# Max-Planck-Institut
# für Mathematik
# in den Naturwissenschaften
# Leipzig

**Interpolation theory and approximation issues in function spaces**

by

*Jan Schneider*

Lecture note no.: 43 2013

# Interpolation theory and approximation issues in function spaces

lecture notes, winter term 2012/13

Dr. Jan Schneider
Max Planck Institute for Mathematics in the Sciences,
Inselstrasse 22, 04103 Leipzig

April 3, 2013

# Contents

# Preface

This one-semester lecture is suitable for graduate students, PhD students and postdocs with background in analysis. It is meant to be a course of two hours per week and aims to present a specific point of view on the terms interpolation and approximation, which are sometimes used with a very different meaning.

In chapter 1 we present the classical real interpolation theory with focus on the so-called $K$- and $J$-methods. Chapter 2 is a very brief excursion to the theory of function spaces, including a short historical survey until the 1970s (Besov spaces). Here we provide the reader with one of the main applications of interpolation theory and the necessary background for chapter 3. This last chapter is dedicated to a very specific aspect of approximation theory in function spaces, related to compactness of operators. Although a self-contained topic, it is also connected to interpolation theory as we will point out.

# Chapter 1

# Interpolation theory

## 1.1 Motivation and basics

### 1.1.1 Introduction

We start by roughly describing the aim of the theory as follows:
Let $\mathcal{A}, \mathcal{B}$ be linear Hausdorff spaces, $A_0, A_1, B_0, B_1$ be Banach spaces with $A_i \subset \mathcal{A}, B_i \subset \mathcal{B}$ for $i = 0, 1$ and $T : \mathcal{A} \to \mathcal{B}$ a linear operator, where the restrictions

$$T|_{A_i} : A_i \to B_i, \quad i = 0, 1$$

are bounded. Then $\{A_0, A_1\}$ and $\{B_0, B_1\}$ are called interpolation pairs of Banach spaces and we ask whether there exist (intermediate) spaces $A \subset \mathcal{A}$ and $B \subset \mathcal{B}$, such that $T|_A : A \to B$ is linear and bounded. If yes, we say $A, B$ have the interpolation property with respect to $\{A_0, A_1\}$ and $\{B_0, B_1\}$.

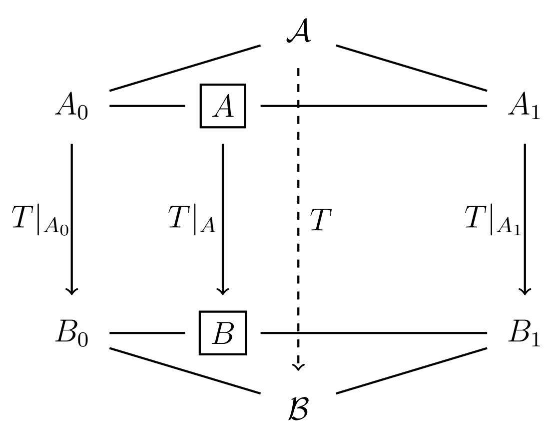


Figure 1.1: Interpolation property

In our treatment of this topic (chapter 1 of the lecture) we concentrate on two very specific (real) interpolation methods for the construction of suitable intermediate spaces.

**Results:** Riesz 1926, Thorin 1939, others 1950+
**Names:** Riesz (1886-1969), Thorin (1912-2004), Lions (1928-2001), Gagliardo(1930-2008), Calderon (1920-1998), Krejn (1917), Peetre (1935)
**Books:** [BS88],[BL76],[Tr78]

**Two examples**

**(a)** $L_{p_\theta}$ has the interpolation property with respect to $\{L_{p_0}, L_{p_1}\}$ for $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $0 < \theta < 1$.

**(b)** $C^1$ does **not** have the interpolation property with respect to $\{C, C^2\}$.

**For (a):**

This is a famous historical result called Convexity Theorem of Riesz and Thorin, it reads as follows

**Theorem 1.1.1 (Riesz/Thorin)** *Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty, p_0 \neq p_1, q_0 \neq q_1$ *and* $A : L_{p_i} \to L_{q_i}$ *be a bounded linear operator for* $i = 0, 1$. *Then for* $0 < \theta < 1$ *the operator* $A : L_{p_\theta} \to L_{q_\theta}$ *is also linear and bounded, where*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

*The operator norm satisfies the estimate*

$$\|A : L_{p_\theta} \to L_{q_\theta}\| \leq \|A : L_{p_0} \to L_{q_0}\|^{1-\theta} \|A : L_{p_1} \to L_{q_1}\|^{\theta}.$$

Note the following geometric interpretation.

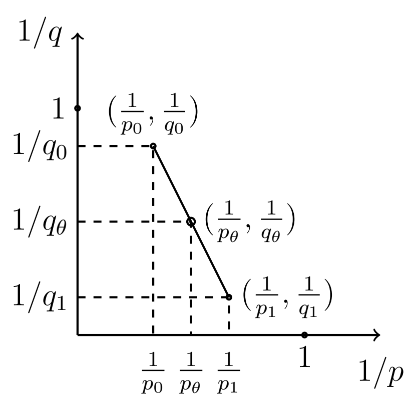


Figure 1.2: Riesz/Thorin Theorem

**Remark 1.1.2** *The original version is more general. There the $L_p$-spaces are defined over complete measure spaces. Thorin's proof uses the so-called three line theorem, compare [BL76](Thm 1.1.1). The name Convexity Theorem comes from the fact that the quantity $M_\theta = \|A : L_{p_\theta} \to L_{q_\theta}\|$ is said to be logarithmically convex. A positive function $f$ is called logarithmically convex, if $g = \log f$ is convex $\Leftrightarrow \quad f(\lambda x + (1-\lambda)y) \le f(x)^\lambda f(y)^{1-\lambda}$.*

We will prove this result later by more abstract tools. Now we motivate the importance of such interpolation results by two famous little applications of the Riesz/Thorin Theorem.
If we denote the Fourier transform by $\mathcal{F}$, we know that $\|\mathcal{F} : L_1 \to L_\infty\| \le (2\pi)^{-\frac{n}{2}}$ and $\|\mathcal{F} : L_2 \to L_2\| = 1$. Now we apply Theorem 1.1.1 to get

**Corollary 1.1.3 (Hausdorff-Young inequality)** *Let $1 \le p \le 2$ and $p'$ given by $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\mathcal{F} : L_p \to L_{p'}$ is a bounded linear operator with*
$\|\mathcal{F} : L_p \to L_{p'}\| \le (2\pi)^{-n(\frac{1}{p}-\frac{1}{2})}$.

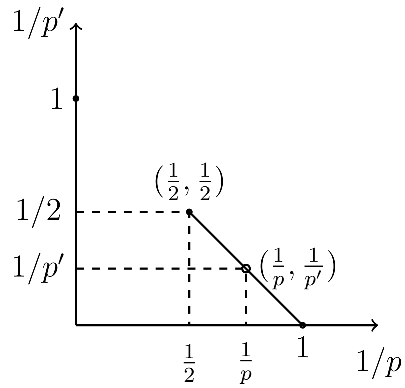


Figure 1.3: Riesz/Thorin for Hausdorff-Young

**Proof** For $p = 1$ and $p' = \infty$ as well as for $p = p' = 2$ the result is known. So let $1 < p < 2$ and we apply Theorem 1.1.1 with $p_0 = 1, q_0 = \infty$, $p_1 = q_1 = 2$ and $\theta$ such that $p_\theta = p$. Then we have $1/p = 1/p_\theta = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - (\theta/2)$

$$\Rightarrow \theta = 2(1 - 1/p) = 2/p' \quad \Rightarrow 1/q_\theta = \frac{1-\theta}{\infty} + \frac{\theta}{2} = 1/p' \Leftrightarrow q_\theta = p'.$$

Now Theorem 1.1.1 gives that $\mathcal{F} : L_p \to L_{p'}$ is linear and bounded with

$$\|\mathcal{F} : L_p \to L_{p'}\| \le \|\mathcal{F} : L_1 \to L_\infty\|^{1-\theta} \|\mathcal{F} : L_2 \to L_2\|^\theta \le (2\pi)^{-n/2(1-\theta)} = (2\pi)^{-n(\frac{1}{p}-\frac{1}{2})}.$$

□

The second application concerns the convolution operator $\mathcal{K}$ given by

$$(\mathcal{K}f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy = \int_{\mathbb{R}^n} k(y)f(x-y)dy = (k * f)(x), \quad x \in \mathbb{R}^n.$$

**Corollary 1.1.4 (Young inequality)** *Let $k \in L_r$ with $1 \leq r \leq \infty$ and $1 \leq p \leq r'$ be given with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'} = 1$. Then $\mathcal{K} : L_p \to L_q$ is linear and bounded with $\|\mathcal{K} : L_p \to L_q\| \leq \|k|L_r\|$, where $q$ is given by $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$.*

**Proof** Step 1: We consider $\mathcal{K} : L_{r'} \to L_\infty$.
By using Hölders inequality we find

$$\|\mathcal{K}f|L_\infty\| = \sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} k(x-y)f(y)dy \right| \leq \sup_{x\in\mathbb{R}^n} \|k(x-\cdot)|L_r\|\cdot\|f|L_{r'}\| \leq \|k|L_r\|\cdot\|f|L_{r'}\|.$$

Step 2: Now we consider $\mathcal{K} : L_1 \to L_r$.
Using the generalized triangle inequality for integrals (see [HLP52], Thm. 202) we get

$$\begin{aligned} \|\mathcal{K}f|L_r\| &= \left\| \int_{\mathbb{R}^n} k(x-y)f(y)dy|L_r \right\| \leq \int_{\mathbb{R}^n} \|k(\cdot - y)f(y)|L_r\|dy \\ &= \int_{\mathbb{R}^n} |f(y)| \cdot \|k(\cdot - y)|L_r\|dy = \|k|L_r\| \cdot \|f|L_1\|. \end{aligned}$$

Step 3:
Let $1 < p < r'$ and we apply Theorem 1.1.1 with $p_0 = r', q_0 = \infty, p_1 = 1, q_1 = r$ and $\theta$ such that $p_\theta = p$. Then we have $1/p = 1/p_\theta = \frac{1-\theta}{r'} + \frac{\theta}{1} = 1/r' + \theta/r$

$$\Rightarrow \theta = r(1/p - 1/r') \quad \Rightarrow 1/q_\theta = \frac{1-\theta}{\infty} + \frac{\theta}{r} = 1/p - 1/r' = 1/q.$$

Now Theorem 1.1.1 gives that $\mathcal{K} : L_p \to L_q$ is linear and bounded with

$$\|\mathcal{K} : L_p \to L_q\| \leq \|\mathcal{K} : L_{r'} \to L_\infty\|^{1-\theta} \|\mathcal{K} : L_1 \to L_r\|^\theta \leq \|k|L_r\|.$$

□

Note again the visualization:

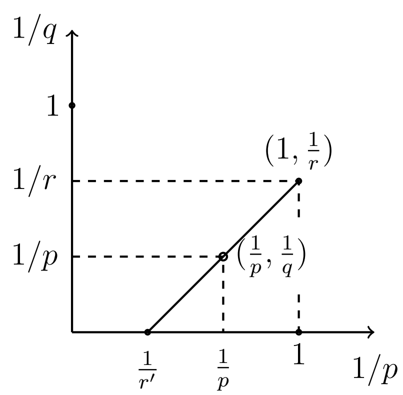


Figure 1.4: Riesz/Thorin for Young

**Remark 1.1.5** *For all $f \in L_p$ we have $\|k * f|L_q\| \leq \|f|L_p\| \cdot \|k|L_r\|$, which often caries the name Young inequality in the literature.*

**For (b):**

In [MS] Mityagin and Semenov construct the following counter example for the space $C^1[-1,1]$.

**Theorem 1.1.6** *Let for $\varepsilon \in (0,1]$ the operator $V_\varepsilon$ on $C[-1,1]$ be given as $(V_\varepsilon f)(x) = \int\limits_{-1}^{1} \frac{x}{x^2+y^2+\varepsilon^2}[f(y)-f(0)]dy$. Then the following is true:*

*(a)* $V_\varepsilon : C[-1,1] \to C^\infty[-1,1]$

*(b)* $\|V_\varepsilon : C[-1,1] \to C[-1,1]\| < 2\pi$ *and* $\|V_\varepsilon : C^2[-1,1] \to C^2[-1,1]\| < 5\pi + 2$ *independent of* $\varepsilon$

*(c) The function $f_\varepsilon(x) = \sqrt{y^2-\varepsilon^2} - \varepsilon$ satisfies $\|f_\varepsilon|C^1[-1,1]\| \leq 2$ independent of $\varepsilon$ and $(V_\varepsilon f_\varepsilon)'(0) > 2\ln(\frac{1}{10\varepsilon})$. Therefore, $\|V_\varepsilon : C^1[-1,1] \to C^1[-1,1]\| > \ln(\frac{1}{10\varepsilon})$.*

Later in Theorem 1.1.16 we learn that, if $C^1$ would have the interpolation property with respect to $\{C, C^2\}$ then there would exist a constant $c > 0$ such that

$$\|V_\varepsilon|\mathcal{L}(C^1)\| \leq c \max(\|V_\varepsilon|\mathcal{L}(C)\|, \|V_\varepsilon|\mathcal{L}(C^2)\|)$$

for all $\varepsilon > 0$. But by assertion (c) of the theorem above, that is not the case.

## 1.1.2 Definitions and notation

In this subsection we skip some of the proofs because they are somehow classical in functional analysis. Nevertheless, we give of course references for them.

**Definition 1.1.7** *Let $A_0, A_1, B_0$ and $B_1$ be (complex) Banach spaces. Then we call $\{A_0, A_1\}$ and $\{B_0, B_1\}$ interpolation pairs (of Banach spaces) if there exist linear Hausdorff spaces $\mathcal{A}, \mathcal{B}$ such that $A_i \hookrightarrow \mathcal{A}$ and $B_i \hookrightarrow \mathcal{B}$ for $i = 0, 1$.*

For each interpolation pair one can form two new spaces that will form the range for the interpolation.

**Lemma 1.1.8** *Let $\{A_0, A_1\}$ be an interpolation pair. Then*

$$A_0 + A_1 = \{a \in \mathcal{A} : \exists a_0 \in A_0, a_1 \in A_1 : a = a_0 + a_1\},$$

*with*

$$\|a|A_0 + A_1\| = \inf_{a = a_0 + a_1, a_i \in A_1} (\|a_0|A_0\| + \|a_1|A_1\|)$$

*and $A_0 \cap A_1$ with*

$$\|a|A_0 \cap A_1\| = \max(\|a|A_0\|, \|a|A_1\|)$$

*are Banach spaces. Furthermore, it holds for $i = 0, 1$ that*

$$A_0 \cap A_1 \hookrightarrow A_i \hookrightarrow A_0 + A_1.$$

By the symbol $\hookrightarrow$ we always mean a continuous embedding. A proof can be found in [BS88] (Ch. 3, Th. 1.3).

**Definition 1.1.9** *Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be interpolation pairs. The set of all linear operators $T : A_0 + A_1 \to B_0 + B_1$, having bounded restrictions $T|_{A_i} : A_i \to B_i$, for $i = 0, 1$, is denoted by $\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$.*

We will write $\mathcal{L}(\{A_0, A_1\})$ for $\mathcal{L}(\{A_0, A_1\}, \{A_0, A_1\})$.

**Lemma 1.1.10** *The set $\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ equipped with the norm*

$$\|T|\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\}) = \max(\|T|\mathcal{L}(A_0, B_0)\|, \|T|\mathcal{L}(A_1, B_1)\|)$$

*is a Banach space continuously embedded into $\mathcal{L}(A_0 + A_1, B_0 + B_1)$.*

For the proof see [BS88] (Ch. 3, Prop. 1.7). Now we can exactly define what an interpolation space "between" two other spaces really means.

**Definition 1.1.11** *Let $\{A_0, A_1\}$ be an interpolation pair.*
*(i) A Banach space $A \in \mathcal{A}$ is called intermediate space with respect to $\{A_0, A_1\}$, if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$.*
*(ii) An intermediate space $A \in \mathcal{A}$ is called interpolation space with respect to $\{A_0, A_1\}$, if in addition $T(A) \subset A$ for all $T \in \mathcal{L}(\{A_0, A_1\})$.*

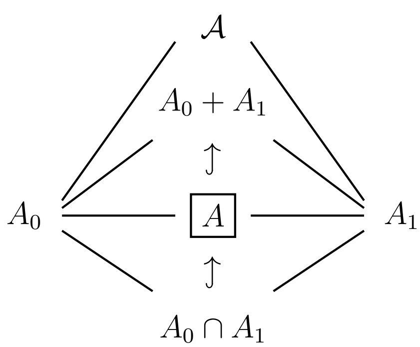


Figure 1.5: Embedded interpolation spaces

If such an interpolation space $A$ exists, such that the restrictions $T|_A$ are even bounded for all operators $T \in \mathcal{L}(\{A_0, A_1\})$, one also says: $A$ has the interpolation property with respect to $\{A_0, A_1\}$ (compare with the Introduction and Figure 1.1). But this boundedness follows immediately with the next lemma.

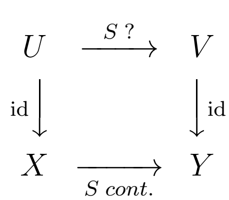


Figure 1.6: Continuity of restrictions

**Lemma 1.1.12** *Let $U, V, X, Y$ be Banach spaces with $U \hookrightarrow X$, $V \hookrightarrow Y$ and $S : U \to V$ an operator from $\mathcal{L}(X, Y)$. Then even $S \in \mathcal{L}(U, V)$ holds.*

The proof in [BS88] (Ch. 3, Lemma 1.9) uses the closed graph theorem, compare Figure 1.6.

**Corollary 1.1.13** *Let $A$ be an interpolation space with respect to $\{A_0, A_1\}$. For all $T \in \mathcal{L}(\{A_0, A_1\})$ then $T|_A \in \mathcal{L}(A)$ holds.*

**Proof** We are using Lemma 1.1.12 with $U = V = A$, $X = Y = A_0 + A_1$ and $S = T$ (compare Figure 1.6). From Definition 1.1.11 we know $A \hookrightarrow A_0 + A_1$ and $T : A \to A$, where Lemma 1.1.10 tells us $T \in \mathcal{L}(A_0 + A_1)$. Then the assumptions of Lemma 1.1.12 are fulfilled and we get $T|_A \in \mathcal{L}(A)$.

□

Now we generalize this concept to our original situation of interest, where $\{A_0, A_1\}$ does not necessarily equal $\{B_0, B_1\}$.

**Definition 1.1.14** *Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be interpolation pairs and $A \in \mathcal{A}, B \in \mathcal{B}$ intermediate spaces. If for all $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ the condition $T(A) \subset B$ holds, then we call $[\{A_0, A_1\}, A]$ interpolation tripel with respect to $[\{B_0, B_1\}, B]$.*

**Corollary 1.1.15** *Let $[\{A_0, A_1\}, A]$ be an interpolation tripel with respect to $[\{B_0, B_1\}, B]$, then for all $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ even $T|_A \in \mathcal{L}(A, B)$ holds.*

**Proof** Here we use Lemma 1.1.12 with $U = A$, $V = B$, $X = A_0 + A_1$, $Y = B_0 + B_1$ and $S = T$ (compare Figure 1.6). Definition 1.1.11 gives $A \hookrightarrow A_0 + A_1$ and $B \hookrightarrow B_0 + B_1$, where from Definition 1.1.14 we know $T : A \to B$. Lemma 1.1.10 tells us $T \in \mathcal{L}(A_0 + A_1, B_0 + B_1)$, therefore, Lemma 1.1.12 gives $T \in \mathcal{L}(A, B)$.

□

The following important result even quantifies this boundedness (remember the considerations after Theorem 1.1.6).

**Theorem 1.1.16** *Let $[\{A_0, A_1\}, A]$ be an interpolation tripel with respect to $[\{B_0, B_1\}, B]$. Then there exists a constant $c > 0$, such that for all $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ the following estimate holds*

$$\|T|_A|\mathcal{L}(A, B)\| \le c \max(\|T|\mathcal{L}(A_0, B_0)\|, \|T|\mathcal{L}(A_1, B_1)\|).$$

**Proof** We want to use Lemma 1.1.12 with $U = \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$, $V = \mathcal{L}(A, B)$, $X = \mathcal{L}(A_0 + A_1, B_0 + B_1)$, $Y = \mathcal{L}(A, B_0 + B_1)$ and $S = R$, where

$$R : \mathcal{L}(A_0 + A_1, B_0 + B_1) \to \mathcal{L}(A, B_0 + B_1) \quad \text{with} \quad R(T) = T|_A.$$

$$\begin{array}{ccc}
\mathcal{L}(\{A_0,A_1\},\{B_0,B_1\}) & \xrightarrow{R\,?} & \mathcal{L}(A,B) \\
\Big\downarrow \text{id} & & \Big\downarrow \text{id} \\
\mathcal{L}(A_0+A_1,B_0+B_1) & \xrightarrow[R\ cont.]{} & \mathcal{L}(A,B_0+B_1)
\end{array}$$

Figure 1.7: Continuity of R

To do so, we have to show the two embeddings: $U \hookrightarrow X$, $V \hookrightarrow Y$ and the mapping properties $R : U \to V$ with $R \in \mathcal{L}(X,Y)$. We start with the latter. Corollary 1.1.15 immediately gives $R : U \to V$. So let $a \in A$, then

$$\|T|_A(a)|B_0+B_1\| \leq \|T|\mathcal{L}(A_0+A_1,B_0+B_1)\| \cdot \|a|A_0+A_1\|.$$

Since $A \hookrightarrow A_0 + A_1$ we know $\|a|A_0+A_1\| \leq c\|a|A\|$ and get

$$\|T|_A|\mathcal{L}(A,B_0+B_1)\| \leq c\|T|\mathcal{L}(A_0+A_1,B_0+B_1)\|,$$

which means $R \in \mathcal{L}(X,Y)$.
The embedding $U \hookrightarrow X$ is a consequence of Lemma 1.1.10. To prove $V \hookrightarrow Y$ let $a \in A$. Then

$$\|T|_A(a)|B_0+B_1\| \leq c\|T|_A(a)|B\| \leq c\|T|_A|\mathcal{L}(A,B)\| \cdot \|a|A\|,$$

because of $B \hookrightarrow B_0 + B_1$ and $T|_A \in \mathcal{L}(A,B)$, but that means

$$\|T|_A|\mathcal{L}(A,B_0+B_1)\| \leq c\|T|_A|\mathcal{L}(A,B)\|,$$

which is the desired estimate.
Now all assumptions of Lemma 1.1.12 are satisfied and we conclude that

$$R : \mathcal{L}(\{A_0,A_1\},\{B_0,B_1\}) \to \mathcal{L}(A,B)$$

is linear and bounded, which means

$$\|T|_A|\mathcal{L}(A,B)\| \leq c\|T|\mathcal{L}(\{A_0,A_1\},\{B_0,B_1\})\| = c\max(\|T|\mathcal{L}(A_0,B_0)\|,\|T|\mathcal{L}(A_1,B_1)\|)$$

. □

**Remark 1.1.17** *The constant $c = c(A,B)$ appearing in the above Theorem is called interpolation constant. In case $c = 1$ the spaces $A, B$ are called exact interpolation spaces.*

## 1.2 Real interpolation methods

In this section we introduce two different but strongly connencted methods to construct interpolation spaces explicitely.

### 1.2.1 The $K$-method

We start by defining the main tool for the construction within this method.

**Definition 1.2.1** *Let $\{A_0, A_1\}$ be an interpolation pair. The $K$-functional of Peetre is defined by*

$$K(t, a; A_0, A_1) = K(t, a) = \inf_{a=a_0+a_1, a_i \in A_i} (\|a_0|A_0\| + t\|a_1|A_1\|)$$

*for $t > 0$ and $a \in A_0 + A_1$.*

Obviously $K(1, a) = \|a|A_0 + A_1\|$ and for each fixed $t > 0$ the function $K(t, a)$ is an equivalent norm in $A_0 + A_1$.

**Lemma 1.2.2** *Let $a \in A_0 + A_1$. Then the function $K(t, a)$ is increasing, continuous and concave. Furthermore,*

$$\min(1, t)\|a|A_0 + A_1\| \le K(t, a) \le \max(1, t)\|a|A_0 + A_1\|. \tag{1.1}$$

**Proof** Clearly, the function is increasing and the estimate is easy to verify. It remains to prove the concavity, i.e.,

$$K((1 - \lambda)t_1 + \lambda t_2, a) \ge (1 - \lambda)K(t_1, a) + \lambda K(t_2, a)$$

for $0 < \lambda < 1$. Let $0 < t_1 < t < t_2 < \infty$, then by definition we have

$$\begin{aligned} \frac{t_2 - t}{t_2 - t_1} K(t_1, a) + \frac{t - t_1}{t_2 - t_1} K(t_2, a) &\le \|a_0|A_0\| + t\|a_1|A_1\| \\ \frac{t_2 - t}{t_2 - t_1} K(t_1, a) + \frac{t - t_1}{t_2 - t_1} K(t_2, a) &\le K(t, a) \end{aligned}$$

after taking the infimum over all representations $a = a_0 + a_1$ on the right-hand side. With $\lambda = \frac{t-t_1}{t_2-t_1}$ we see that $K(t, a)$ is concave. The fact that it is increasing implies continuity.

□

**Geometric interpretation**

We consider the set

$$\Gamma(a) = \{(x_1, x_2) \in \mathbb{R}^2 : \exists a_i \in A_i : a = a_0 + a_1, \|a_i|A_1\| \le x_i \text{ for } i = 0, 1\},$$

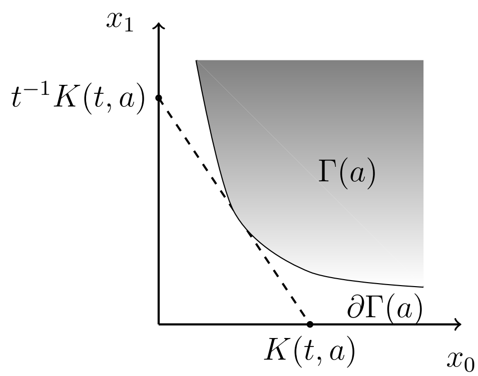


Figure 1.8: Gagliardo diagram

which can easily be verified to be convex. Then we have

$$K(t,a) = \inf_{(x_0,x_1)\in\Gamma(a)} (x_0 + tx_1) = \inf_{(x_0,x_1)\in\partial\Gamma(a)} (x_0 + tx_1),$$

which means that $K(t,a)$ is the zero ($x_0$-intercept) of the tangent to $\partial\Gamma(a)$ (boundary of $\Gamma(a)$) having slope $-t^{-1}$, compare Figure 1.8.
To define the interpolation space constructed by the $K$-method we will use the following expression. For $0 < \theta < 1 \le q \le \infty$ and a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ we set

$$\Phi_{\theta,q}(\varphi) = \begin{cases} \left( \int_0^\infty [t^{-\theta}\varphi(t)]^q \frac{dt}{t} \right)^{1/q} & : \quad q < \infty \\ \sup_{0<t<\infty} t^{-\theta}\varphi(t) & : \quad q = \infty. \end{cases} \tag{1.2}$$

**Definition 1.2.3** *Let $0 < \theta < 1 \le q \le \infty$ and $\{A_0, A_1\}$ be an interpolation pair. Then we define*

$$(A_0, A_1)_{\theta,q} = \Big\{ a \in A_0 + A_1 : \big\| a | (A_0, A_1)_{\theta,q} \big\| = \Phi_{\theta,q}\big(K(\cdot, a)\big) < \infty \Big\}.$$

Explicitely we can write the norm

$$\big\| a | (A_0, A_1)_{\theta,q} \big\| = \begin{cases} \left( \int_0^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t} \right)^{1/q} & : \quad q < \infty \\ \sup_{0<t<\infty} t^{-\theta}K(t,a) & : \quad q = \infty. \end{cases}$$

Here we verify that for $\theta \le 0$ or $\theta \ge 1$ the space $(A_0, A_1)_{\theta,q} = \{0\}$ is not interesting. For example, if $q < \infty$ we find with (1.1)

$$\int_0^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t} \ge K(1,a)^q \left( \int_0^1 t^{(1-\theta)q} \frac{dt}{t} + \int_1^\infty t^{-\theta q} \frac{dt}{t} \right).$$

Since the first integral diverges for $\theta \geq 1$ and the second for $\theta \leq 0$ we see

$$\left\| a|(A_0, A_1)_{\theta,q} \right\| < \infty \Longleftrightarrow 0 = K(1,a) = \|a|A_0 + A_1\| \Longleftrightarrow a = 0.$$

For $q = \infty$ the argument is similar. Now we state that by the above definition we really have an interpolation space, which is the main result of this subsection.

**Theorem 1.2.4** *Let $0 < \theta < 1 \leq q \leq \infty$ and $\{A_0, A_1\}$ be an interpolation pair. Then*
*(i) $(A_0, A_1)_{\theta,q}$ is an interpolation space with respect to $\{A_0, A_1\}$.*
*(ii) The estimate*

$$\left\| T|\mathcal{L}\big((A_0, A_1)_{\theta,q}, (B_0, B_1)_{\theta,q}\big) \right\| \leq \|T|\mathcal{L}(A_0, B_0)\|^{1-\theta} \|T|\mathcal{L}(A_1, B_1)\|^{\theta}$$

*holds for all $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$.*
*(iii) The estimate*

$$K(t,a) \leq ct^{\theta} \left\| a|(A_0, A_1)_{\theta,q} \right\|$$

*holds for all $a \in (A_0, A_1)_{\theta,q}$.*

**Remark 1.2.5** *The $K$-method is called exact, because the constructed interpolation spaces are exact, compare estimate (ii) of the Theorem above and Remark 1.1.17.*

**Proof** step 1: First we prove (iii).
Let $s > 0$, then $s^{-\theta q} = \theta q \int\limits_s^\infty t^{-\theta q} \frac{dt}{t}$ and we can write

$$\begin{aligned} s^{-\theta} K(s,a) = K(s,a) \left( \theta q \int_s^\infty t^{-\theta q} \frac{dt}{t} \right)^{1/q} &\leq \left( \theta q \int_s^\infty t^{-\theta q} K(t,a)^q \frac{dt}{t} \right)^{1/q} \\ &\leq (\theta q)^{1/q} \left\| a|(A_0, A_1)_{\theta,q} \right\| \end{aligned}$$

because $K(\cdot, a)$ is increasing.
step 2: Here we prove (ii).
Let $0 \neq T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ and $a \in A_0 + A_1$. Then by definition

$$K(t, Ta; B_0, B_1) = \inf_{Ta = b_0 + b_1, b_i \in B_i} (\|b_0|B_0\| + t\|b_1|B_1\|).$$

If we now restrict the infimum to $b_i = Ta_i$ for all $a = a_0 + a_1, a_i \in A_i$, we can further estimate

$$\begin{aligned} K(t, Ta; B_0, B_1) &\leq \inf_{a = a_0 + a_1, a_i \in A_i} (\|Ta_0|B_0\| + t\|Ta_1|B_1\|) \\ &\leq \inf_{a = a_0 + a_1, a_i \in A_i} (\|T|\mathcal{L}(A_0, B_0)\| \cdot \|a_0|A_0\| + t\|T|\mathcal{L}(A_1, B_1)\| \cdot \|a_1|A_1\|) \\ &= \|T|\mathcal{L}(A_0, B_0)\| K(\tau, a; A_0, A_1), \end{aligned}$$

where we set $\tau = t\frac{\|T|\mathcal{L}(A_1,B_1)\|}{\|T|\mathcal{L}(A_0,B_0)\|}$. Therefore, we calculate

$$\begin{aligned}\left\|Ta|(B_0,B_1)_{\theta,q}\right\| &= \left(\int_0^\infty [t^{-\theta}K(t,Ta;B_0,B_1)]^q \frac{dt}{t}\right)^{1/q}\\ &\leq \|T|\mathcal{L}(A_0,B_0)\| \left(\int_0^\infty [t^{-\theta}K(\tau,a;A_0,A_1)]^q \frac{dt}{t}\right)^{1/q}\\ &\leq \|T|\mathcal{L}(A_0,B_0)\| \left(\frac{\|T|\mathcal{L}(A_1,B_1)\|}{\|T|\mathcal{L}(A_0,B_0)\|}\right)^\theta \left(\int_0^\infty [\tau^{-\theta}K(\tau,a;A_0,A_1)]^q \frac{d\tau}{\tau}\right)^{1/q}\\ &= \|T|\mathcal{L}(A_0,B_0)\|^{1-\theta}\|T|\mathcal{L}(A_1,B_1)\|^\theta \left\|a|(A_0,A_1)_{\theta,q}\right\|, \qquad (1.3)\end{aligned}$$

which is the desired estimate for (ii).
step 3: Finally we prove the main assertion (i).
It is easy to verify that $\Phi_{\theta,q}(K(\cdot,a))$ is a norm, where the triangle inequality follows from Minkowski's inequality. It remains to show that $(A_0,A_1)_{\theta,q}$ is complete and an intermediate space with respect to $\{A_0,A_1\}$. Then because of (1.3) we have $T((A_0,A_1)_{\theta,q}) \subset (A_0,A_1)_{\theta,q}$ for all $T \in \mathcal{L}(\{A_0,A_1\},\{B_0,B_1\})$, which says by Definition 1.1.11(ii) that $(A_0,A_1)_{\theta,q}$ is an interpolation space.
Therefore, let $(a^n)_n$ be a Cauchy sequence in $(A_0,A_1)_{\theta,q}$. Then because of (iii) $\big(K(t,a^n)\big)_n$ is for every fixed $t$ a Cauchy sequence of positive numbers, in particular $K(1,\cdot) = \|\cdot|A_0+A_1\|$. But that means $(a^n)_n$ is a Cauchy sequence in $A_0+A_1$, which is complete, therefore, there exists an $a \in A_0+A_1$ with $\lim_{n\to\infty} a^n = a$ in $A_0+A_1$. We have to check the convergence in $(A_0,A_1)_{\theta,q}$. Let $q<\infty$ (for $q=\infty$ the proof is analog). Since $(a^n)_n$ is a Cauchy sequence in $(A_0,A_1)_{\theta,q}$ we know

$$\forall \delta > 0\ \exists n_0(\delta) \text{ such that } \forall m > n \geq n_0(\delta) : \left\|a^m - a^n|(A_0,A_1)_{\theta,q}\right\| < \frac{\delta}{2}.$$

Then for $N > 1 > \varepsilon > 0$ we can estimate with (1.1)

$$\begin{aligned}\left(\int_\varepsilon^N [t^{-\theta}K(t,a-a^n)]^q \frac{dt}{t}\right)^{1/q} &\leq \left(\int_\varepsilon^N [t^{-\theta}K(t,a^m-a^n)]^q \frac{dt}{t}\right)^{1/q} + \left(\int_\varepsilon^N [t^{-\theta}K(t,a-a^m)]^q \frac{dt}{t}\right)^{1/q}\\ &\leq \frac{\delta}{2} + \left(\int_\varepsilon^N [t^{-\theta}K(N,a-a^m)]^q \frac{dt}{t}\right)^{1/q}\\ &\leq \frac{\delta}{2} + N\|a-a^m|A_0+A_1\| \left(\int_\varepsilon^N t^{-\theta q} \frac{dt}{t}\right)^{1/q}\\ &< \frac{\delta}{2} + cN\varepsilon^{-\theta}\|a-a^m|A_0+A_1\| < \delta,\end{aligned}$$

since the second summand in the last line is smaller than $\delta/2$ for $m > m_1(\delta, \varepsilon, N)$. If we now consider the limits $N \to \infty$ and $\varepsilon \to 0$ we finally get

$$\left\| a - a^n | (A_0, A_1)_{\theta,q} \right\| < \delta \quad \text{for} \quad n \geq n_0(\delta),$$

which gives the convergence in $(A_0, A_1)_{\theta,q}$ and by triangle inequality it follows $a \in (A_0, A_1)_{\theta,q}$, that establishes the completeness of $(A_0, A_1)_{\theta,q}$.
At the end we check if $(A_0, A_1)_{\theta,q}$ is an intermediate space, i.e., $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow A_0 + A_1$. Therefore, let $a \in A_0 \cap A_1$, then by definition $K(t, a) \leq \min(1, t)\|a|A_0 \cap A_1\|$ and we can estimate

$$\begin{aligned}
\left\| a | (A_0, A_1)_{\theta,q} \right\| &\leq \left( \int_0^1 [t^{-\theta} K(t,a)]^q \frac{dt}{t} \right)^{1/q} + \left( \int_1^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t} \right)^{1/q} \\
&\leq \|a|A_0 \cap A_1\| \left[ \left( \int_0^1 t^{(1-\theta)q} \frac{dt}{t} \right)^{1/q} + \left( \int_1^\infty t^{-\theta q} \frac{dt}{t} \right)^{1/q} \right] \\
&\leq \|a|A_0 \cap A_1\| \left[ \left( \frac{1}{(1-\theta)q} \right)^{1/q} + \left( \frac{1}{\theta q} \right)^{1/q} \right],
\end{aligned}$$

which means $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,q}$. The other inclusion follows from $\|a|A_0 + A_1\| = K(1, a)$ and (iii).

□

Now we know that $(A_0, A_1)_{\theta,q}$ has the desired interpolation property but we can prove even more to shed some light on the meaning of the two indices $\theta$ and $q$.

**Theorem 1.2.6** *Let $\{A_0, A_1\}$ be an interpolation pair and $0 < \theta < 1 \leq q \leq \infty$. Then the following properties hold:*
*(i)* $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$
*(ii) If $q \leq r \leq \infty$, then*

$$(A_0, A_1)_{\theta,1} \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{\theta,r} \hookrightarrow (A_0, A_1)_{\theta,\infty}.$$

*(iii) If $A_0 \hookrightarrow A_1$, then*

$$(A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{\eta,r}$$

*for all $0 < \theta < \eta < 1 \leq q, r \leq \infty$.*
*(iv) If $A_0 = A_1 = A$, then $(A_0, A_1)_{\theta,q} = A$ with*

$$\left\| a | (A_0, A_1)_{\theta,q} \right\| = c^* \|a|A\| \quad \text{where} \quad c^* = \frac{1}{[\theta(1-\theta)q]^{1/q}}.$$

*(v) For all $a \in A_0 \cap A_1$ the estimate*

$$\left\| a | (A_0, A_1)_{\theta,q} \right\| \leq c^* \|a|A_0\|^{1-\theta} \|a|A_1\|^{\theta}.$$

*holds.*

**Proof** Assertion (i) is an easy consequence of $K(t, a; A_0, A_1) = tK(t^{-1}, a; A_1, A_0)$. We prove (ii). From Theorem 1.2.4(iii) we know

$$\big\|a|(A_0, A_1)_{\theta,\infty}\big\| = \sup_{0<t<\infty} t^{-\theta} K(t, a) \leq c\big\|a|(A_0, A_1)_{\theta,r}\big\|.$$

Therefore, $(A_0, A_1)_{\theta,r} \hookrightarrow (A_0, A_1)_{\theta,\infty}$. Now let $1 \leq q \leq r < \infty$, then

$$\big\|a|(A_0, A_1)_{\theta,r}\big\| = \left(\int_0^\infty [t^{-\theta} K(t, a)]^r \frac{dt}{t}\right)^{1/r} = \left(\int_0^\infty [t^{-\theta} K(t, a)]^q [t^{-\theta} K(t, a)]^{r-q} \frac{dt}{t}\right)^{1/r}.$$

We use again Theorem 1.2.4(iii) for the artificial term inside the inegral and get

$$\big\|a|(A_0, A_1)_{\theta,r}\big\| \leq c\big\|a|(A_0, A_1)_{\theta,q}\big\|^{1-q/r} \left(\int_0^\infty [t^{-\theta} K(t, a)]^q \frac{dt}{t}\right)^{1/r} = c\big\|a|(A_0, A_1)_{\theta,q}\big\|,$$

which proves $(A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{\theta,r}$.
Now we check (iii). If $A_0 \hookrightarrow A_1$ then $A_1 = A_0 + A_1$ and we have $K(t, a) \leq t\|a|A_1\|$ for all $a \in A_0 + A_1 = A_1$. Because of (ii) it suffices to show $(A_0, A_1)_{\theta,\infty} \hookrightarrow (A_0, A_1)_{\eta,1}$ for $0 < \theta < \eta < 1$. So let $a \in (A_0, A_1)_{\theta,\infty}$, then we estimate

$$\begin{aligned}
\big\|a|(A_0, A_1)_{\eta,1}\big\| &= \int_0^1 t^{-\eta} K(t, a) \frac{dt}{t} + \int_1^\infty t^{-\eta+\theta} t^{-\theta} K(t, a) \frac{dt}{t} \\
&\leq \|a|A_1\| \int_0^1 t^{1-\eta} \frac{dt}{t} + \sup_{0<t<\infty} t^{-\theta} K(t, a) \int_1^\infty t^{-\eta+\theta} \frac{dt}{t} \\
&\leq c\|a|A_1\| + c'\big\|a|(A_0, A_1)_{\theta,\infty}\big\|.
\end{aligned}$$

From Theorem 1.2.4(i) we know $(A_0, A_1)_{\theta,\infty} \hookrightarrow A_0 + A_1 = A_1$ and therefore $\|a|A_1\| \leq c''\big\|a|(A_0, A_1)_{\theta,\infty}\big\|$, so altogether we get

$$\big\|a|(A_0, A_1)_{\eta,1}\big\| \leq c'''\big\|a|(A_0, A_1)_{\theta,\infty}\big\|,$$

which means $(A_0, A_1)_{\theta,\infty} \hookrightarrow (A_0, A_1)_{\eta,1}$.
To check assertion (iv) we write $K(t, a) = \min(1, t)\|a|A\|$ for all $a \in A_0 = A_1 = A$. Then we estimate

$$\begin{aligned}
\big\|a|(A_0, A_1)_{\theta,q}\|^q &= \int_0^\infty [t^{-\theta} K(t, a)]^q \frac{dt}{t} = \|a|A\|^q \left(\int_0^1 t^{(1-\theta)q} \frac{dt}{t} + \int_1^\infty t^{-\theta q} \frac{dt}{t}\right) \\
&= \|a|A\|^q \left(\frac{1}{(1-\theta)q} + \frac{1}{\theta q}\right),
\end{aligned}$$

which establishes (iv).
We finish by proving (v). For $a \in A_0 \cap A_1$ we define the operator $T_a : \mathbb{C} \to A_0 + A_1$ by $T_a(\lambda) = \lambda a$. Then $\|T_a|\mathcal{L}(\mathbb{C}, A_i)\| = \|a|A_i\|$, for $i = 0, 1$, and by (iv) together with Theorem 1.2.4(ii) we find

$$\begin{aligned} \big\|T_a|\mathcal{L}\big(\mathbb{C}, (A_0, A_1)_{\theta,q}\big)\big\| &= c^* \big\|T_a|\mathcal{L}\big((\mathbb{C}, \mathbb{C})_{\theta,q}, (A_0, A_1)_{\theta,q}\big)\big\| \\ &\leq c^* \|T_a|\mathcal{L}(\mathbb{C}, A_0)\|^{1-\theta} \|T_a|\mathcal{L}(\mathbb{C}, A_1)\|^{\theta} \\ &= c^* \|a|A_0\|^{1-\theta} \|a|A_1\|^{\theta}. \end{aligned}$$

Therefore, we conclude

$$\big\|a|(A_0, A_1)_{\theta,q}\big\| = \big\|T_a|\mathcal{L}\big(\mathbb{C}, (A_0, A_1)_{\theta,q}\big)\big\| \leq c^* \|a|A_0\|^{1-\theta} \|a|A_1\|^{\theta}.$$

□

Note that $q$ can be seen as something like a fine index to adjust the exact "position" of $(A_0, A_1)_{\theta,q}$ between $A_0$ and $A_1$, compare assertion (ii) above.

**Lemma 1.2.7** *Let $\{A_0, A_1\}, \{B_0, B_1\}$ be interpolation pairs with $A_i \hookrightarrow B_i$ for $i = 0, 1$ and $0 < \theta < 1 \leq q \leq \infty$. Then $(A_0, A_1)_{\theta,q} \hookrightarrow (B_0, B_1)_{\theta,q}$.*

**Proof** For $a = a_0 + a_1$ with $a_i \in A_i$ we have by assumption $\|a_i|B_i\| \leq c\|a_i|A_i\|$ for $i = 0, 1$. Then it follows $K(t, a; B_0, B_1) \leq cK(t, a; A_0, A_1)$ which gives the desired embedding.

□

**Application to sequence spaces**

After we got to know our first interpolation method, now we use it in the framework of sequence spaces to prepare later results in function spaces.

**Definition 1.2.8** *Let $A$ be a Banach space, $\sigma \in \mathbb{R}, 1 \leq p \leq \infty$. Then we denote by $l_p^\sigma(A)$ the space of all sequences $\xi = (\xi_j)_{j=0}^\infty$ in $A$ with*

$$\|\xi|l_p^\sigma(A)\| = \left\{ \begin{array}{lcl} \left(\sum_{j=0}^\infty 2^{j\sigma p} \|\xi_j|A\|^p\right)^{1/p} & : & p < \infty \\ \sup_{j\in\mathbb{N}_0} 2^{j\sigma} \|\xi_j|A\| & : & p = \infty \end{array} \right\} < \infty.$$

One can easily verify that $l_p^\sigma(A)$ is a Banach space with $l_p^0(\mathbb{C}) = l_p$. Note that the parameter $\sigma$ will later turn out to be the sequence space analog to the smoothness usually considered in function spaces. Now we formulate our first interpolation result for concrete Banach spaces.

**Theorem 1.2.9** *Let $A$ be a Banach space, $s_0, s_1 \in \mathbb{R}, s_0 \neq s_1, 1 \leq p_0, p_1, p \leq \infty$ and $0 < \theta < 1$. Then for $s = (1-\theta)s_0 + \theta s_1$*

$$\big(l_{p_0}^{s_0}(A), l_{p_1}^{s_1}(A)\big)_{\theta,p} = l_p^s(A).$$

When writing equality between two spaces we always mean it in the sense of equivalent norms.

**Proof** The idea is the following. We want to show that

$$\left(l_\infty^{s_0}(A), l_\infty^{s_1}(A)\right)_{\theta,p} \hookrightarrow l_p^s(A) \hookrightarrow \left(l_1^{s_0}(A), l_1^{s_1}(A)\right)_{\theta,p}.$$

Then because of the well-known monotony $l_r^\sigma(A) \hookrightarrow l_p^\sigma(A)$ for $1 \le r \le p \le \infty$ (analog to the usual $l_p$-spaces), we can by Lemma 1.2.7 reason that

$$\left(l_1^{s_0}(A), l_1^{s_1}(A)\right)_{\theta,p} \hookrightarrow \left(l_{p_0}^{s_0}(A), l_{p_1}^{s_1}(A)\right)_{\theta,p} \hookrightarrow \left(l_\infty^{s_0}(A), l_\infty^{s_1}(A)\right)_{\theta,p},$$

which then proves the desired equality.
step 1: We prove $\left(l_\infty^{s_0}(A), l_\infty^{s_1}(A)\right)_{\theta,p} \hookrightarrow l_p^s(A)$.
First we are interested in the behavior of the $K$-functional for a sequence $\xi = (\xi_j)_{j=0}^\infty \in l_\infty^{s_0} + l_\infty^{s_1} = l_\infty^{\min(s_0,s_1)}$. In fact, we will show

$$K(t,\xi; l_\infty^{s_0}(A), l_\infty^{s_1}(A)) \sim \sup_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\|.$$

By definition we have

$$K(t,\xi; l_\infty^{s_0}(A), l_\infty^{s_1}(A)) = \inf_{\xi=\xi^0+\xi^1} \left( \sup_{j\in\mathbb{N}_0} 2^{js_0}\|\xi_j^0|A\| + t \sup_{j\in\mathbb{N}_0} 2^{js_1}\|\xi_j^1|A\| \right).$$

We choose

$$\hat{\xi}_j^0 = \left\{ \begin{array}{ll} \xi_j: & 2^{js_0} \le t2^{js_1} \\ 0: & \text{else} \end{array} \right\} \quad \text{and} \quad \hat{\xi}^1 = \xi - \hat{\xi}^0,$$

then we can estimate

$$\sup_{j\in\mathbb{N}_0} 2^{js_0}\|\hat{\xi}_j^0|A\| = \sup_{2^{js_0}\le t2^{js_1}} 2^{js_0}\|\xi_j|A\| \le \sup_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\|$$

and

$$t\sup_{j\in\mathbb{N}_0} 2^{js_1}\|\hat{\xi}_j^1|A\| = \sup_{2^{js_0}> t2^{js_1}} t2^{js_1}\|\xi_j|A\| \le \sup_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\|.$$

After taking the infimum over all representations we certainly have

$$K(t,\xi; l_\infty^{s_0}(A), l_\infty^{s_1}(A)) \le 2\sup_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\|. \tag{1.4}$$

On the other hand, for an arbitrary $\xi = \xi^0 + \xi^1$ we find

$$\begin{aligned} \sup_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\| &\le \sup_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\left(\|\xi_j^0|A\| + \|\xi_j^1|A\|\right) \\ &\le \sup_{j\in\mathbb{N}_0} \left(2^{js_0}\|\xi_j^0|A\| + t2^{js_1}\|\xi_j^1|A\|\right) \\ &\le \|\xi^0|l_\infty^{s_0}(A)\| + t\|\xi^1|l_\infty^{s_1}(A)\|, \end{aligned}$$

therefore,

$$\sup_{j\in\mathbb{N}_0} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\| \le K(t,\xi;l_\infty^{s_0}(A),l_\infty^{s_1}(A)). \tag{1.5}$$

Now we use these estimates of the $K$-functionals in the norms. Let $s_0 > s_1$ (w.l.o.g. because of Theorem 1.2.6(i)) and the interval $(0,\infty)$ be decomposed as

$$(0,\infty) = \bigcup_{k=-\infty}^{\infty} \big[2^{(k-1)(s_0-s_1)}, 2^{k(s_0-s_1)}\big).$$

For $\xi \in \big(l_\infty^{s_0}(A), l_\infty^{s_1}(A)\big)_{\theta,p}, p<\infty$ we can write

$$\begin{aligned} \Big\|\xi|\big(l_\infty^{s_0}(A),l_\infty^{s_1}(A)\big)_{\theta,p}\Big\|^p &= \int_0^\infty t^{-\theta p} K(t,\xi;l_\infty^{s_0}(A),l_\infty^{s_1}(A))^p \frac{dt}{t} \\ &= \sum_{k=-\infty}^{\infty} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} t^{-\theta p} K(t,\xi;l_\infty^{s_0}(A),l_\infty^{s_1}(A))^p \frac{dt}{t}. \end{aligned}$$

In each of the small intervals we can estimate $t^{-\theta p} \ge 2^{-\theta p k(s_0-s_1)}$ and with (1.5) we find further

$$\begin{aligned} \Big\|\xi|\big(l_\infty^{s_0}(A),l_\infty^{s_1}(A)\big)_{\theta,p}\Big\|^p &\ge c \sum_{k=-\infty}^{\infty} 2^{-\theta p k(s_0-s_1)} \sup_{j\in\mathbb{N}_0} \min(2^{js_0p}, 2^{k(s_0-s_1)p}2^{js_1p})\|\xi_j|A\|^p \cdot \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} \frac{dt}{t} \\ &\ge c \sum_{k=-\infty}^{\infty} 2^{-\theta p k(s_0-s_1)} \min(2^{ks_0p}, 2^{k(s_0-s_1)p+ks_1p})\|\xi_k|A\|^p \\ &= c\sum_{k=0}^{\infty} 2^{-\theta p k(s_0-s_1)} 2^{ks_0p}\|\xi_k|A\|^p \\ &= c\sum_{k=0}^{\infty} 2^{kp[s_0(1-\theta)+\theta s_1]}\|\xi_k|A\|^p = c\|\xi|l_p^s(A)\|^p. \end{aligned}$$

In case $p=\infty$ the similar estimation runs as follows

$$\begin{aligned} \Big\|\xi|\big(l_\infty^{s_0}(A),l_\infty^{s_1}(A)\big)_{\theta,\infty}\Big\| &= \sup_{t>0} t^{-\theta} K(t,\xi;l_\infty^{s_0}(A),l_\infty^{s_1}(A)) \\ &= \sup_{k\in\mathbb{Z}} \sup_{2^{(k-1)(s_0-s_1)}<t<2^{k(s_0-s_1)}} t^{-\theta} K(t,\xi;l_\infty^{s_0}(A),l_\infty^{s_1}(A)) \\ &\ge c \sup_{k\in\mathbb{Z}} 2^{-\theta k(s_0-s_1)} \sup_{j\in\mathbb{N}_0} \min(2^{js_0}, 2^{k(s_0-s_1)}2^{js_1})\|\xi_j|A\| \\ &\ge c \sup_{k\in\mathbb{Z}} 2^{k[s_0(1-\theta)+\theta s_1]}\|\xi_k|A\| = c\|\xi|l_\infty^s(A)\| \end{aligned}$$

and we conclude with $\left(l^{s_0}_\infty(A), l^{s_1}_\infty(A)\right)_{\theta,p} \hookrightarrow l^s_p(A)$.
step 2: Now we prove $l^s_p(A) \hookrightarrow \left(l^{s_0}_1(A), l^{s_1}_1(A)\right)_{\theta,p}$.
Again we suppose $s_0 > s_1$ (w.l.o.g.) and want to show

$$K(t, \xi, l^{s_0}_1(A), l^{s_1}_1(A)) \sim \sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1})\|\xi|A\|$$

for $\xi \in l^s_p(A)$ $(\hookrightarrow l^{s_0}_1(A) + l^{s_1}_1(A))$. By definition we have

$$K(t, \xi; l^{s_0}_1(A), l^{s_1}_1(A)) = \inf_{\xi=\xi^0+\xi^1} \left( \sum_{j=0}^{\infty} 2^{js_0}\|\xi^0_j|A\| + t\sum_{j=0}^{\infty} 2^{js_1}\|\xi^1_j|A\| \right).$$

Analogously to step 1 we choose $\xi = \hat{\xi}^0 + \hat{\xi}^1$ and estimate

$$\sum_{j=0}^{\infty} 2^{js_0}\|\hat{\xi}^0_j|A\| = \sum_{2^{js_0} \le t2^{js_1}} 2^{js_0}\|\xi_j|A\| \le \sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\|$$

and

$$t\sum_{j=0}^{\infty} 2^{js_1}\|\hat{\xi}^1_j|A\| = \sum_{2^{js_0} > t2^{js_1}} t2^{js_1}\|\xi_j|A\| \le \sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\|,$$

which after taking the infimum over all representations gives

$$K(t, \xi; l^{s_0}_1(A), l^{s_1}_1(A)) \le 2\sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\|. \tag{1.6}$$

On the other hand, for an arbitrary $\xi = \xi^0 + \xi^1$ we find

$$\begin{aligned}\sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\| &\le \sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1})\left(\|\xi^0_j|A\| + \|\xi^1_j|A\|\right) \\ &\le \sum_{j=0}^{\infty} \left(2^{js_0}\|\xi^0_j|A\| + t2^{js_1}\|\xi^1_j|A\|\right) \\ &\le \|\xi^0|l^{s_0}_\infty(A)\| + t\|\xi^1|l^{s_1}_\infty(A)\|,\end{aligned}$$

therefore,

$$\sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1})\|\xi_j|A\| \le K(t, \xi; l^{s_0}_1(A), l^{s_1}_1(A)). \tag{1.7}$$

Now for $p < \infty$ we estimate similar to step 1

$$\begin{aligned}
\left\| \xi | \left( l_1^{s_0}(A), l_1^{s_1}(A) \right)_{\theta,p} \right\|^p &= \int\limits_0^\infty t^{-\theta p} K(t,\xi; l_1^{s_0}(A), l_1^{s_1}(A))^p \frac{dt}{t} \\
&= \sum_{k=-\infty}^{\infty} \int\limits_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} t^{-\theta p} K(t,\xi; l_1^{s_0}(A), l_1^{s_1}(A))^p \frac{dt}{t} \\
&\leq c \sum_{k=-\infty}^{\infty} 2^{-\theta p k(s_0-s_1)} \left[ \sum_{j=0}^{\infty} \min(2^{js_0}, 2^{k(s_0-s_1)+js_1}) \|\xi_j|A\| \right]^p .
\end{aligned}$$

Because of $-\theta(s_0 - s_1) = s - s_0$ and $s_0 > s_1$ we can write further

$$\begin{aligned}
\left\| \xi | \left( l_1^{s_0}(A), l_1^{s_1}(A) \right)_{\theta,p} \right\|^p &\leq c \sum_{k=-\infty}^{\infty} 2^{pks} \left[ \sum_{j=0}^{\infty} \min(2^{(j-k)s_0}, 2^{(j-k)s_1}) \|\xi_j|A\| \right]^p \\
&= c \sum_{k=-\infty}^{\infty} 2^{pks} \left[ \sum_{j=-\infty}^{k} 2^{(j-k)s_0} \|\xi_j|A\| + \sum_{j=k+1}^{\infty} 2^{(j-k)s_1}) \|\xi_j|A\| \right]^p
\end{aligned} \tag{1.8}$$

($\xi_j = 0$ for $j < 0$) Now we apply Hölder's inequality for both the inner sums with $s_0 > \kappa_0 > s > \kappa_1 > s_1$ and get

$$\begin{aligned}
\sum_{j=-\infty}^{k} 2^{(j-k)s_0} \|\xi_j|A\| &= 2^{-ks_0} \sum_{j=-\infty}^{k} 2^{j(s_0-\kappa_0)} 2^{j\kappa_0} \|\xi_j|A\| \\
&\leq 2^{-ks_0} \left( \sum_{j=-\infty}^{k} 2^{j(s_0-\kappa_0)p'} \right)^{1/p'} \left( \sum_{j=-\infty}^{k} 2^{j\kappa_0 p} \|\xi_j|A\|^p \right)^{1/p} \\
&\leq c 2^{-k\kappa_0} \left( \sum_{j=-\infty}^{k} 2^{j\kappa_0 p} \|\xi_j|A\|^p \right)^{1/p} ,
\end{aligned}$$

where the geometric series was estimated as $\sum_{j=-\infty}^{k} 2^{j(s_0-\kappa_0)p'} \leq c 2^{k(s_0-\kappa_0)p'}$. Analogously,

$$\begin{aligned}
\sum_{j=k+1}^{\infty} 2^{(j-k)s_1} \|\xi_j|A\| &= 2^{-ks_1} \sum_{j=k+1}^{\infty} 2^{j(s_1-\kappa_1)} 2^{j\kappa_1} \|\xi_j|A\| \\
&\leq 2^{-ks_1} \left( \sum_{j=k+1}^{\infty} 2^{j(s_1-\kappa_1)p'} \right)^{1/p'} \left( \sum_{j=k+1}^{\infty} 2^{j\kappa_1 p} \|\xi_j|A\|^p \right)^{1/p} \\
&\leq c 2^{-k\kappa_1} \left( \sum_{j=k+1}^{\infty} 2^{j\kappa_1 p} \|\xi_j|A\|^p \right)^{1/p} .
\end{aligned}$$

Included into the estimate (1.8) we use $(a+b)^p \leq 2^p(a^p+b^p)$ and finally get by similar estimates on the appearing geometric sums

$$\begin{aligned}\left\|\xi|\left(l_1^{s_0}(A), l_1^{s_1}(A)\right)_{\theta,p}\right\|^p &\leq c \sum_{k=-\infty}^{\infty} 2^{pks}\left[2^{-k\kappa_0 p}\sum_{j=-\infty}^{k} 2^{j\kappa_0 p}\|\xi_j|A\|^p + 2^{-k\kappa_1 p}\sum_{j=k+1}^{\infty} 2^{j\kappa_1 p}\|\xi_j|A\|^p\right] \\ &\leq c\left[\sum_{j=-\infty}^{\infty} 2^{j\kappa_0 p}\|\xi_j|A\|^p \sum_{k=j}^{\infty} 2^{kp(s-\kappa_0)} + \sum_{j=-\infty}^{\infty} 2^{j\kappa_1 p}\|\xi_j|A\|^p \sum_{k=-\infty}^{j-1} 2^{kp(s-\kappa_1)}\right] \\ &\leq c \sum_{j=-\infty}^{\infty} 2^{jsp}\|\xi_j|A\|^p = c\|\xi|l_p^s(A)\|^p,\end{aligned}$$

which gives $l_p^s(A) \hookrightarrow \left(l_1^{s_0}(A), l_1^{s_1}(A)\right)_{\theta,p}$ and finishes the proof.

□

As we can see in the proof the condition $s_0 \neq s_1$ is essential for Theorem 1.2.9. But what happens, if $s_0 = s_1$? To answer this we consider the case $s_0 = s_1 = 0$ for simplicity (otherwise one easily sets $\eta_j = 2^{js}\xi_j$ and has $\eta \in l_p^0(A) \Leftrightarrow \xi \in l_p^s(A)$). It turns out, that in this case interpolation would in general leave the $l_p$-scale. To formulate the result, we need to embed this scale into a more general one by incorporating something like a fine index here too, compare the note after the proof of Theorem 1.2.6

To do so we introduce the so-called non-increasing rearrangement $\xi^*$ of a sequence $\xi \in l_p(A)$ by

$$\|\xi_0^*|A\| \geq \|\xi_1^*|A\| \geq \|\xi_2^*|A\| \geq \cdots \geq \|\xi_j^*|A\| \geq \cdots \geq 0$$

and define the Lorentz sequence spaces.

**Definition 1.2.10** *Let $A$ be a Banach space and $1 \leq p < \infty, 1 \leq q \leq \infty$. Then we denote by $l_{p,q}(A)$ the space of all sequences $(\xi_j)_{j=0}^{\infty}, \xi_j \in A$ with*

$$\|\xi|l_{p,q}(A)\| = \left\{\begin{array}{lcl} \left(\sum_{j=0}^{\infty}[(j+1)^{1/p-1/q}\|\xi_j^*|A\|]^q\right)^{1/q} & : & q<\infty \\ \sup_{j\in\mathbb{N}_0}(j+1)^{1/p}\|\xi_j^*|A\| & : & q=\infty \end{array}\right\} < \infty.$$

Note that in general $\|\cdot|l_{p,q}(A)\|$ is only a quasi-norm. Nevertheless, $l_{p,p}(A) = l_p(A)$ and

$$\begin{aligned} l_{p,q}(A) &\hookrightarrow l_{p,u}(A) \quad \text{for} \quad 1 \leq p < \infty, 1 \leq q \leq u \leq \infty \\ l_{p,q}(A) &\hookrightarrow l_{r,u}(A) \quad \text{for} \quad 1 \leq p < r < \infty, 1 \leq q, u \leq \infty. \end{aligned}$$

In this sense the second parameter $q$ can be understood as a fine index. Let's now formulate the interpolation result for $s_0 = s_1$.

**Theorem 1.2.11** *Let $A$ be a Banach space, $1 \leq q \leq \infty, 0 < \theta < 1$. Then*

$$(l_1(A), l_\infty(A))_{\theta,q} = l_{\frac{1}{1-\theta},q}(A).$$

The idea of the proof would be the same as for the theorem before. We only sketch it here, the complete proof can be found in [Tr78](1.18.3):
One shows that

$$K(t,\xi;l_1(A),l_\infty(A)) = t\|\xi_0^*|A\|$$

for $0 < t \leq 1$ and

$$K(j,\xi;l_1(A),l_\infty(A)) = \sum_{k=0}^{j-1} \|\xi_k^*|A\|$$

for all $j \in \mathbb{N}$ to prove both embeddings

$$(l_1(A),l_\infty(A))_{\theta,q} \hookrightarrow l_{\frac{1}{1-\theta},q}(A) \quad \text{and} \quad l_{\frac{1}{1-\theta},q}(A) \hookrightarrow (l_1(A),l_\infty(A))_{\theta,q}.$$

Interpreted as an interpolation space, $(l_1(A),l_\infty(A))_{\theta,q}$ with $\|\cdot|(l_1(A),l_\infty(A))_{\theta,q}\|$ is a Banach space, although $\|\cdot|l_{\frac{1}{1-\theta},q}(A)\|$ is in general only a quasi-norm.

### 1.2.2 The $J$-method

**Definition 1.2.12** *Let $\{A_0,A_1\}$ be an interpolation pair. The $J$-functional of Peetre is defined by*

$$J(t,a;A_0,A_1) = J(t,a) = \max(\|a|A_0\|, t\|a|A_1\|)$$

*for $t > 0$ and $a \in A_0 \cap A_1$.*

Analogously to the $K$-functional, $J(1,a) = \|a|A_0 \cap A_1\|$ and for each fixed $t > 0$ the function $J(t,a)$ is an equivalent norm in $A_0 \cap A_1$.

**Lemma 1.2.13** *Let $a \in A_0 \cap A_1$. Then the function $J(t,a)$ is increasing and convex. Furthermore,*

$$\min(1,t)\|a|A_0 \cap A_1\| \leq J(t,a) \leq \max(1,t)\|a|A_0 \cap A_1\| \tag{1.9}$$

*and for $s > 0$ we have*

$$J(t,a) \leq \max(1,t/s)J(s,a) \quad \text{and} \quad K(t,a) \leq \min(1,t/s)J(s,a) \tag{1.10}$$

**Proof** The function is obviously increasing and (1.9) is easy to verify. It remains to prove (1.10) and the convexity, i.e.,

$$J((1-\lambda)t_1 + \lambda t_2, a) \leq (1-\lambda)J(t_1,a) + \lambda J(t_2,a)$$

for $0 < \lambda < 1$. So let $0 < t_1 < t_2 < \infty$, then by definition

$$\begin{aligned} J((1-\lambda)t_1 + \lambda t_2, a) &= \max\Big(\|a|A_0\|, ((1-\lambda)t_1 + \lambda t_2)\|a|A_1\|\Big) \\ &= \max\Big((1-\lambda+\lambda)\|a|A_0\|, ((1-\lambda)t_1 + \lambda t_2)\|a|A_1\|\Big) \\ &\leq \max\Big((1-\lambda)\|a|A_0\|, (1-\lambda)t_1\|a|A_1\|\Big) + \max\Big(\lambda\|a|A_0\|, \lambda t_2\|a|A_1\|\Big) \\ &= (1-\lambda)J(t_1,a) + \lambda J(t_2,a), \end{aligned}$$

where we used the triangle-inequality of the max-norm. To prove (1.10) we write

$$J(t,a) = \max(\|a|A_0\|, s\frac{t}{s}\|a|A_1\|) \leq \max(1, t/s)J(s,a)$$

and for $a \in A_0 \cap A_1$ we have $K(t,a) \leq \|a|A_0\| \leq J(s,a)$ and $K(t,a) \leq t\|a|A_1\| \leq \max(t\|a|A_1\|, t/s\|a|A_0\|) \leq t/sJ(s,a)$ and therefore $K(t,a) \leq \min(1, t/s)J(s,a)$. □

Now we define the interpolation space constructed by the $J$-method using again the functional $\Phi$ from (1.2).($J$ convex $\Rightarrow$ $J$ continuous)

**Definition 1.2.14** *Let $0 < \theta < 1 \leq q \leq \infty$ and $\{A_0, A_1\}$ be an interpolation pair. Then we define*

$$(A_0, A_1)^{\mathcal{J}}_{\theta,q} = \Big\{ a \in A_0 + A_1 : \exists \text{ continuous } u : \mathbb{R}_+ \to A_0 \cap A_1, \\ \text{with } a = \int_0^\infty u(t)\frac{dt}{t} \text{ and } \Phi_{\theta,q}(J(t,u(t))) < \infty \Big\}.$$

*We set*

$$\big\|a|(A_0, A_1)^{\mathcal{J}}_{\theta,q}\big\| = \inf_u \Phi_{\theta,q}(J(\cdot, u(\cdot))).$$

Note that the involved integral $\int_0^\infty u(s)\frac{ds}{s}$ is called Bochner-integral with convergence in $A_0 + A_1$. This concept of an integral for Banach space valued functions is defined very similar to the Lebesgue integral, starting with simple functions and extending it by a limiting procedure. Therefore many properties like linearity etc. are transferable, which we just apply (as for the Lebesgue integral) in calculations. For the precise definition and more details on the properties, see [Yos95](section V.5) or [AE01](section X.2).
Explicitely the norm in Definition 1.2.14 reads as

$$\big\|a|(A_0, A_1)^{\mathcal{J}}_{\theta,q}\big\| = \inf_u \begin{cases} \left(\int_0^\infty [t^{-\theta}J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} & : \quad q < \infty \\ \sup\limits_{0<t<\infty} t^{-\theta}J(t,u(t)) & : \quad q = \infty \end{cases}.$$

**Remark 1.2.15** *The continuity of $u$ is not really necessary. The above definition is also possible with, say, step functions $u$, where the points of discontinuity do not cumulate in $(0,\infty)$, for details see [Tr78](1.6.1).*

Now we state the analog of Theorem 1.2.4 for the $J$-method.

**Theorem 1.2.16** *Let $0 < \theta < 1 \le q \le \infty$ and $\{A_0, A_1\}$ be an interpolation pair. Then*
*(i)* $(A_0, A_1)^{\mathcal{J}}_{\theta,q}$ *is an interpolation space with respect to $\{A_0, A_1\}$.*
*(ii)* *The estimate*

$$\left\| T|\mathcal{L}\big((A_0, A_1)^{\mathcal{J}}_{\theta,q}, (B_0, B_1)^{\mathcal{J}}_{\theta,q}\big) \right\| \le \|T|\mathcal{L}(A_0, B_0)\|^{1-\theta} \|T|\mathcal{L}(A_1, B_1)\|^{\theta}$$

*holds for all $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$.*
*(iii)* *The estimate*

$$\left\| a|(A_0, A_1)^{\mathcal{J}}_{\theta,q} \right\| \le ct^{-\theta} J(t, a)$$

*holds for all $a \in A_0 \cap A_1$.*

**Proof** step 1: First we prove (iii).
For $a \in A_0 \cap A_1$ and $s > 0$ we define

$$u(t) = \frac{a}{\ln 2}\chi_{[s,2s)}(t), \qquad \text{which implies} \qquad \int_0^\infty u(t)\frac{dt}{t} = \frac{a}{\ln 2}\int_s^{2s} \frac{dt}{t} = a.$$

So we found an admissible representation of $a$ (see Remark 1.2.15) and have to check the $\Phi$-functional for it. With 1.10 we find

$$\begin{aligned} \Phi_{\theta,q}(J(t, u(t)))^q &= \int_0^\infty t^{-\theta q} J(t, u(t))^q \frac{dt}{t} \le \int_0^\infty t^{-\theta q} \max(1, t/s)^q J(s, u(t))^q \frac{dt}{t} \\ &= \int_s^{2s} t^{-\theta q} \max(1, t/s)^q J\left(s, \frac{a}{\ln 2}\right)^q \frac{dt}{t} = \frac{s^{-q}}{(\ln 2)^q} J(s, a)^q \int_s^{2s} t^{(1-\theta)q}\frac{dt}{t}, \end{aligned}$$

where we used the fact that $J(t, \cdot)$ is linear. Now we integrate

$$\int_s^{2s} t^{(1-\theta)q} \frac{dt}{t} = \frac{1}{(1-\theta)q} s^{(1-\theta)q}[2^{(1-\theta)q} - 1]$$

and can further estimate

$$\Phi_{\theta,q}(J(t, u(t)))^q \le s^{-\theta q} J(s, a)^q \frac{2^{(1-\theta)q} - 1}{(1-\theta)q(\ln 2)^q} \le cs^{-\theta q} J(s, a)^q,$$

which means (iii).
step 2: Here we prove (i).
Obviously $\| \cdot |(A_0, A_1)^{\mathcal{J}}_{\theta,q}\|$ is a norm. The completeness follows directly from Theorem 1.2.18 below. Here we show

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)^{\mathcal{J}}_{\theta,q} \hookrightarrow A_0 + A_1.$$

From (iii) with $t = 1$ we immediately see

$$\left\| a|(A_0, A_1)^{\mathcal{J}}_{\theta,q} \right\| \leq cJ(1, a) = c\|a|A_0 \cap A_1\|,$$

which is the left inclusion. Now let $a \in (A_0, A_1)^{\mathcal{J}}_{\theta,q}$, then there exists a function $u$ with $a = \int_0^\infty u(t)dt/t$ and with (1.10) we can estimate

$$\begin{aligned}
\|a|A_0 + A_1\| &= \left\| \int_0^\infty u(t)\frac{dt}{t} \Big| A_0 + A_1 \right\| \leq \int_0^\infty \|u(t)|A_0 + A_1\| \frac{dt}{t} \\
&\leq \int_0^\infty K(1, u(t))\frac{dt}{t} \leq \int_0^\infty \min(1, 1/t) J(t, u(t)\frac{dt}{t} \\
&\leq \int_0^1 t^\theta t^{-\theta} J(t, u(t))\frac{dt}{t} + \int_1^\infty t^{-1+\theta} t^{-\theta} J(t, u(t))\frac{dt}{t} \\
&\leq \left( \int_0^1 t^{-\theta q} J(t, u(t))^q \frac{dt}{t} \right)^{1/q} \left( \int_0^1 t^{\theta q'} \frac{dt}{t} \right)^{1/q'} \\
&\qquad + \left( \int_1^\infty t^{-\theta q} J(t, u(t))^q \frac{dt}{t} \right)^{1/q} \left( \int_1^\infty t^{-(1-\theta)q'} \frac{dt}{t} \right)^{1/q'} \leq c \left\| a|(A_0, A_1)^{\mathcal{J}}_{\theta,q} \right\|,
\end{aligned}$$

where we used Hölder's inequality. That proves the right inclusion. As for the $K$-method, to conclude that $(A_0, A_1)^{\mathcal{J}}_{\theta,q}$ is not only itermediate but even an interpolation space we need the estimate in (ii), compare the discussion at the beginning of step 3 in the proof of Theorem 1.2.4

step 3: Now we prove (ii).

Let $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$, then for $a \in (A_0, A_1)^{\mathcal{J}}_{\theta,q}$ there exists a function $u : \mathbb{R}_+ \to A_0 \cap A_1$ with $a = \int_0^\infty u(s)ds/s$. Because $u$ is continuous, also $Tu : \mathbb{R}_+ \to B_0 \cap B_1$ is continuous and we verify $Ta = \int_0^\infty Tu(s)ds/s$ in $B_0 + B_1$. We have

$$\begin{aligned}
J(s, Tu(s); B_0, B_1) &= \max(\|Tu(s)|B_0\|, s\|Tu(s)|B_1\|) \\
&\leq \|T|\mathcal{L}(A_0, B_0)\| \max\left( \|u(s)|A_0\|, s\frac{\|T|\mathcal{L}(A_1, B_1)\|}{\|T|\mathcal{L}(A_0, B_0)\|} \|u(s)|A_1\| \right) \\
&= \|T|\mathcal{L}(A_0, B_0)\| J(\tau, u(s); A_0, A_1),
\end{aligned}$$

for $\tau = s\|T|\mathcal{L}(A_1,B_1)\| \big/ \|T|\mathcal{L}(A_0,B_0)\|$. Inserted into the Φ-functional we get

$$\begin{aligned}
\Phi_{\theta,q}(J(s,Tu(s))) &= \left(\int_0^\infty [s^{-\theta}J(s,Tu(s);B_0,B_1)]^q \frac{ds}{s}\right)^{1/q} \\
&\leq \|T|\mathcal{L}(A_0,B_0)\| \left(\int_0^\infty [s^{-\theta}J(\tau,u(s);A_0,A_1)]^q \frac{ds}{s}\right)^{1/q} \\
&\leq \|T|\mathcal{L}(A_0,B_0)\| \left(\frac{\|T|\mathcal{L}(A_0,B_0)\|}{\|T|\mathcal{L}(A_1,B_1)\|}\right)^{-\theta} \left(\int_0^\infty [\tau^{-\theta}J(\tau,\tilde{u}(\tau);A_0,A_1)]^q \frac{d\tau}{\tau}\right)^{1/q} \\
&= \|T|\mathcal{L}(A_0,B_0)\|^{1-\theta}\|T|\mathcal{L}(A_1,B_1)\|^{\theta}\Phi_{\theta,q}(J(\tau,\tilde{u}(\tau))),
\end{aligned}$$

with $\tilde{u}(\tau) := u\left(\tau\frac{\|T|\mathcal{L}(A_0,B_0)\|}{\|T|\mathcal{L}(A_1,B_1)\|}\right) = u(s)$ and $a = \int_0^\infty u(s)\frac{ds}{s} = \int_0^\infty \tilde{u}(\tau)\frac{d\tau}{\tau}$. That means, for any fixed representation $a = \int_0^\infty u(s)ds/s$ we can estimate

$$\left\|Ta|(B_0,B_1)^{\mathcal{J}}_{\theta,q}\right\| \leq \Phi_{\theta,q}(J(s,Tu(s))) \leq \|T|\mathcal{L}(A_0,B_0)\|^{1-\theta}\|T|\mathcal{L}(A_1,B_1)\|^{\theta}\Phi_{\theta,q}(J(\tau,\tilde{u}(\tau))).$$

Finally, after taking the infimum over all such representations of $a$ we have

$$\left\|Ta|(B_0,B_1)^{\mathcal{J}}_{\theta,q}\right\| \leq \|T|\mathcal{L}(A_0,B_0)\|^{1-\theta}\|T|\mathcal{L}(A_1,B_1)\|^{\theta}\left\|a|(A_0,A_1)^{\mathcal{J}}_{\theta,q}\right\|,$$

which gives the desired assertion and finishes the proof.

□

Now we have constructed the two interpolation spaces $(A_0,A_1)_{\theta,q}$ and $(A_0,A_1)^{\mathcal{J}}_{\theta,q}$ explicitely by two different methods. Naturally, the question arises how these spaces are related. To answer this, we need the following Lemma.

**Lemma 1.2.17 (Fundamental Lemma of Interpolation Theory)** *Let $a \in A_0 + A_1$ with $\lim_{t\to 0} K(t,a) = \lim_{t\to\infty}\frac{K(t,a)}{t} = 0$. Then for every $\varepsilon > 0$ there exists a representation $a = \sum_{j=-\infty}^{\infty} u_j$, with $u_j \in A_0 \cap A_1$ for all $j \in \mathbb{Z}$, which converges in $A_0 + A_1$ and in addition*

$$J(2^j,u_j) \leq 3(1+\varepsilon)K(2^j,a)$$

*is satisfied for all $j \in \mathbb{Z}$.*

**Proof** Let $\varepsilon > 0$ be given. We choose two sequences $(a_{i,j})_{j=-\infty}^{\infty} \subset A_i, i = 0,1$ such that for all $j \in \mathbb{Z}$

$$a = a_{0,j} + a_{1,j} \quad \text{with} \quad \|a_{0,j}|A_0\| + 2^j\|a_{1,j}|A_1\| \leq (1+\varepsilon)K(2^j,a)$$

hold. This is always possible by the definition of the $K$-functional. Because of $\lim_{t\to 0} K(t,a) = \lim_{t\to\infty} \frac{K(t,a)}{t} = 0$ we deduce

$$\lim_{j\to-\infty} \|a_{0,j}|A_0\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|a_{1,j}|A_1\| = 0. \tag{1.11}$$

Now we set

$$u_j = a_{0,j} - a_{0,j-1} = a_{1,j-1} - a_{1,j} \in A_0 \cap A_1 \quad \text{for all} \quad j \in \mathbb{Z}$$

and calculate

$$a - \sum_{j=-m}^{k} u_j = a - \sum_{j=-m}^{k} (a_{0,j} - a_{0,j-1}) = a - a_{0,k} + a_{0,-m-1} = a_{1,k} + a_{0,-m-1}.$$

To prove convergence in $A_0 + A_1$ we conclude

$$\left\| a - \sum_{j=-m}^{k} u_j \middle| A_0 + A_1 \right\| = K\left(1, a - \sum_{j=-m}^{k} u_j\right) \le \|a_{0,-m-1}|A_0\| + \|a_{1,k}|A_1\| \longrightarrow 0$$

for $m, k \to \infty$ by (1.11). Furthermore, using our assumption on the chosen sequences, for the $J$-functional we can estimate

$$\begin{aligned} J(2^j, u_j) &= \max(\|u_j|A_0\|, 2^j\|u_j|A_1\|) \\ &\le \max\left(\|a_{0,j}|A_0\| + \|a_{0,j-1}|A_0\|, 2^j\|a_{1,j-1}|A_1\| + 2^j\|a_{1,j}|A_1\|\right) \\ &\le \max\left((1+\varepsilon)K(2^j,a) + (1+\varepsilon)K(2^{j-1},a), 2(1+\varepsilon)K(2^{j-1},a) + (1+\varepsilon)K(2^j,a)\right) \\ &\le (1+\varepsilon)\left(K(2^j,a) + 2K(2^{j-1},a)\right) \le 3(1+\varepsilon)K(2^j,a). \end{aligned}$$

□

Now we can state how the two spaces $(A_0, A_1)_{\theta,q}$ and $(A_0, A_1)^{\mathcal{J}}_{\theta,q}$ are related.

**Theorem 1.2.18 (Equivalence Theorem)** *Let* $0 < \theta < 1 \le q \le \infty$ *and* $\{A_0, A_1\}$ *be an interpolation pair. Then*

$$(A_0, A_1)_{\theta,q} = (A_0, A_1)^{\mathcal{J}}_{\theta,q}$$

*with equivalent norms.*

**Proof** We consider $q < \infty$.
step 1: First we prove $(A_0, A_1)^{\mathcal{J}}_{\theta,q} \hookrightarrow (A_0, A_1)_{\theta,q}$.
For $a \in (A_0, A_1)^{\mathcal{J}}_{\theta,q}$ there exists a function $u$ with $a = \int_0^\infty u(t)\frac{dt}{t}$, therefore, we have with (1.10)

$$K(s,a) \le \int_0^\infty K(s,u(t))\frac{dt}{t} \le \int_0^\infty \min(1,1/\tau)J(s\tau,u(s\tau))\frac{d\tau}{\tau} =: \psi(s),$$

where $t = \tau s$ for $s > 0$. Now we can estimate

$$\begin{aligned}
\Big\| a|(A_0,A_1)_{\theta,q}\Big\| &= \Phi_{\theta,q}(K(\cdot,a)) \le \Phi_{\theta,q}(\psi(\cdot)) \\
&= \left(\int_0^\infty \Big(\int_0^\infty s^{-\theta}\min(1,1/\tau)J(s\tau,u(s\tau))\frac{d\tau}{\tau}\Big)^q \frac{ds}{s}\right)^{1/q} \\
&\le \int_0^\infty \min(1,1/\tau)\Big(\int_0^\infty s^{-\theta q}J(s\tau,u(s\tau))^q\frac{ds}{s}\Big)^{1/q}\frac{d\tau}{\tau} \\
&= \int_0^\infty \min(1,1/\tau)\tau^{\theta}\Big(\int_0^\infty t^{-\theta q}J(t,u(t))^q\frac{dt}{t}\Big)^{1/q}\frac{d\tau}{\tau} \\
&= \Phi_{\theta,q}(J(t,u(t)))\left(\int_0^1 \tau^{\theta}\frac{d\tau}{\tau} + \int_1^\infty \tau^{\theta-1}\frac{d\tau}{\tau}\right) \le c\Phi_{\theta,q}(J(t,u(t))),
\end{aligned}$$

where we used the generalized triangle inequality for integrals, which reads as

$$\left(\int\Big(\int f(x,y)dy\Big)^k dx\right)^{1/k} \le \int\Big(\int f^k(x,y)dx\Big)^{1/k} dy$$

for $k \ge 1$ (see [HLP52], Th.202). After taking the infimum over all representations of $a$ we derived the desired estimate.
step 2: Now we prove $(A_0,A_1)_{\theta,q} \hookrightarrow (A_0,A_1)^{\mathcal{J}}_{\theta,q}$.
For $a \in (A_0,A_1)_{\theta,q}$ we have by Theorem 1.2.4 (iii)

$$K(t,a) \le ct^{\theta}\Big\|a|(A_0,A_1)_{\theta,q}\Big\|, \qquad \text{therefore,} \quad \lim_{t\to 0} K(t,a) = \lim_{t\to\infty}\frac{K(t,a)}{t} = 0.$$

Now Lemma 1.2.17 gives for every $\varepsilon > 0$ a representation $a = \sum_{j=-\infty}^{\infty} u_j$ with $J(2^j,u_j) \le 3(1+\varepsilon)K(2^j,a)$. Now we define the function

$$u(t) = \frac{1}{\ln 2}\sum_{j=-\infty}^{\infty} u_j\chi_{[2^j,2^{j+1})}(t),$$

which is a step function where the points of discontinuity do not cumulate in $(0,\infty)$ (compare Remark 1.2.15) and it allows to write

$$a = \sum_{j=-\infty}^{\infty} u_j = \sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}} u(t)\frac{dt}{t} = \int_0^\infty u(t)\frac{dt}{t}.$$

So we estimate with (1.10)

$$\begin{aligned}\Phi_{\theta,q}(J(t,u(t)))^q &= \int_0^\infty t^{-\theta q} J(t,u(t))^q \frac{dt}{t} = \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} t^{-\theta q} J(t,u(t))^q \frac{dt}{t} \\ &\leq c \sum_{j=-\infty}^{\infty} 2^{-j\theta q} J(2^j, u_j)^q \leq c(1+\varepsilon)^q \sum_{j=-\infty}^{\infty} 2^{-j\theta q} K(2^j, a)^q \\ &\leq c(1+\varepsilon)^q \int_0^\infty t^{-\theta q} K(t,a)^q \frac{dt}{t} = c(1+\varepsilon)^q \Big\| a|(A_0,A_1)_{\theta,q} \Big\|^q,\end{aligned}$$

which, after taking the infimum over all representations of $a$, gives the desired estimate, when $\varepsilon \downarrow 0$. In case $q = \infty$ the arguments are absolutely analog.

□

From now on we also denote $(A_0, A_1)^{\mathcal{J}}_{\theta,q}$ by $(A_0, A_1)_{\theta,q}$. Similar to the $K$- and $J$-method there are many other real interpolation methods, say the $L$-method, the Mean-methods or the Trace-method, all leading to the same interpolation spaces $(A_0, A_1)_{\theta,q}$. In contrast to that, the so-called complex interpolation in general leads to different spaces $[A_0, A_1]_\theta$, for more details see [Tr78](sections 1.4-1.10).

## 1.3 Further properties of $(A_0, A_1)_{\theta,q}$

For later use we want to characterize the norm $\| \cdot |(A_0, A_1)_{\theta,q}\|$ in terms of certain sequence space norms. These spaces are similar to those in Definition 1.2.8 with $A = \mathbb{C}$. In some sense this will lead to the discrete versions of the $K$- and $J$-methods and is also called discrete method.

**Definition 1.3.1** *Let $0 < \theta < 1 \leq q \leq \infty$. Then we denote by $\lambda^{\theta,q}$ the space of all complex sequences $(a_k)_{k=-\infty}^\infty$ with*

$$\|(a_k)_k|\lambda^{\theta,q}\| = \left\{ \begin{array}{ll} \left(\sum_{k=-\infty}^\infty 2^{-k\theta q}|a_k|^q\right)^{1/q}: & p < \infty \\ \sup_{k\in\mathbb{Z}} 2^{-k\theta}|a_k| \quad : & p = \infty \end{array} \right\} < \infty.$$

The following result will be a helpful tool.

**Theorem 1.3.2** *Let $0 < \theta < 1 \leq q \leq \infty$ and $\{A_0, A_1\}$ be an interpolation pair.*

*(i)* $a \in (A_0, A_1)_{\theta,q}$ *if and only if* $\left(K(2^k, a)\right)_{k=-\infty}^\infty \in \lambda^{\theta,q}$. *Furthermore,*

$$2^{-\theta}(\ln 2)^{1/q}\| \left(K(2^k,a)\right)_k |\lambda^{\theta,q}\| \leq \Big\|a|(A_0,A_1)_{\theta,q}\Big\| \leq 2(\ln 2)^{1/q}\| \left(K(2^k,a)\right)_k |\lambda^{\theta,q}\|.$$

*(ii)* $a \in (A_0, A_1)_{\theta,q}$ *if and only if there exists a sequence* $(u_j)_{j=-\infty}^{\infty} \subset A_0 \cap A_1$ *with* $a = \sum_{j=-\infty}^{\infty} u_j$ *in* $A_0 + A_1$ *and* $(J(2^j, u_j))_{j=-\infty}^{\infty} \in \lambda^{\theta,q}$. *Furthermore,*

$$\left\| a|(A_0, A_1)_{\theta,q} \right\| \sim \inf_{a=\sum u_j} \| \left( J(2^j, u_j) \right)_j |\lambda^{\theta,q}\|.$$

These results can be found in [BL76](Lemma 3.1.3 and Lemma 3.2.3).
For the sake of completeness we now collect some results concerning density and duality issues. For $j = 0, 1$ we will denote by $\overset{\circ}{A}_j$ the closure of $A_0 \cap A_1$ in $\| \cdot |A_j\|$ and by $\overset{\circ}{A}_{\theta,\infty}$ the closure of $A_0 \cap A_1$ in $\| \cdot |(A_0, A_1)_{\theta,\infty}\|$.

**Theorem 1.3.3** *Let* $0 < \theta < 1 \le q < \infty$ *and* $\{A_0, A_1\}$ *be an interpolation pair.*

*(i)* $A_0 \cap A_1$ *is dense in* $(A_0, A_1)_{\theta,q}$.

*(ii)* $(A_0, A_1)_{\theta,q} = \left(\overset{\circ}{A}_0, A_1\right)_{\theta,q} = \left(A_0, \overset{\circ}{A}_1\right)_{\theta,q} = \left(\overset{\circ}{A}_0, \overset{\circ}{A}_1\right)_{\theta,q}$.

*(iii)* *For* $a \in A_0 + A_1$, *we have*

$$a \in \overset{\circ}{A}_{\theta,\infty} \iff \lim_{t\to 0} t^{-\theta} K(t, a) = \lim_{t\to\infty} t^{-\theta} K(t, a) = 0.$$

**Proof** We prove (i).
By part (ii) of the previous theorem we know for $a \in (A_0, A_1)_{\theta,q}$ there exists $(u_j)_{j=-\infty}^{\infty} \subset A_0 \cap A_1$ with $a = \sum_{j=-\infty}^{\infty} u_j$ and $\left(\sum_{j=-\infty}^{\infty} 2^{-j\theta q} J(2^j, u_j)^q\right)^{1/q} < \infty$. Therefore,

$$\left\| a - \sum_{j=-m}^{m} u_j |(A_0, A_1)_{\theta,q} \right\| \le \left( \sum_{|j|>m} 2^{-j\theta q} J(2^j, u_j)^q \right)^{1/q} \longrightarrow 0 \quad \text{for } m \to \infty.$$

Here we prove (ii).
Obviously, $A_0 \cap A_1 = \overline{A_0 \cap A_1}^{\|\cdot|A_0\cap A_1\|} \hookrightarrow \overline{A_0 \cap A_1}^{\|\cdot|A_j\|} = \overset{\circ}{A}_j \hookrightarrow \overline{A_j}^{\|\cdot|A_j\|} = A_j$ for $j = 0, 1$. Therefore, we have by Lemma 1.2.7

$$(A_0 \cap A_1, A_0 \cap A_1)_{\theta,q} \hookrightarrow \left(\overset{\circ}{A}_0, \overset{\circ}{A}_1\right)_{\theta,q} \hookrightarrow \left\{ \begin{array}{c} \left(\overset{\circ}{A}_0, A_1\right)_{\theta,q} \\ \left(A_0, \overset{\circ}{A}_1\right)_{\theta,q} \end{array} \right\} \hookrightarrow (A_0, A_1)_{\theta,q}.$$

On the other hand we know from part (iv) of Theorem 1.2.6 and part (i) above

$$(A_0 \cap A_1, A_0 \cap A_1)_{\theta,q} = \overline{A_0 \cap A_1}^{\|\cdot|(A_0,A_1)_{\theta,q}\|} = (A_0, A_1)_{\theta,q},$$

which proves the equality.

Now we prove (iii) starting with " $\Rightarrow$ ".
Let $a \in \overset{\circ}{A}_{\theta,\infty} = \overline{A_0 \cap A_1}^{\|\cdot|(A_0,A_1)_{\theta,\infty}\|}$ and $\varepsilon > 0$. Then there exists $u \in A_0 \cap A_1$ with $\|a - u|(A_0, A_1)_{\theta,\infty}\| < \varepsilon$. Now we estimate with Theorem 1.2.4 and (1.10)

$$\begin{aligned} K(t,a) &\leq K(t,a-u) + K(t,u) \leq ct^\theta \|a - u|(A_0,A_1)_{\theta,\infty}\| + \min(1,t)J(1,u) \\ &< c\varepsilon t^\theta + \min(1,t)J(1,u). \end{aligned}$$

Therefore we have

$$\lim_{t\to 0} t^{-\theta} K(t,a) \leq c\varepsilon + J(1,u) \lim_{t\to 0} t^{1-\theta} = c\varepsilon$$

and

$$\lim_{t\to\infty} t^{-\theta} K(t,a) \leq c\varepsilon + J(1,u) \lim_{t\to\infty} t^{-\theta} = c\varepsilon$$

for arbitrary small $\varepsilon > 0$, which proves the first direction.
Finally we prove " $\Leftarrow$ ".
So let $a \in A_0 + A_1$ with $\lim_{t\to 0} t^{-\theta} K(t,a) = \lim_{t\to\infty} t^{-\theta} K(t,a) = 0$. Then for sure the weaker conditions $\lim_{t\to 0} K(t,a) = \lim_{t\to\infty} K(t,a)/t = 0$ hold and for $\varepsilon > 0$ by Lemma 1.2.17 there exists $(u_j)_{j=-\infty}^{\infty} \subset A_0 \cap A_1$ with $a = \sum u_j$ and $J(2^j, u_j) \leq 3(1+\varepsilon)K(2^j, a)$. Therefore we have $(J(2^j, u_j))_{j=-\infty}^{\infty} \in \lambda^{\theta,\infty}$ and can estimate with Theorem 1.3.2(ii)

$$\left\| a - \sum_{j=-m}^{m} u_j | (A_0, A_1)_{\theta,\infty} \right\| \leq \sup_{|j|>m} 2^{-j\theta} J(2^j, u_j) \leq 3(1+\varepsilon) \sup_{|j|>m} 2^{-j\theta} K(2^j, a) \longrightarrow 0$$

for $m \to \infty$, which proves $a \in \overline{A_0 \cap A_1}^{\|\cdot|(A_0,A_1)_{\theta,\infty}\|}$.

□

Here the condition $q < \infty$ is essential, $A_0 \cap A_1$ is in general not dense in $(A_0, A_1)_{\theta,\infty}$.
Let's come to the question how duality behaves under interpolation. As usual we denote by $A'$ the dual of a Banach space $A$ equipped with the norm

$$\|\psi|A'\| = \sup_{0\neq a\in A} \frac{|\psi(a)|}{\|a|A\|}.$$

**Remark 1.3.4** *The following statements were proved in the 1960's, see [Tr78](section 1.11.2): Let $\{A_0, A_1\}$ be an interpolation pair such that $A_0 \cap A_1$ is dense in $A_i$ for $i = 0, 1$. Then*

$$(A_0 \cap A_1)' = A_0' + A_1' \quad \textit{with} \quad \|\psi|A_0' + A_1'\| = \sup_{0\neq a\in A_0\cap A_1} \frac{|\psi(a)|}{\|a|A_0 \cap A_1\|}$$

*and*

$$(A_0 + A_1)' = A_0' \cap A_1' \quad \textit{with} \quad \|\psi|A_0' \cap A_1'\| = \sup_{0\neq a\in A_0+A_1} \frac{|\psi(a)|}{\|a|A_0 + A_1\|}.$$

*Therefore, we have for an interpolation pair* $\{A_0, A_1\}$

$$A_0' \cap A_1' = (A_0 + A_1)' \hookrightarrow A_j' \hookrightarrow (A_0 \cap A_1)' = A_0' + A_1',$$

*which means that also* $\{A_0', A_1'\}$ *is an interpolation pair.*

In the sequel we use the notation $(A_0, A_1)'_{\theta,q} = [(A_0, A_1)_{\theta,q}]'$.

**Theorem 1.3.5** *Let* $0 < \theta < 1$ *and* $\{A_0, A_1\}$ *be an interpolation pair such that* $A_0 \cap A_1$ *is dense in* $A_i$ *for* $i = 0, 1$.
*(i) For* $1 \leq q < \infty$ *and* $q'$ *given by* $1/q + 1/q' = 1$ *we have*

$$(A_0, A_1)'_{\theta,q} = (A_0', A_1')_{\theta,q'}.$$

*(ii) For* $q = \infty$ *we have* $(\overset{\circ}{A}_{\theta,\infty})' = (A_0', A_1')_{\theta,1}$.

For the proof see [Tr78](Thm.1.11.2)

## 1.4 The Reiteration Theorem

The main question of this section is: What happens if we take an interpolation space as an endpoint for a second interpolation process? To answer this we need to classify intermediate spaces.

**Definition 1.4.1** *Let* $\{A_0, A_1\}$ *be an interpolation pair,* $E$ *a Banach space with* $A_0 \cap A_1 \hookrightarrow E \hookrightarrow A_0 + A_1$ *and* $0 \leq \theta \leq 1$.
*(i)* $E$ *belongs to the class* $\mathcal{K}(\theta; A_0, A_1)$ *if there exists* $c > 0$ *such that*

$$K(t, a; A_0, A_1) \leq ct^\theta \|a|E\|$$

*for all* $a \in E$ *and* $0 < t < \infty$.
*(ii)* $E$ *belongs to the class* $\mathcal{J}(\theta; A_0, A_1)$ *if there exists* $c > 0$ *such that*

$$\|a|E\| \leq ct^{-\theta} J(t, a; A_0, A_1)$$

*for all* $a \in A_0 \cap A_1$ *and* $0 < t < \infty$.

For a fixed pair $\{A_0, A_1\}$ we write $\mathcal{K}(\theta)$ instead of $\mathcal{K}(\theta; A_0, A_1)$.

**Remark 1.4.2** *As introduced earlier, E is called an intermediate space with respect to* $\{A_0, A_1\}$. *Following Theorem 1.2.4 and Theorem 1.2.16, we verify that for* $0 < \theta < 1 \leq q \leq \infty$

$$(A_0, A_1)_{\theta,q} \in \mathcal{K}(\theta), \quad (A_0, A_1)^{\mathcal{J}}_{\theta,q} \in \mathcal{J}(\theta)$$

*and therefore by the Equivalence Theorem:* $(A_0, A_1)_{\theta,q} \in \mathcal{K}(\theta) \cap \mathcal{J}(\theta)$.
*Furthermore, for* $a \in A_0$ *we have* $K(t, a) \leq \|a|A_0\| \leq J(t, a)$, *which means* $A_0 \in \mathcal{K}(0) \cap \mathcal{J}(0)$. *On the other hand for* $a \in A_1$ *we know* $K(t, a) \leq t\|a|A_1\| \leq J(t, a)$, *so* $A_1 \in \mathcal{K}(1) \cap \mathcal{J}(1)$.

We have seen some examples for the above classes and now we want to characterize them.

**Theorem 1.4.3** *Let $\{A_0, A_1\}$ be an interpolation pair, $E$ a Banach space with $A_0 \cap A_1 \hookrightarrow E \hookrightarrow A_0 + A_1$ and $0 < \theta < 1$.*
*(a) The following assertions are equal:*
  *(i) $E \in \mathcal{K}(\theta; A_0, A_1)$*
  *(ii) $A_0 \cap A_1 \hookrightarrow E \hookrightarrow (A_0, A_1)_{\theta,\infty}$*
*(b) The following assertions are equal:*
  *(i) $E \in \mathcal{J}(\theta; A_0, A_1)$*
  *(ii) $(A_0, A_1)_{\theta,1} \hookrightarrow E \hookrightarrow A_0 + A_1$*
  *(iii) There exists $c > 0$ such that $\|a|E\| \leq c\|a|A_0\|^{1-\theta}\|a|A_1\|^{\theta}$ for all $a \in A_0 \cap A_1$.*

**Proof** step 1: First we prove (a).
By definition we immediately see

$$\begin{aligned} E \in \mathcal{K}(\theta) \quad &\iff \quad A_0 \cap A_1 \hookrightarrow E, \quad \sup_{t>0} t^{-\theta} K(t, a) \leq c\|a|E\| \\ &\iff \quad A_0 \cap A_1 \hookrightarrow E \hookrightarrow (A_0, A_1)_{\theta,\infty}. \end{aligned}$$

step 2: Now we prove (b).
Here we first recall that by Theorem 1.3.2(ii) $a \in (A_0, A_1)_{\theta,1}$ if and only if, there exists a sequence $(u_j)_{j=-\infty}^{\infty} \subset A_0 \cap A_1$ with $a = \sum_{j=-\infty}^{\infty} u_j$ and

$$\left\| a|(A_0, A_1)_{\theta,1} \right\| \sim \inf_{a=\sum u_j} \sum_{j=-\infty}^{\infty} 2^{-j\theta} J(2^j, u_j).$$

We start with (i)⇒(ii): By Definition 1.4.1(ii) we know for $E \in \mathcal{J}(\theta)$ that $E \hookrightarrow A_0 + A_1$ and if we choose $t = 2^j$ for each summand $u_j$ from above we have $\|u_j|E\| \leq c2^{-j\theta} J(2^j, u_j)$. So for $a \in (A_0, A_1)_{\theta,1}$ we can write

$$\|a|E\| \leq \sum_{j=-\infty}^{\infty} \|u_j|E\| \leq c \sum_{j=-\infty}^{\infty} 2^{-j\theta} J(2^j, u_j)$$

for all such representations, taking the infimum gives the desired estimate for the emdedding $(A_0, A_1)_{\theta,1} \hookrightarrow E$.
We proceed with the converse (ii)⇒(i).
Let $a \in A_0 \cap A_1$, then for each $m \in \mathbb{Z}$ there is the representation

$$a = \sum_{j=-\infty}^{\infty} u_j^{(m)} \quad \text{with} \quad u_j^{(m)} = \begin{cases} a & : \quad j = m \\ 0 & : \quad j \neq m. \end{cases}$$

Because of $(A_0, A_1)_{\theta,1} \hookrightarrow E$ we can estimate with Theorem 1.3.2(ii) and (1.10)

$$\begin{aligned} \|a|E\| &\leq c\Big\|a|(A_0,A_1)_{\theta,1}\Big\| \leq c \inf_{m\in\mathbb{Z}} \sum_{j=-\infty}^{\infty} 2^{-j\theta} J(2^j, u_j^{(m)}) \\ &= c \inf_{m\in\mathbb{Z}} 2^{-m\theta} J(2^m, a) \leq c \inf_{m\in\mathbb{Z}} 2^{-m\theta} \max(1, 2^m/t) J(t,a) \\ &= c \inf_{m\in\mathbb{Z}} \max\Big[(2^{-m}t)^\theta, (2^{-m}t)^{\theta-1}\Big] t^{-\theta} J(t,a) \end{aligned}$$

for an arbitrarily fixed $t > 0$. Now we choose $m$ with $2^{m-1} \leq t \leq 2^m$, then the term $(2^{-m}t)^{\theta-1}$ dominates the maximum and we arrive at

$$\|a|E\| \leq c2^{1-\theta} t^{-\theta} J(t,a),$$

which gives the desired estimate.
Now we prove (i)⇒(iii).
Let $0 \neq a \in A_0 \cap A_1$ and $\tau = \frac{\|a|A_0\|}{\|a|A_1\|}$. Then by definition we know for $E \in \mathcal{J}(\theta)$

$$\begin{aligned} \|a|E\| &\leq c\tau^{-\theta} \max(\|a|A_0\|, \tau\|a|A_1\|) \\ &= c\max(\|a|A_0\|^{1-\theta}\|a|A_1\|^\theta, \|a|A_0\|^{1-\theta}\|a|A_1\|^\theta) = c\|a|A_0\|^{1-\theta}\|a|A_1\|^\theta, \end{aligned}$$

which finishes the argument.
For the converse (iii)⇒(i) let $a \in A_0 \cap A_1$. Then

$$\|a|E\| \leq ct^{-\theta}\|a|A_0\|^{1-\theta}(t\|a|A_1\|)^\theta \leq ct^{-\theta}\max(\|a|A_0\|, t\|a|A_1\|) = ct^{-\theta}J(t,a),$$

which is the desired estimate and finishes the proof.

□

Here we formulate the main assertion of this section.

**Theorem 1.4.4 (Reiteration Theorem)** *Let $\{A_0, A_1\}$ and $\{E_0, E_1\}$ be interpolation pairs such that $E_j \in \mathcal{K}(\theta_j; A_0, A_1) \cap \mathcal{J}(\theta_j; A_0, A_1)$ for $j = 0, 1$ with $0 \leq \theta_0 < \theta_1 \leq 1$ and let $0 < \eta < 1 \leq q \leq \infty$ with $\theta := (1-\eta)\theta_0 + \eta\theta_1$. Then*

$$(E_0, E_1)_{\eta,q} = (A_0, A_1)_{\theta,q}$$

*with equivalent norms. In particular, for $1 \leq q_0, q_1 \leq \infty$*

$$\Big((A_0,A_1)_{\theta_0,q_0}, (A_0,A_1)_{\theta_1,q_1}\Big)_{\eta,q} = (A_0,A_1)_{\theta,q}.$$

Before proving this result we take a look at the following visualization.

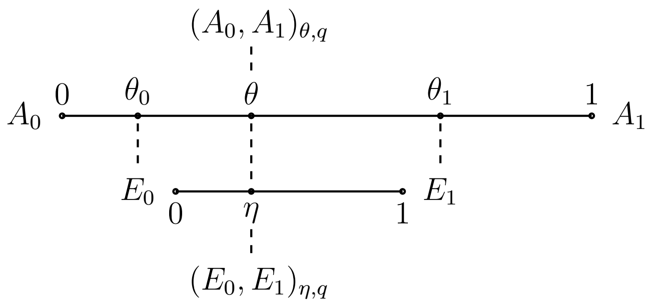


Figure 1.9: Reiteration

**Proof** Step 1: We prove $(E_0, E_1)_{\eta,q} \hookrightarrow (A_0, A_1)_{\theta,q}$
Let $a \in (E_0, E_1)_{\eta,q} \hookrightarrow E_0 + E_1 \hookrightarrow A_0 + A_1$ where $a = e_0 + e_1$ with $e_j \in E_j \in \mathcal{K}(\theta_j; A_0, A_1)$ for $j = 0, 1$. Then by Definition 1.4.1 we have

$$\begin{aligned} K(t, a; A_0, A_1) &\leq K(t, e_0; A_0, A_1) + K(t, e_1; A_0, A_1) \\ &\leq ct^{\theta_0}\Big(\|e_0|E_0\| + t^{\theta_1-\theta_0}\|e_1|E_1\|\Big) \\ \Longrightarrow K(t, a; A_0, A_1) &\leq ct^{\theta_0} K(t^{\theta_1-\theta_0}, a; E_0, E_1), \end{aligned}$$

where we took the infimum over all representations $a = e_0 + e_1$ on the right-hand side. Now for $s = t^{\theta_1-\theta_0}$ we can estimate the norm

$$\begin{aligned} \Big\|a|(A_0, A_1)_{\theta,q}\Big\| &= \left(\int_0^\infty t^{-\theta q} K(t, a; A_0, A_1)^q \frac{dt}{t}\right)^{1/q} \\ &\leq c\left(\int_0^\infty t^{(\theta_0-\theta)q} K(t^{\theta_1-\theta_0}, a; E_0, E_1)^q \frac{dt}{t}\right)^{1/q} \\ &= c\left(\int_0^\infty s^{-\frac{\theta-\theta_0}{\theta_1-\theta_0}q} K(s, a; E_0, E_1)^q \frac{ds}{s}\right)^{1/q} \\ &= c\left(\int_0^\infty s^{-\eta q} K(s, a; E_0, E_1)^q \frac{ds}{s}\right)^{1/q} = c\Big\|a|(E_0, E_1)_{\eta,q}\Big\|, \end{aligned}$$

because of $\theta - \theta_0 = \eta(\theta_1 - \theta_0)$.
Step 2: We prove the converse $(A_0, A_1)_{\theta,q} \hookrightarrow (E_0, E_1)_{\eta,q}$.
For $a \in (A_0, A_1)^{\mathcal{J}}_{\theta,q}$ there exists a function $u : \mathbb{R}_+ \longrightarrow A_0 \cap A_1$ with $a =$

$\int_0^\infty u(s)ds/s$. Therefore, we can write with (1.10)

$$
\begin{aligned}
t^{\theta_0} K(t^{\theta_1-\theta_0}, a; E_0, E_1) &\leq \int_0^\infty t^{\theta_0} K(t^{\theta_1-\theta_0}, u(s); E_0, E_1) \frac{ds}{s} \\
&\leq \int_0^\infty t^{\theta_0} \min\left(1, (t/s)^{\theta_1-\theta_0}\right) J(s^{\theta_1-\theta_0}, u(s); E_0, E_1) \frac{ds}{s}.
\end{aligned}
$$

Because of $E_j \in \mathcal{J}(\theta_j; A_0, A_1)$ for $j = 0, 1$ the $J$-functional on the right-hand side can be estimated as

$$
\begin{aligned}
J(s^{\theta_1-\theta_0}, u(s); E_0, E_1) &= \max\left(\|u(s)|E_0\|, s^{\theta_1-\theta_0}\|u(s)|E_1\|\right) \\
&\leq c \max(s^{-\theta_0} J(s, u(s); A_0, A_1), s^{\theta_1-\theta_0-\theta_1} J(s, u(s); A_0, A_1)) \\
&= c s^{-\theta_0} J(s, u(s); A_0, A_1).
\end{aligned}
$$

So we can continue to estimate

$$
\begin{aligned}
t^{\theta_0} K(t^{\theta_1-\theta_0}, a; E_0, E_1) &\leq c \int_0^\infty t^{\theta_0} \min\left(1, (t/s)^{\theta_1-\theta_0}\right) s^{-\theta_0} J(s, u(s); A_0, A_1) \frac{ds}{s} \\
&= c \int_0^\infty \min\left((t/s)^{\theta_0}, (t/s)^{\theta_1}\right) J(s, u(s); A_0, A_1) \frac{ds}{s} \\
&= c \int_0^\infty \min(\tau^{-\theta_0}, \tau^{-\theta_1}) J(t\tau, u(t\tau); A_0, A_1) \frac{d\tau}{\tau},
\end{aligned}
$$

where $s = t\tau$. Using this for the norm in $(E_0, E_1)_{\eta,q}$ we get

$$
\begin{aligned}
\left\| a|(E_0, E_1)_{\eta,q} \right\| &= \left( \int_0^\infty s^{-\eta q} K(s, a; E_0, E_1)^q \frac{ds}{s} \right)^{1/q} \\
&= c \left( \int_0^\infty t^{-\theta q} t^{\theta_0 q} K(t^{\theta_1-\theta_0}, a; E_0, E_1)^q \frac{dt}{t} \right)^{1/q} \\
&\leq c \left( \int_0^\infty t^{-\theta q} \left( \int_0^\infty \min(\tau^{-\theta_0}, \tau^{-\theta_1}) J(t\tau, u(t\tau); A_0, A_1) \frac{d\tau}{\tau} \right)^q \frac{dt}{t} \right)^{1/q}.
\end{aligned}
$$

Using again the generalized triangle inequality for integrals ([HLP52], Th.202) we can go on

$$\begin{aligned}
\left\| a | (E_0, E_1)_{\eta,q} \right\| &\leq c \int_0^\infty \min(\tau^{-\theta_0}, \tau^{-\theta_1}) \left( \int_0^\infty t^{-\theta q} J(t\tau, u(t\tau); A_0, A_1)^q \frac{dt}{t} \right)^{1/q} \frac{d\tau}{\tau} \\
&= c \int_0^\infty \min(\tau^{-\theta_0}, \tau^{-\theta_1}) \tau^{\theta} \left( \int_0^\infty s^{-\theta q} J(s, u(s); A_0, A_1)^q \frac{ds}{s} \right)^{1/q} \frac{d\tau}{\tau} \\
&= c \Phi_{\theta,q}(J(s, u(s); A_0, A_1)) \int_0^\infty \min(\tau^{\theta-\theta_0}, \tau^{\theta-\theta_1}) \frac{d\tau}{\tau} \\
&\leq c \Phi_{\theta,q}(J(s, u(s); A_0, A_1)),
\end{aligned}$$

which, after taking the infimum over all such representations of $a$, gives the desired inequality.

□

**Remark 1.4.5** *As one can see from the proof, the assumption* $\theta_0 \neq \theta_1$ *is essential (in case* $\theta_0 > \theta_1$ *see Theorem 1.2.6). If* $\theta_0 = \theta = \theta_1$ *one needs a different method of proof, see [BL76] (Theorems 3.5.4 and 5.2.4). Here the fine index* $q$ *comes into the picture and the result reads as follows: For* $0 < \theta < 1 \leq q_0, q_1 \leq \infty$

$$\Big( (A_0, A_1)_{\theta,q_0}, (A_0, A_1)_{\theta,q_1} \Big)_{\eta,q} = (A_0, A_1)_{\theta,q}, \qquad \text{with} \quad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}.$$

Now we can complement our investigations on Lorentz spaces from the end of subsection 1.2.1 and see that this scale of sequence spaces is in some sense closed under interpolation.

**Corollary 1.4.6** *Let* $A$ *be a Banach space,* $1 < p_0, p_1 < \infty$ *with* $p_0 \neq p_1$ *and* $0 < \eta < 1 \leq q_0, q_1, q \leq \infty$*. Then*

$$\Big( l_{p_0,q_0}(A), l_{p_1,q_1}(A) \Big)_{\eta,q} = l_{p,q}(A) \quad \text{with} \quad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1},$$

*in particular,*

$$\Big( l_{p_0}(A), l_{p_1}(A) \Big)_{\eta,p} = l_p(A).$$

**Proof** We use the Reiteration Theorem and Theorem 1.2.11 with $A_0 = l_1(A)$, $A_1 = l_\infty(A)$ and $\theta_j$ chosen such that $\theta_j = 1 - 1/p_j$ for $j = 0, 1$. Then by Theorem 1.2.11 we have

$$(A_0, A_1)_{\theta_j,q_j} = \Big( l_1(A), l_\infty(A) \Big)_{1-1/p_j,q_j} = l_{p_j,q_j}(A), \quad 1 \leq q_j \leq \infty \quad \text{for } j = 0, 1.$$

Using this interpolation result together with the Theorem above we get

$$\begin{aligned}\Big(l_{p_0,q_0}(A), l_{p_1,q_1}(A)\Big)_{\eta,q} &= \Big(\Big(l_1(A), l_\infty(A)\Big)_{1-1/p_0,q_0}, \Big(l_1(A), l_\infty(A)\Big)_{1-1/p_1,q_1}\Big)_{\eta,q} \\ &= \Big(l_1(A), l_\infty(A)\Big)_{\theta,q} = l_{\frac{1}{1-\theta},q}(A),\end{aligned}$$

where the last equality comes from Theorem 1.2.11 again and $\theta$ was chosen such that

$$\theta = (1-\eta)\theta_0 + \eta\theta_1 \quad \Longleftrightarrow \quad \frac{1}{1-\theta} = p.$$

□

For the sake of completeness we add here the following result.

**Theorem 1.4.7 (Holmstedt's formula)** *Let $\{A_0, A_1\}$ be an interpolation pair, $0 \le \theta_0 < \theta_1 \le 1 \le q_0, q_1 \le \infty$ with $\lambda := \theta_1 - \theta_0$ and $E_j = (A_0, A_1)_{\theta_j,q_j}$ for $j = 0, 1$. Then for all $t > 0$ and $a \in A_0 + A_1$ we have*

$$K(t, a; E_0, E_1) \sim \Big(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s, a; A_0, A_1)) q_0 \frac{ds}{s}\Big)^{1/q_0} + t\Big(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s, a; A_0, A_1)) q_1 \frac{ds}{s}\Big)^{1/q_1}.$$

For the proof see [BL76](Thm. 3.6.1).

## Retraction, coretraction and compact operators

For later use we need some mechanism to transfer our knowledge from the level of sequence spaces to function spaces. Here we shortly provide the necessary results and also a connection to the approximation issues concerning compact operators that we will treat in chapter 3.
In chapter 2 we will work with the following equality

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| = \left\|(\mathcal{F}^{-1}\varphi_j\mathcal{F}f)_{j=0}^{\infty}|l^s_q(L_p(\mathbb{R}^n))\right\|, \tag{1.12}$$

where on the left-hand side we have the norm of a function $f$ in a certain Besov space (compare section 2.2) and on the right-hand side the norm is taken in the sequence space $l^s_q(A)$ (recall Definition 1.2.8), where the elements of the sequence are certain $L_p$-functions, derived by manipulating $f$ with Fourier transform $\mathcal{F}$ and other operations. This is an example of what we have in mind when saying to transfer between sequence space and function space levels and it motivates the following concept.

**Definition 1.4.8** *Let $A, B$ be Banach spaces and $R \in \mathcal{L}(A, B)$. Then $R$ is called retraction, if there exists $S \in \mathcal{L}(B, A)$ with $R \circ S = id_B$. (Then $S$ is called coretraction)*

**Theorem 1.4.9** *Let* $\{A_0, A_1\}$ *and* $\{B_0, B_1\}$ *be interpolation pairs,* $R \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ *and* $S \in \mathcal{L}(\{B_0, B_1\}, \{A_0, A_1\})$ *with*

$$R|_{A_0} \circ S|_{B_0} = id_{B_0}, \quad R|_{A_1} \circ S|_{B_1} = id_{B_1}, \quad \textit{then}$$

*(i)* $(SR)|_{(A_0,A_1)_{\theta,q}}$ *is a projection in* $(A_0, A_1)_{\theta,q}$.

*(ii)* $S$ *is an isomorphism from* $(B_0, B_1)_{\theta,q}$ *onto* $Im\Big((SR)|_{(A_0,A_1)_{\theta,q}}\Big) \subset (A_0, A_1)_{\theta,q}$.

A proof in the more general framework of functors and categories can be found in [Tr78](Thm. 1.2.4).

**Remark 1.4.10** *What we can gain from the Theorem is that for a known interpolation space* $(A_0, A_1)_{\theta,q}$ *with respect to* $\{A_0, A_1\}$ *(for example: sequence spaces) and another interpolation pair* $\{B_0, B_1\}$ *(for example: function spaces), we can construct the unknown space* $(B_0, B_1)_{\theta,q}$ *(up to isomorphism) by using suitable operators* $R, S$. *That is the transfer mechanism we were looking for and we will use it in connection with* (1.12) *to interpolate Besov spaces in subsection 2.3.2.*

Since in the last chapter we will deal with measuring the degree of compactness of operators (mainly embeddings), we state here how compactness behaves under interpolation.

**Theorem 1.4.11** *(i) Let* $\{A_0, A_1\}$ *be an interpolation pair,* $B$ *a Banach space and* $T \in \mathcal{L}(\{A_0, A_1\}, \{B, B\})$ *with*

$$T : A_0 \longrightarrow B \ \textit{compact} \quad \textit{and} \quad T : A_1 \longrightarrow B \ \textit{continuous.}$$

*Then is* $T : A \longrightarrow B$ *compact, for* $A \in \mathcal{K}(\theta, A_0, A_1)$ *with* $0 < \theta < 1$.

*(ii) Let* $A$ *be a Banach space,* $\{B_0, B_1\}$ *an interpolation pair and* $S \in \mathcal{L}(\{A, A\}, \{B_0, B_1\})$ *with*

$$S : A \longrightarrow B_0 \ \textit{compact} \quad \textit{and} \quad S : A \longrightarrow B_1 \ \textit{continuous.}$$

*Then is* $S : A \longrightarrow B$ *compact, for* $B \in \mathcal{J}(\theta, B_0, B_1)$ *with* $0 < \theta < 1$.

The proof can be found in [BL76](Thm. 3.8.1). Note that it is enough to assume compactness only for one of the restrictions $T|_{A_i}$ ($i = 0$ or $i = 1$), visualized in Figure 1.10.

**Remark 1.4.12** *The more general question under which conditions on* $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ *the operator* $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ *is compact was finally answered by M. Cwikel 1992, who showed that the compactness of one of the restrictions* $T : A_i \longrightarrow B_i$, $i = 0, 1$, *is sufficient.*

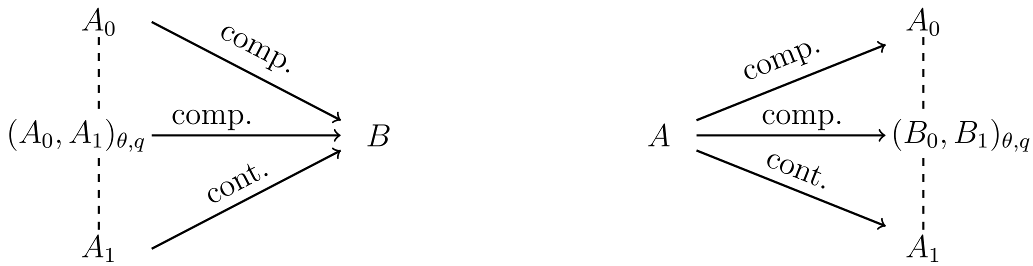


Figure 1.10: Compactness and interpolation

**Corollary 1.4.13** *Let* $\{A_0, A_1\}$ *be an interpolation pair with compact embedding* $A_0 \hookrightarrow A_1$, $0 < \theta_0 < \theta_1 < 1 \leq q_0, q_1 \leq \infty$. *Then the embedding* $(A_0, A_1)_{\theta_0,q_0} \hookrightarrow (A_0, A_1)_{\theta_1,q_1}$ *is compact.*

**Proof** We consider the identity operator id, for which id : $A_0 \longrightarrow A_1$ is compact and id : $A_1 \longrightarrow A_1$ is continuous. Since $(A_0, A_1)_{\theta_0,q_0} \in \mathcal{K}(\theta_0; A_0, A_1)$, Theorem 1.4.11(i) tells us that id : $(A_0, A_1)_{\theta_0,q_0} \longrightarrow A_1$ is compact. Since id : $(A_0, A_1)_{\theta_0,q_0} \longrightarrow (A_0, A_1)_{\theta_0,q_0}$ is continuous, part (ii) of Theorem 1.4.11 would give the compactness of id : $(A_0, A_1)_{\theta_0,q_0} \longrightarrow (A_0, A_1)_{\theta_1,q_1}$, if $(A_0, A_1)_{\theta_1,q_1} \in \mathcal{J}(\eta; A_1, (A_0, A_1)_{\theta_0,q_0})$ for $\eta \in (0, 1)$. But that follows by the Reiteration Theorem 1.4.4, i.e.,

$$(A_0, A_1)_{\theta_1,q_1} = (A_1, (A_0, A_1)_{\theta_0,q_0})_{\eta,q_1} \text{ for } \theta_1 = (1-\eta)+\eta\theta_0 \Longleftrightarrow \eta = \frac{1-\theta_1}{1-\theta_0} \in (0, 1).$$

□

In chapter 3 we are especially interested in the so-called entropy numbers $e_n(T)$ of an operator $T \in \mathcal{L}(X, Y)$, which are defined as the smallest radius $\varepsilon > 0$ such that one can cover the image of the unit ball $T(B_X)$ by $2^{n-1}$ $\varepsilon$-balls in $Y$. They characterize the compactness of $T$, because of

$$e_n(T) \longrightarrow 0 \Longleftrightarrow T \text{ is compact.}$$

The question arises, if one can determine the entropy numbers of $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ by the knowledge of $e_n(T : A_i \longrightarrow B_i)$, $i = 0, 1$ in the sense

$$e_{n+m-1}\left(T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}\right) \leq c e_n(T : A_0 \longrightarrow B_0)^{1-\theta} e_m(T : A_1 \longrightarrow B_1)^{\theta}.$$

But in [EN10] Edmunds and Netrusov showed that such an estimate is not true in general. Weaker results are known since the late 1970s. In chapter 3 we will state the following assertion in analogy to Theorem 1.4.11:
If $T \in \mathcal{L}(\{A_0, A_1\}, \{B, B\})$ and $A \in \mathcal{K}(\theta; A_0, A_1)$, then

$$e_{n+m-1}\left(T : A \longrightarrow B\right) \leq c e_n(T : A_0 \longrightarrow B)^{1-\theta} e_m(T : A_1 \longrightarrow B)^{\theta}.$$

(similar for $T \in \mathcal{L}(\{A, A\}, \{B_0, B_1\})$)

# Chapter 2

# Function spaces

## 2.1 A brief history

This whole section is based on [Tr92] and gives a very short historical overview about the development of certain function spaces mainly considering the question how to measure smoothness. Therefore, this overview is by no means complete and leaves many important branches of the theory out of consideration. The main interest lies in the evolution of spaces of functions featuring certain smoothness properties, until the spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ were defined via the unifying Fourier analytical approach in the 1970s. Nowadays these spaces are the fundament of modern theory of function spaces although the evolution did not stop and many generalizations in different directions have been proved to be reasonable and sucessful. In sections 2.1 and 2.2 there will be no proofs but only references for them.

### 2.1.1 Basic spaces

Let $\Omega$ be a domain in $\mathbb{R}^n$. Then, by definition, $C(\Omega)$ is the collection of all complex-valued bounded and uniformly continuous functions in $\Omega$, equipped with the norm

$$\|f|C(\Omega)\| = \sup_{x\in\Omega} |f(x)|.$$

$C(\Omega)$ is a Banach space. Now, let $k \in \mathbb{N}$, then

$$C^k(\Omega) = \{f \in C(\Omega) : D^\alpha f \in C(\Omega) \quad \text{if} \quad |\alpha| \le k\}$$

are Banach spaces equipped with the norm

$$\|f|C^k(\Omega)\| = \sum_{|\alpha|\le k} \|D^\alpha f|C(\Omega)\|.$$

Here we used standard notations: $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex with $\alpha_j \in \mathbb{N}_0$. Furthermore, we set

$$|\alpha| = \sum_{j=1}^{n} \alpha_j \quad \text{and} \quad D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x).$$

Let $dx$ stand for the Lebesgue measure in $\mathbb{R}^n$, then we put

$$\|f|L_p(\Omega)\| = \left( \int_\Omega |f(x)|^p dx \right)^{1/p}, \quad \text{for} \quad 0 < p < \infty,$$

with the usual modification

$$\|f|L_\infty(\Omega)\| = \sup_{x \in \Omega} |f(x)|.$$

By definition, $L_p(\Omega)$ with $0 < p \leq \infty$, is the collection of all complex-valued Lebesgue-measurable functions in $\Omega$ such that $\|f|L_p(\Omega)\|$ is finite (remember that we are actually talking about representatives of equavalence classes here). $L_p(\Omega)$ are Banach spaces for $1 \leq p \leq \infty$, otherwise only quasi-Banach spaces. Let us recall what is meant by a quasi-Banach space $A$. The only difference to the definition of a Banach space is that here the triangle inequality has to be satisfied up to a constant, i.e.,

$$\|a_1 + a_2|A\| \leq c(\|a_1|A\| + \|a_2|A\|).$$

In the above definitions one immediately can substitute $\Omega$ by $\mathbb{R}^n$ to define the spaces on the whole euclidean $n$-space. From now on we stick to the spaces on $\mathbb{R}^n$ and write only $C$ or $L_p$. If it comes to further spaces on domains, we will define them explicitly where they are needed. These types of spaces were thoroughly investigated at the beginning of the last century. From the point of view of possible applications, for example to PDE's, the spaces $L_1(\Omega), L^\infty(\Omega)$ and $C^k(\Omega)$ are not so well-suited (see [Tr83], section 2.2.3. for more details) and one was interested in good substitutes.

### 2.1.2 Hölder-Zygmund spaces

Often nowadays the universe of function spaces is visualized by the $(s, 1/p)$-diagram below. Till now we can identify only certain points on the axes with spaces we already defined. It will be the aim of this section to provide many definitions that help to fill the whole diagram.
The first step to fill the gaps between the spaces $C^k$ with $k \in \mathbb{N}_0$ are the Hölder spaces $C^s$ with $0 < s \neq$ integer. Let $0 < \sigma < 1$, then we set

$$\|f|C^\sigma\| = \|f|C\| + \sup \frac{|f(x) - f(y)|}{|x - y|^\sigma},$$

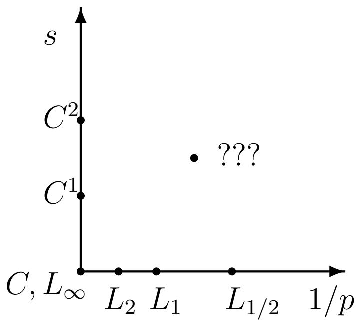


Figure 2.1: Function space universe

where the supremum is taken over all $x, y \in \mathbb{R}^n$ with $x \neq y$. For $s \in \mathbb{R}$ we put

$$s = [s] + \{s\} = [s]^- + \{s\}^+$$

where $[s], [s]^- \in \mathbb{Z}$ are the integer parts of $s$ with $0 \leq \{s\} < 1$ and $0 < \{s\}^+ \leq 1$.

**Definition 2.1.1** *For $0 < s \notin \mathbb{N}$ the Hölder spaces are defined by*

$$C^s = \Big\{ f \in C : \|f|C^s\| = \|f|C^{[s]}\| + \sum_{|\alpha|=[s]} \|D^\alpha f|C^{\{s\}}\| < \infty \Big\}.$$

**Remark 2.1.2** *Hölder spaces have been investigated by Schauder and others since the mid-thirties of the last century using them for boundary value problems for second order elliptic differential equations.*

Another way of measuring smoothness are the following differences of functions

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + hj), \quad k \in \mathbb{N}; h, x \in \mathbb{R}^n,$$

which can also be expressed iteratively

$$\Delta_h^1 f(x) = f(x+h) - f(x) \quad \text{and} \quad \Delta_h^{k+1} = \Delta_h^1 \Delta_h^k.$$

**Definition 2.1.3** *For $s > 0$ the Zygmund spaces are defined by*

$$\mathcal{C}^s = \Big\{ f \in C : \|f|\mathcal{C}^s\| = \|f|C^{[s]^-}\| + \sum_{|\alpha|=[s]^-} \sup_{0 \neq h \in \mathbb{R}^n} |h|^{-\{s\}^+} \|\Delta_h^2 D^\alpha f|C\| < \infty \Big\}.$$

**Remark 2.1.4** *Zygmund discovered in 1945 that sometimes instead of $\Delta_h^1$ second differences are the better choice, compare the reference given in [Tr92] (section 1.2.2 Remark 3).*

The name Hölder-Zygmund spaces is justified by the following theorem.

**Theorem 2.1.5** *If $0 < s \neq$ integer then $C^s = \mathcal{C}^s$, but if $s = k \in \mathbb{N}$ then $C^k \subset \mathcal{C}^k$ with $C^k \neq \mathcal{C}^k$.*

### 2.1.3 Sobolev spaces

The spaces named after S.L.Sobolev and introduced by him in the mid-thirties of the last century are perhaps the best known function spaces beyond the spaces we discussed so far. Since 1950 they count as some of the key ingredients when dealing with PDE's. Here we enter the world of generalized (weak) derivatives and tempered distributions in $S'(\mathbb{R}^n)$ (to recall the basics of this theory, we refer to [Tr72], sections 1.4 and 1.5).

**Definition 2.1.6** *Let $1 < p < \infty$ and $k \in \mathbb{N}_0$, then the Sobolev spaces are defined by*

$$W_p^k = \{f \in L_p : D^\alpha f \in L_p \text{ for } |\alpha| \le k\}.$$

**Remark 2.1.7** *If one equips the space with the norm*

$$\|f|W_p^k\| = \sum_{|\alpha|\le k} \|D^\alpha f|L_p\|,$$

*then it becomes a Banach space and even a Hilbert space for $p = 2$.*

Note that $W_p^0 = L_p$ and for $k - n/p \ge m - n/q$ there is the famous Sobolev embedding $W_p^k \hookrightarrow W_q^m$, visualized as follows:

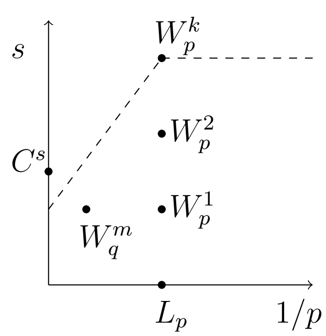


Figure 2.2: Sobolev embedding

Here the left part of the dashed line has slope $n$, meaning that $W_p^k$ can be embedded into all spaces lying below it. If one considers spaces on bounded domains then the embedding even exists into spaces lying below the right part of the dashed line.

### 2.1.4 Besov spaces

In the middle of the last century on one hand there was the continuous scale $\mathcal{C}^s$, with $s > 0$; on the other hand the discrete scales $W_p^k$, with $1 < p < \infty$ and $k \in \mathbb{N}_0$. It was a natural task to fill the gaps between $L_p, W_p^1, W_p^2, \ldots$. Here we describe two major attempts that started around this time.
The first one, combining the ideas of Hölder spaces and Sobolev spaces, started

with S.M.Nikol'skij in 1951, who introduced the space $B^s_{p,\infty}$, for $0 < s \notin \mathbb{N}$ and $1 < p < \infty$, as the collection of all $f \in L_p$ such that

$$\|f|B^s_{p,\infty}\| = \|f|W^{[s]}_p\| + \sum_{|\alpha|=[s]} \sup_{0\neq h\in\mathbb{R}^n} |h|^{\{s\}}\|\Delta^1_h D^\alpha f|L_p\|$$

is finite. To replace the sup-norm here by an $L_p$-norm was the suggestion of N.Aronszajn, L.N. Slobodeckij and E.Gagliardo about 5 years later: For $0 < s \notin \mathbb{N}$ and $1 < p < \infty$ the space $B^s_{p,p}$ is the collection of all $f \in L_p$ such that

$$\|f|B^s_{p,p}\| = \|f|W^{[s]}_p\| + \sum_{|\alpha|=[s]} \left(\int_{\mathbb{R}^n} |h|^{-\{s\}p}\|\Delta^1_h D^\alpha f|L_p\|^p \frac{dh}{|h|^n}\right)^{1/p}$$

is finite. Finally Besov in 1959/60 combined Zygmund's idea, i.e., to use higher differences, with Sobolev's approach to extend the previous scales to all values $s > 0$ and a third parameter $1 \leq q < \infty$.

**Definition 2.1.8** *The Besov spaces are defined by*

$$B^s_{p,q} = \{f \in L_p : \|f|B^s_{p,q}\| < \infty\}, \quad \textit{where}$$

$$\|f|B^s_{p,q}\| = \|f|W^{[s]^-}_p\| + \sum_{|\alpha|=[s]^-} \left(\int_{\mathbb{R}^n} |h|^{-\{s\}^+q}\|\Delta^2_h D^\alpha f|L_p\|^q \frac{dh}{|h|^n}\right)^{1/q}.$$

**Remark 2.1.9** *For $p = 2$ we have $W^k_2 = B^k_{2,2}$ for $k \in \mathbb{N}_0$. In sharp contrast to that, if $p \neq 2$ then $W^k_p$ does not coincide with any $B^\sigma_{r,q}$. But we always have $(W^{k_0}_p, W^{k_1}_p)_{\theta,q} = B^s_{p,q}$ with $s = k_0(1-\theta) + k_1\theta$ as we will prove in section 2.3.2 showing that Besov spaces are a good choice when dealing with interpolation.*

The second attempt to fill the gaps between $L_p, W^1_p, W^2_p, \ldots$ is based on the Fourier transform. We recall that by $S'$ we denote the collection of all complex-valued tempered distributions on $\mathbb{R}^n$. Then the Fourier transform $\hat{f}$ of $f \in S'$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx, \quad \xi \in \mathbb{R}^n,$$

where $x\xi = \sum_{i=1}^n x_i\xi_i$ and the inverse Fourier transform $\check{f}$ is given by the same formula with $i$ instead of $-i$. Remember that earlier we also used the symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ for the Fourier transform and its inverse. As well-known facts, we just mention that both $\mathcal{F}$ and $\mathcal{F}^{-1}$ map $S'$ onto itself and $L_2$ isometrically onto itself ($\|f\|_2 = \|\hat{f}\|_2$). We also recall

$$D^\alpha(\mathcal{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha\varphi(x)), \quad \xi^{|\alpha|}(\mathcal{F}\varphi(x)) = (-i)^{|\alpha|}\mathcal{F}(D^\alpha\varphi(x))$$

for Schwartz functions $\varphi$ and the formula of Plancherel:

$$\|D^\alpha f|L_2\| = \|\mathcal{F}(D^\alpha f)|L_2\| = \|\xi^\alpha \mathcal{F}f(\xi)|L_2\|$$

for $f \in W_2^k$ and $|\alpha| \le k$. With the observation that $|\xi^\alpha| \le (1+|\xi|^2)^{k/2}$ one can prove

$$\|f|W_2^k\| \sim \|(1+|\xi|^2)^{k/2}\hat{f}(\xi)|L_2\| = \|((1+|\xi|^2)^{k/2}\hat{f}(\xi))^\vee|L_2\| \qquad (2.1)$$

for $k \in \mathbb{N}_0$ and similar

$$\|f|B_{2,2}^s\| \sim \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)|L_2\| = \|((1+|\xi|^2)^{s/2}\hat{f}(\xi))^\vee|L_2\| \qquad (2.2)$$

if $0 < s \notin \mathbb{N}$, which is a characterization of the spaces on the left-hand side without using any derivatives. Furthermore, it shows that $B_{2,2}^s$ already fills the gaps between the spaces $W_2^k$ in a natural way. To overcome the shortcoming mentioned in Remark 2.1.9 one may now have the idea to replace the $L_2$-norm in (2.1) and (2.2) by an $L_p$-norm. The next definition goes back to N.Aronszajn, K.T.Smith and A.P.Calderon.

**Definition 2.1.10** *Let $1 < p < \infty$ and $s \in \mathbb{R}$ then the fractional Sobolev spaces (or Bessel potential spaces) are defined by*

$$H_p^s = \{f \in S' : ((1+|\xi|^2)^{s/2}\hat{f})^\vee \in L_p\}.$$

**Remark 2.1.11** *Note that the extension is done not only for $1 < p < \infty$ but also for $s \le 0$. If $H_p^s$ is equipped with the norm $\|((1+|\xi|^2)^{s/2}\hat{f})^\vee|L_p\|$ it becomes a Banach space and the operator $I_s : f \mapsto ((1+|\xi|^2)^{s/2}\hat{f})^\vee$ is a lift that maps $H_p^\sigma$ isomorphically onto $H_p^{\sigma-s}$.*

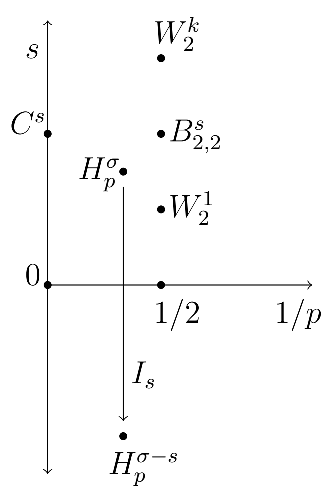


Figure 2.3: Function space lift

The justification of the approach above are the so-called Fourier multiplier theorems, a key concept of modern function space theory, established by S.G.Michlin and L.Hörmander aroud 1960. To motivate Definition 2.1.12 below, we take a look at one side of the equality $H_2^k = W_2^k$ (or of equation (2.1)):

$$\begin{aligned}
\|f|W_2^k\| &= \sum_{|\alpha|\le k} \|D^\alpha f|L_2\| = \sum_{|\alpha|\le k} \|\xi^\alpha \mathcal{F} f|L_2\| \\
&\le c\|(1+|\xi|^2)^{k/2}\mathcal{F} f|L_2\| \\
&= c\|\mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F} f|L_2\| = c\|f|H_2^k\|.
\end{aligned}$$

This is an easy consequence of Plancherels formula. But what happens if $p \neq 2$, where Plancherels formula is not available? What we can do is

$$\begin{aligned}
\|f|W_p^k\| &= \sum_{|\alpha|\le k} \|D^\alpha f|L_p\| = \sum_{|\alpha|\le k} \|D^\alpha(\mathcal{F}^{-1}\mathcal{F})f|L_p\| \\
&= \sum_{|\alpha|\le k} \|\mathcal{F}^{-1}(\xi^\alpha \mathcal{F} f)|L_p\|
\end{aligned}$$

and we would like to estimate further

$$\sum_{|\alpha|\le k} \|\mathcal{F}^{-1}(\xi^\alpha \mathcal{F} f)|L_p\| \le c\|\mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F} f|L_p\| = c\|f|H_p^k\|.$$

So, the question is: When does something like

$$\|\mathcal{F}^{-1}(\xi^\alpha \mathcal{F} f)|L_p\| \le c\|\mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F} f|L_p\|$$

hold, or in the rewritten form

$$\left\|\mathcal{F}^{-1}\frac{\xi^\alpha}{(1+|\xi|^2)^{k/2}}\mathcal{F}\mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F} f|L_p\right\| \le c\|\mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F} f|L_p\| \quad ?$$

If we set $m(\xi) = \frac{\xi^\alpha}{(1+|\xi|^2)^{k/2}}$ and $\varrho(\xi) = \mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F} f$ then the last line simplies to

$$\|\mathcal{F}^{-1} m \mathcal{F}\varrho|L_p\| \le c\|\varrho|L_p\|,$$

which leads to the following definition.

**Definition 2.1.12** *A function $m \in L_\infty$ is called Fourier multiplier for $L_p$, $1 \le p \le \infty$, if there exists $c > 0$ such that*

$$\|\mathcal{F}^{-1} m \mathcal{F} f|L_p\| \le c\|f|L_p\|$$

*holds for all $f \in L_p$ (Then we write $m \in \mathcal{M}(L_p)$).*

**Theorem 2.1.13 (Michlin/Hörmander)** *Let $1 < p < \infty$ and $m \in L_\infty$. Then $m \in \mathcal{M}(L_p)$ if*

$$\sup_{|\alpha|\le 1+[n/2]} \sup_{\xi\in\mathbb{R}^n} |\xi|^{|\alpha|}|D^\alpha m(\xi)| < \infty.$$

This result can be used to prove part (i) of the following Theorem, which solves the problem in Remark 2.1.9 and examines the connections between Sobolev, fractional Sobolev, Besov and even Hölder-Zygmund spaces.

**Theorem 2.1.14** *(i) Let $1 < p < \infty$ and $k \in \mathbb{N}_0$, then $W_p^k = H_p^k$.*

*(ii) Let $1 < p < \infty$ and $s, \sigma \in \mathbb{R}$, then $I_s H_p^\sigma = H_p^{\sigma - s}$.*

*(iii) Let $1 < p < \infty$, $1 \le q \le \infty$, $\sigma > 0$, $s \in \mathbb{R}$ and $\sigma - s > 0$, then $I_s B_{p,q}^\sigma = B_{p,q}^{\sigma - s}$ and $I_s \mathcal{C}^\sigma = \mathcal{C}^{\sigma - s}$.*

*(iv) Let $s > 0$, then $H_2^s = B_{2,2}^s$.*

Looking at *(iii)* lead to some speculations. Are the assumptions $\sigma > 0$ and $\sigma - s > 0$ really necessary or do also spaces $B_{p,q}^{\sigma - s}$ with $\sigma - s \le 0$ make sense ? In fact, they do and formula *(iii)* strongly suggests to look for Fourier analytical characterizations of $B_{p,q}^s$ and $\mathcal{C}^s$. As a prototype for that, we want to formulate a so called Littlewood-Paley Theorem. But before, we introduce one of the keys to the Fourier analytical setting in function spaces, the resolution of unity:
Let $\{\varphi_j\}_{j=0}^\infty$ be a sequence of $C^\infty$ functions on $\mathbb{R}^n$ with

$$\begin{aligned} \operatorname{supp} \varphi_0 &\subset \{\xi \in \mathbb{R}^n : \quad |\xi| \le 2\} \\ \operatorname{supp} \varphi_k &\subset \{\xi \in \mathbb{R}^n : \quad 2^{k-1} \le |\xi| \le 2^{k+1}\} \quad k \in \mathbb{N}, \end{aligned}$$

$$\sup_{\xi \in \mathbb{R}^n, j \in \mathbb{N}_0} 2^{j|\alpha|} |D^\alpha \varphi_j(\xi)| < \infty \quad \text{for any multi-index} \quad \alpha$$

and

$$\sum_{j=0}^\infty \varphi_j(\xi) = 1 \quad \text{if} \quad \xi \in \mathbb{R}^n.$$

If all the properties above are fulfilled we call the sequence $\{\varphi_j\}_{j=0}^\infty$ a smooth resolution of unity. A typical example of such a sequence is the following.

**Example 2.1.15** *Let $\varphi_0$ be an arbitrary $C^\infty$-function on $\mathbb{R}^n$ with $\operatorname{supp} \varphi_0 \subset \{\xi \in \mathbb{R}^n : \quad |\xi| < 2\}$ and $\varphi_0(\xi) = 1$ if $|\xi| \le 1$. Then the system $\{\varphi_j\}_{j=0}^\infty$ with $\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$ for $j \in \mathbb{N}$ has the desired properties.*

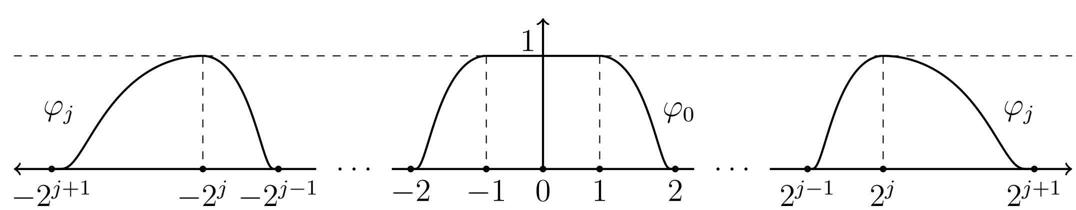


Figure 2.4: Smooth resolution of unity

It is not hard to varify that such a bump function $\varphi_j$ is a Fourier multiplier for $L_p$ by checking the condition in Theorem 2.1.13. From the properties of a smooth resolution of unity we know

$$\sup_{\xi\in\mathbb{R}^n} |\xi|^{|\alpha|}|D^\alpha\varphi_j(\xi)| \le c_\alpha 2^{-j|\alpha|} \sup_{2^{j-1}<|\xi|<2^{j+1}} |\xi|^{|\alpha|} \le c_\alpha.$$

**Theorem 2.1.16** *(Littlewood/Paley) Let $\{\varphi_j\}_{j=0}^\infty$ be a smooth resolution of unity, $s\in\mathbb{R}$ and $1<p<\infty$ then*

$$H_p^s = \left\{ f\in S' : \left\| \Big( \sum_{j=0}^\infty 2^{2js} |(\varphi_j\hat{f})^\vee(\cdot)|^2 \Big)^{1/2} \Big| L_p \right\| < \infty \right\}$$

*in the sense of equivalent norms.*

Again, Fourier multipliers are essential in the proof.

**Remark 2.1.17** *This is the above mentioned prototype of a Fourier analytical characterization of function spaces. The key of this approach is the decomposition of $f$ (or actually $\hat{f}$) into dyadic pieces. Because $(\varphi_j\hat{f})^\vee(x)$ is an entire analytic function in $\mathbb{R}^n$ (by the famous Paley-Wiener-Schwartz-Theorem) the measurement in $L_p$ makes sense.*

Now the idea is the following: Any $f\in S'$ can be represented as

$$f = (\hat{f})^\vee = \left( \sum_{j=0}^\infty \varphi_j \hat{f} \right)^\vee = \sum_{j=0}^\infty (\varphi_j\hat{f})^\vee.$$

The last result suggests to measure smoothness of $f$ by investigating the sequence $\{2^{js}(\varphi_j\hat{f})^\vee(x)\}$ with respect to $L_p(l_q)$-norms. And what happens if we change the order of $L_p$ and $l_q$? In fact, this is even simpler as we will see in the last theorem of this section.
Finally we have paved the way to give a Fourier analytical characterization of classical Besov spaces that we will use for definition in the next section as a basis for all that comes in this lecture concerning Besov spaces. Note here the connenction to equation (1.12) which we used to motivate the concept of retraction and coretraction at the end of chapter 1.

**Theorem 2.1.18** *(i)Let $s>0$, $1<p<\infty$ and $1\le q\le\infty$, then*

$$B_{p,q}^s = \left\{ f\in S' : \left( \sum_{j=0}^\infty 2^{jsq} \|(\varphi_j\hat{f})^\vee | L_p\|^q \right)^{1/q} < \infty \right\}$$

*in the sense of equivalent norms (with the usual modification for $p=\infty$).*
*(ii)Let $s>0$, then*

$$\mathcal{C}^s = \left\{ f\in S' : \sup_{x\in\mathbb{R}^n, j\in\mathbb{N}_0} 2^{js} |(\varphi\hat{f})^\vee(x)| < \infty \right\}$$

*in the sense of equivalent norms.*

**Remark 2.1.19** *This characterization of Besov spaces is due to Peetre in 1967 and gives the starting point for a Fourier analytical definition. Note that for* $s > 0$ *one has* $B^s_{\infty,\infty} = \mathcal{C}^s$.

Here we end our trip through history of function spaces (around 1970), because we will work throughout the lecture with spaces not more complicated than those from Definition 2.2.1 below. But it must be mentioned that there is a lot more that happened in that area since the late sixties, for the interested reader we refer to [Tr92].

## 2.2 The scales $B^s_{pq}$ and $F^s_{pq}$

For the sake of completeness we give here a more general Fourier-analytical Definition of the two function space scales that are nowadays most prominent in this area, the Besov spaces and the Triebel-Lizorkin spaces.

**Definition 2.2.1** *Let* $\{\varphi_j\}_{j=0}^\infty$ *be a smooth resolution of unity.*
*(Besov spaces) For* $s \in \mathbb{R}$, $0 < p, q \leq \infty$ *we define*

$$B^s_{p,q} = \left\{ f \in S' : \left( \sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \hat{f})^\vee | L_p\|^q \right)^{1/q} < \infty \right\}$$

*(with the usual modification for* $q = \infty$*).*
*(Triebel-Lizorkin spaces) For* $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ *we define*

$$F^s_{p,q} = \left\{ f \in S' : \left\| \left( \sum_{j=0}^\infty 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p \right\| < \infty \right\}$$

*(with the usual modification for* $q = \infty$*).*

Note that the order of taking the $L_p$-norm and the weighted $l_q$-norm is just interchanged. For $1 \leq p, q$ these spaces are Banach spaces (otherwise quasi-Banach spaces) and independent of the choice of $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms.
From now on we restrict ourselves to the investigation of Besov spaces.

### 2.2.1 Embeddings

Besides the well-known fact that

$$S \hookrightarrow B^s_{p,q} \hookrightarrow S' \quad \text{for} \quad s \in \mathbb{R}, 0 < p, q \leq \infty$$

and $S$ even dense in $B^s_{p,q}$ is $(p, q < \infty)$, we want to formulate more interesting embeddings here.

**Theorem 2.2.2** *(i) Let $1 < p < \infty$ and $s \in \mathbb{R}$, then*

$$B^s_{p,1} \hookrightarrow H^s_p \hookrightarrow B^s_{p,\infty}, \quad \textit{in particular} \quad B^0_{p,1} \hookrightarrow L_p \hookrightarrow B^0_{p,\infty}.$$

*(ii) Let $0 < p, q_1, q_2 \le \infty$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 > s_2$, then*

$$B^{s_1}_{p,q_1} \hookrightarrow B^{s_2}_{p,q_2}.$$

*(iii) Let $0 < p \le \infty$, $0 < q_1 \le q_2 \le \infty$ and $s \in \mathbb{R}$, then*

$$B^s_{p,q_1} \hookrightarrow B^s_{p,q_2}.$$

*(iv) Let $0 < p_1 \le p_2 \le \infty$, $0 < q \le \infty$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 > s_2$, then*

$$B^{s_1}_{p_1,q} \hookrightarrow B^{s_2}_{p_2,q} \quad \textit{if} \quad s_1 - \frac{n}{p_1} \ge s_2 - \frac{n}{p_2}.$$

The proof for all these embeddings can be found in [Tr83].

## 2.3 Interpolation in function spaces

Now we apply our knowledge on interpolation of Banach Spaces to concrete function spaces starting with the basic $L_p$-spaces, trying first to recover the Riesz/Thorin Theorem from the very beginning.

### 2.3.1 $L_p$-spaces

As we have discussed after Theorem 1.2.9, when we interpolate two spaces of type $l_p^\sigma$ with the same parameter $\sigma$ the resulting interpolation space is in general not in this scale anymore. We had to introduce the Lorentz sequence spaces $l_{p,q}$ to formulate the result. Now in the continuous world the same problem occurs. The interpolation of two different $L_p$-spaces might leave this scale and we need to introduce the so-called Lorentz spaces to give a complete answer.
Let $(X, \mathcal{X}, \mu)$ a complete measure space with a $\sigma$-finite, positive measure $\mu$ on $(X, \mathcal{X})$ and $A$ a Banach space. Then we extend the scale $L_p(A) = L_p(A; X, \mathcal{X}, \mu)$ to all functions $f : X \longrightarrow A$ that are $p$-integrable with respect to $\mu$ $(1 \le p \le \infty)$ with

$$\|f|L_p(A)\| = \left\{ \begin{array}{lcl} \left( \int\limits_X \|f(x)|A\|^p \mu(dx) \right)^{1/p} & : & 0 < p < \infty \\ \operatorname{ess\,sup}_{x \in X} \|f(x)|A\| & : & p = \infty \end{array} \right\} < \infty.$$

Analog to the discrete case, now we need the notion of non-increasing rearrengement of a function. Roughly this is just a reordering of function values in two steps. First we set for $f \in L_1(A) + L_\infty(A)$ the distribution function of $f$ to be

$$\varrho(f, \sigma) = \mu(\{x \in X : \|f(x)|A\| > \sigma\}) \quad \text{for} \quad \sigma > 0.$$

Now we can define the central construction in this area (see [BS88]).

**Definition 2.3.1** *Let $f \in L_1(A)+L_\infty(A)$, then its non-increasing rearrangement $f^* : (0,\infty) \longrightarrow [0,\mu(X)]$ is defined by*

$$f^*(t) = \inf\{\sigma > 0 : \varrho(f,\sigma) \le t\} \quad \text{for} \quad t > 0.$$

Let us first have a look to an illustrative example. In case $(X, \mathcal{X}, \mu) = (\mathbb{R}^n, \mathcal{B}, \lambda_n)$ (Borel $\sigma$-algebra and Lebesgue measure in $\mathbb{R}^n$) and $A = \mathbb{R}$ we consider step functions of the form

$$f(x) = \sum_j a_j \chi_{A_j}(x) \quad \text{with} \quad A_j \cap A_k = \emptyset,\ j \neq k.$$

Then the rearrangement is very simple:

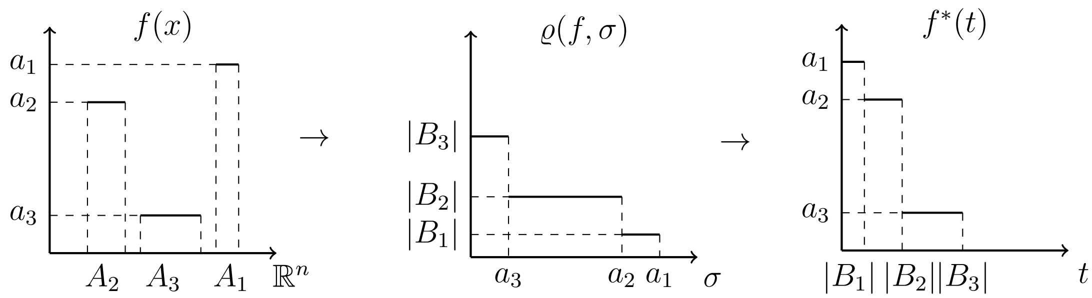


Figure 2.5: Example for rearrangement

Here we set $B_l = \bigcup_{j=1}^{l} A_j$.

**Remark 2.3.2** *As the name suggests $f^*(t)$ as well as $\varrho(f,\sigma)$ are monotone decreasing, right-continuous and $\lim_{\sigma\to\infty} \varrho(f,\sigma) = 0$ hold. Furthermore, if $\mu(X) < \infty$ then $f^*(t) = 0$ for $t > \mu(X)$. Obviously for $f \in L_\infty(A)$ we have $f^*(0) = \lim_{t\downarrow 0} f^*(t) = \|f|L_\infty(A)\|$ .*

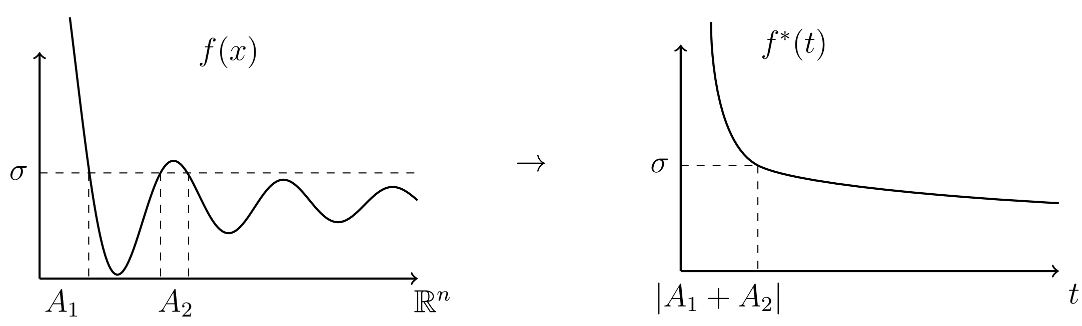


Figure 2.6: Measure preserving

The imagination that $f^*$ is just a reordering of function values even in the continuous case is supported by the fact that it is measure preserving, i.e.,

$$|\{t > 0 : f^*(t) > \sigma\}| = \mu(\{x \in X : \|f(x)|A\| > \sigma\}), \tag{2.3}$$

compare Figure 2.6. For $\sigma_2 \geq \sigma_1 > 0$ it follows

$$|\{t > 0 : \sigma_2 > f^*(t) > \sigma_1\}| = \mu(\{x \in X : \sigma_2 > \|f(x)|A\| > \sigma_1\}). \tag{2.4}$$

Observations like this make the non-increasing rearrangement to a powerful tool, for detailed discription we refer again to [BS88].
Now we can define the continuous analog to the Lorentz sequence spaces.

**Definition 2.3.3** *Let $A$ be a Banach space, $1 < p < \infty$ and $1 \leq q \leq \infty$. Then we define the Lorentz space $L_{p,q}(A)$ consisting of all functions $f \in L_1(A)+L_\infty(A)$ with*

$$\|f|L_{p,q}(A)\| = \left\{ \begin{array}{lcl} \left( \int\limits_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t} \right)^{1/q} & : & q < \infty \\ \sup_{0<t<\infty} t^{1/p} f^*(t) & : & q = \infty \end{array} \right\} < \infty.$$

**Remark 2.3.4** *As in the discrete case, we have $L_{p,p} = L_p$ and certain monotony in the second index: $L_{p,q} \hookrightarrow L_{p,u}$ for $1 \leq q \leq u \leq \infty$. (Note that embeddings with respect to the first index depend on the underlying measure space)*

We are ready to formulate the main assertion of this subsection, which is the perfect analog to Theorem 1.2.11.

**Theorem 2.3.5** *Let $A$ be a Banach space and $0 < \theta < 1 \leq q \leq \infty$. Then*

$$(L_1(A), L_\infty(A))_{\theta,q} = L_{\frac{1}{1-\theta},q}(A).$$

**Proof** Step 1: First we show

$$K(t, a; L_1(A), L_\infty(A)) = \int_0^t f^*(\tau)d\tau \quad \text{for} \quad f \in L_1(A) + L_\infty(A).$$

We set $t_- = \inf\{s > 0 : f^*(s) = f^*(t)\} \leq t$ and define $\mathcal{A}_t^* = \{x \in X : \|f(x)|A\| > f^*(t)\}$ and $\mathcal{B}_t^* = \{x \in X : \|f(x)|A\| = f^*(t)\}$. Looking at (2.3) we verify $\mu(\mathcal{A}_t^*) = t_-$ and for $\mu(\mathcal{B}_t^*) \geq t - t_- > 0$ we can write with (2.4)

$$\int_0^t f^*(\tau)d\tau = \int_0^{t_-} f^*(\tau)d\tau + \int_{t_-}^t f^*(\tau)d\tau = \int_{\mathcal{A}_t^*} \|f(x)|A\|\mu(dx) + \|f(y)|A\|(t - t_-)$$

for an $y \in \mathcal{B}_t^*$.
Let now $f = f^0 + f^1$ with $f^0 \in L_1(A)$ and $f^1 \in L_\infty(A)$, then we choose $y^0 \in \mathcal{B}_t^*$

such that $\|f^0(y^0)|A\| = \min\{\|f^0(x)|A\| : x \in \mathcal{B}_t^*\}$ and estimate

$$\begin{aligned}
\int\limits_0^t f^*(\tau)d\tau &\le \int\limits_{\mathcal{A}_t^*} \|f^0(x)|A\|\mu(dx) + \|f^0(y^0)|A\|(t-t_-) \\
&\quad + \int\limits_{\mathcal{A}_t^*} \|f^1(x)|A\|\mu(dx) + \|f^1(y^0)|A\|(t-t_-) \\
&\le \int\limits_{\mathcal{A}_t^*} \|f^0(x)|A\|\mu(dx) + \|f^0(y^0)|A\| \int\limits_{\mathcal{B}_t^*} \mu(dx) \\
&\quad + \|f^1|L_\infty(A)\|\mu(\mathcal{A}_t^*) + \|f^1|L_\infty(A)\|(t-t_-) \\
&\le \int\limits_{\mathcal{A}_t^*} \|f^0(x)|A\|\mu(dx) + \int\limits_{\mathcal{B}_t^*} \|f^0(x)|A\|\mu(dx) \\
&\quad + \|f^1|L_\infty(A)\|t_- + \|f^1|L_\infty(A)\|(t-t_-) \\
&\le \|f^0|L_1(A)\| + t\|f^1|L_\infty(A).\|
\end{aligned}$$

After taking the infimum over all such decompositions $f = f_0 + f_1$ we end up with

$$\int_0^t f^*(\tau)d\tau \le K(t, a; L_1(A), L_\infty(A)).$$

For the converse inequality we decompose $f$ for fixed $t$ in a special way, i.e.,

$$\hat{f}^0(x) = \left\{ \begin{array}{lcl} f(x) - \frac{f(x)}{\|f(x)|A\|} f^*(t) & : & x \in \mathcal{A}_t^* \\ 0 & : & \text{else} \end{array} \right\} \quad \text{and} \quad \hat{f}^1 = f - \hat{f}^0.$$

Then we can write

$$\begin{aligned}
\|\hat{f}^0|L_1(A)\| &= \int\limits_X \|\hat{f}^0(x)|A\|\mu(dx) = \int\limits_{\mathcal{A}_t^*} \|f(x)|A\| \left(1 - \frac{f^*(t)}{\|f(x)|A\|}\right) \mu(dx) \\
&= \int\limits_{\mathcal{A}_t^*} \|f(x)|A\|\mu(dx) - f^*(t)\mu(\mathcal{A}_t^*) = \int\limits_0^{t_-} f^*(\tau)d\tau - f^*(t)t_-,
\end{aligned}$$

where we used (2.4) in the last line. On the other hand we have

$$\begin{aligned}
\|\hat{f}^1|L_\infty(A)\| &= \sup_{x\in X} \|\hat{f}^1(x)|A\| \\
&= \max\left\{ \sup_{x\in\mathcal{A}_t^*} \left\| f(x) - \left(f(x) - \frac{f(x)}{\|f(x)|A\|} f^*(t)\right)|A\right\|, \sup_{x\in X\setminus\mathcal{A}_t^*} \|f(x)|A\| \right\} \\
&\le \max(f^*(t), f^*(t)).
\end{aligned}$$

Now by definition of the $K$-functional we arrive at

$$\begin{aligned} K(t,a;L_1(A),L_\infty(A)) &\le \|\hat{f}^0|L_1(A)\| + t\|\hat{f}^1|L_\infty(A)\| \\ &\le \int_0^{t_-} f^*(\tau)d\tau + f^*(t)(t-t_-) \\ &\le \int_0^{t_-} f^*(\tau)d\tau + \int_{t_-}^{t} f^*(\tau)d\tau = \int_0^t f^*(\tau)d\tau, \end{aligned}$$

which is the desired estimate and establishes

$$K(t,a;L_1(A),L_\infty(A)) = \int_0^t f^*(\tau)d\tau.$$

Step 2: We prove $(L_1(A),L_\infty(A))_{\theta,q} \hookrightarrow L_{\frac{1}{1-\theta},q}(A)$.
Because of $\int_0^t f^*(\tau)d\tau \ge tf^*(t)$ we estimate for $q<\infty$

$$\begin{aligned} \left\|f|(L_1(A),L_\infty(A))_{\theta,q}\right\|^q &= \int_0^\infty t^{-\theta q} K(t,f;L_1(A),L_\infty(A))^q \frac{dt}{t} \\ &\ge \int_0^\infty t^{(1-\theta)q} f^*(t)^q \frac{dt}{t} = \left\|f|L_{\frac{1}{1-\theta},q}(A)\right\|^q. \end{aligned}$$

The same argument runs for $q=\infty$.
Step 3: Finally we prove the converse, i.e., $L_{\frac{1}{1-\theta},q}(A) \hookrightarrow (L_1(A),L_\infty(A))_{\theta,q}$.
First let $q<\infty$, then by our knowledge about the $K$-functional we can write with $\tau = st$

$$\begin{aligned} \left\|f|(L_1(A),L_\infty(A))_{\theta,q}\right\| &= \left(\int_0^\infty t^{-\theta q} K(t,f;L_1(A),L_\infty(A))^q \frac{dt}{t}\right)^{1/q} \\ &= \left(\int_0^\infty t^{-\theta q}\left(\int_0^t f^*(\tau)d\tau\right)^q \frac{dt}{t}\right)^{1/q} \\ &= \left(\int_0^\infty t^{(1-\theta)q}\left(\int_0^1 f^*(st)ds\right)^q \frac{dt}{t}\right)^{1/q} \end{aligned}$$

and by using the generalized triangle inequality for integrals again ([HLP52] Thm. 202), we further estimate

$$
\begin{aligned}
\Big\| f|(L_1(A), L_\infty(A))_{\theta,q} \Big\| &\leq \int_0^1 \left( \int_0^\infty t^{(1-\theta)q} f^*(st)^q \frac{dt}{t} \right)^{1/q} ds \\
&= \int_0^1 s^{-(1-\theta)} \left( \int_0^\infty \tau^{(1-\theta)q} f^*(\tau)^q \frac{d\tau}{\tau} \right)^{1/q} ds \\
&= \Big\| f|L_{\frac{1}{1-\theta},q}(A) \Big\| \int_0^1 s^{-(1-\theta)} ds \leq c \Big\| f|L_{\frac{1}{1-\theta},q}(A) \Big\|.
\end{aligned}
$$

For $q = \infty$ the argument runs very similar, so we established the converse inclusion and therefore the desired equality in the above Theorem.

□

As we did in the discrete case (compare Corollary 1.4.6) we can now use the Reiteration Theorem to state a result covering the whole scale of Lorentz spaces.

**Corollary 2.3.6** *Let $A$ be a Banach space, $1 < p_0, p_1 < \infty$ with $p_0 \neq p_1$ and $0 < \eta < 1 \leq q_0, q_1, q \leq \infty$. Then*

$$
\Big( L_{p_0,q_0}(A), L_{p_1,q_1}(A) \Big)_{\eta,q} = L_{p,q}(A) \quad \text{with} \quad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1},
$$

*in particular,*

$$
\Big( L_{p_0}(A), L_{p_1}(A) \Big)_{\eta,p} = L_p(A).
$$

This is the perfect analog to Corollary 1.4.6 and also the proof can be done in the same way with $A_0 = L_1(A)$ and $A_1 = L_\infty(A)$. Note that the famous Theorem of Riesz/Thorin (Theorem 1.1.1) is now just a special case of the above result.

### 2.3.2 Sobolev- and Besov spaces

Now we come to the interpolation of Sobolev- and Besov spaces where we heavily benefit from our preparations in the sequence spaces. We recall that by definition we have for $f \in B^s_{p,q}(\mathbb{R}^n)$

$$
\|f|B^s_{p,q}(\mathbb{R}^n)\| = \Big\| ((\varphi \hat{f})^\vee)_{j=0}^\infty | l^s_q(L_p(\mathbb{R}^n)) \Big\| < \infty.
$$

Again we will omit to write $\mathbb{R}^n$ in what follows. We still use both notations $\hat{f}, f^\vee$ as well as $\mathcal{F}f, \mathcal{F}^{-1}f$ for the Fourier transform of $f$ and its inverse in parallel. Now we consider the map

$$
S_\varphi : B^s_{p,q} \longrightarrow l^s_q(L_p) \quad \text{with} \quad S_\varphi f = ((\varphi \hat{f})^\vee)_{j=0}^\infty.
$$

Since $S$ is obviously linear and bounded, i.e. $S \in \mathcal{L}(B^s_{p,q}, l^s_q(L_p))$, it could be a coretraction from $B^s_{p,q}$ onto $l^s_q(L_p)$, according to Definition 1.4.8, if one finds the corresponding retraction $R \in \mathcal{L}(l^s_q(L_p), B^s_{p,q})$. This result is given now.

**Lemma 2.3.7** *Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $(\varphi_j)_{j=0}^\infty, (\psi_j)_{j=0}^\infty$ two smooth resolutions of unity with $\psi_j(\xi) = 1$ for $\xi \in \operatorname{supp} \varphi_j$, $j \in \mathbb{N}_0$. Then is*

$$R_\psi : l^s_q(L_p) \longrightarrow B^s_{p,q} \quad \text{with} \quad R_\psi\Big((h_j)_{j=0}^\infty\Big) = \sum_{j=0}^\infty \mathcal{F}^{-1}\psi_j \mathcal{F} h_j$$

*a retraction in $\mathcal{L}(l^s_q(L_p), B^s_{p,q})$ and the above operator $S_\varphi$ is the corresponding coretraction.*

**Proof** Obviously $R_\psi$ is linear, so first we show the boundedness. So let $(h_k)_{k=0}^\infty \in l^s_q(L_p)$ $(q < \infty)$, then

$$\Big\| R_\psi\Big((h_k)_{k=0}^\infty\Big) | B^s_{p,q}\| = \left( \sum_{j=0}^\infty 2^{jsq} \Big\| \mathcal{F}^{-1}\varphi_j \mathcal{F}\Big( \sum_{k=0}^\infty \mathcal{F}^{-1}\psi_k \mathcal{F} h_k\Big) | L_p\Big\|^q \right)^{1/q}.$$

Because both $(\varphi_j)_{j=0}^\infty$ and $(\psi_k)_{k=0}^\infty$ are smooth resolutions of unity, by definition, the function $\varphi_j$ only overlaps with at most $\psi_{j-1}, \psi_j, \psi_{j+1}$, therefore,

$$\mathcal{F}^{-1}\varphi_j \mathcal{F}\Big( \sum_{k=0}^\infty \mathcal{F}^{-1}\psi_k \mathcal{F} h_k\Big) = \sum_{k=j-1}^{j+1} \mathcal{F}^{-1}\varphi_j\psi_k \mathcal{F} h_k.$$

Since $\varphi_j\psi_{j-1}, \varphi_j\psi_j, \varphi_j\psi_{j+1}$ are Fourier multipliers (see the arguments after Example 2.1.15) we can further estimate

$$\begin{aligned}
\Big\| R_\psi\Big((h_k)_{k=0}^\infty\Big) | B^s_{p,q}\| &\leq c \left( \sum_{j=0}^\infty 2^{jsq} \Big( \|h_{j-1}|L_p\|^q + \|h_j|L_p\|^q + \|h_{j+1}|L_p\|^q \Big) \right)^{1/q} \\
&\leq c \left( (2^{-sq} + 1 + 2^{sq}) \sum_{j=0}^\infty 2^{jsq} \|h_j|L_p\|^q \right)^{1/q} \\
&= c \Big\| (h_j)_{j=0}^\infty | l^s_q(L_p) \Big\|,
\end{aligned}$$

which is the desired estimate.

We still need to show that $R_\psi \circ S_\varphi = id_{B^s_{p,q}}$. By definition we have for $f \in B^s_{p,q}$

$$R_\psi(S_\varphi f) = \sum_{j=0}^\infty \mathcal{F}^{-1}\psi_j \mathcal{F}\mathcal{F}^{-1}\varphi_j \mathcal{F} f = \sum_{j=0}^\infty \mathcal{F}^{-1}\varphi_j \mathcal{F} f = f,$$

because $\psi_j(\xi) = 1$ for $\xi \in \operatorname{supp} \varphi_j$, which finishes the proof.

□

Since $B^s_{p,q} \hookrightarrow S'$ for the whole range of parameters, the pair $\{B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\}$ is an interpolation pair of Banach spaces, if $1 \le p_i, q_i \le \infty$ and $s_i \in \mathbb{R}$ for $i = 0, 1$. Then the above Lemma allows us to apply Theorem 1.4.9(ii) to state that

$$S_\varphi : \left(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\right)_{\theta,q} \longrightarrow \left(l^{s_0}_{q_0}(L_{p_0}), l^{s_1}_{q_1}(L_{p_1})\right)_{\theta,q}$$

is an isomorphism onto a closed subspace of $\left(l^{s_0}_{q_0}(L_{p_0}), l^{s_1}_{q_1}(L_{p_1})\right)_{\theta,q}$ for $0 < \theta < 1 \le q \le \infty$, i.e.,

$$\left\| f | \left(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\right)_{\theta,q} \right\| \sim \left\| ((\varphi \hat{f})^\vee)^\infty_{j=0} | \left(l^{s_0}_{q_0}(L_{p_0}), l^{s_1}_{q_1}(L_{p_1})\right)_{\theta,q} \right\|. \tag{2.5}$$

Now we have developed all the tools to easily proof interpolation results for Besov spaces using the corresponding sequence spaces.

**Theorem 2.3.8** *Let $0 < \theta < 1 \le p, q_0, q_1, q \le \infty$ and $s_0, s_1 \in \mathbb{R}$ with $s_0 \ne s_1$. Then*

$$\left(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1}\right)_{\theta,q} = B^s_{p,q} \quad \textit{with} \quad s = (1-\theta)s_0 + \theta s_1.$$

**Proof** We use Theorem 1.2.9 with $A = L_p$ to get

$$\left(l^{s_0}_{q_0}(L_p), l^{s_1}_{q_1}(L_p)\right)_{\theta,q} = l^s_q(L_p).$$

Now (2.5) already ensures

$$\left\| f | \left(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1}\right)_{\theta,q} \right\| \sim \left\| ((\varphi \hat{f})^\vee)^\infty_{j=0} | l^s_q(L_p) \right\| \sim \|f | B^s_{p,q}\|.$$

□

Thanks to our preparations also further results can be proven that fast.

**Theorem 2.3.9** *Let $0 < \theta < 1 \le p, q_0, q_1 \le \infty$ with $q_0 \ne q_1$ and $s \in \mathbb{R}$. Then*

$$\left(B^s_{p,q_0}, B^s_{p,q_1}\right)_{\theta,q} = B^s_{p,q} \quad \textit{with} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

**Proof** Here we use Corollary 1.4.6 (and the considerations before Definition 1.2.10) to get

$$\left(l^s_{q_0}(A), l^s_{q_1}(A)\right)_{\theta,q} = l^s_q(A).$$

With $A = L_p$ the result follows by the same argument as above.

□

With a slightly more general argument (compare [Tr78] Thm. 2.4.1.) one can even prove the following

**Theorem 2.3.10** *Let $0 < \theta < 1 \le q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$ and $1 < p_0, p_1 < \infty$ with $p_0 \neq p_1$ and*

$$\frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} =: \frac{1}{p}$$

*Then for $s = (1-\theta)s_0 + \theta s_1$*

$$\left(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\right)_{\theta,p} = B^s_{p,p}.$$

The last result of this section shows that Besov spaces naturally appear when interpolating within the scale of Sobolev spaces.

**Theorem 2.3.11** *Let $0 < \theta < 1 \le q \le \infty$, $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$ and $k \in \mathbb{N}$. Then*

$$\left(H^{s_0}_p, H^{s_1}_p\right)_{\theta,q} = B^s_{p,q} \quad \text{with} \quad s = (1-\theta)s_0 + \theta s_1,$$

*in particular for $s > 0$*

$$\left(L_p, H^s_p\right)_{\theta,q} = B^{\theta s}_{p,q} \quad \text{and} \quad \left(L_p, W^k_p\right)_{\theta,q} = B^{\theta k}_{p,q}$$

**Proof** We use Theorem 2.3.8 and Lemma 1.2.7 with the embedding (i) from Theorem 2.2.2 to get

$$\begin{aligned} B^s_{p,q} = \left(B^{s_0}_{p,1}, B^{s_1}_{p,1}\right)_{\theta,q} &\hookrightarrow \left(H^{s_0}_p, H^{s_1}_p\right)_{\theta,q} \\ &\hookrightarrow \left(B^{s_0}_{p,\infty}, B^{s_1}_{p,\infty}\right)_{\theta,q} = B^s_{p,q}. \end{aligned}$$

Because of $L_p = H^0_p$ and $W^k_p = H^k_p$ the other results follow as special cases.

□

# Chapter 3

# Quantitative approximation aspects in function spaces

To motivate the content of this chapter we start with a rough sketch of an idea. For more details and extensive investigations on the mechanism described below we refer to [ET96] (sect. 5.2.4), [Tr97] (sect. 30) or [Tr01] (sect. 19).
Suppose we consider the problem of an oscillating membrane ($\Omega \subset \mathbb{R}^2$) which is fixed at the boundary ($C^\infty$-boundary $\partial\Omega$) and it's deviation $u(x,t)$ (at time $t$) described by

$$\begin{aligned} \Delta u(x,t) &= m(x)\frac{\partial^2 u(x,t)}{\partial^2 t}, \quad x \in \Omega, t \geq 0 \\ u(y,t) &= 0, \quad y \in \partial\Omega, t \geq 0, \end{aligned} \tag{3.1}$$

where $\Delta$ is the Laplace operator and $m(x)$ is a mass density on $\Omega$ (compare the beginning of section 30 in [Tr97]). If we are interested in the eigenfrequencies $\lambda \in \mathbb{R}$ of this vibrating system, we follow the usual strategy and insert the ansatz function $u(x,t) = \exp(i\lambda t)v(x)$ into (3.1) to separate the variables. We get $-\Delta v(x) = \lambda^2 m(x)v(x)$, which is nothing else than an eigenvalue problem for the operator $B = (-\Delta)_D^{-1} \circ m(\cdot)$, where the subscript $D$ indicates the Dirichlet boundary condition. For a positive eigenvalue $\mu$ of $B$ we have the corresponding eigenfrequency $\lambda = \mu^{-1/2}$, so it is of interest to investigate the eigenvalue distribution of such operators $B$. To consider a slightly more general situation we set

$$Bf = (b_2 \circ (-\Delta)_D^{-1} \circ b_1)f,$$

where $b_1, b_2$ are multiplication operators being elements from some $L_p$-spaces. The situation is visualized in the following diagram:

$$\begin{array}{ccccc} L_p & & \xrightarrow{\quad B \quad} & & L_p \\ {\scriptstyle b_1}\big\downarrow & & & & \big\downarrow{\scriptstyle b_2} \\ L_q & \xrightarrow{(-\Delta)_D^{-1}} & \overset{\circ}{W}{}^2_q & \xrightarrow{\text{id}} & L_r \end{array}$$

Here all the spaces are defined over $\Omega$ and $\overset{\circ}{W}{}^2_q$ is a Sobolev space of functions vanishing at the boundary. The operator id is the embedding $\overset{\circ}{W}{}^2_q (\Omega) \hookrightarrow L_r(\Omega)$. For a suitable choice of parameters $p, q, r$ we can assume the operators $B$ and id to be compact (see section 5.2.4 in [ET96]).
Suppose we are concerned with investigating the asymptotic behavior of eigenvalues of such compact operators $B$, then we will learn in the next section that there are several concepts to do so by taking a little detour. Such concepts somehow measure the degree of compactness of an operator in a quantitative way. One of these concepts are the entropy numbers $e_n(B)$ (shortly introduced at the end of chapter 1) and a famous result by Carl, published in [CT80], connects them directly to the eigenvalues $\mu(B)$:

$$\mu_k(B) \leq \sqrt{2} e_k(B).$$

We will discuss entropy numbers in more detail in section 3.1. This result (precisely formulated as our Theorem 3.1.22) tells us that it makes sense to study the entropy numbers of compact operators to gain knowledge about their eigenvalue distribution. Moreover, if the operator is decomposed as in the diagram above, by some nice properties of entropy numbers (compare Theorem 3.1.12) it is even sufficient to study $e_n(\text{id} : \overset{\circ}{W}{}^2_q \to L_r)$.
We take a closer look at this embedding by using another key concept in modern function space theory: decomposition techniques. Similar to the retraction and coretraction transfer mechanism we used for interpolation (compare section 2.3.2), these decomposition techniques, roughly speaking, translate questions in function spaces to the level of sequence spaces using operators similar to $S, R$ from Lemma 2.3.7. We are not going into any details here but refer for an extensive study of this subject to the books by Triebel and related literaure cited there. With such tools at hand, the above embedding $\text{id} : \overset{\circ}{W}{}^2_q \to L_r$ can further be decomposed as:

$$\begin{array}{ccc} \overset{\circ}{W}{}^2_q & \xrightarrow{\text{id}} & L_r \\ {\scriptstyle T}\big\downarrow & & \big\uparrow{\scriptstyle \tilde{T}} \\ l_s(X) & \xrightarrow{\tilde{\text{id}}} & l_t(Y) \end{array},$$

where $X, Y$ are weighted $l_p$-spaces similar to those from Definition 1.2.8 and $\tilde{\text{id}}$ is again an embedding between such nested sequence spaces ($l_s(X) \hookrightarrow l_t(Y)$ for

suitable parameters $s, t$). As described above, by general properties of the entropy numbers one can reduce the estimates for $e_n(\mathrm{id})$ to those for $e_n(\tilde{\mathrm{id}})$.
After describing all this heavy machinery we conclude that estimates for entropy numbers of compact embeddings between certain sequence spaces are helpful to determine the asymptotic behavior of the eigenvalue distribution in concrete applications. Note that this is just one very specific motivation to study compact embeddings in sequence spaces.

## 3.1 Kolmogorov-, approximation- and entropy numbers

### 3.1.1 Definitions and properties

According to our motivation we treat these numbers as different possibilities to characterize and measure the compactness of operators. Before we do so, we discuss some preparations. Here we refer for more details and the omitted proofs to [Pink85](section II.1)

A well-known notion in approximation theory is the so-called best approximation. For a Banach space $X$ and a subspace $U_n \subset X$ with $\dim U_n < n$ one usually defines

$$E(f, U_n) = \inf_{g \in U_n} \|f - g\|$$

to measure what is the minimal error when approximating $f \in X$ by elements in $U_n$. Let's consider a whole subset $A \subset X$ and define

$$E(A, U_n) = \sup_{f \in A} E(f, U_n) = \sup_{f \in A} \inf_{g \in U_n} \|f - g\|$$

to measure the quality of approximating the "worst" element in $A$ by the concrete subspace $U_n$. And now we go one step further and ask for the best subspace $U_n$ to approximate $A$. This leads to the concept of Kolmogorov $n$-widths.

**Definition 3.1.1 (Kolmogorov)** *Let $n \in \mathbb{N}$ and $X$ be a (real or complex) Banach space with $A \subset X$. Then*

$$d_n(A, X) = \inf_{\substack{U_n \subset X \\ \dim U_n < n}} \sup_{f \in A} \inf_{g \in U_n} \|f - g\|$$

*is called the Kolmogorov $n$-width of $A$ in $X$.*

**Remark 3.1.2** *In case there is such a subspace $U_n$ with $d_n(A, X) = E(A, U_n)$, then $U_n$ is called optimal.*

As an example we state here one of the earliest results in this area. Let $L_2 = L_2[0, 2\pi]$ be the usual space of square integrable functions on $[0, 2\pi]$ normed by $\|f\|^2 = 1/2\pi \int_0^{2\pi} |f(x)|^2 dx$ and let

$$\tilde{W}_2^r = \{f : f^{(r-1)} \text{ abs. cont.}, f^{(r)} \in L_2, f^{(i)}(0) = f^{(i)}(2\pi) \text{ for } i = 0, 1, \ldots, r\}$$

be the Sobolev space of $2\pi$-periodic (real-valued) functions. We denote its unit ball by $\tilde{B}_2^r$ and consider $d_n = d_n(\tilde{B}_2^r, L_2)$.

**Theorem 3.1.3 (Kolmogorov (1936))**

$$d_{2n-1} = d_{2n} = n^{-r} \quad \textit{for all } n \in \mathbb{N}.$$

*Furthermore, the space of trigonometric polynomials of degree at most* $n - 1$

$$T_{n-1} = \text{span}\{1, \sin x, \cos x, \ldots, \sin(n-1)x, \cos(n-1)x\}$$

*is an optimal subspace.*

We will use these $n$-widths to define the Kolmogorov numbers for linear operators. But first, for the sake of completeness, we collect some properties of $d_n(A, X)$. If the reference space $X$ is fixed, we write $d_n(A)$ instead of $d_n(A, X)$. Furthermore, we denote by

$$E(A, B) = \sup_{f \in A} \inf_{g \in B} \|f - g\|$$

the distance between $A$ and $B$ and by

$$\mathcal{K}(A) = \bigcap_{convex K \supset A} K$$

the convex hull of A.

**Theorem 3.1.4** *Let $X$ be a Banach space and $A \subset X$. Then for all $n \in \mathbb{N}$*

*(i)* $d_n(A) \geq d_{n+1}(A)$

*(ii)* $d_n(\alpha A) = \alpha d_n(A) \quad \forall \alpha \in \mathbb{R}$

*(iii)* $d_n(A) - E(A, B) \leq d_n(B) \leq d_n(A) \quad \forall B \subset A$

*(iv)* $d_n(A) = d_n(\bar{A}), \quad d_n(A) = d_n(\mathcal{K}(A))$

*(v)* $d_n(A, X) \geq d_n(A, Y)$ *for any Banach space* $Y \supset X$.

From properties (ii),(iii) it also follows that $d_n(A) = d_n(\mathcal{B}(A))$ for the balanced hull $\mathcal{B}(A) = \{\alpha x : x \in A, |\alpha| \leq 1\}$. Therefore, we will assume in the following that $A$ is bounded, convex and symmetric to the origin. Since the following result is of special interest for us, we prove it explicitely.

**Theorem 3.1.5** *Let $X$ be a Banach space and $A \subset X$. Then is $A$ compact if and only if $A$ is bounded and $d_n(A) \longrightarrow 0$ for $n \to \infty$.*

**Proof** First we prove "$\Rightarrow$". Let $A$ be compact. Then $A$ is bounded and for every $\varepsilon > 0$ there exists a finite $\varepsilon$-net, i.e.

$$x_1, \dots, x_N \in X \quad \text{such that } \forall x \in A : \min_{i=1,\dots,N} \|x - x_i\| \leq \varepsilon.$$

If we denote by $U_N = \operatorname{span}\{x_1, \dots, x_N\}$, then $\dim U_N \leq N$ and by definition $d_{N+1}(A) \leq \varepsilon$. Property (i) of Theorem 3.1.4 tells us that $d_n(A) \leq \varepsilon$ for all $n > N$, which means $d_n(A) \to 0$.
To prove "$\Leftarrow$" we assume that $A$ is bounded, which means $d_1(A) = \sup_{x \in A} \|x\| < \infty$, and we assume $d_n(A) \to 0$. So, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ the following holds: There exists an $n$-dimensional subspace $U_n$ such that for all $x \in A$ there is a $y \in U_n$ with $\|x - y\| < \varepsilon$. Hence, $\|y\| \leq \|x - y\| + \|x\| < \varepsilon + d_1(A)$. The set $M$ of all such $y$ is a bounded subset of a finite dimensional space and therefore compact. Hence, it exists a finite $\varepsilon$-net $\{m_1, \dots, m_k\}$ for it, i.e.

$$\forall y \in M : \min_{i=1,\dots,k} \|y - m_i\| \leq \varepsilon,$$

which is by triangle inequality a $2\varepsilon$-net for $A$. Therefore, $A$ is compact.

□

We denote by $B_X$ ($\bar{B}_X$) the open (closed) unit ball of a Banach space $X$.

**Theorem 3.1.6** *Let $n \in \mathbb{N}$ and $A$ be a Banach space.*
*(i) If $\dim X \geq n \in \mathbb{N}$, then*

$$d_k(\bar{B}_X, X) = 1 \quad \textit{for } k = 1, \dots, n.$$

*(ii) If $V_n \subset X$ is a $n$-dimensional subspace, then*

$$d_k(\bar{B}_{V_n}, X) = 1 \quad \textit{for } k = 1, \dots, n.$$

Now we are prepared to introduce the Kolmogorov numbers mentioned above.

**Definition 3.1.7** *Let $X, Y$ be Banach spaces, $T \in \mathcal{L}(X, Y)$ and $n \in \mathbb{N}$. Then*

$$d_n(T) = d_n(T(\bar{B}_X), Y) = \inf_{\substack{U_n \subset Y \\ \dim U_n < n}} \sup_{\|x\|_X \leq 1} \inf_{y \in U_n} \|Tx - y\|$$

*is called the $n$-th Kolmogorov number of $T$.*

As an example we assume $X = Y$. In case $\dim X \geq n$ and $T = id$ is the identity operator on $X$, part (i) of Theorem 3.1.6 gives $d_k(id) = 1$ for $k \leq n$. In case $U$ is a $n$-dimensional subspace and $T = P$ is the projection on $U$, then part (ii) gives

$d_n(P) = 1$ for $k \leq n$.
In what follows we denote by

$$\mathcal{R}(T) = \{y \in Y : \exists x \in X : Tx = y\} \subset Y$$

the range of $T$ and set $\operatorname{rank} T = \dim \mathcal{R}(T)$.

**Definition 3.1.8** *Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X, Y)$.*

*(i) $T$ is called compact, $T \in \mathcal{K}(X, Y)$, if $T(\bar{B}_X)$ is compact in $Y$.*

*(ii) $T$ is called finite, if $\operatorname{rank} T < \infty$. We denote the closure of the set of all finite operators by*

$$\mathcal{F}(X, Y) = \{T : \exists (T_n)_n \text{ with } \operatorname{rank} T_n \leq n : \|T - T_n\| \to 0\}.$$

Note that $\mathcal{F}(X, Y) \subset \mathcal{K}(X, Y)$, where in general there is no equality, compare [Piet87](section 10).
Next we want to prove some properties of the Kolmogorov numbers. To do so, we need the following result, which can be found in [CS90](Lemma 2.1.2).

**Lemma 3.1.9** *Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X, Y)$ with $\operatorname{rank} T \geq n$. Then there exist a Banach space $W$ with $\dim W = n$ and operators $S \in \mathcal{L}(W, X)$, $R \in \mathcal{L}(Y, W)$ such that $RTS = id_W$:*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle S}\uparrow & & \downarrow{\scriptstyle R} \\ W & \xrightarrow{\mathrm{id}_W} & W \end{array}$$

**Theorem 3.1.10 (Properties of $d_n(T)$)** *Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X, Y)$.*
$(\mathbf{M_d})$ $\|T\| = d_1(T) \geq d_2(T) \geq \cdots \geq 0$

$(\mathbf{A_d})$ *Let $m, n \in \mathbb{N}$ and $S \in \mathcal{L}(X, Y)$. Then*

$$d_{m+n-1}(S + T) \leq d_m(S) + d_n(T), \quad \text{in particular } |d_n(S) - d_n(T)| \leq \|S - T\|.$$

$(\mathbf{P_d})$ *Let $m, n \in \mathbb{N}$ and $S \in \mathcal{L}(Y, W)$. Then*

$$d_{m+n-1}(ST) \leq d_m(S) d_n(T).$$

$(\mathbf{R_d})$ $d_n(T) = 0 \Longleftrightarrow \operatorname{rank} T < n$.

$(\mathbf{N_d})$ *If* $\dim X \geq n$ *then* $d_n(\mathrm{id}_X) = 1$.

$(\mathbf{K_d})$ $T \in \mathcal{K}(X, Y) \Longleftrightarrow \lim_{n\to\infty} d_n(T) = 0$.

**Proof** $(\mathbf{M_d})$: The monotony is clear by definition and we have

$$d_1(T) = \sup_{x \in \bar{B}_X} \|Tx\|_Y = \|T\|.$$

$(\mathbf{A_d})$: For $m, n \in \mathbb{N}$ and $\varepsilon > 0$ there exist suspaces $U_m, V_n \subset Y$ with $\dim U_m < m$ and $\dim V_n < n$ such that for all $x \in \bar{B}_X$ one can find elements $u_m^x \in U_m$ and $v_n^x \in V_n$ with

$$\|Sx - u_m^x\|_Y < d_m(S) + \varepsilon \quad \text{and} \quad \|Tx - v_n^x\|_Y < d_n(S) + \varepsilon.$$

We take two such spaces $U_m, V_n$ and set $W_{m,n} = U_m + V_n \subset Y$, then we know $\dim W_{m,n} < m + n - 1$ and for all $x \in \bar{B}_X$ one can find an element $w_{m,n}^x = u_m^x + v_n^x \in W_{m,n}$ with

$$\|(S+T)x - w_{m,n}^x\|_Y \leq \|Sx - u_m^x\|_Y + \|Tx - v_n^x\|_Y \leq d_m(S) + d_n(T) + 2\varepsilon.$$

That means there exist a space $W = W_{m,n} \subset Y$ with $\dim W < m + n - 1$ such that

$$\sup_{\|x\|_X \leq 1} \inf_{w \in W} \|(S+T)x - w\|_Y \leq d_m(S) + d_n(T) + 2\varepsilon.$$

If we now take the infimum over all spaces $W \subset Y$ with $\dim W < m + n - 1$ on the left-hand side and let $\varepsilon \to 0$, we conclude

$$d_{m+n-1}(S+T) \leq d_m(S) + d_n(T).$$

In particular one has the following

$$\left.\begin{array}{l} d_n(S) \leq d_1(S-T) + d_n(T) \Rightarrow d_n(S) - d_n(T) \leq \|S-T\| \\ d_n(T) \leq d_1(T-S) + d_n(S) \Rightarrow d_n(T) - d_n(S) \leq \|T-S\| \end{array}\right\} \Rightarrow |d_n(S) - d_n(T)| \leq \|S-T\|.$$

$(\mathbf{P_d})$ : For $m, n \in \mathbb{N}$ and $\varepsilon > 0$ there exist suspaces $U_n \subset Y$ and $V_m \subset W$ with $\dim U_n < n$ and $\dim V_m < m$ such that for all $x \in \bar{B}_X$, $y \in \bar{B}_Y$ one can find elements $u_n^x \in U_n$ and $v_m^y \in V_m$ with

$$\|Tx - u_n^x\|_Y < d_n(T) + \varepsilon \quad \text{and} \quad \|Sy - v_m^y\|_Y < d_m(S) + \varepsilon.$$

For an element $x \in \bar{B}_X$ we set

$$y(x) = \frac{Tx - u_n^x}{d_n(T) + \varepsilon} \in B_Y$$

as a particular choice in the estimates above to see that there is an element

$$w_{m,n} = w_{m,n}(x) := Su_n^x + (d_n(T) + \varepsilon) v_m(y(x)) \in S(U_n) + V_m =: W_{m,n} \subset W,$$

with

$$\left\| S\left(\frac{Tx - u_n^x}{d_n(T)+\varepsilon}\right) - \frac{w_{m,n} - Su_n^x}{d_n(T)+\varepsilon}\right\|_W < d_m(S)+\varepsilon,$$

which is equivalent to

$$\|STx - w_{m,n}\|_W < (d_m(S)+\varepsilon)(d_n(T)+\varepsilon).$$

So, we found a subspace $W_{m,n} \subset W$ with $\dim W_{m,n} < m+n-1$ and

$$\sup_{\|x\|_X \le 1} \inf_{w \in W_{m,n}} \|STx - w\|_W < (d_m(S)+\varepsilon)(d_n(T)+\varepsilon).$$

Finally we let $\varepsilon$ tend to zero and conclude $d_{m+n-1}(ST) \le d_m(S)d_n(T)$.
$(\mathbf{N_d})$ : That is the example discussed after Definition 3.1.7.
$(\mathbf{K_d})$ : This is given by Theorem 3.1.5 with $A = T(\bar{B}_X)$.
$(\mathbf{R_d})$ : The sufficiency is clear with $U_n = \mathcal{R}(T)$. So let $\operatorname{rank} T \ge n$, then we know be Lemma 3.1.9 that there exists a $n$-dimensional Banach space $W$ and operators $S \in \mathcal{L}(W,X)$, $R \in \mathcal{L}(Y,W)$ such that $RTS = \mathrm{id}_W$. Properties $(N_d)$ and $(P_d)$ give

$$1 = d_n(RTS) \le \|R\| d_n(T) \|S\|,$$

therefore $d_n(T) > 0$.

□

For more details on Kolmogorov numbers, explicit calculations for specific operators and applications, see [Pink85, Piet87]. These numbers characterize the compactness of a linear operator quantitatively. Definition 3.1.8 suggests that there are also other concepts to do so, here we introduce two alternatives.

**Definition 3.1.11** *Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X,Y)$.*
*(i) Then*

$$e_n(T) = \inf\left\{\varepsilon > 0 : \exists y_1, \ldots, y_{2^{n-1}} \text{ with } T(\bar{B}_X) \subset \bigcup_{i=1}^{2^{n-1}} \{y_i + \varepsilon \bar{B}_Y\}\right\}$$

*is called $n$-th (dyadic) entropy number of $T$.*
*(ii) Then*

$$a_n(T) = \inf\{\|T - L\| : L \in \mathcal{L}(X,Y) \text{ with } \operatorname{rank} L < n\}$$

*is called $n$-th approximation number of $T$.*

Similar to the Kolmogorov numbers one can introduce (inner and outer) entropy numbers for arbitrary sets first, for details see [CS90](chapter 1). Note that the entropy numbers are a very geometric concept:

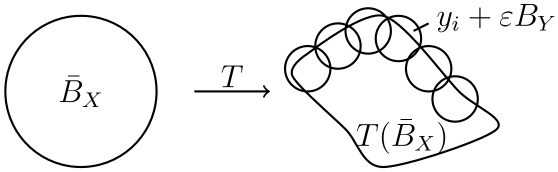


Figure 3.1: Covering of $T(\bar{B}_X)$

Compare to the short discussion at the end of chapter 1.
For the two concepts in the above definition similar properties as for the Kolmogorov numbers are fulfilled. We start with the entropy numbers.

**Theorem 3.1.12 (Properties of $e_n(T)$)** *Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X,Y)$.*
($\mathbf{M_e}$) $\|T\| = e_1(T) \geq e_2(T) \geq \cdots \geq 0$

($\mathbf{A_e}$) *Let $m, n \in \mathbb{N}$ and $S \in \mathcal{L}(X,Y)$. Then*

$$e_{m+n-1}(S+T) \leq e_m(S) + e_n(T), \quad \textit{in particular } \; |e_n(S) - e_n(T)| \leq \|S-T\|.$$

($\mathbf{P_e}$) *Let $m, n \in \mathbb{N}$ and $S \in \mathcal{L}(Y,W)$. Then*

$$e_{m+n-1}(ST) \leq e_m(S)e_n(T).$$

($\mathbf{K_e}$) $T \in \mathcal{K}(X,Y) \Longleftrightarrow \lim_{n\to\infty} e_n(T) = 0.$

**Proof** ($\mathbf{M_e}$): The monotony is clear by definition. Because of $\|Tx\|_Y \leq \|T\|\|x\|_X$ we have $T(\bar{B}_X) \subset \|T\|\bar{B}_Y \subset \{y_1 + \|T\|\bar{B}_Y\}$ for $y_1 = 0$, which means $e_1(T) \leq \|T\|$.
For $\varepsilon > e_1(T) \geq 0$ there is a point $y_1 \in Y$ with $T(\bar{B}_X) \subset \{y_1 + \varepsilon\bar{B}_Y\}$. So, for $x \in \bar{B}_X$ we can find $\eta_1, \eta_2 \in \bar{B}_Y$ with $Tx = y_1 + \varepsilon\eta_1$ and $T(-x) = -Tx = y_1 + \varepsilon\eta_2$. Therefore, $2Tx = \varepsilon(\eta_1 - \eta_2)$, which gives $\|Tx\|_Y \leq \frac{\varepsilon}{2}(\|\eta_1\|_Y + \|\eta_2\|_Y) \leq \varepsilon$. Taking the supremum over all such $x$ gives $\|T\| \leq \varepsilon$ and the infimum over all such $\varepsilon$ yields $\|T\| \leq e_1(T)$.
($\mathbf{A_e}$): Let $\lambda > e_n(T)$ and $\mu > e_m(S)$. Then there are points $y_1, \ldots, y_N, z_1, \ldots, z_M \in Y$ with $N \leq 2^{n-1}, M \leq 2^{m-1}$ and

$$T(\bar{B}_X) \subset \bigcup_{i=1}^{N}\{y_i + \lambda\bar{B}_Y\}, \quad S(\bar{B}_X) \subset \bigcup_{j=1}^{M}\{z_i + \mu\bar{B}_Y\}.$$

That means for a particular point $x \in \bar{B}_X$ we can find points $y_i, z_j \in Y$ with $Tx \in \{y_i + \lambda\bar{B}_Y\}$ and $Sx \in \{z_j + \mu\bar{B}_Y\}$, therefore we have

$$(S+T)x \in \{y_i + z_j + (\lambda + \mu)\bar{B}_Y\}.$$

Since that is true for all $x \in \bar{B}_X$ we end up with

$$(S+T)(\bar{B}_X) \subset \bigcup_{i=1}^{N}\bigcup_{j=1}^{M}\{y_i + z_j + (\lambda+\mu)\bar{B}_Y\},$$

where the number of such points $y_i + z_j$ is at most $NM \leq 2^{(n+m-1)-1}$. That means $e_{m+n-1}(S+T) \leq \lambda + \mu$ and after taking the infimum over all such $\lambda, \mu$ we see $e_{m+n-1}(S+T) \leq e_m(S) + e_n(T)$.
($\mathbf{P_e}$): Here the arguments are analog.
($\mathbf{K_e}$): This is clear by definition, because $T(\bar{B}_X)$ is compact in $Y$, if and only if, for all $\varepsilon > 0$ there exists a finite $\varepsilon$-net, compare the proof of Theorem 3.1.5.

□

**Theorem 3.1.13** *Let $X$ be a real Banach space with* $\dim X = m < \infty$. *Then*

$$2^{-\frac{n-1}{m}} \leq e_n(\mathrm{id}_X) \leq 4 \cdot 2^{-\frac{n-1}{m}} \qquad \textit{for all } n \in \mathbb{N}.$$

A proof can be found in [EE87]. Now we state a helpful connenction of these numbers with interpolation theory, compare again the end of chapter 1.

**Theorem 3.1.14** *Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be interpolation pairs, $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ and $0 < \theta < 1$.*
*(i) For $A \in \mathcal{K}(\theta; A_0, A_1)$ and a Banach space $B$ there exists a constant $c > 0$ such that for all $k_0, k_1 \in \mathbb{N}$*

$$e_{k_0+k_1-1}(T : A \longrightarrow B) \leq c e_{k_0}(T : A_0 \longrightarrow B)^{1-\theta} e_{k_1}(T : A_1 \longrightarrow B)^{\theta}.$$

*(ii) For $B \in \mathcal{J}(\theta; B_0, B_1)$ and a Banach space $A$ with $T \in \mathcal{L}(A, B_0 \cap B_1)$ there exists a constant $c > 0$ such that for all $k_0, k_1 \in \mathbb{N}$*

$$e_{k_0+k_1-1}(T : A \longrightarrow B) \leq c e_{k_0}(T : A \longrightarrow B_0)^{1-\theta} e_{k_1}(T : A \longrightarrow B_1)^{\theta}.$$

For the proof see [ET96](Thm. 1.3.2)

**Theorem 3.1.15 (Properties of $a_n(T)$)** *Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X, Y)$.*
($\mathbf{M_a}$) $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq 0$

($\mathbf{A_a}$) *Let $m, n \in \mathbb{N}$ and $S \in \mathcal{L}(X, Y)$. Then*

$$a_{m+n-1}(S+T) \leq a_m(S) + a_n(T), \quad \textit{in particular} \ |a_n(S) - a_n(T)| \leq \|S - T\|.$$

($\mathbf{P_a}$) *Let $m, n \in \mathbb{N}$ and $S \in \mathcal{L}(Y, W)$. Then*

$$a_{m+n-1}(ST) \leq a_m(S) a_n(T).$$

($\mathbf{R_a}$) $a_n(T) = 0 \Longleftrightarrow \operatorname{rank} T < n$.

($\mathbf{N_a}$) *If* $\dim X \geq n$ *then* $a_n(\mathrm{id}_X) = 1$.

($\mathbf{K_a}$) $\lim_{n\to\infty} a_n(T) = 0 \Longrightarrow T \in \mathcal{K}(X, Y)$.

**Proof** ($\mathbf{M_a}$): Again the monotony is clear and because $\operatorname{rank} L < 1$ means $L = 0$, it follows $a_1(T) = \|T\|$.
($\mathbf{A_a}$): Let $\lambda > a_n(T)$ and $\mu > a_m(S)$. Then there exist operators $L, R \in \mathcal{L}(X,Y)$ with $\operatorname{rank} L \leq n-1$, $\operatorname{rank} R \leq m-1$ and $\|T-L\| < \lambda$, $\|T-R\| < \mu$. For $M = L + R \in \mathcal{L}(X,Y)$ we have $\operatorname{rank} M \leq n + m - 2$ and

$$\|(S+T) - M\| \leq \|S - R\| + \|T - L\| < \lambda + \mu,$$

which means $a_{m+n-1}(S+T) < \lambda + \mu$ and after taking the infimum over all such $\lambda, \mu$ we see $a_{m+n-1}(S+T) \leq a_m(S) + a_n(T)$.
($\mathbf{P_a}$): The strategy is similar. Let $\lambda > a_n(T)$ and $\mu > a_m(S)$. Then there exist operators $L \in \mathcal{L}(X,Y)$, $R \in \mathcal{L}(Y,W)$ with $\operatorname{rank} L \leq n-1$, $\operatorname{rank} R \leq m-1$ and $\|T-L\| < \lambda$, $\|T-R\| < \mu$. For $M = RT + SL - RL \in \mathcal{L}(X,W)$ we have

$$\operatorname{rank} M \leq \operatorname{rank}(SL) + \operatorname{rank}(R(T-L)) \leq \operatorname{rank} L + \operatorname{rank} R \leq n + m - 2$$

and

$$\|ST - M\| + \|(S-R)(T-L)\| \leq \|S-R\|\|T-L\| < \lambda\mu,$$

which means $a_{m+n-1}(ST) < \lambda\mu$ and after taking the infimum over all such $\lambda, \mu$ we see $a_{m+n-1}(S+T) \leq a_m(S)a_n(T)$.
($\mathbf{N_a}$): For sure we know $a_n(\mathrm{id}_X) \leq a_1(\mathrm{id}_X) = \|\mathrm{id}_X\| = 1$. So let $\dim X \geq n$ and $L \in \mathcal{L}(X,X)$ with $\operatorname{rank} L < n$. Then we find an element $x_0 \neq 0$ in $X$ with $L(x_0) = 0$ and $\|x_0\|_X = 1$. Therefore, we can estimate

$$1 = \|x_0\|_X = \|L(x_0) - x_0\|_X \leq \sup_{\|x\|_X \leq 1} \|(L - \mathrm{id}_X)x\|_X = \|L - \mathrm{id}_X\|_X$$

and after taking the infimum over all such operators $L$, we get $a_n(\mathrm{id}_X) \geq 1$.
($\mathbf{R_a}$): The direction $'' \Leftarrow''$ is clear by definition. So let $\operatorname{rank} T \geq n$, then by Lemma 3.1.9 there exist a $n$-dimensional Banach space $W$ and operators $S \in \mathcal{L}(W,X)$, $R \in \mathcal{L}(Y,W)$, such that $RTS = \mathrm{id}_W$. Using properties $(N_a)$ and $(P_a)$ we find

$$1 = a_n(\mathrm{id}_W) = a_n(RTS) \leq \|R\| a_n(T) \|S\|,$$

which implies $a_n(T) > 0$.
($\mathbf{K_a}$): This is another way of writing $\mathcal{F}(X,Y) \subset \mathcal{K}(X,Y)$, compare Definition 3.1.8.

□

**Example:** $D_\sigma$

Remember our goal : investigation of id : $l_p(\sigma) \longrightarrow l_p$ (weighted $l_p$-spaces). Similar to the spaces $l_p^\sigma$ (Definition 1.2.8) they are normed by

$$\|x|l_p(\sigma)\| = \left( \sum_{k=1}^{\infty} \sigma_k^p |\xi_k|^p \right)^{1/p} \qquad (p < \infty).$$

Equivalently one can investigate an operator $D_\sigma : l_p \longrightarrow l_p$ that incorporates the weight as elementwise multiplication

$$D_\sigma : x = (\xi_k)_k \longmapsto (\sigma_k \xi_k)_k, \quad 1 \le p \le \infty.$$

So, we consider a monotone decreasing non-negative sequence $(\sigma_k)_k$ as weight sequence and state:

**Theorem 3.1.16** *(i)* *For* $n \in \mathbb{N}$ *:* $\quad d_n(D_\sigma) = a_n(D_\sigma) = \sigma_n.$

*(ii)* *For* $n \in \mathbb{N}$ *:* $\quad 1 \le \sup_{k\in\mathbb{N}} 2^{\frac{n-1}{k}} (\sigma_1 \cdots \sigma_k)^{-1/k} e_n(D_\sigma) \le 6.$

**Proof** Part (ii) is a famous result by Gordon, König and Schütt, see [GKS87]. For part (i) we give the proof here: Consider the following operators

$$\begin{aligned} D_{\sigma,n-1} : l_p \longrightarrow l_p, \quad D_{\sigma,n-1} &: (\xi_k)_k \longmapsto (\sigma_1\xi_1, \ldots, \sigma_{n-1}\xi_{n-1}, 0, 0, \ldots) \Longrightarrow \operatorname{rank} D_{\sigma,n-1} < n \\ \mathrm{id}_n : l_p^n \longrightarrow l_p, \quad id_n &: (\xi_k)_k \longmapsto (\xi_1, \ldots, \xi_n, 0, 0, \ldots) \Longrightarrow \|id_n\| = 1 \\ P_n : l_p \longrightarrow l_p^n, \quad P_n &: (\xi_k)_k \longmapsto (\xi_1, \ldots, \xi_n) \Longrightarrow \|P_n\| = 1 \\ D_\sigma^{(n)} : l_p^n \longrightarrow l_p^n, \quad D_\sigma^{(n)} &: (\xi_k)_k \longmapsto (\sigma_1\xi_1, \ldots, \sigma_n\xi_n). \end{aligned}$$

Because of $\|D_\sigma - D_{\sigma,n-1}\| = \sigma_n$ with $\operatorname{rank} D_{\sigma,n-1} < n$ we find $a_n(D_\sigma) \le \sigma_n$. In case $\sigma_n = 0$ of course $a_n(D_\sigma) = 0$ follows. So, let's suppose $\sigma_n > 0$, then $D_\sigma^{(n)} = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ with $\det D_\sigma^{(n)} = \sigma_1 \cdots \sigma_n > 0$, therefore it's invertible and we see $\|(D_\sigma^{(n)})^{-1}\| = \sigma_n^{-1}$. Using the propeties $(N_a)$ and $(P_a)$ we can estimate

$$1 = a_n(\mathrm{id}_{l_p^n}) = a_n((D_\sigma^{(n)})^{-1} D_\sigma^{(n)}) \le \|(D_\sigma^{(n)})^{-1}\| a_n(D_\sigma^{(n)}) = \sigma_n^{-1} a_n(D_\sigma^{(n)}),$$

therefore $a_n(D_\sigma^{(n)}) \ge \sigma_n$. On the other hand we have $D_\sigma^{(n)} = P_n D_\sigma id_n$, visualized as

$$\begin{array}{ccc} l_p^n & \xrightarrow{D_\sigma^{(n)}} & l_p^n \\ {\scriptstyle \mathrm{id}_n}\downarrow & & \uparrow{\scriptstyle P_n} \\ l_p & \xrightarrow[D_\sigma]{} & l_p \end{array}$$

So, using $(P_a)$ again we conclude

$$\sigma_n \le a_n(D_\sigma^{(n)}) = a_n(P_n D_\sigma \,\mathrm{id}_n) \le \|P_n\| a_n(D_\sigma) \|\,\mathrm{id}_n\,\| = a_n(D_\sigma) \le \sigma_n \quad \Leftrightarrow a_n(D_\sigma) = \sigma_n.$$

Analog arguments run for $d_n(D_\sigma)$.

$\square$

**A unifying concept: $s$-numbers**

This approach is due to Pietsch, see [Piet78](section 11) and [Piet87].
Let $T \in \mathcal{L}(X,Y)$, then the elements of a sequence $(s_n(T))_n$ are called $s$-numbers, if all the following properties are satisfied:

($\mathbf{M_s}$) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$

($\mathbf{A_s^*}$) For $n \in \mathbb{N}$ and $S \in \mathcal{L}(X,Y)$ always:

$$s_n(S+T) \leq s_n(S) + \|T\|, \text{ in particular } |s_n(S) - s_n(T)| \leq \|S-T\|.$$

($\mathbf{P_s^*}$) For $n \in \mathbb{N}$, $S \in \mathcal{L}(Y,W)$ and $R \in \mathcal{L}(V,X)$ always:

$$s_n(STR) \leq \|S\| s_n(T) \|R\|.$$

($\mathbf{R_s}$) $s_n(T) = 0 \Longleftrightarrow \operatorname{rank} T < n$.

($\mathbf{N_s}$) For $X = l_2^n$ always: $s_n(\mathrm{id}_X) = 1$.

Obvioulsy $(d_n(T))_n$ and $(a_n(T))_n$ are sequences of $s$-numbers. For $T \in \mathcal{L}(X,Y)$ other examples are

$$\text{Gelfand numbers: } c_n(T) = \inf\{\|TJ_M\| : \operatorname{codim}(M) < n\}$$

(where $J_M = id : M \longrightarrow X$ is the canonical injection from the subspace $M \subset X$),

$$\text{Weyl numbers: } x_n(T) = \inf\{a_n(TS) : S \in \mathcal{L}(l_2, X), \|S\| \leq 1\},$$

$$\text{Chang numbers: } y_n(T) = \sup\{a_n(RT) : R \in \mathcal{L}(Y, l_2), \|R\| \leq 1\},$$

and

$$\text{Hilbert numbers: } h_n(T) = \sup\{a_n(RTS), S \in \mathcal{L}(l_2, X), R \in \mathcal{L}(Y, l_2), \|R\|, \|S\| \leq 1\}.$$

$s$-numbers are called additiv, if for $m, n \in \mathbb{N}$ and $S, T \in \mathcal{L}(X,Y)$ always

$$s_{m+n-1}(S+T) \leq s_m(S) + s_n(T)$$

holds. Analoguously they are called multiplicative, if for $m, n \in \mathbb{N}$, $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,W)$ always

$$s_{m+n-1}(ST) \leq s_m(S) s_n(T).$$

holds. As we have seen, $(d_n(T))_n$ and $(a_n(T))_n$ fulfill these two properties.
The entropy numbers $(e_n(T))_n$ are not $s$-numbers, because property $(R_s)$ is not satisfied, compare Theorem 3.1.13.

### 3.1.2 Mutual estimates and the connection to eigenvalues

In the following we just collect some known results.

**Theorem 3.1.17** *For $T \in \mathcal{L}(X,Y)$:*

$$h_n(T) \le x_n(T) \le c_n(T) \le a_n(T) \quad \text{and} \quad h_n(T) \le y_n(T) \le d_n(T) \le a_n(T).$$

*In case $X, Y$ are Hilbert spaces, all instances of $s$-numbers coincide (and are equal to the so called singular values of $T$).*

The following picture visualizes the above estimates again, where the arrows and the dashed line correspond to certain duality relations that we explain on the next page.

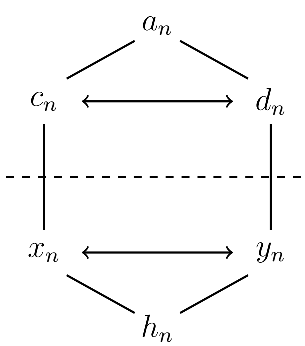


Figure 3.2: Relations between $s$-numbers

For more details about specific $s$-numbers including the proofs of the above relations see [Piet87]. The question arises how the entropy numbers are related. Note that no general "pointwise" estimate between approximation- and entropy numbers are possible since for $\operatorname{rank} T < n$ we know $a_n(T) = 0$ but in general $e_n(T) \neq 0$, compare Theorem 3.1.13. On the other hand, in some situations $e_n(T) \le c a_n(T)$ is possible. Nevertheless, if we consider weighted means, the following is known:

**Theorem 3.1.18** *Let $X, Y$ be Banach spaces, $T \in \mathcal{L}(X,Y)$ and $0 < p < \infty$. Then there exists $c(p) > 0$ such that for all $m \in \mathbb{N}$*

$$\sup_{k=1,\dots,m} k^{1/p} e_k(T) \le c \sup_{k=1,\dots,m} k^{1/p} a_k(T).$$

Here one can ask whether suitable substitutions for the supremum are possible:

**Theorem 3.1.19** *Let $X, Y$ be Banach spaces, $T \in \mathcal{L}(X,Y)$ and $0 < p \le \infty$, $0 < r < \infty$. Then there exists $c(p,r) > 0$ such that for all $m \in \mathbb{N}$*

$$\left( \sum_{k=1}^{m} k^{r/p-1} e_k^r(T) \right)^{1/r} \le c \left( \sum_{k=1}^{m} k^{r/p-1} a_k^r(T) \right)^{1/r}.$$

For the proofs, see [CS90](section 3.1).
In the Hilbert space situation (see [CS90](Thm.3.4.2)), one finds for $T \in \mathcal{L}(H_1, H_2)$:

$$1 \leq \sup_{k\in\mathbb{N}} 2^{\frac{n-1}{k}} (a_1(T)\cdots a_k(T))^{-1/k} e_n(T) \leq 14,$$

which in particular means $a_n(T) \leq 2e_n(T)$.
What about a comparison of Kolmogorov- and entropy numbers?

**Theorem 3.1.20** *Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X,Y)$. Then for all $n \in \mathbb{N}$:*

$$d_n(T) \leq \left(\prod_{k=1}^{n} d_k(T)\right)^{1/n} \leq n \inf_{k\in\mathbb{N}} 2^{\frac{k-1}{n}} e_k(T) \leq 2ne_n(T).$$

The proof is given by [CS90](Thm.3.2.1) and [EE87](Prop.II.3.8).
Another aspect of interest here is the concept of duality: If $T'$ is the dual operator of $T$ the following can be stated:

$$x_n(T') = y_n(T), \quad y_n(T') = x_n(T), \quad h_n(T') = h_n(T)$$

for arbitrary operators (compare Figure 3.2 below the dashed line) and

$$c_n(T') = d_n(T), \quad d_n(T') = c_n(T), \quad a_n(T') = a_n(T)$$

in general for compact operators only (above the dashed line in Figure 3.2). We refer again to [Piet87]. Duality assertions for entropy numbers are more complicated.
Let's now formulate the exact relations of approximation- and entropy numbers to eigenvalues of compact operators. We know that for $T \in \mathcal{K}(X,X)$ the sequence of the eigenvalues $(\mu_k)_k$ of $T$ can be ordered with respect to its algebraic multiplicity:

$$\|T\| \geq |\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_k| \longrightarrow 0.$$

In case of a self adjoint and compact operator $T$ in a Hilbert space $X = H$. we get $|\mu_k| = a_k$ for all $k \in \mathbb{N}$. Without $T$ being self adjoint we can state the following:

**Theorem 3.1.21 (Weyl-inequalities)** *Let $H$ be a Hilbert space and $T \in \mathcal{K}(H,H)$ with Eigenvalues $(\lambda_k(T))_k$. Then*

$$\prod_{k=1}^{n} |\lambda_k(T)| \leq \prod_{k=1}^{n} a_k(T), \qquad \textit{for all } n \in \mathbb{N}$$

*and for $p > 0$*

$$\sum_{k=1}^{n} |\lambda_k(T)|^p \leq \sum_{k=1}^{n} a_k(T)^p, \qquad \textit{for all } n \in \mathbb{N}.$$

In the case $\dim H = n$ we even have equality in the first line. For the proof see [CS90](Thm. 4.4.2). If we are in the more general Banach space situation, i.e. $T \in \mathcal{K}(X, X)$, then

$$\left(\sum_{k=1}^{n} |\lambda_k(T)|^p\right)^{1/p} \leq C_p \left(\sum_{k=1}^{n} a_k^p(T)\right)^{1/p}, \quad \text{for all } n \in \mathbb{N}, 0 < p < \infty,$$

compare [EE87]. Also estimates between eigenvalues and other $s$-numbers are known, sometimes even with optimal constants, see for example [H05].
As a final statement in this section we want to present the connection of entropy numbers with Eigenvalues of compact operators by the famous inequality of Carl.

**Theorem 3.1.22 (Carl's inequality)** *Let $X$ be a Banach space and $T \in \mathcal{K}(X, X)$ with Eigenvalues $(\mu_k)_k$. Then*

$$|\mu_k| \leq \left(\prod_{j=1}^{k} |\mu_j|\right)^{1/k} \leq \inf_{m \in \mathbb{N}} 2^{m/2k} e_m(T) \leq \sqrt{2} e_k(T), \quad \text{for all } k \in \mathbb{N}.$$

For the proof see [CS90](section 4.2) and [CT80]. Note that Carl's inequality connects a very geometric concept, the entropy numbers, with the algebraic eigenvalues pointwise. This remarkable observation was the starting point for a whole industry on entropy numbers, which we pointed out in our motivation at the beginning of the chapter.

## 3.2 Estimates for concrete operators

The results in the last section justify to investigate entropy numbers of all sorts of compact operators, especially embeddings between certain weighted sequence spaces related to function spaces (by some kind of decomposition technique) that are relevant for applications.
By the nice factorization properties of entropy numbers, compare Theorem 3.1.12, it makes sense to consider even the simplest examples of such spaces, namely $l_p^n$. So we start with a famous result from 1984.

**Theorem 3.2.1** *Let $1 \leq p \leq q \leq \infty$, then*

$$e_k(id : l_p^n \longrightarrow l_q^n) \sim \begin{cases} 1 & : \quad 1 \leq k \leq \log n \\ \left(\frac{\log(1+n/k)}{k}\right)^{1/p-1/q} & : \quad \log n \leq k \leq n \\ 2^{-(k-1)/n} n^{1/q-1/p} & : \quad k \geq n. \end{cases}$$

See [Sch84] for the origin. Besides embeddings in finite-dimensional spaces $l_p^n$, diagonal operators $D_\sigma$ (as from Theorem 3.1.16) are examples for basic ingredients in this area. To show some concrete results we consider a very specific situation in terms of the weights $(\sigma_n)$.

**Theorem 3.2.2** *Let $1 \leq p, q \leq \infty$, $\alpha > \max(1/q - 1/p, 0)$ and $\delta \in \mathbb{R}$. If $\sigma_n \sim n^{-\alpha}(1 + \log n)^{\delta}$, then*

$$e_n(D_\sigma : l_p \longrightarrow l_q) \sim n^{1/q-1/p-\alpha}(1 + \log n)^{\delta}.$$

If the decay of the weight sequence is only logarithmic the result looks as follows

**Theorem 3.2.3** *Let $1 \leq p < q \leq \infty$, $\delta > 0$ and set $\delta* = 1/p - 1/q$. If $\sigma_n \sim (1 + \log n)^{-\delta}$, then*

$$e_n(D_\sigma : l_p \longrightarrow l_q) \sim \begin{cases} n^{1/q-1/p}(1 + \log n)^{1/p-1/q-\delta} & : \quad \delta \geq \delta* \\ n^{-\delta} & : \quad \delta \leq \delta * . \end{cases}$$

These results are due to Kühn, see [K01]. The author was able to generalize these results with respect to a more general class of weight sequences, compare [S05]. Let's call a monotone decreasing sequence $(\sigma_k)$ of positive real numbers admissible, if there exist numbers $J \in \mathbb{N}$ and $d_1, d_2 > 0$ such that

$$d_1\sigma_{2^j} \leq \sigma_{2^{j+1}} \leq d_2\sigma_{2^j} \quad \text{for all} \quad j \geq J.$$

**Theorem 3.2.4** *Let $1 \leq p, q \leq \infty$ and $(\sigma)_n$ be admissible with $-\log d_2 > \max(1/q - 1/p, 0)$. Then*

$$e_n(D_\sigma : l_p \longrightarrow l_q) \sim n^{1/q-1/p}\sigma_n.$$

In the last years many papers appeared giving a quite complete picture about entropy numbers of embeddings in sequence spaces (references at the end).
Results of the above type also for other operators acting between more complicated sequence spaces (nested and weighted) were used to answer questions similar to our motivating problem. As an example we state the following closing result concerning the operator $Bf = b_2(-\Delta)_D^{-1}b_1 f$ as in the motivation to this section.

**Theorem 3.2.5** *Suppose that $p, r_1, r_2 \in [1, \infty]$ and*

$$1/p + 1/r_1 < 1, \quad n(1/p - 1/r_2) > 0, \quad n - (1/r_1 - 1/r_2) > 0.$$

*For $b_1 \in L_{r_1}$ and $b_2 \in L_{r_2}$ the operator $B : L_p \longrightarrow L_p$ is compact and*

$$\mu_k(B) \leq c\|b_1|L_{r_1}\| \cdot \|b_2|L_{r_2}\| \cdot k^{-2/n}.$$

The original proof can be found in [ET96] (Prop.5.2.4) not yet using the decomposition technique and entropy results in sequence spaces. Nowadays such a reduction gives a considerable simplification of the problem and many people work on related questions using this machinery.

# Bibliography


[AE01] Amann, H. and Escher, J.: Analysis III. Birkhäuser, Basel 2001.

[BS88] Bennett, C. and Sharpley, R.: Interpolation of operators. Academic Press, Boston 1988.

[BL76] Bergh, J. and Löfström, L.: Interpolation spaces. Springer, Berlin 1976.

[CS90] Carl, B. and Stephani, I.: Entropy, compactness and the approximation of operators. Cambridge Univ. Press, Cambridge 1990.

[CT80] Carl, B. and Triebel, H.: Inequalities between eigenvalues, entropy numbers and related quantities of compact operators in Banach spaces. Math. Ann. 251, 129-133.

[EE87] Edmunds, D.E. and Evans, W.D.: Spectral theory and differential operators. Oxford Univ. Press 1987.

[EN10] Edmunds, D.E. and Netrusov, Yu.: Entropy numbers and interpolation. Mathemat. Annalen, Volume 351, Number 4 (2011), 963-977.

[ET96] Edmunds, D.E. and Triebel, H.: Function spaces, entropy numbers, differential operators. Cambridge Univ. Press, Cambridge 1996.

[GKS87] Gordon, Y.; König, H. and Schütt, C.: Geometric and probabilistic estimates for entropy and approximation numbers of operators. J. Approx. Theory 49 (1987), 219-239.

[HLP52] Hardy, G.; Littlewood, J.E. and Polya, G.: Inequalities. Cambridge Univ. Press, 2nd edition 1952.

[HApp] Haroske, D.D.: Approximationstheorie. Lecture notes at FSU Jena 2004 (version of 2012), 133 pages.

[HInt] Haroske, D.D.: Interpolationstheorie. Lecture notes at FSU Jena 2003 (version of 2012), 84 pages.

[H05] Hinrichs, A.: Optimal Weyl inequalities in Banach spaces. Proc. Amer. Math. Soc. 134 (2005), 731-735.

[K01] Kühn, T.: Entropy numbers of diagonal operators of logarithmic type. Georg. Math. J., Vol.8(2001), Nr.2, 307-318.

[MS] Mityagin, B.S. and Semenov, E.M.: $C^k$ is not an interpolation space between $C$ and $C^n, 0 < k < n$. Sov. Math.Dokl., 17:778-782, 1976.

[Piet78] Pietsch, A.: Operator ideals. vol 16 of Mathem. Monogr., Dt. Verlag Wiss., Berlin 1978.

[Piet87] Pietsch, A.: Eigenvalues and $s$-numbers. Akad. Verlagsgesellschaft Geest & Portig, Leipzig 1987.

[Pink85] Pinkus, A.: $n$-width in approximation theory. volume 3/7 of EMG, Springer, Berlin 1985.

[S05] Schneider, J.: Entropiezahlen für Einbettungen von Folgenräumen mit Gewichten. Diploma thesis, FSU Jena, 2005.

[Sch84] Schütt, C.: Entropy numbers of diagonal operators between symmetric Banach spaces. J. Approx. Theory 40 (1984), 121-128.

[Tr72] Triebel, H.: Higher Analyis. Barth, Leipzig 1992.

[Tr78] Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. 2nd edition, Barth, Leipzig 1995.

[Tr83] Triebel, H.: Theory of Function Spaces. Geest & Portig, Leipzg 1983, Birkhäuser, Basel 1983.

[Tr92] Triebel, H.: Theory of Function Spaces II. Birkhäuser, Basel 1992.

[Tr97] Triebel, H.: Fractals and Spectra. Birkhäuser, Basel 1997.

[Tr01] Triebel, H.: The structure of functions. Birkhäuser, Basel 2001.

[Yos95] Yosida, K.: Functional Analysis. Springer, Berlin 1995.

**Further literature**

For the interested reader, here we list the groups and there main contributors who actively work in the area of entropy numbers and their applications (up to our knowledge):

**Jena:** Triebel, Carl, Pietsch, Haroske, Leopold, Sickel **Leipzig:** Kühn
**Poznan:** Skrzypczak
**Sussex:** Edmunds **Cardiff:** Evans **Bristol:** Netrusov
**Madrid:** Cobos **Aveiro:** Caetano