation operators,

$$\chi_\alpha, \chi_\beta^* \rightarrow A_\alpha, A_\beta^\dagger \tag{3.12}$$

which naturally leads to equations of the form:

$$\mathcal{F}_N = \{(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}|0> \Big|_{n_1+n_2=N}\} \tag{3.13}$$

$$\hat{x}_i = A^\dagger \sigma_k A \tag{3.14}$$

Differently than in the classical case, we define maps:

$$A_{MN} : \mathcal{F}_N \to \mathcal{F}_M$$

$$\left[\frac{M}{2}\right] \otimes \left[\frac{N}{2}\right] = \left[\frac{|M-N|}{2}\right] \oplus \cdots \oplus \left[\frac{M+N}{2}\right] \tag{3.15}$$

Indeed what we sustain are projective modules:

$$\langle\psi_k| = (\bar{\chi}_0^k, \ldots, \sqrt{\begin{pmatrix} k \\ j \end{pmatrix}} \bar{\chi}_0^{k-j}\chi_1^j, \ldots \chi_1^k) \tag{3.16}$$

$$\exists P_k = |\psi_k\rangle\langle\psi_k|$$

3.3 Nc $\mathbb{T}^2$

Another example that quickened big interest is the noncommutative torus. As a quite simple compactification it reveals many useful applications in field theory considerations. The periodicity is of capital importance:

$$S^1 = \mathbb{R}/2\pi\mathbb{Z} \Longleftrightarrow x \simeq x + 2\pi R \tag{3.17}$$

Quantization For the purpose of quantization, we take the space $\mathcal{H} = L^2(S^1, d\varphi)$ together with the unitary operators

$$(\hat{U}\phi)(\varphi) = e^{i\varphi}\phi(\varphi)\,, \tag{3.18}$$

which lead to the equivalence relation:

$$U\hat{X}U^\dagger = \hat{X} + 2\pi R\mathbf{1},\ UU^\dagger = U^\dagger U = \mathbf{1}, \tag{3.19}$$

and to the representation in terms of ϕ

$$\hat{X} = 2\pi i R\frac{\partial}{\partial\varphi} + A(\varphi) \tag{3.20}$$

Compactification on $\mathbb{T}^2$: Since (3.19) holds, the willing reader is invited to retrace that an invariant on the torus is $U_1 U_2 U_1^\dagger U_2^\dagger$, satisfying

$$[U_1 U_2 U_1^\dagger U_2^\dagger, X_j] = 0, \tag{3.21}$$

What follows is the famous torus equation together with a representation similar to (3.20)

$$\begin{aligned} \Rightarrow U_1 U_2 &= e^{2i\pi\theta} U_2 U_1 \,, & (3.22) \\ X_j &= 2\pi R_j \frac{\partial}{\partial \varphi_j} + A_j(\varphi_1, \varphi_2) & (3.23) \end{aligned}$$

$$\mathbb{T}^2_\theta = \left\{ \sum c_{\underline{n}} (U_1)^{n_1} (U_2)^{n_2} \right\} / (3.22), \ \sum_{\underline{n}} |c_{\underline{n}}|^2 < \infty \tag{3.24}$$

A nice (i.e. practical) representation goes back to Julian Schwinger (1960); he used the so-called *clock* and *shift operators* with $\theta = \frac{1}{N}$. In the quantum group language, a space endowed with the structure $xy = qyx$ is known as *Manin plane* and there these operators look as follows

$$U_1 = \begin{pmatrix} a & & & \\ & a^2 & & \\ & & \ddots & \\ & & & a^N \end{pmatrix}, \ U_2 = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \tag{3.25}$$

with $U_1^N = U_2^N = \mathbf{1}$ and $a = q^{-1}$.

Chapter 4

QFT on $\mathbb{R}^4_\theta$

4.1 Euclidean formulation - Moyal Space

The central relation in noncommutative quantum field theory is

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$$

with $\theta^{\mu\nu} = -\theta^{\nu\mu}$, saying that coordinates of the considered space-time are replaced by operators, which obey the stated commutator-relation. Clearly, this is reminiscent of the famous commutator between position and momentum operator in standard quantum mechanics. The one studied here is the simplest thinkable deformation that satises the anti-symmetry of the commutator and the hermiticity of the operators $\hat{x}^\mu$. As a further simplication, the components $\theta^{\mu\nu}$ are real numbers of dimension length squared: allowing space-dependent components would gather some more general effects, which are not treated in these notes. As in standard quantum mechanics, this commutator implies a Heisenberg uncertainty relation, here containing of position uncertainties. As a result, space-time points no longer exist, but are replaced by cells of a dimension of at least $|\theta^{\mu\nu}|$.

Operators Hermann Weyl generalized the usual quantization method in the year 1931, associating a quantum operator to a usual phase space function. Because his procedure works for more general commutator relations than the ones that came up in quantum mechanics, it also can be applied for noncommutative geometry. Given the premise of using just functions of the Schwartz class, i.e. the functions of which all derivatives vanish at infinity, one can define the *Weyl symbol* :

$$W[\phi] = \int \mathrm{d}p\, e^{ip_\mu \hat{x}^\mu} \tilde{\phi}(p) = \int \mathrm{d}p\, u_p \tilde{\phi}(p), \tag{4.1}$$

where $\tilde{\phi}(p) \in \mathcal{S}(\mathbb{R}^4)$ denoting the Fourier transform of $\phi(x)$, the $\hat{x}$ s are the operators and u_p are the so-called *Weyl operators* satisfying

$$u_p u_q = e^{ip\theta q} u_q u_p \tag{4.2}$$

Then the differential calculus takes the following form:

$$\partial_\mu u_p = ip_\mu u_p = -i[\tilde{x}_\mu, u_p] \tag{4.3}$$

with $\tilde{x}_\mu = \theta^{-1}_{\mu\nu} x^\nu$.
Furthermore, they generate an ∞-dimensional Lie algebra

$$[u_p, u_q] = 2i\sin(p\theta q) u_{p+q} \tag{4.4}$$

Star product Instead of the non-commutative operators it is possible, and in many cases more convenient, to use a product of smooth functions in $\mathcal{S}(\mathbb{R}^4)$ providing the noncommutatvity. The simplest case of such a product is the associative, noncommutative Groenewold-Moyal *Star Product*:

$$\begin{aligned}(f\star g)(x) &= e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)\Big|_{x=y}, \\ &= const\int \mathrm{d}y \int \mathrm{d}z\, f(x+y)g(x+z)e^{iy\theta^{-1}z} \\ &= const\int \mathrm{d}p \int \mathrm{d}z\, f(x+\theta p)g(x+z)e^{ipz}\end{aligned} \tag{4.5}$$

With this tool it becomes possible to obtain a noncommutative theory from a commutative one in writing the ordinary commutative functions, but implementing this non-local product instead of the usual point-wise one.

Rules:

$$f \star_\theta g = g \star_{-\theta} f$$

A short calculation shows that

$$\int \mathrm{d}x (f\star_\theta g)(x) = \int \mathrm{d}x (g \star_\theta f)(x)$$

This can be seen when using the Fourier representation of the star product. Moreover we have

$$\int \mathrm{d}x f_1 \star \cdots \star f_N\,(x) = \int \prod_{j=1}^{N} (\mathrm{d}p_j\, \tilde{f}_j(p_j))\delta(\sum_i^N p_i) e^{i\sum_{l<k} p_l \theta p_m} \tag{4.6}$$

Another important property allows leaving out one star in any product in an integral:

$$
\begin{aligned}
\int \mathrm{d}x\ (f \star g)(x) &= \int \mathrm{d}x\ e^{\frac{i}{2}\theta^{ij}\partial_i^x \partial_j^y} f(x) g(y)\,|_{x=y} \\
&= \int \mathrm{d}x \int \mathrm{d}y \int \mathrm{d}p \int \mathrm{d}k\ e^{-\frac{i}{2}\theta^{ij}p_i k_j}\, e^{ipx} \tilde{f}(p)\, e^{iky} \tilde{g}(k-p) \\
&= (2\pi)^2 \int \mathrm{d}p \int \mathrm{d}k\ e^{-\frac{i}{2}\theta^{ij}p_i k_j} \delta(p+k) \tilde{f}(p)\, \tilde{g}(k-p) \\
&= \int \mathrm{d}x\ e^{ipx} f(x)\, e^{-ipy} g(y)\,|_{x=y} = \int \mathrm{d}x\ f(x)\, g(x)
\end{aligned}
$$

ϕ^4 on $\mathbb{R}^4$: As stated before, the noncommutative ϕ^4 theory now simply evolves when inserting star instead of usual products:

$$
S[\phi] = \int \mathrm{d}^4 x\ \left\{ \frac{1}{2} \left(\partial_\mu \phi \star \partial^\mu \phi\right)(x) + \frac{m^2}{2} (\phi \star \phi)(x) + \frac{\lambda}{4} (\phi \star \phi \star \phi \star \phi)(x) \right\} \tag{4.7}
$$

If we now analyse perturbation theory, a new kind of graphs appear: the ribbon graphs. Think of the first order Wick contractions of the two point function

$$
\langle 0 | \phi(z_1)\, \phi(z_2)\, \frac{\lambda}{4} \int \mathrm{d}^4 x\ \phi \star \phi \star \phi \star \phi\, | 0 \rangle
$$

Due to the noncommutativity of the vertex (4.7), as can be seen in (4.6), the contractions on the one hand produce graphs equivalent to those of commutative field theory (*planar* graphs) and, additionally, these *nonplanar* graphs that inhabit different asymptotical behaviour.

4.2 IR/UV mixing

Let us quantify these first order corrections on an easy graph like the tadpole. There, two third of the contributions are equal to those coming out when considering the commutative theory. Thus, these planar graphs suffer from the same UV divergences as can bee seen by implementing a UV cutoff Λ:

$$
\Gamma_{pl} = \frac{g^2}{3} \int^{\Lambda} \frac{\mathrm{d}^4 k}{k^2 + m^2} \sim \Lambda^2, \tag{4.8}
$$

One of the most puzzling effects of the last years of noncommutative perturbative quantum field theories shows up when one regards the divergences of the nonplanar graphs. The nonplanar tadpole is given by

$$\begin{aligned}\Gamma_{npl} &= \frac{g^2}{6}\int^{\Lambda}\frac{\mathrm{d}^4k}{k^2+m^2}e^{ik\theta p} \qquad (4.9)\\ &= \int \mathrm{d}^4k\int_0^\infty \mathrm{d}\alpha\, e^{-\alpha(k^2+m^2)}e^{ik\theta p} = \lim_{\Lambda\to\infty}\int_0^\infty \mathrm{d}\alpha\, e^{-\alpha m^2-\frac{\theta p\theta p}{4\alpha}-\frac{1}{\Lambda^2\alpha}}\end{aligned}$$

$$\Gamma_{pl} \simeq \frac{g^2}{48\pi^2}(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}+\mathcal{O}(1)), \qquad (4.10)$$

$$\Gamma_{npl} \simeq \frac{g^2}{96\pi^2}(\Lambda^2_{eff} - m^2\ln\frac{\Lambda^2_{eff}}{m^2}+\mathcal{O}(1)) \qquad (4.11)$$

with

$$\Lambda^2_{eff} = \frac{1}{1/\Lambda^2+(\theta p)^2}$$

and $(\theta p)^2 := -p_i(\theta^2)^{ij}p_j \geq 0$.

We will show now that the noncommutativity destroys the commutation between UV and IR limits. Obviously, first performing the limit $p \to 0$ and then $\Lambda \to \infty$ restores the "old" divergences that are renormalizable due to known schemes. Now taking the limit $\Lambda \to \infty$ lets Γ_{npl} be UV **finite** if $(\theta p)^2 \neq 0$. Minwalla, van Raamsdonk and Seiberg were the first to mention that in their famous paper of 1999 [7].
On the other hand, if one considers the limit $p \to 0$ afterwards, the UV divergence is recovered, but in the small-p, i.e. IR region. Such a connection between short and long scales is up until now only known from String Theory, which has demonstrable connections with noncommutative quantum field theory (Seiberg-Witten [10], Schomerus [8]). What makes things even worse is that in iterating these contributions one can realize arbitrary high divergences; this was also remarked in the paper by Minwalla, van Raamsdonk and Seiberg.

4.3 Renormalization of nc scalar QFT on $\mathbb{R}^4_\theta$

The relation stated between long and short distances results in the non-renormalizability of the theory (4.7). Hence, there was added a term into that action modifying long distances and led to the

<u>Theorem:</u> *Grosse, Wulkenhaar* [2]
The quantum field theory governed by the action

$$S_{GW} = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}\tilde{x}_\mu\phi \star \tilde{x}^\mu\phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\right) \tag{4.12}$$

is perturbatively renormalizable to all orders in λ.

Remark: this Lagrangian obeys to the so-called *Langmann-Szabo duality*, what can be comprehended by applying (4.3):

$$\mathcal{FT} : \phi(x) \to \tilde{\phi}(p) \Rightarrow S[\phi; \mu, \lambda, \Omega] \to \Omega^2 S[\phi; \frac{\mu}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}] \tag{4.13}$$

($\mathcal{FT}$ ... cyclic labelled Fourier transform)
Model is selfdual at $\Omega = 1$!
The proof inhabits the

Reformulation as dynamical matrix model: For simplicity, $D = 2$: $[x_1, x_2] = i\theta \to a = x_1 + ix_2$, $a^\dagger$; take basis (the *matrix basis*)

$$f_{mn} = (a^\dagger)^{\star m} e^{-a^\dagger a} a^{\star n}.$$

We have

$$f_{mn} \star f_{pq} = \delta_{np} f_{mq}, \quad \int f_{mn} = \delta_{m0}\delta_{n0} \tag{4.14}$$

Rewriting the action in terms of this basis means that there are no more oscillations

$$S = \frac{1}{2}\sum_{m,n,p,q}\left(\phi_{mn}\Delta_{nm,pq}\phi_{qp} + \frac{\lambda}{4}\phi_{mn}\phi_{np}\phi_{pq}\phi_{qm}\right) \tag{4.15}$$

Properties:

- $\Delta_{mn,pq} = 0$, if $m + p \neq n + q$ ($SO(2) \times SO(2)$ symmetry)
-

$$\begin{aligned} \Delta_{mn,pq} &= \delta_{np}\delta_{mp}((m + n + 1)\frac{1+\Omega^2}{\theta} + \mu^2) \\ &\quad -\frac{1-\Omega^2}{\theta}(\delta_{n+1,p}\delta_{m+1,q}\sqrt{pq} + \delta_{n-1,p}\delta_{m-1,q}\sqrt{mn}) \end{aligned} \tag{4.16}$$

- $D = 4$:

$$
\begin{aligned}
(\Delta^{-1})_{\substack{mm\,mm\\ 0\,0,0\,0}} &\simeq \frac{\text{const}}{\sqrt{m+1+\Omega^2(m+1)^2}}, &(4.17)\\
(\Delta^{-1})_{\substack{m_1\,m_1\,\,00\\ m_2 m_2,00}} &\simeq \frac{\text{const}}{m_1+m_2+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m_1+m_2} &(4.18)
\end{aligned}
$$

The authors follow the renormalization group equation

$$
Z_\Lambda[J] = \int \prod d\phi\, e^{-S_\Lambda[\phi,J]}, \tag{4.19}
$$

where

$$
\begin{aligned}
S_\Lambda[\phi,J] &= \sum \frac{1}{2}\phi\Delta^\Lambda\phi + L_\Lambda[\phi] + \langle\phi,J\rangle_\Lambda, &(4.20)\\
L_\infty[\phi] &= \frac{\lambda}{4}\phi\phi\phi\phi. &(4.21)
\end{aligned}
$$

and require

$$
\frac{\partial}{\partial\Lambda} Z_\Lambda[J] = 0\,.
$$

Inserting a useful ansatz into the Polchinski Equation reveals the correct propagator.
The topology of ribbon graphs drawn on Riemann surfaces of genus g with B holes is described by the following equation:

$$
2 - 2g = L - I + V,
$$

where $V = \#$ vertices, $I = \#$ double line propagators, $L = \#$ single line loops (for closed external lines). This enforces us to derive a power counting rule

$$
|A^\Lambda| \le (\sqrt{\theta}\Lambda)^{4-N+4(1-B-2g)}\,\mathrm{Pol}^{2V-N/2}[\ln\frac{\Lambda_0}{\Lambda_R}] \tag{4.22}
$$

In their <u>conclusion</u>, the authors state that the only relevant/marginal quantities occur for $B = 1$ and $g = 0$, $N = 2, 4$.

There exist some further proofs for the renormalizability of the modified (*"vulcanized"*) NC ϕ^4 theory, most notably the one of V. Rivasseau, R. Gurau, J. Magnen and F. Vignes-Tourneret, where they use *multi-scale analysis* involving the *Mehler kernel.*

Chapter 5

Taming the Landau Ghost

The proof of the renormalizability of the theory (4.12) was a motivation for going a step further and concerning the summability of this theory.

In the original theory one starts with the decomposition of the covariance of the Gauß process and the field into independent random variables: Let $C_p(x)$ be covariant for a field with momentum $\sim M^p$

$$C_p(x) = \int_{M^{-2p}}^{M^{-2p+2}} \frac{\mathrm{d}\alpha}{\alpha^{D/2}} e^{-m^2\alpha} e^{-\frac{x^2}{4\alpha}} \tag{5.1}$$

There exists a bound $|C_p(x)| \leq kM^{(D-2)p} e^{-M^p|x|}$. If we now sum up the first ρ steps we obtain:

$$C_{\leq\rho}(x) = \sum_{p=0}^{\rho} C_p(x),$$

writing $\Phi_\rho = \sum_{p=0}^{\rho} \phi_p(x)$ (field with frequencies $\leq \rho$).
The renormalization group idea goes as follows: write $\Phi_p = \phi_p + \Phi_{p-1}$, where ϕ_p are the fluctuations - integrate out and iterate:

$$e^{-S_{j-1}[\phi_{j-1}]} = Z_{j-1}[\Phi_{j-1}] = \int d\mu_j(\phi_j) e^{-S[\phi_j + \Phi_{j-1}]} \tag{5.2}$$

This procedure gives maps for the couplings:

$$\left.\begin{array}{l} \lambda_{j-1} \cong \lambda_j - \beta\lambda_j^2 \\ \lambda_j \cong \lambda_{j-1} + \beta\lambda_{j-1}^2 \end{array}\right\} \Rightarrow \lambda_j \cong \frac{\lambda_0}{1 - \lambda_0 \beta j} \tag{5.3}$$

from there one can immediately see the Landau singularity (or *Landau ghost*)

$$\lambda_{ren} = \lambda_0 \text{ fix} \quad \Rightarrow \quad \lambda_{bare} \to \infty, \tag{5.4}$$

$$\lambda_\infty = \lambda_{bare} \text{ fix} \quad \Rightarrow \quad \lambda_{ren} \to 0\,. \tag{5.5}$$

Calculating β-function for renormalizable nc ϕ^4: Applying these methods to the renormalizable theory has been done by H.G. and R. Wulkenhaar up to first order. We note that: $Z = 1 - \mathcal{O}(\lambda) \Rightarrow$

$$\frac{d\lambda_j}{dj} = \alpha(1-\Omega_j)\lambda_j^2, \ \frac{d\Omega_j}{dj} = \beta(1-\Omega_j)\lambda_j \tag{5.6}$$

This has rich consequences: at $\Omega = 1$ we reach a fixed point: $\beta_\lambda = \beta_\Omega = 0$ The next breakthrough was done by the Rivasseau group: they managed to show that the β-function vanishes up to all orders [13]! This means that the theory, whether "physical" enough due to the Euclidean instead of the Minkowski approach or not, may at least exist and may also lead to a constructive noncommutative quantum field theory, which is nontrivial! The Rivasseau group is working on the case right now and we are very anxious to its results ...

Bibliography

[1] R. S. Doran and V.A. Belfi. Characterizations of c*-algebras: The gelfand-naimark theorems. *Pure and Applied Mathematics 101, New York: Marcel-Dekker*, 1986.

[2] H. Grosse and R. Wulkenhaar. Renormalization of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base. *arXiv:hep-th/0401128v2*, 2004.

[3] R. Kerner M. Dubois-Violette and J. Madore. Non-commutative differential geometry of matrix algebras. *J. Math. Phys. 31 (1990)*, pages 316–322, 1988.

[4] J. Madore. An introduction to noncommutative differential geometry and its applications. *Cambridge University Press*, 1995.

[5] B. Riemann. On the hypotheses which lie at the bases of geometry. *Nature, Vol.* ***VIII*** *Nos. 183, 184, pp. 1417, 36, 37.*, 1873.

[6] K. Fredenhagen S. Doplicher and J. Roberts. The quantum structure of spacetime at the planck scale and quantum fields. *arXiv:hep-th/0303037*, 2003.

[7] M. van Raamsdonk S. Minwalla and N. Seiberg. Noncommutative perturbative dynamics. *arXiv:hep-th/9912072v2*, 1999.

[8] V. Schomerus. D-branes and deformation quantization. *arXiv:hep-th/9903205*, 1999.

[9] M. Schweda and St. Denk. Time ordered perturbation theory for nonlocal interactions; applications to ncqft. *arXiv:hep-th/0306101*, 2003.

[10] N. Seiberg and E. Witten. String theory and noncommutative geometry. *arXiv:hep-th/9908142v3*, 1999.

[11] H.S. Snyder. Quantized spacetime. *Phys. Rev. 71 38*, 1947.

[12] Filk T. Divergences in a field theory on quantum space. *Phys.Lett. B376 (1996) 53.*, 1996.

[13] R. Gurau V. Rivasseau, M. Disertori and J. Magnen. Vanishing of beta function of non commutative ϕ_4^4 theory to all orders. *arXiv:hep-th/0612251*, 2006.