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An introduction to generalized Young measures

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AN INTRODUCTION TO GENERALIZED YOUNG MEASURES

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A common strategy for approaching difficult problems in analysis is to approximate them with simpler problems. These can be either viscosity approximations or other regularizations of the equations, finite dimensional (numerical) discretizations, linearizations of nonlinear problems, or minimizing sequences for variational problems, to recall a few. The hope is then that some of the good properties of the approximate solutions can be transferred to a solution of the original problem.

Very roughly speaking, this is usually done in two steps: First, one ensures a compactness property of the family of approximate solutions, then one proves that a cluster point is indeed a solution of the hard problem. If the compactness is achieved in a strong topology, then it is likely that a suitable limiting procedure can be performed. However, it is rarely the case that one can obtain such good approximations and, instead, compactness in a weak topology is to be expected. From a practical point of view, weak sequential topologies are consistent with physical measurements and usually follow from relatively simple energy estimates. Consequently, one needs to understand the interaction between the weakly convergent sequences of approximate solutions and the nonlinearities arising inherently in a difficult problem.

In fact, it is important to understand the gap between weak and strong convergence, as the former induces severe restrictions on the admissible nonlinearities. Also, depending on each problem, the construction of the approximate problems may be dramatically different. In this course, we will illustrate the above procedure using both PDE examples, as well as examples from the calculus of variations.

We will work under the so-called **Murat–Tartar framework for compensated compactness**, which is a broad framework that encompass many interesting problems in continuum mechanics. A brief outline of our setup is as follows:

$$\text{(CC)} \qquad \textbf{Given} \begin{cases} v_j \rightharpoonup v & \text{(weakly)} \\ \mathscr{A} v_j = f_j \text{ and } f_j \to f & \text{(strongly)} \\ \Phi \colon \mathbb{V} \to \mathbb{R} \text{ is continuous} & \text{(nonlinearity),} \end{cases} \qquad \textbf{calculate the limit(s) of } (\Phi(v_j)).$$

When using the term *nonlinearity*, we refer in principle to a non affine function, but we will not be strict with this terminology; we can think of affine functions as degenerate nonlinearities. Here $\mathscr{A} v_j = f_j$ are systems of linear PDEs which represent the *compensation* in the problem (isolating a part of the weakly convergent sequence that converges strongly) and we think of the linear differential operator $\mathscr{A}$ as a *compensation operator*.

To illustrate the scheme above with a rough example, consider the problem of minimizing the energy functional

$$\mathscr{E}[v] = \int_\Omega \Phi(v(x))\mathrm{d}x$$

over a subset of $\mathrm{L}^2(\Omega, \mathbb{V})$, assuming that Φ has quadratic growth from above and below. Such an example is reminiscent of the Dirichlet energy, which corresponds to the case when v is a gradient and $\Phi = |\bullet|^2$. We briefly recall the direct method in the calculus of variations:

(a) Since Φ is bounded below, there exists a minimizing sequence v_j.
(b) Since $\Phi \geqslant c|\bullet|^2$ the minimizing sequence is bounded in L^2. In particular it has a weakly convergent subsequence $v_{j_i} \rightharpoonup v$.
(c) If we can show that $\mathscr{E}$ is weakly sequentially lower semi-continuous, which implies

$$\liminf_{i\to\infty} \int_\Omega \Phi(v_{j_i}(x))\mathrm{d}x \geqslant \int_\Omega \Phi(v(x))\mathrm{d}x,$$

then we can conclude that v is a solution (minimizer).

The requirement in step (c) is very delicate and will be studied in detail over the duration of the course. In particular, we will aim to persuade the reader that Young measures are particularly effective tools to describe the restrictions that the compensation operator $\mathscr{A}$ places on the nonlinearity Φ, when working in the framework (CC).

1. Background

1.1. **Weak/strong convergence/compactness in L^p spaces.** We begin by recalling the definition of Lebesgue spaces. Let $\mathbb{V} \simeq \mathbb{R}^N$ be a finite dimensional inner product space, $\Omega \subset \mathbb{R}^n$ be a Borel set (often open, bounded, with Lipschitz boundary), and $1 \leqslant p \leqslant \infty$. The space $\mathrm{L}^p(\Omega, \mathbb{V})$ is the set of measurable functions $v \colon \Omega \to \mathbb{V}$ such that

$$\|v\|^p_{\mathrm{L}^p(\Omega)} := \int_\Omega |v(x)|^p \mathrm{d}x \text{ for } 1 \leqslant p < \infty \quad \text{and} \quad \|v\|_{\mathrm{L}^\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega} |v(x)|.$$

We also recall the definition of, respectively, *strong* and *weak(ly-*)* convergence, namely:

(a) For all $1 \leqslant p \leqslant \infty$, we say that $v_j \to v$ in $\mathrm{L}^p(\Omega, \mathbb{V})$ if and only if $\|v_j - v\|_{\mathrm{L}^p(\Omega)} \to 0$.

(b) For $1 \leqslant p < \infty$, we say that $v_j \rightharpoonup v$ in $\mathrm{L}^p(\Omega, \mathbb{V})$ ($\overset{*}{\rightharpoonup}$ if $p = \infty$) if and only if

$$\int_\Omega \langle v_j, \tilde v\rangle \mathrm{d}x \to \int_\Omega \langle v, \tilde v\rangle \mathrm{d}x \quad \text{for all } \tilde v \in \mathrm{L}^{p'}(\Omega, \mathbb{V}),$$

where $p' = p/(p-1)$.

We will characterize both, starting with strong convergence:

Theorem 1.1 (Vitali convergence theorem)**.** *Let $1 \leqslant p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a Borel set of finite Lebesgue measure, and let v_j, $v \in \mathrm{L}^p(\Omega, \mathbb{V})$. Then $v_j \to v$ in $\mathrm{L}^p(\Omega, \mathbb{V})$ if and only if both of the following conditions hold:*

(a) $v_j \to v$ in $\mathscr{L}^n$-measure, *meaning that for all $\varepsilon > 0$ we have that*

$$\mathscr{L}^n(\{x \in \Omega \colon |v_j(x) - v(x)| > \varepsilon\}) \to 0 \quad \textit{as } j \to \infty.$$

(b) (v_j) *is* p-uniformly integrable, *meaning that*

$$\sup_j \int_{\{|v_j|>t\}} |v_j(x)|^p \mathrm{d}x \to 0 \quad \textit{as } t \to \infty.$$

If $p = 1$, we will just say that the sequence is *uniformly integrable*, or *equi-integrable*. If $\mathscr{L}^n(\Omega) = \infty$, a third condition is needed, namely *tightness* of (v_j), see, e.g., [9, Thm. 2.24]. Before we prove this cornerstone sharp extension of the dominated convergence theorem, we will first discuss the two conditions, starting with two simple examples to show that each assumption is necessary:

Example 1.2 (Oscillation). Let v_1 be the 2-periodic extension of $\mathbf{1}_{(0,1)} - \mathbf{1}_{(1,2)}$ to $\mathbb{R}$ and let $v_j(x) := v_1(jx)$. It is then clear that $(v_j) \subset \mathrm{L}^p(0, 2)$ is p-uniformly integrable for any $p < \infty$ and not too difficult to see that (v_j) has no pointwisely convergent subsequence, so that (a) fails for (v_j) by Proposition B.1.

Example 1.3 (L^2-concentration). We let $v_j := \sqrt{j}\mathbf{1}_{(0,1/j)}$, which clearly converges in measure to 0 but is not 2-uniformly integrable.

Convergence in measure is closely related to pointwise convergence and can be characterized using a metric on the space of measurable functions, see Proposition B.1.

Uniform integrability will prove to be one of the most important concepts in this course, so we will now give it a fair treatment:

Theorem 1.4 (On uniform integrability)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $\mathcal{F} \subset \mathrm{L}^1(\Omega, \mathbb{V})$ be a family of functions. The following are equivalent:*

(a) *$\mathcal{F}$ is uniformly integrable, i.e.,*

$$\sup_{v\in\mathcal{F}} \int_{\{|v|>t\}} |v(x)| \mathrm{d}x \to 0 \quad \textit{as } t \to \infty.$$

(b) *For all $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\sup_{v\in\mathcal{F}} \int_A |v(x)| \mathrm{d}x < \varepsilon \quad \textit{for all Borel } A \subset \Omega \textit{ with } \mathscr{L}^n(A) < \delta.$$

(c) *There exists a nondecreasing convex function $\theta \colon [0, \infty) \to [0, \infty)$ such that*

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty \quad \textit{and} \quad \sup_{v\in\mathcal{F}} \int_\Omega \theta(|v(x)|) \mathrm{d}x < \infty.$$

The statement in (c) is the so-called De la Valée Poussin criterion for uniform integrability, which is very important because it essentially states that in order to have uniform integrability, a norm bound in L^1 must be improved to an Orlicz space bound. Throughout this section, we will denote by $\mu := \mathscr{L}^n$ the n-dimensional Lebesgue measure. One reason for this is that most claims extend to finite, non-atomic, positive Borel measures μ on a measure space Ω.

Proof. (a) $\Longrightarrow$ (c). For each $k \in \mathbb{N}$, let $j_k \geqslant k$ be such that

$$\sup_{v\in\mathcal{F}} \int_{|v|>j_k} |v|\mathrm{d}\mu < 2^{-k-1}.$$

We further note that, since $\sum_{i>j_k} \mathbf{1}_{(i,\infty)}(t) \leqslant t\mathbf{1}_{(j_k,\infty)}(t)$, we have

$$\sum_{i>j_k} \mu(|v|>i) \leqslant \int_{|v|>j_k} |v|\mathrm{d}\mu < 2^{-k-1}.$$

Define $\theta_i := \#\{k\colon j_k < i\}$ for $i \in \mathbb{N}$, so $\theta_i \uparrow \infty$. We next define an increasing, convex function

$$\theta(t) := \int_0^t \tilde\theta(s)\mathrm{d}s, \quad \text{where} \quad \tilde\theta := \sum_{i=0}^\infty \theta_i \mathbf{1}_{[i,i+1)}.$$

Moreover, for $t \geqslant 2k$ we have

$$\frac{\theta(t)}{t} = \frac{\theta(t)-\theta(0)}{t} \geqslant \frac{\theta(2k)-\theta(0)}{2k} = \frac{1}{2k}\sum_{i=1}^{2k}\theta_{i-1} \geqslant \frac{1}{2k}k\theta_{k-1} = \frac{\theta_{k-1}}{2} \to \infty \quad \text{as } k\to\infty,$$

so that indeed θ is superlinear. It remains to prove the Orlicz-type bound. Let $v \in \mathcal{F}$ and write

$$\theta(|v|) = \sum_{k=0}^{i}\theta_k + \int_i^{|v|}\theta_i \mathrm{d}t \leqslant \sum_{k=0}^{i+1}\theta_k \quad \text{when } i \leqslant |v| < i+1,$$

so

$$\begin{aligned}\int_\Omega \theta(|v|)\mathrm{d}\mu &= \sum_{i=0}^\infty \int_{i\leqslant|v|<i+1}\theta(|v|)\mathrm{d}\mu \leqslant \sum_{i=0}^\infty\sum_{k=0}^{i+1}\theta_k\mu(i\leqslant v<i+1) = \sum_{k=0}^\infty\sum_{i=k}^\infty \theta_k\mu(i\leqslant v<i+1)\\ &= \sum_{k=0}^\infty \theta_k\mu(|v|\geqslant k) = \sum_{k=0}^\infty\sum_{\{l\colon j_l<k\}}\mu(|v|\geqslant k) = \sum_{l=0}^\infty\sum_{k>j_l}\mu(|v|\geqslant k) < \sum_{l=0}^\infty 2^{-l-1} = 1,\end{aligned}$$

which concludes the proof of this implication.

(c) $\Longrightarrow$ (b). Convexity of θ will not be used in the proof. There is no loss of generality in assuming that $\int_\Omega \theta(|v|)\mathrm{d}\mu \leqslant 1$ for $v \in \mathcal{F}$. Let $\varepsilon > 0$ and $t_\varepsilon < 0$ be such that $\theta(t) \geqslant 2t/\varepsilon$ whenever $t \geqslant t_\varepsilon$. Let $A \subset \Omega$ be a Borel set with $\mu(A) < \delta$, where $\delta < \varepsilon/(2t_\varepsilon)$. Then

$$\begin{aligned}\int_A |v|\mathrm{d}\mu &\leqslant \int_{A\cap\{|v|\leqslant t_\varepsilon\}}|v|\mathrm{d}\mu + \int_{A\cap\{|v|>t_\varepsilon\}}|v|\mathrm{d}\mu\\ &\leqslant t_\varepsilon\mu(A) + \frac{\varepsilon}{2}\int_\Omega\theta(|v|)\mathrm{d}\mu < \varepsilon.\end{aligned}$$

(b) $\Longrightarrow$ (a). Let $\varepsilon \in (0,1]$ and $v \in \mathcal{F}$. By assumption, there exists $\delta > 0$ such that

$$\mu(A) < \delta \implies \int_A |v|\mathrm{d}\mu < \varepsilon \quad \text{for all } v \in \mathcal{F}. \tag{1.1}$$

We fix $v \in \mathcal{F}$ and prove a uniform bound: decompose $\Omega = \bigcup_{j=1}^m A_j$, where A_j are Borel sets with $\mu(A_j) < \delta$. In particular, it follows that

$$\int_\Omega |v|\mathrm{d}\mu \leqslant \sum_{j=0}^m \int_{A_j}|v|\mathrm{d}\mu < m\varepsilon \leqslant m,$$

so that by Chebyshev's inequality we have, for $t > m/\delta$, that

$$\mu(|v|>t) \leqslant \frac{1}{t}\int_{|v|>t}|v|\mathrm{d}\mu < \frac{m}{t} < \delta.$$

Therefore, we can take $A = \{|v| > t\}$ in (1.1) to conclude. □

Useful applications of the De la Valée Poussin criterion include the following:

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $(v_j) \subset \mathrm{L}^1(\Omega,\mathbb{V})$ be a sequence of functions.Then:*

(a) *If*

$$\sup_j \int_\Omega |v_j(x)|\log(1+|v_j(x)|)\mathrm{d}x < \infty,$$

then (v_j) is uniformly integrable. In particular, bounded sequences in $\mathrm{L}\log\mathrm{L}$ are uniformly integrable (hence relatively weakly compact in L^1, see Theorem 1.12(b) *below).*

(b) *Let $1 \leqslant q < p$. If $\mathcal{F}$ is bounded in $\mathrm{L}^p(\Omega,\mathbb{V})$, then $\mathcal{F}$ is q-uniformly integrable.*

We will also give an important sufficiency criterion for uniform integrability:

Theorem 1.6 (Vitali–Hahn–Saks)**.** *Let $\Omega\subset\mathbb{R}^n$ be open and bounded and $(v_j)\subset \mathrm{L}^1(\Omega,\mathbb{V})$ be a sequence of functions. Suppose that*

$$\lim_{j\to\infty}\int_A v_j(x)\mathrm{d}x \quad \textit{exists in } \mathbb{V} \textit{ for all Borel } A\subset\Omega.$$

Then (v_j) is uniformly integrable.

Proof. There is no loss of generality in assuming that $\mathbb{V}=\mathbb{R}$, otherwise argue componentwise. We write $\mathbb{B}(\Omega)$ for the Borel σ-algebra of subsets of Ω.

For A, $B\in\mathbb{B}(\Omega)$, we define $\mathrm{d}(A,B):=\mu(A\triangle B)=\int_\Omega|\mathbf{1}_A-\mathbf{1}_B|\mathrm{d}\mu$. One can then check that d is a complete metric on $\mathbb{B}(\Omega)/\!\sim$, where $A\sim B$ if and only if $\mu(A\triangle B)=0$.

Let $\varepsilon>0$. For $k\in\mathbb{N}$, define

$$S_k:=\left\{A\in\mathbb{B}(\Omega)/\!\sim\colon \sup_{i,j\geqslant k}\left|\int_A(v_i-v_j)\mathrm{d}\mu\right|\leqslant\varepsilon\right\},$$

which is a closed subset of $\mathbb{B}(\Omega)/\!\sim$. We claim that $\mathbb{B}(\Omega)/\!\sim=\bigcup_{k=0}^\infty S_k$. This is the case since, for each $A\in\mathbb{B}(\Omega)$, we have that $(\int_A v_j\mathrm{d}\mu)$ has a limit, hence is Cauchy, so that $A\in S_k$ for k sufficiently large.

By the Baire category theorem, there exists k such that S_k has non-empty interior. Hence, we can find $A\in S_k$ and $r>0$ such that

$$\mathrm{d}(A,B)<r\implies \sup_{i,j\geqslant k}\left|\int_B(v_i-v_j)\mathrm{d}\mu\right|\leqslant\varepsilon.$$

Since finite families of integrable functions are uniformly integrable, we can choose $\delta\in(0,r]$ such that $\left|\int_B v_j\mathrm{d}\mu\right|\leqslant\varepsilon$ whenever $\mu(B)<\delta$ for all $1\leqslant j\leqslant k$.

Next, let $B\in\mathbb{B}(\Omega)$ have $\mu(B)<\delta$. We claim that $\int_B|v_j|\leqslant 6\varepsilon$, which suffices to conclude. To this end, define $B_1:=A\cup B$ and $B_2:=A\setminus B$, so that $B=B_1\setminus B_2$, $\mathrm{d}(A,B_1)<\delta$, and $\mathrm{d}(A,B_2)<\delta$. Hence, for $j\geqslant k$ we have

$$\begin{aligned}\left|\int_B v_j\mathrm{d}\mu\right|&\leqslant\left|\int_B v_k\mathrm{d}\mu\right|+\left|\int_B(v_j-v_k)\mathrm{d}\mu\right|=\left|\int_B v_k\mathrm{d}\mu\right|+\left|\int_{B_1}(v_j-v_k)\mathrm{d}\mu-\int_{B_2}(v_j-v_k)\mathrm{d}\mu\right|\\&\leqslant\varepsilon+\left|\int_{B_1}(v_j-v_k)\mathrm{d}\mu\right|+\left|\int_{B_2}(v_j-v_k)\mathrm{d}\mu\right|\leqslant 3\varepsilon.\end{aligned}$$

This would complete the proof if the sequence (v_j) would have constant sign. To cover the case of signed sequences, apply the previous inequality with B replaced by $B\cap\{v_j\lesseqgtr 0\}$:

$$\int_B|v_j|\mathrm{d}\mu=\left|\int_{B\cap\{v_j<0\}}v_j\mathrm{d}\mu\right|+\left|\int_{B\cap\{v_j>0\}}v_j\mathrm{d}\mu\right|\leqslant 6\varepsilon,$$

which completes the proof. □

Remark 1.7. In fact, the statement of Theorem 1.1 holds whenever we consider the Lebesgue spaces with respect to a finite, positive, non-atomic measure μ on a measure space Ω. To make this extension available, recall that we will write $\mu:=\mathscr{L}^n$ in the proof.

Proof of VCT, Theorem 1.1. There is no loss of generality in assuming that $v=0$ and, further, that $p=1$.

We first prove strong convergence. Suppose that $v_j\to v$ in (μ-)measure and (v_j) is p-uniformly integrable. Let $\varepsilon>0$. By uniform integrability, we can choose $t>0$ such that

$$\int_{|v_j|>t}|v_j|\mathrm{d}\mu\leqslant\frac{\varepsilon}{2}\quad\text{for all } j.$$

Letting $0<s<t$ to be determined, we write

$$\int_{|v_j|\leqslant t}|v_j|\mathrm{d}\mu\leqslant\int_{s<|v_j|\leqslant t}|v_j|\mathrm{d}\mu+\int_{|v_j|\leqslant s}|v_j|\mathrm{d}\mu\leqslant t\mu(|v_j|>s)+s\mu(\Omega).$$

We choose $s\leqslant\varepsilon/(4\mu(\Omega))$ and j sufficiently large so that $\mu(|v_j|>s)\leqslant\varepsilon/(4t)$ to conclude that

$$\int_\Omega|v|\mathrm{d}\mu\leqslant\int_{|v_j|>t}|v_j|\mathrm{d}\mu+\int_{s<|v_j|\leqslant t}|v_j|\mathrm{d}\mu+\int_{|v_j|\leqslant s}|v_j|\mathrm{d}\mu\leqslant\varepsilon$$

for sufficiently large j.

Conversely, assume that $v_j \to 0$ in $\mathrm{L}^1(\Omega;\mu,\mathbb{V})$. Let $\varepsilon > 0$. First, by Chebyshev's inequality, we have

$$\mu(|v_j| > \varepsilon) \leqslant \frac{1}{\varepsilon}\int_\Omega |v_j|\mathrm{d}\mu \to 0 \quad \text{as } j \to \infty,$$

which implies convergence in measure. For uniform integrability, we have that there exists $k \in \mathbb{N}$ such that

$$\int_{|v_j|>t} |v_j|\mathrm{d}\mu \leqslant \int_\Omega |v_j|\mathrm{d}\mu \leqslant \varepsilon$$

for $j \geqslant k$ and all $t > 0$. By choosing t large enough we can ensure that $\int_{|v_j|>t} |v_j|\mathrm{d}\mu \leqslant \varepsilon$ for all $j < k$. The proof is complete. □

Having covered strong convergence in detail, we remark that it obviously implies weak convergence by Hölder's inequality. We begin exploring the gap between the two notions. We make an additional observation that the sequences presented in Examples 1.2 and 1.3 are also weakly(-*) convergent. This motivates the following:

Definition 1.8 (Oscillating and concentrating sequences)**.** *Let $1 \leqslant p \leqslant \infty$ and $v_j \rightharpoonup v$ in $\mathrm{L}^p(\Omega,\mathbb{V})$ ($\overset{*}{\rightharpoonup}$ if $p = \infty$). We say that:*

(a) (v_j) oscillates *if $v_j \not\to v$ in measure.*
(b) *For $p < \infty$, then* (v_j) concentrates in L^p *if (v_j) is not p-uniformly integrable.*

These concepts will be of particular importance in the later sections. For now, we give a description of weak(ly-*) convergence and compactness in L^p:

Proposition 1.9 (On convergence of averages, $p > 1$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $1 < p \leqslant \infty$. Then $(v_j) \subset \mathrm{L}^p(\Omega,\mathbb{V})$ converges weakly in $\mathrm{L}^p(\Omega,\mathbb{V})$ to v (weakly-* if $p = \infty$) if and only if both of the following conditions hold:*

(a) $\int_{Q\cap\Omega} v_j \mathrm{d}x \to \int_{Q\cap\Omega} v\mathrm{d}x$ *for all cubes $Q \subset \mathbb{R}^n$.*
(b) $\sup_j \|v_j\|_{\mathrm{L}^p(\Omega)} < \infty$.

This fact follows easily since, with the restriction $p > 1$, we have that $\operatorname{span}\{\mathbf{1}_Q : Q \subset \Omega \text{ cube}\}$ is dense in $\mathrm{L}^{p/(p-1)}(\Omega,\mathbb{V})$. It turns out that this restriction is phenomenological:

Exercise 1.10. Find a sequence (v_j) that is uniformly bounded in $\mathrm{L}^1(0,1)$ and satisfies $\int_a^b v_j \mathrm{d}t \to \int_a^b v\mathrm{d}t$ for all $a, b \in [0,1]$ and some $v \in \mathrm{L}^1(0,1)$, but $v_j \not\rightharpoonup v$ in $\mathrm{L}^1(0,1)$.

We have the following correction of Proposition 1.9 for $p = 1$:

Proposition 1.11 (On convergence of averages, $p = 1$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then $(v_j) \subset \mathrm{L}^1(\Omega,\mathbb{V})$ converges weakly in $\mathrm{L}^1(\Omega,\mathbb{V})$ to v if and only if both of the following conditions hold:*

(a) $\int_Q v_j \mathrm{d}x \to \int_Q v\mathrm{d}x$ *for all cubes $Q \subset \Omega$.*
(b) (v_j) *is uniformly integrable.*

This is indeed a strengthening of the assumptions of Proposition 1.9 for $p = 1$: by the proof of Theorem 1.4, we have that uniform integrability implies uniform boundedness in L^1 (on finite measure spaces).

The proof of Proposition 1.11 follows easily from the following criterion for weak compactness:

Theorem 1.12 (On weak(ly-*) compactness in L^p)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $1 \leqslant p \leqslant \infty$, and $\mathcal{F} \subset \mathrm{L}^p(\Omega,\mathbb{V})$ be a family of functions. Then $\mathcal{F}$ is relatively weakly sequentially compact (weakly-* for $p = \infty$) if and only if*

(a) *For $1 < p \leqslant \infty$:* $\sup_{v\in\mathcal{F}} \|v\|_{\mathrm{L}^p(\Omega)} < \infty$.
(b) *For $p = 1$: $\mathcal{F}$ is uniformly integrable.*

The statement in (b) is the Dunford–Pettis theorem, for a proof see [2, Thm. 1.38]. The statement in in (a) follows from the Banach–Alaoglu theorem, see the abstract compactness principle in Theorem A.2. Obviously, this criterion is much weaker than the one for strong compactness outlined in Theorem A.3.

However, we will not be interested in weak compactness in L^1, due to the following reason: From the De la Valée Poussin criterion, Theorem 1.4(c), and the Dunford–Pettis theorem 1.12(b), we see that weak compactness in L^1 is equivalent with boundedness in a smaller space. We will be interested in the setting where we have integrands Φ of p-growth acting on sequences (v_j) that converge weakly in L^p. In this context, the sequence of nonlinear compositions $(\Phi(v_j))$ can only be assumed to be bounded in L^1.

In this case, one expects weakly-* compactness in a space of measures, which are, roughly speaking, dual to spaces of continuous functions (see Theorems B.9, B.10, and B.11). To record the concentration effects of (v_j), we will use a sphere compactification of $\mathbb{V}$, which naturally leads to working with functionals belonging to the space $\mathrm{C}_b(B_{\mathbb{V}})$, the space of bounded continuous functions on the *open* unit ball in $\mathbb{V}$. Even though the dual of $\mathrm{C}_b(B_{\mathbb{V}})$ is understood (Theorem B.9), the space itself is not separable [9, Thm. 1.194], so we cannot apply the Banach–Alaoglu Theorem A.2 to realize the concentration effects as weakly-* cluster points of (v_j) when identified with elements of $\mathrm{C}_b(B_{\mathbb{V}})^*$.[1] Instead, we will work in $\mathrm{C}(\bar{B}_{\mathbb{V}})$, which is separable and whose dual is $\mathcal{M}(\bar{B}_{\mathbb{V}})$; the discrepancy will result in significant technical difficulties to be overcome.

Until then, we record the following consequence of Theorems B.9, B.11, and A.2:

Proposition 1.13 (On weakly-* compactness of measures). *Let $\Omega\subset\mathbb{R}^n$ be open and bounded and $(v_j)\subset \mathrm{L}^1(\Omega,\mathbb{V})$ be such that $\sup_j\|v_j\|_{\mathrm{L}^1(\Omega)}<\infty$. Then there exists $\mu\in\mathcal{M}(\bar{\Omega},\mathbb{V})$ such that*

(a) $v_j\overset{*}{\rightharpoonup}\mu$ *in* $\mathcal{M}(\bar{\Omega},\mathbb{V})\simeq \mathrm{C}(\bar{\Omega},\mathbb{V})^*$.
(b) $v_j\overset{*}{\rightharpoonup}\mu\llcorner\Omega$ *in* $\mathcal{M}(\Omega,\mathbb{V})\simeq \mathrm{C}_0(\Omega,\mathbb{V})^*$.

Example 1.14. To illustrate the failure of weak convergence (uniform integrability) of a weakly-* convergent sequence of L^1 functions, we consider $v_j=j\mathbf{1}_{(0,1/j)\cup(1,1+1/j)}\in\mathrm{L}^1(0,2)$. This example also exhibits the discrepancy between (a) and (b) in Proposition 1.13, as we have

$$v_j\overset{*}{\rightharpoonup}\delta_0+\delta_1 \text{ in } \mathcal{M}[0,2]\quad\text{and}\quad v_j\overset{*}{\rightharpoonup}\delta_1 \text{ in } \mathcal{M}(0,2).$$

In this case, $\mu=\delta_0+\delta_1$ charges the boundary $\{0,2\}$.

We conclude this section by recalling the elementary remark that $\mathrm{C}(\bar{\Omega})$ can be identified with the subspace of $\mathrm{C}_b(\Omega)$ consisting of uniformly continuous function. By this we mean

$$\mathrm{C}(\bar{\Omega})\simeq \mathrm{BUC}(\Omega), \tag{1.2}$$

both endowed with the supremum norm and the isomorphism being given by restriction/unique extension.

1.2. **Linear partial differential operators and their algebra.** In this section we will lay out some of the basic facts about differential operators that we will need when dealing with questions arising from (CC). In later sections, we will work to show that the strongly convergent, PDE constrained parts of the sequence, can be separated from the weak convergence effects of oscillation and concentration. For now, we be begin with defining the linear partial differential operators we work with, which are of the form

$$\mathscr{A}v:=\sum_{|\alpha|=\ell}A_\alpha\partial^\alpha v\quad\text{for } v\colon\mathbb{R}^n\to\mathbb{V},\quad \text{where } A_\alpha\in\mathrm{Lin}(\mathbb{V},\mathbb{W}) \text{ whenever } |\alpha|=\ell. \tag{1.3}$$

Here $\mathbb{V}$, $\mathbb{W}$ are finite dimensional inner product spaces[2] and we consider multi-indices $\alpha\in\mathbb{N}_0^n$. Therefore, we work with *vectorial* linear partial differential operators on $\mathbb{R}^n$ from $\mathbb{V}$ to $\mathbb{W}$ that are homogeneous and have constant coefficients (which are essentially matrices).

Extensive information about linear differential equations can be obtained by looking at the Fourier transform in full space or at the Fourier series in the periodic setting. Recall the Schwarz space of *rapidly decreasing functions*

$$\mathscr{S}(\mathbb{R}^n,\mathbb{V}):=\left\{v\in\mathrm{C}^\infty(\mathbb{R}^n,\mathbb{V})\colon \sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta v(x)|<\infty \text{ for all } \alpha,\,\beta\in\mathbb{N}_0^n\right\},$$

for which we define the Fourier transform

$$\mathscr{F}v\equiv\hat{v}\colon\xi\in\mathbb{R}^n\mapsto\int_{\mathbb{R}^n}v(x)\mathrm{e}^{-\mathrm{i}\,\xi\cdot x}\mathrm{d}x\quad\text{for } v\in\mathscr{S}(\mathbb{R}^n,\mathbb{V}).$$

In this case, we have

$$\widehat{\mathscr{A}v}(\xi)=(-\mathrm{i})^\ell\mathscr{A}(\xi)\hat{v}(\xi),\quad\text{where } \mathscr{A}(\xi):=\sum_{|\alpha|=\ell}\xi^\alpha A_\alpha\in\mathrm{Lin}(\mathbb{V},\mathbb{W}) \text{ for } \xi\in\mathbb{R}^n.$$

We will refer to this as the *(Fourier) symbol map* or *characteristic polynomial* of $\mathscr{A}$. In the periodic setting, we denote by $\mathbb{T}^n$ the n-dimensional torus, which is identified with $Q:=[0,1]^n$. We will use the same notation for the Fourier coefficients

$$v(x)=\sum_{\xi\in\mathbb{Z}^n}\hat{v}(\xi)\mathrm{e}^{\mathrm{i}\,\xi\cdot x}\quad\text{for } x\in\mathbb{T}^n,\ v\in\mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}),\quad\text{where } \hat{v}(\xi):=\int_{\mathbb{T}^n}v(y)\mathrm{e}^{-\mathrm{i}\,\xi\cdot y}\mathrm{d}y.$$

[1]Convergence in $\mathrm{C}_b(\Omega)^*$ is termed *narrow convergence* and is used in probability theory, where it is often referred to as *convergence in distribution* for probability measures.

[2]Technically, normed structure would suffice to be able to define these operators, but we feel that this would make the notation needlessly cumbersome.

The strongest form of the differential constraint in (CC) is to simply consider $\mathscr{A} v_j = 0$, which we will generally try to reduce to. Motivated for instance by the Fourier series, we now determine the plane wave exact solutions. By this we mean one-dimensional objects such that

$$\mathscr{A}\,[f(x\cdot\xi)z] = 0 \iff f^{(\ell)}(x\cdot\xi)\mathscr{A}(\xi)z = 0, \tag{1.4}$$

where $\xi \in \mathbb{R}^n$, $z \in \mathbb{V}$, $f \in \mathrm{C}^\ell(\mathbb{R})$ (which is easily relaxed to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$). As we will see in the introduction to Section 4, the pairs (ξ, z) such that $\mathscr{A}(\xi)z = 0$ holds prescribe the allowed directions of oscillation that $\mathscr{A}$-free plane waves are allowed to take. This motivates defining the *wave cone* of $\mathscr{A}$ by

$$\Lambda_{\mathscr{A}} := \bigcup_{\xi\in S^{n-1}} \ker \mathscr{A}(\xi).$$

The following lemma presents another aspect of $\Lambda_{\mathscr{A}}$:

Lemma 1.15. *We have that*

$$\operatorname{span}\Lambda_{\mathscr{A}} = \{v(x)\colon x \in \mathbb{T}^n,\ v \in \mathrm{L}^2(\mathbb{T}^n,\mathbb{V}),\ \mathcal{A}v = 0 \text{ in } \mathscr{D}'(\mathbb{T}^n,\mathbb{W})\}.$$

Proof. To prove the first inclusion, it is enough to check that if $\mathscr{A}(\xi)z = 0$ for $z \in \mathbb{V}$, $\xi \in S^{n-1}$, then z lies in the right hand side. This follows by taking $v(x) := f(x\cdot\xi)z$ where $f \in \mathrm{C}^\infty_c(0,1)$ is such that $f = 1$ in a neighbourhood of $1/2$. Then the periodic extension of v to $\mathbb{R}^n$ can be identified with a function in $\mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V})$ and $v(x) = z$ for x near $\xi/2$.

Conversely, we write the Fourier series

$$0 = \mathscr{A}v(x) = \mathrm{i}^\ell \sum_{\xi\in\mathbb{Z}^n} \mathscr{A}(\xi)\hat{v}(\xi)\mathrm{e}^{\mathrm{i}\,\xi\cdot x},$$

so that each Fourier coefficient $\hat{v}(\xi)$ is in $\Lambda_{\mathscr{A}}$. We conclude by looking at the Fourier series of v. □

The lemma above motivates one of our main standing assumptions, namely that the wave cone of $\mathscr{A}$ spans $\mathbb{V}$:

$$\operatorname{span}\Lambda_{\mathscr{A}} = \mathbb{V}. \tag{SC}$$

This is a mild non degeneracy assumption that we will keep in mind; we will occasionally try to explain how certain results can be extended in its absence. For now, we give some examples of differential operators:

Example 1.16. The following are all linear, homogeneous differential operators with constant coefficients:

(a) $\mathscr{A} \equiv 0$: this is the unconstrained case, which will be studied in detail in Section 3. In this case, we have that $\Lambda_{\mathscr{A}} = \mathbb{V}$.

(b) If $\mathscr{A}$ is (overdetermined) *elliptic*, meaning that $\ker\mathscr{A}(\xi) = \{0\}$ for all $\xi \neq 0$. In this case, we have by Calderón–Zygmund theory that $\mathscr{A}v_j \to \mathscr{A}v$ in $\mathrm{W}^{-\ell,p}_{\mathrm{loc}}$ implies that $v_j \to v$ in $\mathrm{L}^p_{\mathrm{loc}}$ when $1 < p < \infty$. We will prove this, assuming for simplicity that $v = 0$ and $v_j \in \mathrm{C}^\infty_c(\Omega,\mathbb{V})$ for some bounded open set $\Omega \subset \mathbb{R}^n$. We remark that the injectivity of $\mathscr{A}(\xi)$ is equivalent to the existence of a left inverse of this matrix, for instance $\mathscr{A}^\dagger(\xi) := [\mathscr{A}^*(\xi)\mathscr{A}(\xi)]^{-1}\mathscr{A}^*(\xi)$ for $\xi \neq 0$. We can then compute in Fourier space:

$$\widehat{\mathscr{A}v_j}(\xi) = (-\,\mathrm{i})^\ell\mathscr{A}(\xi)\hat{v}_j(\xi), \quad \text{so } \hat{v}_j(\xi) = \mathrm{i}^\ell\,\mathscr{A}^\dagger(\xi)\widehat{\mathscr{A}v_j}(\xi) = \mathrm{i}^\ell\,\mathscr{A}^\dagger\left(\frac{\xi}{|\xi|}\right)\frac{\widehat{\mathscr{A}v_j}(\xi)}{|\xi|^\ell},$$

so that by the Hörmander–Mikhlin multiplier theorem (see Theorem D.1) we have

$$\|v_j\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant c\left\|\mathscr{F}^{-1}\left(\frac{\widehat{\mathscr{A}v_j}(\xi)}{|\xi|^\ell}\right)\right\|_{\mathrm{L}^p(\mathbb{R}^n)} = \|\mathscr{A}v_j\|_{\mathrm{W}^{-\ell,p}(\mathbb{R}^n)} \to 0.$$

In this case, there are no weak convergence effects.

(c) An important case in the calculus of variations is given by the case of gradients, i.e., taking $v_j = Du_j$ in (CC). In this case we have $\mathscr{A}v = 0$, for $\mathscr{A} = \operatorname{curl}$. To be precise, consider matrix valued fields $v\colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$ and define

$$\operatorname{curl} v := (\partial_i v_{hj} - \partial_j v_{hi})_{i,j=1,\ldots,n;\,h=1,\ldots,N}.$$

It is then easy to check that $\ker\mathscr{A}(\xi) = \operatorname{im}\mathscr{B}(\xi)$ for $\xi \neq 0$, where $\mathscr{B} = D$. Since $\widehat{Du}(\xi) = -\,\mathrm{i}\,\hat{u}(\xi)\otimes\xi$ for $u \in \mathscr{S}(\mathbb{R}^n,\mathbb{R}^N)$, we have that

$$\Lambda_{\mathscr{A}} = \{a\otimes b\colon a \in \mathbb{R}^N,\ b \in \mathbb{R}^n\},$$

the cone of rank one matrices.

A similar first order operator such that $\ker\mathscr{A}(\xi) = \operatorname{im}\mathscr{B}(\xi)$ for $\xi \neq 0$ can be constructed for the case of higher order gradients, $\mathscr{B} = D^k$.

(d) Consider $\mathscr{A} = \operatorname{div}$, defined on matrix fields $v\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$. In this case, we have $\mathscr{A}(\xi)M = M\xi$ for $M \in \mathbb{R}^{n\times n}$, $\xi \in \mathbb{R}^n$. In this case,

$$\Lambda_{\mathscr{A}} = \{M \in \mathbb{R}^{n\times n}\colon M\xi = 0 \text{ for some } \xi \in S^{n-1}\} = \{M \in \mathbb{R}^{n\times n}\colon \det M = 0\}.$$

(e) Let $\mathscr{A}(v, \tilde v) = (\operatorname{div} v, \operatorname{curl} \tilde v)$ for v, $\tilde v\colon \mathbb{R}^n \to \mathbb{R}^n$, in which case $\mathscr{A}(\xi)(z, \tilde z) = (z\cdot\xi, \tilde z \times \xi)$, where we wrote $\tilde z \times \xi = (z_i\xi_j - z_j\xi_i)_{i,j=1,\ldots,n}$. It is then the case that if $\mathscr{A}(\xi)(z,\tilde z) = 0$, then $\tilde z \parallel \xi$, so $z \cdot \tilde z = 0$. We have that

$$\Lambda_{\mathscr{A}} = \{(z, \tilde z) \in \mathbb{R}^{2n}\colon z\cdot\tilde z = 0\}.$$

(f) Exterior derivatives of differential forms can also be expressed as first order, linear, homogeneous partial differential operators. Many of the examples above fit in this framework.

(g) The Saint Venant compatibility operator is defined for $v\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ by

$$\mathscr{A} v = (\partial^2_{ij} v_{hk} + \partial^2_{hk} v_{ij} - \partial^2_{ih} v_{jk} - \partial^2_{jk} v_{ih})_{i,j,h,k=1,\ldots n}.$$

This operator is such that $\mathscr{A} v = 0$ implies $v = \mathcal{E}u := \frac12\left(Du + (Du)^{\mathrm{t}}\right)$ in simply connected domains. Here $u\colon \mathbb{R}^n \to \mathbb{R}^n$. In fact, we have that $\ker \mathscr{A}(\xi) = \operatorname{im} \mathscr{B}(\xi)$ for $\xi \neq 0$ where $\mathscr{B} = \mathcal{E}$, so that

$$\Lambda_{\mathscr{A}} = \left\{a \odot b := \tfrac12 (a\otimes b + b \otimes a)\colon a,\, b \in \mathbb{R}^n\right\}.$$

(h) Maxwell's equations are $\mathscr{A}(E, B, D, H) = 0$, for E, D, B, $H\colon \mathbb{R}^{1+3} \to \mathbb{R}^3$, where

$$\mathscr{A}(E, D, B, H) = (\partial_t B + \operatorname{curl} E,\ \operatorname{div} D,\ \partial_t D - \operatorname{curl} H,\ \operatorname{div} B).$$

In the static case, when the unknowns are independent of time, this example reduces to the classical div-curl case in (e).

(i) For $v\colon \mathbb{R}^n \to \mathbb{R}^n$, let $\mathscr{A} v = (\partial_i v_j)_{i\neq j}$. Although this operator contains many of the entries of the gradient (which is elliptic, i.e., falls under example (b)), it is quite wildly behaved, as we will see later. This is already visible in the case $n = 2$, when $\mathscr{A} v = (\partial_1 v_2,\ \partial_2 v_1)$.

We already mentioned that there will be a need to reduce to exact constraints $\mathscr{A} v = 0$. The main technical difficulties arising subsequently are that the linear PDEs are not stable under cut off and blow up, so that if one wants to manipulate exact solutions effectively, one can do very well with being able to project L^p fields on $\ker \mathscr{A}$ boundedly. We briefly outline how we will do this in the case when $v \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{V})$, by arguing pointwisely in Fourier space. Let

$$\widehat{\pi_{\mathscr{A}} v}(\xi) := \operatorname{Proj}_{\ker \mathscr{A}(\xi)} \hat v(\xi) \quad \text{for } \xi \in \mathbb{R}^n, \tag{1.5}$$

so that, by Plancherel's theorem

$$\inf\{\|v - v_0\|^2_{\mathrm{L}^2}\colon v_0 \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{V}),\ \mathscr{A} v_0 = 0 \ \text{ in } \mathscr{D}'(\mathbb{R}^n, \mathbb{W})\} \geqslant \int_{\mathbb{R}^n} |\hat v(\xi) - \widehat{\pi_{\mathscr{A}} v}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^n} |v - \pi_{\mathscr{A}} v|^2 \mathrm{d}x,$$

so that, as long as $\pi v \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{V})$, we see that $\pi_{\mathscr{A}} v$ is a minimizer of the left hand side. Since the L^2 multipliers are the L^∞ functions and the matrices $\operatorname{Proj}_{\ker \mathscr{A}(\xi)}$ have operator norm at most one, it follows from (1.5), that one can always define the projection operator $\pi_{\mathscr{A}}$ on $\mathrm{L}^2(\mathbb{R}^n, \mathbb{V})$.

Nevertheless, to work on different L^p spaces, the boundedness of the 0-homogeneous multiplier

$$P_{\mathscr{A}}(\xi) := \operatorname{Proj}_{\ker \mathscr{A}(\xi)} \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}$$

is no longer sufficient. One would expect some differentiability away from zero. We have the following:

Lemma 1.17. *Let $\mathscr{A}$ be an operator as in* (1.3). *The following are equivalent:*

(a) *$\mathscr{A}$ has* constant rank, *i.e., there exists an integer r such that* $\operatorname{rank} \mathscr{A}(\xi) = r$ *for all $\xi \neq 0$.*
(b) *$P_{\mathscr{A}}\colon \mathbb{R}^n \setminus \{0\} \to \operatorname{Lin}(\mathbb{V}, \mathbb{V})$ is a rational function.*
(c) *$P_{\mathscr{A}}\colon \mathbb{R}^n \setminus \{0\} \to \operatorname{Lin}(\mathbb{V}, \mathbb{V})$ is smooth.*
(d) *$P_{\mathscr{A}}\colon \mathbb{R}^n \setminus \{0\} \to \operatorname{Lin}(\mathbb{V}, \mathbb{V})$ is continuous.*

The only implications that requires attention are (a) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (a) since (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) are clear. The proof of (d) $\Longrightarrow$ (a) follows from the following elementary algebraic fact:

Lemma 1.18. *There exists $\varepsilon > 0$ depending only on the norm on* $\operatorname{Lin}(\mathbb{V}, \mathbb{V})$ *such that if P_1, P_2 are orthogonal projections on $\mathbb{V}$, then $|P_1 - P_2| < \varepsilon$ implies that* $\operatorname{rank} P_1 = \operatorname{rank} P_2$.

For details, see [5, Cor. 10.4.2]. To conclude the proof of Lemma 1.17, we show that (a) $\Longrightarrow$ (b). This follows as a consequence of the proof of the following:

Lemma 1.19. *Let* $M \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$. *Then*

$$\mathrm{Proj}_{\mathrm{im}\, M} = -\frac{1}{a_0}\sum_{i=1}^{r} a_i M^i,$$

where $r = \operatorname{rank} M$ *and* a_i *are the coefficients of the characteristic polynomial* $\chi_M(t) = t^{N-r}\sum_{i=0}^r a_i t^i$, *where, in particular,* $a_r = 1$ *and* $a_0 \neq 0$.

Proof. Let $\lambda_i \in \mathbb{R}$, $i = 1,\dots,r$ be the (possibly repeating) non-zero eigenvalues of M. We know that there exist an orthogonal matrix $P \in \mathrm{O}(N)$ such that $M = P^{\mathrm{t}}DP$, where $M = \mathrm{diag}(\lambda_1,\dots,\lambda_r,0,\dots,0)$. We infer that $Q(t) := (t-\lambda_1)\dots(t-\lambda_r) = \sum_{i=0}^r a_i t^i$ so that

$$(-1)^r\lambda_1\dots\lambda_r I_N + \sum_{i=1}^r a_i D^i = (D-\lambda_1 I_N)\dots(D-\lambda_r I_N) = (-1)^r\lambda_1\dots\lambda_r \mathrm{diag}(\underbrace{0,\dots,0}_{r \text{ times}},\underbrace{1,\dots,1}_{N-r \text{ times}})$$

where the second inequality follows by direct computation. We conclude that, since $a_0 = (-1)^r\lambda_1\dots\lambda_r \neq 0$ as $\operatorname{rank} D = r$,

$$\sum_{i=1}^r a_i D^i = -(-1)^r\lambda_1\dots\lambda_r\mathrm{diag}(\underbrace{0,\dots,0}_{r \text{ times}},\underbrace{1,\dots,1}_{N-r \text{ times}}) = -a_0\mathrm{Proj}_{\mathrm{im}\, D}.$$

We conclude by multiplying with P^{t} to the left and by P to the right and using the fact that $P^{\mathrm{t}}\mathrm{Proj}_{\mathrm{im}\, D}P = \mathrm{Proj}_{\mathrm{im}\, M}$ (which is an elementary exercise). □

Proof of Lemma 1.17. In view of the considerations above, it remains to prove that (a) implies (b). We write $M = \mathscr{A}^*(\xi)\mathscr{A}(\xi)$, which can be identified with a symmetric $N\times N$ matrix, $N := \dim\mathbb{V}$. In view of Lemma 1.19, since M is (2ℓ)-homogeneous in ξ, one can check that a_i is $(2\ell(N-i))$-homogeneous in ξ, so that we can conclude that $\mathrm{Proj}_{\mathrm{im}\, M}$ can be expressed as the ratio of two $(2\ell N)$-homogeneous polynomials. Also a_0 is non-zero for all $\xi \neq 0$, which, together with the fact that $P_{\mathscr{A}} = I_N - \mathrm{Proj}_{\mathrm{im}\, M}$, concludes the proof. □

From the proof above we can infer a very important property of constant rank operators:

Lemma 1.20 (Algebraic potentials for CR operators)**.** *Let* $\mathscr{A}$ *be a constant rank operator as in* (1.3). *Then there exists a linear, homogeneous partial differential operator* $\mathscr{B}$ *on another finite dimensional inner product space* $\mathbb{U}$ *such that*

$$\ker\mathscr{A}(\xi) = \mathrm{im}\,\mathscr{B}(\xi) \quad \textit{for all } \xi \in \mathbb{R}^n\setminus\{0\}.$$

In particular, $\mathscr{B}$ *also has constant rank.*

Proof. Ideally, we would choose $\mathscr{B}(\xi) = P_{\mathscr{A}}(\xi)$, but, from the proof of Lemma 1.17, we see that $P_{\mathscr{A}}$ is zero-homogeneous, so not a polynomial in general. In this case, $\mathscr{B}$ would not define a differential operator. Instead, we read from the proof of Lemma 1.19 that, under the identification $\mathbb{V}\simeq\mathbb{R}^N$,

$$P_{\mathscr{A}}(\xi) = I_N - \left(-\frac{1}{a_0(\xi)}\sum_{i=1}^r a_i[\mathscr{A}^*(\xi)\mathscr{A}(\xi)]^i\right) = \frac{Q(\xi)}{a_0(\xi)},$$

where Q, a_0 are $(2\ell N)$-homogeneous polynomials, with $a_0 \neq 0$ away from zero and Q is $\mathbb{R}^{N\times N}$-valued. We can then choose

$$\mathscr{B}(\xi) := Q(\mathscr{A}^*(\xi)\mathscr{A}(\xi)),$$

which defines a homogeneous linear partial differential operator. □

It is quite interesting that this result holds independently of the spanning cone condition (SC). The converse is also true, since rank is a lower semi-continuous map on matrices, but we do not pursue this here. We also remark that, in general, the degree of $\mathscr{B}$ can be very large, but this cannot be avoided. In fact, it may be that $\mathscr{A}$ or $\mathscr{B}$ have order one, but the other operator has large order, see e.g., Example 1.16(c), (g); see also Example 1.23 below.

We will also give a strengthening of Lemma 1.19 which will have consequence in our analysis; in particular, it will enable us to turn the exact relation in Lemma 1.20 into an exact relation for smooth periodic fields. To this end, we recall the definition of the *Moore–Penrose generalized inverse* $A^\dagger \in \mathbb{R}^{m\times N}$ of a matrix $A \in \mathbb{R}^{N\times m}$, which can be defined explicitly by

$$A^\dagger := \begin{cases} \left(A\big|_{(\ker A)^\perp}\right)^{-1} & \text{on } \mathrm{im}\, A\\ 0 & \text{on } (\mathrm{im}\, A)^\perp,\end{cases}$$

or, equivalently, either geometrically as the unique matrix such that

$$A^\dagger A = \mathrm{Proj}_{\mathrm{im}\,A^*}, \quad AA^\dagger = \mathrm{Proj}_{\mathrm{im}\,A},$$

or algebraically by the relations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

It is elementary to see that the three definitions are equivalent and will use this fact without mention in the sequel. For more properties of generalized inverses we refer the reader to the monograph [5]. Here we will refer to $A^\dagger$ as the *pseudoinverse* of A, although this is not standard.

Theorem 1.21 (Decell)**.** *Let $A \in \mathbb{R}^{N\times m}$. Then*

$$A^\dagger = -\frac{1}{a_0}\sum_{i=1}^{r} a_i A^*(AA^*)^{i-1},$$

where the coefficients a_i are as in Lemma 1.19 for $M = AA^*$.

The important consequence of this is that the pseudoinverse of the symbol map of a constant rank operator is an appropriate multiplier.

Corollary 1.22. *Let $\mathscr{A}$ be a constant rank operator of order ℓ. Then $\mathscr{A}^\dagger(\bullet) \in \mathrm{C}^\infty(\mathbb{R}^n\setminus\{0\}, \mathrm{Lin}(\mathbb{W},\mathbb{V}))$ is a $(-\ell)$-homogeneous rational function.*

Proof of Theorem 1.21. Apply Lemma 1.19 to $M = AA^*$ to get

$$AA^\dagger = \mathrm{Proj}_{\mathrm{im}\,A} = -\frac{1}{a_0}\sum_{i=1}^{r} a_i (AA^*)^i = -\frac{1}{a_0}\sum_{i=1}^{r} a_i AA^*(AA^*)^{i-1},$$

so each term on the right hand side lies in $\mathrm{im}\,A$. We apply $A^\dagger$ to the left to get

$$A^\dagger = A^\dagger AA^\dagger = -\frac{1}{a_0}\sum_{i=1}^{r} a_i A^\dagger AA^*(AA^*)^{i-1} = -\frac{1}{a_0}\sum_{i=1}^{r} a_i A^*(AA^*)^{i-1},$$

which completes the proof. □

Before we conclude this subsection with an analytic consequence of Corollary 1.22, we display some choices of $\mathscr{A}$ and $\mathscr{B}$ as per Example 1.16:

Example 1.23. One can compute explicitly to see that Examples 1.16(h) and (i) are not of constant rank. As for the others it is easy to see that for $\mathscr{A} = 0$ we can take $\mathscr{B} = I_{\mathbb{V}}$; if $\mathscr{A}$ is elliptic, we have that $\mathscr{B} = 0$. The other examples require more computation:

- For (c), we have that $\mathscr{A} = \mathrm{curl}$ and $\mathscr{B} = D$.
- For (d), we have that $\mathscr{A} = \mathrm{div}$ and $\mathscr{B} = \mathrm{curl}^*$, the adjoint of the curl operator; this can be deduced by duality from the previous example.
- For (g), we have $\mathscr{B} = \mathcal{E}$, the symmetrized gradient, and $\mathscr{A} = \mathrm{curl}\,\mathrm{curl}$, the Saint Venant operator.

We show that *periodic* $\mathscr{A}$-free fields are in the image of $\mathscr{B}$:

Lemma 1.24. *Let $\mathscr{A}$, $\mathscr{B}$ be as in Lemma 1.20. Let $v \in \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{V})$ be such that $\mathscr{A}v = 0$ and $\int v = 0$. Then there exists $u \in \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{U})$ such that $v = \mathscr{B}u$.*

The same holds true with for $\mathscr{A}$-free fields $v \in \mathscr{S}(\mathbb{R}^n, \mathbb{V})$; however, by taking $\mathscr{A}$ to be elliptic, it is immediate to see that the implication

$$\mathscr{A}v = 0 \text{ in } \Omega \implies v = \mathscr{B}u$$

fails in general. In particular, to see how far is $\ker\mathscr{A}$ from $\mathrm{im}\,\mathscr{B}$ on domains, one needs to enforce special boundary conditions or add a penalization $v = \mathscr{B}u + h$.

Proof. We can write

$$v(x) = \sum_{0\neq\xi\in\mathbb{Z}^n} \hat{v}(\xi)\mathrm{e}^{\mathrm{i}\,x\cdot\xi},$$

so that $\mathscr{A}v = 0$ implies $\hat{v}(\xi) \in \ker\mathscr{A}(\xi) = \mathrm{im}\,\mathscr{B}(\xi)$ for $0 \neq \xi \in \mathbb{Z}^n$, so that

$$\mathscr{B}(\xi)\mathscr{B}^\dagger(\xi)\hat{v}(\xi) = \mathrm{Proj}_{\mathrm{im}\,\mathscr{B}(\xi)}\hat{v}(\xi) = \hat{v}(\xi).$$

We can then define, writing k for the order of $\mathscr{B}$,

$$u(x) := (-\mathrm{i})^k \sum_{0\neq\xi\in\mathbb{Z}^n} \mathscr{B}^\dagger(\xi)\hat{v}(\xi)\mathrm{e}^{\mathrm{i}\,x\cdot\xi},$$

so $u \in \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{V})$, as the coefficients of both u and v have arbitrary polynomial decay, and $\mathscr{B}u = v$. □

2. Young measures for oscillation and concentration

We now introduce the formal setup for our energy functionals and weakly convergent sequences. We **freeze $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ to be bounded and open with $\mathscr{L}^n(\partial\Omega) = 0$.** Certain, if not most, statements have corresponding variants at the endpoints as well, but we will only seldom focus on that. We look at weakly convergent sequences of vector fields

$$v_j \rightharpoonup v \quad \text{in } \mathrm{L}^p(\Omega, \mathbb{V}),$$

which are therefore uniformly bounded in L^p. We consider non-autonomous integrands, by which we mean $\Phi \in \mathrm{C}(\Omega \times \mathbb{V})$, that satisfy a p-growth condition

$$|\Phi(x, z)| \leqslant c(1 + |z|)^p \quad \text{for } x \in \Omega,\ z \in \mathbb{V}.$$

In this case, the sequence $(\Phi(\,\bullet\,, v_j))$ is bounded in $\mathrm{L}^1(\Omega)$, hence has some subsequence (that we do not relabel) converging weakly-* in the space of measures

$$\Phi(\,\bullet\,, v_j) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\bar{\Omega}),$$

by Proposition 1.13. In particular, we can identify the limit of the energy as

$$\int_\Omega \Phi(x, v_j(x))\mathrm{d}x \to \mu(\bar{\Omega}) \quad \text{as } j \to \infty,$$

but this is far from satisfactory as both the subsequence and limit measure depend on the integrand Φ.

The functional analytic entry point to circumvent this issue will be to view $(x, v_j(x))$ as a sequence of measures acting on the continuous integrands $\Phi \in \mathrm{C}(\Omega \times \mathbb{V})$.[3] We would like to have a handle of the Φ-moment of the limiting distribution of $(x, v_j(x))$. We can think of $(\Phi(x, v_j(x)))$ as one-point statistics for this distribution.

We would like to use our energy bounds to obtain weakly-* compactness when testing with integrands Φ which lie in the natural space

$$\mathbb{G}_p(\Omega, \mathbb{V}) := \left\{ \Phi \in \mathrm{C}(\Omega \times \mathbb{V})\colon \|\Phi\|_{\mathbb{G}_p(\Omega)} := \sup_{(x,z)} \frac{|\Phi(x, z)|}{(1 + |z|)^p} < \infty \right\}.$$

This space can be equivalently described by using the sphere compactification of $\mathbb{V}$:

$$\mathbb{V} \ni z \mapsto \tilde{z} := \frac{z}{1 + |z|} \in B_{\mathbb{V}} \quad \text{with inverse} \quad B_{\mathbb{V}} \ni \tilde{z} \mapsto z := \frac{\tilde{z}}{1 - |\tilde{z}|} \in \mathbb{V},$$

which we can use to define the linear map

$$(T_p\Phi)(x, \tilde{z}) := (1 - |\tilde{z}|)^p\, \Phi\left(x, \frac{\tilde{z}}{1 - |\tilde{z}|}\right) \quad \text{for } x \in \Omega,\ \tilde{z} \in B_{\mathbb{V}},$$

which captures the behavior of the p-growth integrands in the following sense:

Lemma 2.1. *We have that $T_p\colon \mathbb{G}_p(\Omega, \mathbb{V}) \to \mathrm{C}_b(\Omega \times B_{\mathbb{V}})$ is a linear isometric isomorphism. Therefore, so is the adjoint operator $T_p^*\colon\ \mathrm{C}_b(\Omega \times B_{\mathbb{V}})^* \to \mathbb{G}_p(\Omega, \mathbb{V})^*$.*

The proof follows by algebraically checking that $\|\Phi\|_{\mathbb{G}_p(\Omega)} = \|T_p\Phi\|_{\mathrm{L}^\infty(\Omega\times B_{\mathbb{V}})}$. For later use, we record the explicit expression of the inverse, defined for $\Psi \in \mathrm{C}_b(\Omega \times B_{\mathbb{V}})$ by

$$(T_p^{-1}\Psi)(x, z) = (1 + |z|)^p\, \Psi\left(x, \frac{z}{1 + |z|}\right) \quad \text{for } x \in \Omega,\ z \in \mathbb{V}.$$

We clarify in what sense do L^p functions act on integrands:

Lemma 2.2 (Elementary Young measures)**.** *Let $v \in \mathrm{L}^p(\Omega, \mathbb{V})$. Then v can be identified with a bounded linear functional on $\mathbb{G}_p(\Omega, \mathbb{V})$ via*

$$\langle \boldsymbol{\varepsilon}_v, \Phi\rangle_{\mathbb{G}_p^*, \mathbb{G}_p} := \int_\Omega \Phi(x, v(x))\mathrm{d}x \quad \textit{for } \Phi \in \mathbb{G}_p(\Omega, \mathbb{V}). \tag{2.1}$$

We say that $\boldsymbol{\varepsilon}_v$ is an *elementary Young measure.* To prove the lemma, it is easy to see that

$$\|\boldsymbol{\varepsilon}_v\|_{\mathbb{G}_p(\Omega)^*} = \int_\Omega (1 + |v(x)|)^p\mathrm{d}x.$$

It is important to stress here, however, that **the application $v \mapsto \boldsymbol{\varepsilon}_v$ is not linear**. This is to say that, for instance, sums of (elementary) Young measures are not (elementary) Young measures in general. This alludes to the fact that the topological space of Young measures will not be a linear space.

[3]To be specific, $V_j\colon x \in \Omega \mapsto (x, v_j(x))$ acts on Φ as the push-forward $V_{j\#}(\mathscr{L}^n \llcorner \Omega)$, but we will not focus on this aspect.

Considering now a uniformly bounded sequence $(v_j) \subset \mathrm{L}^p(\Omega,\mathbb{V})$, we can infer that the sequence of elementary Young measures $(\boldsymbol{\varepsilon}_{v_j})$ is bounded in $\mathbb{G}_p(\Omega,\mathbb{V})^* \simeq \mathrm{C}_b(\Omega\times B_{\mathbb{V}})^*$, which is thus the dual of a non-separable Banach space. In particular, we lack an appropriate compactness principle to obtain (weakly-*) cluster points of $(\boldsymbol{\varepsilon}_{v_j})$. If this was rectified, then we could hope that such a point may encode the limiting probability distributions of values of (v_j) close to points in Ω.

One way to go about this is to look at the subspace of $\mathbb{G}_p(\Omega,\mathbb{V})$ that is isomorphic to $\mathrm{BUC}(\Omega\times B_{\mathbb{V}})$ (recall Proposition 1.13(a) and Equation (1.2)). We can write this subspace down explicitly as

$$\mathbb{E}_p(\Omega,\mathbb{V}) \coloneqq T_p^{-1}\left(\mathrm{C}(\bar\Omega\times \bar B_{\mathbb{V}})\right).$$

We can make its properties more transparent with the following:

Lemma 2.3 (On p-admissible integrands)**.** *We have that*

$$\begin{aligned}\mathbb{E}_p(\Omega,\mathbb{V}) &= \{\Phi\in\mathrm{C}(\Omega\times\mathbb{V})\colon T_p\Phi\in\mathrm{BUC}(\Omega\times B_{\mathbb{V}})\}\\ &=\left\{\Phi\in\mathrm{C}(\bar\Omega\times\mathbb{V})\colon \Phi_p^\infty(x,z)\coloneqq\lim_{t\to\infty}\frac{\Phi(x,tz)}{t^p}\in\mathbb{R}\ \textit{locally uniformly for } (x,z)\in\bar\Omega\times\mathbb{V}\right\}.\end{aligned}$$

For $\Phi\in\mathbb{E}_p(\Omega,\mathbb{V})$, we will term Φ_p^∞ as the p-recession integrand *of Φ. It satisfies*

$$\Phi_p^\infty\in\mathbb{E}_p(\Omega,\mathbb{V})\quad\textit{and}\quad \Phi_p^\infty(x,tz)=t^p\Phi_p^\infty(x,z)\quad \textit{for } t\geqslant 0,\ x\in\bar\Omega,\ z\in\mathbb{V},$$

so Φ_p^∞ is itself a p-admissible integrand and is, in addition, positively p-homogeneous.

It is an elementary exercise in analysis to prove this. The p-recession integrands will enable us to capture the L^p concentration. We also record the following immediate consequence of Lemma 2.1:

Lemma 2.4. *We have that $T_p\colon\mathbb{E}_p(\Omega,\mathbb{V})\to\mathrm{C}(\bar\Omega\times\bar B_{\mathbb{V}})$ is a linear isometric isomorphism. Therefore, so is the adjoint operator $T_p^*\colon\ \mathrm{C}(\bar\Omega\times\bar B_{\mathbb{V}})^*\to\mathbb{E}_p(\Omega,\mathbb{V})^*$.*

We are now in a position to obtain a satisfactory weakly-* compactness result for elementary Young measures, which can be viewed as dual elements of $\mathbb{E}_p$ by restricting their action from (2.1). We introduce the notation

$$\langle\!\langle \boldsymbol{\varepsilon}_v,\Phi\rangle\!\rangle \coloneqq \langle\boldsymbol{\varepsilon}_v,\Phi\rangle_{\mathbb{E}_p^*,\mathbb{E}_p}=\int_\Omega\Phi(x,v(x))\mathrm{d}x\quad\text{for } v\in\mathrm{L}^p(\Omega,\mathbb{V}),\ \Phi\in\mathbb{E}_p(\Omega,\mathbb{V}).$$

As a consequence of the isomorphism in Lemma 2.4, of the separability of $\mathrm{C}(\overline{\Omega\times B_{\mathbb{V}}})$, and of the weakly-* compactness principle in Theorem A.2, we have the following crucial basic result for this course:

Proposition 2.5 (Young measures, abstract definition)**.** *Let $(v_j)\subset\mathrm{L}^p(\Omega,\mathbb{V})$ satisfy $\sup_j\|v_j\|_{\mathrm{L}^p(\Omega)}<\infty$. Then there exists $\boldsymbol{\nu}\in\mathbb{E}_p(\Omega,\mathbb{V})^*$ such that $\boldsymbol{\varepsilon}_{v_j}\overset{*}{\rightharpoonup}\boldsymbol{\nu}$ in $\mathbb{E}_p(\Omega,\mathbb{V})^*$ along a subsequence. We say that such $\boldsymbol{\nu}$ is a* p-Young measure *(or simply* Young measure*) and that (v_j)* generates *$\boldsymbol{\nu}$.*

As it stands, the definition of a Young measure is hardly useful. We now formulate a measure theoretic argument to substantiate our claims above that the Young measures can be viewed as space parametrized limiting probability distributions, see Definition 2.6 below.

For now, we will derive two necessary conditions of $\boldsymbol{\nu}\in\mathbb{E}_p^*$ to be a Young measure, when viewed as a measure on $\bar\Omega\times\bar B_{\mathbb{V}}$. To this end, write

$$\mu\coloneqq(T_p^*)^{-1}\boldsymbol{\nu}\in\mathrm{C}(\overline{\Omega\times B_{\mathbb{V}}})^*\simeq\mathcal{M}(\overline{\Omega\times B_{\mathbb{V}}}).$$

Bear in mind that $\boldsymbol{\nu}$ is generated by a bounded L^p sequence, so we would not expect that μ can be any measure in $\mathcal{M}(\bar\Omega\times\bar B_{\mathbb{V}})$. Let $\Phi\in\mathbb{E}_p$ and write

$$\begin{aligned}\int_\Omega\Phi(x,v_j(x))\mathrm{d}x=\langle\!\langle\boldsymbol{\varepsilon}_{v_j},\Phi\rangle\!\rangle\overset{j\to\infty}{\longrightarrow}\langle\!\langle\boldsymbol{\nu},\Phi\rangle\!\rangle\coloneqq\langle\boldsymbol{\nu},\Phi\rangle_{\mathbb{E}_p^*,\mathbb{E}_p}=\langle T_p^*\mu,\Phi\rangle_{\mathbb{E}_p^*,\mathbb{E}_p}=\langle\mu,T_p\Phi\rangle_{\mathcal{M},\mathrm{C}}=\int_{\bar\Omega\times\bar B_{\mathbb{V}}}T_p\Phi\mathrm{d}\mu\\ =\int_{\bar\Omega\times B_{\mathbb{V}}}(1-|\tilde z|)^p\,\Phi\left(x,\frac{\tilde z}{1-|\tilde z|}\right)\mathrm{d}\mu(x,\tilde z)+\int_{\bar\Omega\times S_{\mathbb{V}}}\Phi_p^\infty(x,\tilde z)\mathrm{d}\mu(x,\tilde z),\end{aligned}$$

where $S_{\mathbb{V}}=\partial B_{\mathbb{V}}$ denotes the unit sphere in $\mathbb{V}$. Writing in this way we can already get a glimpse of the fact that the recession integrand will capture the concentration effects of (v_j) as $|z|\to\infty$, i.e., $|\tilde z|\to 1$.

By testing the previous relation first with $\Phi\geqslant 0$ and then with $\Phi=\varphi\otimes\mathbf{1}_{\mathbb{V}}$ for $\varphi\in\mathrm{C}(\bar\Omega)$ (in which case $\Phi_p^\infty\equiv 0$), we obtain

$$\mu\geqslant 0\text{ in }\mathcal{M}(\bar\Omega\times\bar B_{\mathbb{V}})\quad\text{and}\quad\int_{\bar\Omega}\varphi(x)\mathrm{d}x=\int_{\bar\Omega\times B_{\mathbb{V}}}\varphi(x)(1-|\tilde z|)^p\mathrm{d}\mu(x,\tilde z). \tag{2.2}$$

We will show that these two conditions in fact characterize Young measures. It turns out that we will be able to say much more, namely that Young measures can be parametrized as in the following:

Definition 2.6 (Young measures, measure theoretic definition)**.** *Let* $\boldsymbol{\nu} = ((\nu_x)_{x\in\Omega},\ \lambda,\ (\nu_x^\infty)_{x\in\bar\Omega})$ *be so that*

(a) $(\nu_x)_{x\in\Omega} \subset \mathcal{M}_1^+(\mathbb{V})$ *is* $\mathscr{L}^n$*-measurable; this is the* oscillation measure*.*
(b) $\lambda \in \mathcal{M}^+(\bar\Omega)$*; this is the* concentration measure*.*
(c) $(\nu_x^\infty)_{x\in\bar\Omega} \subset \mathcal{M}_1^+(S_{\mathbb{V}})$ *is* λ*-measurable; this is the* concentration angle measure*.*
(d) *the* p-th moment condition $\int_\Omega\int_{\mathbb{V}} |z|^p \mathrm{d}\nu_x(z)\mathrm{d}x < \infty$ *must hold.*

We denote the set of such parametrized measures by $\mathrm{Y}^p(\Omega,\mathbb{V})$*.*

To be a bit pedantic, in the definition above we should specify $\nu_x \in \mathcal{M}^1_+(\mathbb{V})$ (resp. $\nu_x^\infty \in \mathcal{M}^1_+(S_{\mathbb{V}})$) for $\mathscr{L}^n$-a.e. (resp. λ-a.e.) $x\in\bar\Omega$. The basic terminology revolving around parametrized measures is collected in Definition B.5. We can view a parametrized measure $\boldsymbol{\nu}\in\mathrm{Y}^p(\Omega,\mathbb{V})$ as an element of $\mathbb{E}_p^*$ via the action

$$\langle\!\langle \boldsymbol{\nu},\Phi\rangle\!\rangle := \int_\Omega \langle \nu_x, \Phi(x,\,\boldsymbol{\cdot}\,)\rangle \mathrm{d}x + \int_{\bar\Omega}\langle \nu_x^\infty, \Phi_p^\infty(x,\,\boldsymbol{\cdot}\,)\rangle \mathrm{d}\lambda(x) \quad \text{for } \Phi\in\mathbb{E}_p(\Omega,\mathbb{V}). \tag{2.3}$$

It is then easy to see that indeed $\boldsymbol{\nu}\in\mathbb{E}_p(\Omega,\mathbb{V})^*$ since

$$\langle\!\langle \boldsymbol{\nu},\Phi\rangle\!\rangle \leqslant \|\Phi\|_{\mathbb{E}_p(\Omega)}\left(\int_\Omega \langle \nu_x, (1+|\,\boldsymbol{\cdot}\,|)^p\rangle \mathrm{d}x + \lambda(\bar\Omega)\right)$$

and we can use the moment condition. We next show that the two definitions of Young measures that we gave above (abstract and measure theoretic) are consistent:

Theorem 2.7 (Fundamental Theorem of Young measures)**.** *We have that*

$$\mathrm{Y}^p(\Omega,\mathbb{V}) = T_p^*\left\{\mu\in\mathcal{M}^+(\bar\Omega\times\bar B_{\mathbb{V}})\colon \int_{\bar\Omega\times B_{\mathbb{V}}}\varphi(x)(1-|\tilde z|)^p\mathrm{d}\mu(x,\tilde z) = \int_{\bar\Omega}\varphi(x)\mathrm{d}x \text{ for } \varphi\in\mathrm{C}(\bar\Omega)\right\}. \tag{2.4}$$

This set also coincides with the set of Young measures, as defined in Proposition 2.5.

Once we prove Theorem 2.7, we will mainly work with the measure theoretic Definition 2.6, as it is explicit and displays the oscillation and concentration effects as parametrized measures. We record the following immediate consequence, that Y^p is (sequentially) weakly-* closed in $\mathbb{E}_p^*$ and satisfies a compactness property that fits our aim:

Corollary 2.8 (Compactness for Young measures)**.** *Let* $1<p<\infty$ *and* $\Omega\subset\mathbb{R}^n$ *be open and bounded. Suppose that* $(v_j)\subset\mathrm{L}^p(\Omega,\mathbb{V})$ *is such that* $\sup_j\|v_j\|_{\mathrm{L}^p(\Omega)}<\infty$*. Then there exists* $\boldsymbol{\nu}\in\mathrm{Y}^p(\Omega,\mathbb{V})$ *such that* $\boldsymbol{\varepsilon}_{v_j}\overset{*}{\rightharpoonup}\boldsymbol{\nu}$ *in* $\mathbb{E}_p(\Omega,\mathbb{V})^*$ *along a subsequence, i.e.,*

$$\int_\Omega \Phi(x,v_j(x))\mathrm{d}x \to \int_\Omega\int_{\mathbb{V}}\Phi(x,z)\mathrm{d}\nu_x(z)\mathrm{d}x + \int_{\bar\Omega}\int_{S_{\mathbb{V}}}\Phi_p^\infty(x,\tilde z)\mathrm{d}\nu_x^\infty(\tilde z)\mathrm{d}\lambda(x) \quad \text{for } \Phi\in\mathbb{E}_p(\Omega,\mathbb{V}). \tag{2.5}$$

Moreover, $\mathrm{Y}^p(\Omega,\mathbb{V})$ *is convex and weakly-* closed in* $\mathbb{E}_p(\Omega,\mathbb{V})^*$*.*

The convergence in (2.5) can be interpreted loosely as the fact that "integrands of p-growth are continuous with respect to weak convergence in L^p, when embedded in the space of p-Young measures". With this in mind, we emphasize that Corollary 2.8 is the main foundational result for this course, underpinned by most background results, as well as the functional analytic construction we presented so far.

Proof of FTYM, Theorem 2.7. We first prove (2.4). To see that Y^p is contained in the right hand side, we proceed as for the proof of (2.2) and test (2.3) first with $\Phi\geqslant 0$, then with $\Phi=\varphi\otimes\mathbf{1}_{\mathbb{V}}$, where $\varphi\in\mathrm{C}(\bar\Omega)$.

We now show that measures $\mu\in\mathcal{M}(\bar\Omega\times\bar B_{\mathbb{V}})$ that satisfy (2.2) can be identified with elements of $\mathrm{Y}^p(\Omega,\mathbb{V})$. To this end, we use the disintegration Theorem B.6 to write

$$\mu=\gamma\otimes\eta_x \quad \text{for } \gamma\text{-measurable } (\eta_x)_{x\in\bar\Omega}\subset\mathcal{M}_1^+(\bar B_{\mathbb{V}}).$$

We also record the Radon–Nýkodim decomposition $\gamma=\gamma^a\mathscr{L}^n\llcorner\Omega+\gamma^s$, where $\gamma^s\perp\mathscr{L}^n$ and $\gamma^a\in\mathrm{L}^1(\Omega)$.

We first rewrite (2.2) as

$$\int_{\bar\Omega}\varphi(x)\underbrace{[\langle\eta_x,(1-|\,\boldsymbol{\cdot}\,|)^p\rangle\gamma^a(x)-1]\mathrm{d}x}_{\text{AC part}}+\int_{\bar\Omega}\varphi(x)\underbrace{\langle\eta_x,(1-|\,\boldsymbol{\cdot}\,|)^p\rangle\mathrm{d}\gamma^s(x)}_{\text{singular part}}=0 \quad \text{for all } \varphi\in\mathrm{C}(\bar\Omega).$$

In particular, $\langle\eta_x,(1-|\,\boldsymbol{\cdot}\,|)^p\rangle=0$ for γ^s-a.e. $x\in\bar\Omega$, so that η_x is concentrated on $S_{\mathbb{V}}$ for γ^s-a.e. $x\in\bar\Omega$.

Let now $\Phi\in\mathbb{E}_p(\Omega,\mathbb{V})$, so that $T_p\Phi=\Phi_p^\infty$ on $\bar\Omega\times S_{\mathbb{V}}$, and look at

$$\begin{aligned}\langle T_p\Phi(x,\,\boldsymbol{\cdot}\,),\eta_x\rangle\gamma &= \left(\int_{B_{\mathbb{V}}}+\int_{S_{\mathbb{V}}}\right)T_p\Phi(x,\tilde z)\mathrm{d}\eta_x(\tilde z)(\gamma^a(x)\mathscr{L}^n\llcorner\Omega+\gamma^s)\\ &= \left(\gamma^a(x)\int_{B_{\mathbb{V}}}T_p\Phi(x,\tilde z)\mathrm{d}\eta_x(\tilde z)\right)\mathscr{L}^n\llcorner\Omega+\left(\int_{S_{\mathbb{V}}}\Phi_p^\infty(x,\tilde z)\mathrm{d}\eta_x(\tilde z)\right)\gamma.\end{aligned}$$

This formula leads to defining the candidates for the triple:

$$\langle\nu_x, f\rangle := \gamma^a(x)\int_{B_{\mathbb{V}}} f(\tilde z)\mathrm{d}\eta_x(\tilde z) \quad \text{for } f\in \mathrm{C}_0(\mathbb{V})$$
$$\lambda := \eta_x(S_{\mathbb{V}})\gamma$$
$$\langle\nu_x^\infty, f^\infty\rangle := \fint_{S_{\mathbb{V}}} f(\tilde z)\mathrm{d}\eta_x(\tilde z) \quad \text{for } f^\infty\in \mathrm{C}(S_{\mathbb{V}}).$$

It is then clear that $\nu_x\in\mathcal{M}^+(\mathbb{V})$, $\lambda\in\mathcal{M}^+(\bar\Omega)$, $\nu_x^\infty\in\mathcal{M}_1^+(S_{\mathbb{V}})$, and

$$\langle T_p\Phi(x,\bullet),\eta_x\rangle\gamma = \langle\nu_x,\Phi(x,\bullet)\rangle\mathscr{L}^n\llcorner\Omega + \langle\nu_x^\infty,\Phi_p^\infty(x,\bullet)\rangle\lambda.$$

To conclude, we must prove that ν_x is a probability measure and that the moment condition holds. To show that $\nu_x(\mathbb{V})=1$, we write for $\varphi\in\mathrm{C}(\bar\Omega)$ that

$$\int_\Omega\varphi(x)\langle\nu_x,\mathbf{1}_{\mathbb{V}}\rangle\mathrm{d}x = \int_{\bar\Omega}\varphi(x)\langle(1-|\bullet|)^p,\eta_x\rangle\mathrm{d}\gamma(x) = \int_{\bar\Omega\times B_{\mathbb{V}}}\varphi(x)(1-|\tilde z|)^p\mathrm{d}\mu(x,\tilde z) = \int_\Omega\varphi(x)\mathrm{d}x,$$

where the last equality follows from (2.2). For the moment condition, we simply test with $\Phi(x,z)=(1+|z|)^p$, in which case $T_p\Phi=\mathbf{1}_{\bar\Omega\times\bar B_{\mathbb{V}}}$. We obtain

$$\langle\mu,\mathbf{1}_{\bar\Omega\times\bar B_{\mathbb{V}}}\rangle = \int_\Omega\langle\nu_x,(1+|\bullet|)^p\rangle\mathrm{d}x + \lambda(\bar\Omega),$$

which concludes the proof of this inclusion.

To conclude, we prove that the set of Young measures coincides with the right hand side of (2.4). We already showed that the images of Young measures under $(T_p^*)^{-1}$ satisfy (2.2). To prove the converse, let $\mathcal{E}_p$ denote the set of elementary Young measures. We claim that the weakly-* closure of $\mathcal{E}_p$ is convex. To see this consider $(v_i)_{i=1}^k\subset\mathrm{L}^p(\Omega,\mathbb{V})$ and define $\boldsymbol{\nu}:=\sum_{i=1}^k t_i\boldsymbol{\varepsilon}_{v_i}$ for $t_i\geqslant 0$, $\sum_{i=1}^k t_i=1$. We will show that $\boldsymbol{\nu}$ can be approximated weakly-* in $\mathbb{E}_p^*$ by a sequence of elementary Young measures.

We begin by defining characteristic functions $\chi_j^i\overset{*}{\rightharpoonup} t_i$ in $\mathrm{L}^\infty(\Omega)$ as $j\to\infty$. These can be constructed as follows: Consider a partition of the unit cube $Q:=(0,1)^n$ into disjoint subsets A_i such that $\mathscr{L}^n(A_i)=t_i$, $i=1,\dots k$ and let χ^i be the Q-periodic extension of $\mathbf{1}_{A_i}$ to $\mathbb{R}^n$. It is then elementary to show that $\chi_j^i:=\chi^i(j\bullet)\overset{*}{\rightharpoonup} t_i$ in $\mathrm{L}^\infty(\Omega)$ (see also the discussion of the Riemann–Lebesgue lemma in Example 2.11).

Let $\Phi\in\mathbb{E}_p(\Omega,\mathbb{V})$ and define

$$\tilde v_j := \sum_{i=1}^k\chi_j^i v_i, \quad \text{so that} \quad \Phi(x,\tilde v_j(x)) = \sum_{i=1}^k\chi_j^i(x)\Phi(x,v_i(x)),$$

where we use the fact that $(\chi_j^i)_{i=1}^k$ are characteristic functions of disjoint sets. Therefore

$$\langle\!\langle\boldsymbol{\varepsilon}_{\tilde v_j},\Phi\rangle\!\rangle = \sum_{i=1}^k\int_\Omega\chi_j^i(x)\Phi(x,v_i(x))\mathrm{d}x \to \sum_{i=1}^k\int_\Omega t_i\Phi(x,v_i(x))\mathrm{d}x = \langle\!\langle\boldsymbol{\nu},\Phi\rangle\!\rangle,$$

which indeed implies that $\bar{\mathcal{E}}_p$ is convex, so it can be described as an intersection of halfspaces, see Lemma C.2. To this end, we first give an explicit description of the images of elementary Young measures. Let $v\in\mathrm{L}^p(\Omega,\mathbb{V})$, $\mu_v:=(T_p^{-1})^*\boldsymbol{\varepsilon}_v$, and $\Psi\in\mathbb{E}_p(\Omega,\mathbb{V})$. Then

$$\begin{aligned}\langle\mu_v,\Psi\rangle &= \langle(T_p^{-1})^*\boldsymbol{\varepsilon}_v,\Psi\rangle = \langle\!\langle\boldsymbol{\varepsilon}_v,T_p^{-1}\Psi\rangle\!\rangle = \int_\Omega T_p^{-1}\Psi(x,v(x))\mathrm{d}x = \int_\Omega(1+|v(x)|)^p\Psi\left(x,\frac{v(x)}{1+|v(x)|}\right)\mathrm{d}x\\ &= \int_\Omega\frac{\Psi(x,Sv(x))}{(1-|Sv(x)|)^p}\mathrm{d}x = \int_\Omega\int_{\bar B_{\mathbb{V}}}\Psi(x,\tilde z)\mathrm{d}\delta_{(1-|Sv(x)|)^{-p}Sv(x)}(\tilde z)\mathrm{d}x,\end{aligned}$$

where we wrote $Sz:=\tilde z=(1+|z|)^{-1}z$ for the sphere compactification mapping $z\in\mathbb{V}$ to $\tilde z\in B_{\mathbb{V}}$. It follows that, in the notation of Theorem B.6, we have

$$\mu_v = \mathrm{d}x\otimes\delta_{\frac{Sv(x)}{(1-|Sv(x)|)^p}}. \tag{2.6}$$

We will try to eliminate the the dependence on v in the description of

$$\overline{\mathrm{co}}\left(\{\mu_v\colon v\in\mathrm{L}^p(\Omega,\mathbb{V})\}\right) = \left\{\mu\in\mathcal{M}(\bar\Omega\times\bar B_{\mathbb{V}})\colon \text{there exist }\Psi,\,t\text{ s.t.}\inf_{v\in\mathrm{L}^p(\Omega,\mathbb{V})}\langle\mu_v,\Psi\rangle\geqslant t\implies\langle\mu,\Psi\rangle\geqslant t\right\},$$

where we used Lemma C.2 to write down the right hand side. Using (2.6), we see that

$$\inf_{v\in\mathrm{L}^p(\Omega,\mathbb{V})}\langle\mu_v,\Psi\rangle = \int_\Omega\varphi_0(x)\mathrm{d}x, \quad \text{where } \varphi_0(x):=\inf_{\tilde z\in B_{\mathbb{V}}}\frac{\Psi(x,\tilde z)}{(1-|\tilde z|)^p}.$$

We also define $\Psi_0(x,\tilde z) := \Psi(x,\tilde z) - \varphi_0(x)(1-|\tilde z|)^p \geqslant 0$. We claim that

$$\overline{\mathrm{co}}\,(\{\mu_v\colon v\in \mathrm{L}^p(\Omega,\mathbb{V})\}) \supset \left\{\mu\in\mathcal{M}(\bar\Omega\times\bar B_{\mathbb{V}})\colon \Psi_0\geqslant 0,\ \int_\Omega \varphi_0\mathrm{d}x\geqslant t \implies \langle \mu, \varphi_0\otimes(1-|\,\boldsymbol{\cdot}\,|)^p+\Psi_0\rangle \geqslant t\right\}.$$

To see this, let $\mu\in\mathcal{M}(\bar\Omega\times\bar B_{\mathbb{V}})$ lie in the right hand side. We let $\Psi\in \mathrm{C}(\bar\Omega\times\bar B_{\mathbb{V}})$ and $t\in\mathbb{R}$ be such that $\langle\mu_v,\Psi\rangle\geqslant t$ for all $v\in\mathrm{L}^p(\Omega,\mathbb{V})$. Let φ_0 and Ψ_0 be defined as above, so that $\Psi_0\geqslant 0$ and $\langle \mathrm{d}x,\varphi_0\rangle\geqslant t$. By definition of μ, we have that

$$\langle\mu,\Psi\rangle = \langle \mu, \varphi_0\otimes(1-|\,\boldsymbol{\cdot}\,|)^p+\Psi_0\rangle\geqslant t,$$

which proves that μ lies in the convex hull of $\{\mu_v\}$.

To conclude, let $\mu\in\mathcal{M}^+(\bar\Omega\times\bar B_{\mathbb{V}})$ be such that

$$\int_{\bar\Omega\times B_{\mathbb{V}}}\varphi(x)(1-|\tilde z|)^p\mathrm{d}\mu(x,\tilde z) = \int_{\bar\Omega}\varphi(x)\mathrm{d}x \quad \text{for } \varphi\in\mathrm{C}(\bar\Omega).$$

Letting φ_0 be such that $\langle \mathrm{d}x,\varphi_0\rangle\geqslant t$ and $\Psi_0\geqslant 0$, we write

$$\langle \mu, \varphi_0\otimes(1-|\,\boldsymbol{\cdot}\,|)^p+\Psi_0\rangle = \langle\mathrm{d}x,\varphi_0\rangle + \langle\mu,\Psi_0\rangle\geqslant t+0=t.$$

This proves that $T_p^*\mu\in T_p^*\overline{\mathrm{co}}(\{\mu_v\}) = \overline{\mathrm{co}}(\mathcal{E}_p)=\bar{\mathcal{E}}_p$ and completes the proof of the theorem. □

We next aim to give generic examples to illustrate how can one test the relation (2.3) suitably and identify the oscillation and concentration effects. We first substantiate the claim that strong convergence leads to trivial Young measures:

Lemma 2.9 (On strong convergence)**.** *Let $v_j\to v$ in $\mathrm{L}^p(\Omega,\mathbb{V})$. Then (v_j) generates $\boldsymbol{\varepsilon}_v$.*

Proof. Let $\Phi\in\mathbb{E}_p(\Omega,\mathbb{V})$. By the Vitali convergence theorem 1.1 and Proposition B.1, we have that v_j converges $\mathscr{L}^n$-a.e. to v on a subsequence. Since Φ is continuous, we obtain that $(\Phi(\,\boldsymbol{\cdot}\,,v_j))$ also converges a.e., hence in measure. We also have that (v_j) being p-uniformly integrable implies that $(\Phi(\,\boldsymbol{\cdot}\,,v_j))$ is uniformly integrable. We conclude by Vitali's theorem that $\Phi(\,\boldsymbol{\cdot}\,,v_j)\to\Phi(\,\boldsymbol{\cdot}\,,v_j)$ in $\mathrm{L}^1(\Omega)$, hence $\langle\!\langle \boldsymbol{\varepsilon}_{v_j},\Phi\rangle\!\rangle\to\langle\!\langle\boldsymbol{\varepsilon}_v,\Phi\rangle\!\rangle$, which completes the proof. □

We also record the fact that we can use Young measures to identify a weak limit:

Lemma 2.10 (Barycenter of a Young measure)**.** *Let $v_j\rightharpoonup v$ in $\mathrm{L}^p(\Omega,\mathbb{V})$ generate $\boldsymbol{\nu}\in\mathrm{Y}^p(\Omega,\mathbb{V})$. Then*

$$v(x)=\bar\nu_x := \int_{\mathbb{V}} z\mathrm{d}\nu_x(z)\quad \textit{for } \mathscr{L}^n\textit{-a.e. } x\in\Omega,$$

which is the center of mass (expectation) of the probability measure ν_x. We say that $\bar{\boldsymbol{\nu}}:=\bar\nu_{\boldsymbol{\cdot}}\in\mathrm{L}^p(\Omega,\mathbb{V})$ is the barycenter *of the Young measure $\boldsymbol{\nu}$.*

Proof. We test (2.3) with $\Phi(x,z):=\varphi(x)z_i$ for $\varphi\in\mathrm{C}(\bar\Omega)$, so $\Phi_p^\infty\equiv 0$ (as $p>1$), to obtain

$$\int_\Omega\varphi(x)v_j(x)\mathrm{d}x\to\int_\Omega\int_{\mathbb{V}}\varphi(x)z\mathrm{d}\nu_x(z)\mathrm{d}x,$$

which is sufficient to conclude. □

We remark in passing that the statement of Lemma 2.10 also holds for $p=1$, although for a sligtly different reason. In the case of linear growth integrands, it would be more natural from the point of view of compactness to consider a sequence of weakly-* convergent measures, in which case the natural of the barycenter (which is obtained by testing with $\Phi(x,z)=\varphi(x)z_i$) is

$$\bar{\boldsymbol{\nu}} := \bar\nu_{\boldsymbol{\cdot}}\,\mathscr{L}^n\,\llcorner\,\Omega + \bar\nu^\infty_{\boldsymbol{\cdot}}\lambda$$

In that case, we have that $v_j\overset{*}{\rightharpoonup} v$ in $\mathcal{M}(\bar\Omega,\mathbb{V})$ implies that $v=\bar{\boldsymbol{\nu}}$. This is consistent with Lemma 2.10 for $p=1$, since weak convergence in L^1 implies uniform integrability (Proposition 1.11), in which case $\lambda\equiv 0$.

Before we list the examples, we will indicate how one can test the duality relation (2.3) to obtain information about each part of the triple $\boldsymbol{\nu}=((\nu_x)_{x\in\Omega},\,\lambda,\,(\nu_x^\infty)_{x\in\bar\Omega})$ generated by $(v_j)\subset\mathrm{L}^p(\Omega,\mathbb{V})$:

(a) To identify the oscillation measures (ν_x), we test with $\Phi(x,z):=\varphi(x)F(z)$, with $\varphi\in\mathrm{C}_c(\Omega)$ and $F\in\mathrm{C}_c(\mathbb{V})$. Then $\Phi_p^\infty\equiv 0$ and

$$\langle\!\langle\boldsymbol{\varepsilon}_{v_j},\varphi\otimes F\rangle\!\rangle = \int_\Omega\varphi(x)F(v_j(x))\mathrm{d}x\to\int_\Omega\varphi(x)\langle\nu_x,F\rangle\mathrm{d}x.$$

Since in this case $(F(v_j))$ is uniformly integrable, we can assume by the Dunford–Pettis Theorem 1.12(b) that $F(v_j)\rightharpoonup\langle\nu_{\boldsymbol{\cdot}},F\rangle$ in $\mathrm{L}^1(\Omega)$, so we can alternatively take $\varphi=\mathbf{1}_{Q\cap\Omega}$ for cubes $Q\subset\mathbb{R}^n$, see Proposition 1.11.

(b) To identify the concentration (location) measure λ, we test with the $\Phi(x,z) \coloneqq \varphi(x)|z|^p$ with $\varphi \in \mathrm{C}(\bar\Omega)$. Then $\Phi_p^\infty = \Phi$ and

$$\int_\Omega \varphi(x)|v_j(x)|^p \mathrm{d}x = \langle\!\langle \boldsymbol{\varepsilon}_{v_j}, \varphi \otimes |\,\boldsymbol{\cdot}\,|^p \rangle\!\rangle \to \langle\!\langle \nu, \varphi \otimes |\,\boldsymbol{\cdot}\,|^p \rangle\!\rangle = \int_\Omega \varphi(x)\langle \nu_x, |\,\boldsymbol{\cdot}\,|^p\rangle \mathrm{d}x + \int_{\bar\Omega} \varphi \mathrm{d}\lambda,$$

which implies that

$$\lambda = \mathcal{M}(\bar\Omega)\text{-w*-}\lim_{j\to\infty} (|v_j|^p \mathscr{L}^n \llcorner \Omega) - (\langle \nu_x, |\,\boldsymbol{\cdot}\,|^p\rangle \mathscr{L}^n \llcorner \Omega),$$

so, once we know the oscillation measure, the concentration measure is quite explicit.

(c) To identify the concentration angle measure (ν_x^∞), we simply test with p-homogeneous integrands. Let $\varphi \in \mathrm{C}(\bar\Omega)$ and $F \in \mathrm{C}(S_{\mathbb{V}})$, which we then extend homogeneously to

$$\tilde F(z) \coloneqq \begin{cases} |z|^p F\left(\frac{z}{|z|}\right) & z \neq 0 \\ 0 & z = 0 \end{cases},$$

so we can test with $\Phi(x,z) \coloneqq \varphi(x)\tilde F(z)$, so that $\Phi_p^\infty = \Phi$ and

$$\int_\Omega \varphi(x)\tilde F(v_j(x))\mathrm{d}x = \langle\!\langle \boldsymbol{\varepsilon}_{v_j}, \varphi \otimes \tilde F \rangle\!\rangle \to \langle\!\langle \nu, \varphi \otimes \tilde F \rangle\!\rangle = \int_\Omega \varphi(x)\langle \nu_x, \tilde F\rangle \mathrm{d}x + \int_{\bar\Omega} \varphi(x)\langle \nu_x^\infty, F\rangle \mathrm{d}\lambda(x),$$

so we can determine the action $\langle \nu_x^\infty, F\rangle$ for all $F \in \mathrm{C}(S_{\mathbb{V}})$ and λ-a.e. $x \in \bar\Omega$.

We also remark that when testing with integrands of the form $\Phi = \varphi \otimes F$, we need only consider functions of x and z that are dense in their respective spaces.

We are now prepared to discuss the examples, in all of which we will have that the sequence $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ generates $\boldsymbol{\nu} = ((\nu_x)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar\Omega})$. We begin with a generalization of Example 1.2:

Example 2.11 (Pure oscillation, Riemann–Lebesgue lemma). Write $Q \coloneqq (0,1)^n$ and let $v \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{V})$ be Q-periodic. Set $v_j(x) \coloneqq v(jx)$, so that one obtains after a not entirely trivial argument that

$$\int_Q \varphi(x)F(v_j(x))\mathrm{d}x \to \int_Q \varphi(x)F(v(x))\mathrm{d}x \quad \text{for } \varphi \in \mathrm{C}_c(Q),\ F \in \mathrm{C}_c(\mathbb{V}),$$

hence $\nu_x = v_\#(\mathscr{L}^n \llcorner Q)$ (recall Definition B.7). A similar computation using Item (b) above yields $\lambda = 0$. We can also use Lemma 2.10 to see that $v_j \rightharpoonup \fint_Q v\mathrm{d}x$ in $\mathrm{L}^p(Q, \mathbb{V})$. Altogether, we showed that

$$\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} ((v_\#\mathscr{L}^n \llcorner Q)_{x\in Q},\, 0,\, \texttt{n/a}) \quad \text{in } \mathbb{E}_p(Q, \mathbb{V})^*.$$

We will refer to a Young measure that is independent of $x \in \bar\Omega$ as a *homogenous Young measure*.

In the next example, we will look at a Young measure generated by a weakly-* convergent sequence, so it falls in a setup that we did not cover explicitly. However, the example is straightforward enough to not need further explanation:

Example 2.12 (Concentration in the limit). Let $v_j \coloneqq j\mathbf{1}_{(0,1/j)}$ which converges weakly-* to δ_0 in $\mathcal{M}[-1,1]$ (see also Example 1.14). It is then obvious that

$$\int_{-1}^1 \varphi(x)F(v_j(x))\mathrm{d}x \to \int_{-1}^1 \varphi(x)F(0)\mathrm{d}x \quad \text{for } \varphi \in \mathrm{C}_c(-1,1),\ F \in \mathrm{C}_c(\mathbb{R}),$$

so that $\nu_x = \delta_0$. Performing the computation in Item (b) with $p = 1$ leads to $\lambda = \delta_0$. As for the concentration angle measures, the one dimensional case $\mathbb{V} = \mathbb{R}$ is quite trivial: we must have $\nu_x^\infty = a(x)\delta_{-1} + b(x)\delta_1$, where $a + b = 1$ for λ-a.e. x. In the notation of Item (c), we have the integrands

$$F(z) \coloneqq \begin{cases} A & \text{if } z = -1 \\ B & \text{if } z = 1 \end{cases}, \quad \text{so } \tilde F(z) = \begin{cases} -Az & \text{if } z \leqslant 0 \\ Bz & \text{if } z > 0 \end{cases},$$

which leads, after performing the computation, to $\varphi(0)B = \varphi(0)(aA + bB)$, so $a = 0$, $b = 1$, hence

$$\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} \left((\delta_0)_{x\in(-1,1)},\, \delta_0,\, (\delta_1)_{x\in[-1,1]}\right) \quad \text{in } \mathbb{E}_1(-1,1)^*.$$

In the L^p case $(1 < p < \infty)$, we set $\tilde v_j(x) = j^{1/p}\mathbf{1}_{(0,1/j)}$ and can show by a similar reasoning that

$$\boldsymbol{\varepsilon}_{\tilde v_j} \overset{*}{\rightharpoonup} \left((\delta_0)_{x\in(-1,1)},\, \delta_0,\, (\delta_1)_{x\in[-1,1]}\right) \quad \text{in } \mathbb{E}_p(-1,1)^*.$$

The main difference is that the concentration effect is not visible in the weak limit, $\tilde v_j \rightharpoonup 0$ in $\mathrm{L}^p(-1,1)$. We have already seen this sequence for $p = 2$ in Example 1.3.

Next, we will display a concentration effect that is not visible in the weakly-* limit also for $p = 1$:

Example 2.13 (Concentration effect). Let $v_j := j^{1/p}\mathbf{1}_{(-1/j,0)} - j^{1/p}\mathbf{1}_{(0,1/j)}$. With similar considerations as in Example 2.12, we see that

$$\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} \Big((\delta_0)_{x\in(-1,1)},\, 2\delta_0,\, \big(\tfrac{1}{2}(\delta_{-1}+\delta_1)\big)_{x\in[-1,1]}\Big) \quad \text{in } \mathbb{E}_p(-1,1)^*.$$

Much as before, with $1 < p < \infty$, we have that $v_j \rightharpoonup 0$ in $\mathrm{L}^p(-1,1)$. Unlike in the previous Example 2.12, we have that also for $p = 1$, $v_j \overset{*}{\rightharpoonup} 0$ in $\mathcal{M}[-1,1]$. In particular, the sequence concentrates in two directions in the same location, and the concentration effect of magnitude 2 is cancelled out and not visible in the absolutely continuous limit.

So far, we only exhibited concentration at points. In the following example we showcase the fact that the concentration location measure can be diffused, $\lambda = \mathscr{L}^1 \llcorner (0,1)$.

Example 2.14 (Diffuse concentration). Let

$$v_j := j^{\frac{1}{p}} \sum_{i=0}^{j-1} \mathbf{1}_{\left(\frac{i}{j},\frac{i}{j}+\frac{1}{j^2}\right)}.$$

It is easy to check that in this case $\nu_x = \delta_0$ (similarly to Example 2.12). In fact, as we will soon see in Theorem 2.17, this is due to the fact that $v_j \to 0$ in $\mathscr{L}^n$-measure. By Item (b), we see that

$$\lambda = \mathcal{M}[0,1]\text{-w*-}\lim_{j\to\infty} j \sum_{i=0}^{j-1} \mathbf{1}_{\left(\frac{i}{j},\frac{i}{j}+\frac{1}{j^2}\right)} = \mathscr{L}^1 \llcorner (0,1).$$

To determine the concentration direction measures, we use notation similar to Example 2.12:

$$\nu_x^\infty = a(x)\delta_{-1} + (1-a(x))\delta_1 \quad \text{and} \quad F(z) := \begin{cases} A & \text{if } z=-1 \\ B & \text{if } z=1 \end{cases}, \quad \text{so } \tilde{F}(z) = \begin{cases} A|z|^p & \text{if } z\leqslant 0 \\ Bz^p & \text{if } z>0 \end{cases},$$

The relation in Item (c) leads after computation to

$$B\int_0^1 \varphi \,\mathrm{d}x = \int_0^1 \varphi(x)\,(a(x)A + (1-a(x))B)\,\mathrm{d}x \quad \text{for } \varphi \in \mathrm{C}[0,1].$$

In particular, we obtain that $a = 0$ Lebesgue-a.e., so that

$$\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} \big((\delta_0)_{x\in(0,1)},\, \mathscr{L}^n \llcorner (0,1),\, (\delta_1)_{x\in[0,1]}\big) \quad \text{in } \mathbb{E}_p(0,1)^*.$$

Similarly to Example 2.13, we have that for $1 < p < \infty$, $v_j \rightharpoonup 0$ in $\mathrm{L}^p(0,1)$, whereas $v_j \overset{*}{\rightharpoonup} \mathscr{L}^1 \llcorner (0,1)$ in $\mathcal{M}[0,1]$ if $p = 1$. Obviously, in the latter case, weak convergence in L^1 fails; this can be seen as a consequence of the fact that $\lambda \not\equiv 0$ (see Theorem 2.18 below).

A modification of Example 2.14 gives the following multi-dimensional example:

Example 2.15. Let

$$v_j(x) := j^{\frac{1}{p}} \sum_{i=0}^{j-1} \mathbf{1}_{\left(\frac{i}{j},\frac{i}{j}+\frac{1}{j^2}\right)}(x)\,\big(\cos(2\pi j^2 x),\, \sin(2\pi j^2 x)\big),$$

in which case

$$\boldsymbol{\varepsilon}_{v_j} \overset{*}{\rightharpoonup} \big((\delta_0)_{x\in(0,1)},\, \mathscr{L}^n \llcorner (0,1),\, (\mathscr{H}^1 \llcorner S^1)_{x\in[0,1]}\big) \quad \text{in } \mathbb{E}_p\left((0,1),\mathbb{R}^2\right)^*.$$

We conclude this segment of examples with a direct proof of the fact that any $\lambda \in \mathcal{M}^+(\bar{\Omega})$ can arise as a concentration location measure (of course, this also follows from the generation part of Theorem 2.7).

Example 2.16 (On the arbitrariness of concentration measures). We begin with a generalization of Examples 2.12 and 2.13. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $x_0 \in \Omega$, and $v \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ have $\|v\|_{\mathrm{L}^p} = 1$. We define $v_j(x) := j^{n/p} v(j(x-x_0))$ for $x \in \Omega$, so that $\|v_j\|_{\mathrm{L}^p(\Omega)} = 1$ for sufficiently large j. Much like in the previous three examples, we have that $\nu_x = \delta_0$. It is not too difficult then to see that $\lambda = \delta_{x_0}$ and

$$\nu_{x_0} = \left(\int_{\mathbb{R}^n} (v^+)^p \,\mathrm{d}x\right)\delta_{-1} + \left(\int_{\mathbb{R}^n} (v^-)^p \,\mathrm{d}x\right)\delta_1.$$

Let now $\lambda \in \mathcal{M}^+(\bar{\Omega})$ be approximated weakly-* by measures

$$\lambda_i := t_1\delta_{x_1} + \ldots + t_i\delta_{x_i}, \quad t_l \geqslant 0,\, t_1 + \ldots + t_i = \lambda(\bar{\Omega}).$$

By choosing sequences $v_j^i(x) := j^{n/p} v^i(j(x-x_i))$ starting with $v^i \in \mathrm{C}_c^\infty(\mathbb{R}^n)^+$, we can generate

$$\boldsymbol{\varepsilon}_{v_j^i} \overset{*}{\rightharpoonup} \boldsymbol{\nu}_i := ((\delta_0)_{x\in\Omega},\, \lambda_i,\, (\delta_1)_{x\in\bar{\Omega}}) \quad \text{in } \mathbb{E}_p(\Omega)^* \text{ as } j\to\infty,$$

so that $\boldsymbol{\nu}_i \in \mathrm{Y}^p(\Omega)$. It is then a matter of checking definitions to see that $\boldsymbol{\nu}_i \overset{*}{\rightharpoonup} \boldsymbol{\nu} := ((\delta_0)_{x\in\Omega},\, \lambda,\, (\delta_1)_{x\in\bar\Omega})$ in $\mathbb{E}_p(\Omega)^*$, so that indeed $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega)$, proving our assertion.

We will next formalize two important ideas that are already noticeable in the examples above:

Theorem 2.17 (No oscillation – convergence in measure). *Let $(v_j) \subset \mathrm{L}^p(\Omega,\mathbb{V})$ generate $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega,\mathbb{V})$. Then $v_j \to v$ in $\mathscr{L}^n$-measure if and only if $v(x) = \bar\nu_x$ and $\nu_x = \delta_{\bar\nu_x}$ for $\mathscr{L}^n$-a.e. $x \in \Omega$.*

Moreover, let $\tilde v_j \in \mathrm{L}^0(\Omega,\mathbb{V})$ be such that $v_j - \tilde v_j \to 0$ in $\mathscr{L}^n$-measure. Then

$$\int_\Omega \varphi(x)F(\tilde v_j(x))\mathrm{d}x \to \int_\Omega \varphi(x)\langle \nu_x, F\rangle \mathrm{d}x \quad \text{for } \varphi \in \mathrm{C}(\bar\Omega),\ F \in \mathrm{C}_c(\mathbb{V}). \tag{2.7}$$

The converse to the second part is false, as can be seen from looking at $v_j(x) := \sin(jx)$ and $\tilde v_j(x) := \sin(2jx)$, which are both covered by Example 2.11 and generate the same Young measure, but $v_j - \tilde v_j$ does not converge to zero in measure.

Theorem 2.18 (No concentration – uniform integrability). *Let $(v_j) \subset \mathrm{L}^p(\Omega,\mathbb{V})$ generate $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega,\mathbb{V})$ and $\Phi \in \mathbb{E}_p(\Omega,\mathbb{V})$. Then $(\Phi(\,\bullet\,, v_j))$ is uniformly integrable if and only if*

$$\langle |\Phi_p^\infty(x,\bullet)|, \nu_x^\infty\rangle = 0 \quad \text{for } \lambda\text{-a.e. } x \in \bar\Omega. \tag{2.8}$$

In particular, (v_j) is p-uniformly integrable if and only if $\lambda \equiv 0$.

Furthermore, if $(\tilde v_j) \subset \mathrm{L}^p(\Omega,\mathbb{V})$ generate $\tilde{\boldsymbol{\nu}} \in \mathrm{Y}^p(\Omega,\mathbb{V})$ is such that $(v_j - \tilde v_j)$ is p-uniformly integrable, then $\lambda_{\boldsymbol{\nu}} = \lambda_{\tilde{\boldsymbol{\nu}}} (=: \lambda)$ and $\nu_x^\infty = \tilde\nu_x^\infty$ for λ-a.e. $x \in \bar\Omega$.

Here we can include $p = 1$ without any modification.

Remark 2.19 (On Exercise 1.10). As a consequence of Theorem 2.18, we can give an answer to Exercise 1.10: One checks directly that the averages converge for the sequence (v_j) defined in Example 2.14. The sequence is then not p-uniformly integrable since $\lambda \not\equiv 0$ by Theorem 2.18 (this can also be checked separately). Finally, this implies that (v_j) cannot converge weakly in L^1 by Proposition 1.11.

Proof of Theorem 2.17. We first assume that $v_j \to v$ in $\mathscr{L}^n$-measure and test $\boldsymbol{\nu}$ against $\Phi = \varphi \otimes F$, $\varphi \in \mathrm{C}_c(\Omega)$, $F \in \mathrm{C}_c(\mathbb{V})$. We have that $(\Phi(\,\bullet\,, v_j))$ is uniformly integrable and converges in measure, so that by Vitali's convergence theorem, we have that

$$\int_\Omega \varphi(x)F(v_j(x))\mathrm{d}x \to \int_\Omega \varphi(x)F(v(x))\mathrm{d}x.$$

On the other hand, we have that

$$\langle\!\langle \boldsymbol{\nu}, \Phi\rangle\!\rangle = \int_\Omega \varphi(x)\langle \nu_x, F\rangle\mathrm{d}x,$$

so that $F(v(x)) = \langle \nu_x, F\rangle$ for $\mathscr{L}^n$-a.e. $x \in \Omega$, which proves this implication.

Conversely, suppose that $\nu_x = \delta_{v(x)}$, where $v(x) = \bar\nu_x$, for $\mathscr{L}^n$-a.e. $x \in \Omega$. We claim that $v_j \to v$ in $\mathscr{L}^n$-measure. We have that $v \in \mathrm{L}^p(\Omega,\mathbb{V})$ by the moment condition (see also Lemma 2.10). Let $\varepsilon > 0$ and $\tilde v \in \mathrm{C}_c(\Omega,\mathbb{V})$ be such that $\|v - \tilde v\|^p_{\mathrm{L}^p(\Omega)} \leqslant \varepsilon$. We will test $\boldsymbol{\nu}$ with the integrand $\Phi(x,z) := \min\{1, |z - \tilde v(x)|^p\}$, which is clearly p-admissible and $\Phi_p^\infty \equiv 0$. We have

$$\int_\Omega \min\{1, |v_j(x) - \tilde v(x)|^p\}\mathrm{d}x = \langle\!\langle \boldsymbol{\varepsilon}_{v_j}, \Phi\rangle\!\rangle \to \langle\!\langle \boldsymbol{\nu}, \Phi\rangle\!\rangle = \int_\Omega \min\{1, |v(x) - \tilde v(x)|^p\}\mathrm{d}x \leqslant \varepsilon.$$

Therefore $\int_\Omega \min\{1, |v_j(x) - v(x)|^p\}\mathrm{d}x \to 0$, so that $\min\{1, |v_j - v|\} \to 0$ in measure, which suffices to conclude the proof of the equivalence.

It remains to prove (2.7), to which end we let $\varphi \in \mathrm{C}(\bar\Omega)$ and $F \in \mathrm{C}_c(\mathbb{V})$. Since F is uniformly continuous, we infer that $F(v_j) - F(\tilde v_j) \to 0$ in measure, so that we can use Vitali's theorem again to deduce that

$$\int_\Omega \varphi(x)F(\tilde v_j(x))\mathrm{d}x = \int_\Omega \varphi(x)F(v_j(x))\mathrm{d}x + \int_\Omega \varphi(x)\left(F(\tilde v_j(x)) - F(v_j(x))\right)\mathrm{d}x \to \int_\Omega \varphi(x)\langle \nu_x, F\rangle\mathrm{d}x,$$

and the proof is complete. □

Proof of Theorem 2.18. We first observe that we can assume without loss of generality that $\Phi \geqslant 0$, since $(\Phi(\,\bullet\,, v_j))$ is uniformly integrable if and only if $(|\Phi(\,\bullet\,, v_j)|)$ is and $|\Phi_p^\infty| = |\Phi|_p^\infty$.

Secondly, consider the truncated integrands $\Phi_t := \min\{\Phi, t\}$ for $t \geqslant 0$. Let $\varphi \in \mathrm{C}(\bar\Omega)^+$, so that we can infer $\varphi(\Phi - \Phi_t) \in \mathbb{E}_p$ and $(\varphi(\Phi - \Phi_t))_p^\infty = \varphi\Phi_p^\infty$ and we have

$$\int_\Omega \varphi(x)\langle \nu_x^\infty, \Phi_p^\infty(x,\bullet)\rangle\mathrm{d}\lambda(x) = \lim_{j\to\infty}\int_\Omega \varphi(\Phi - \Phi_t)(\,\bullet\,, v_j)\mathrm{d}x - \int_\Omega \varphi(x)\langle \nu_x, (\Phi - \Phi_t)(x,\bullet)\rangle\mathrm{d}x.$$

We have that $\Phi - \Phi_t \downarrow 0$ pointwisely, so that we can infer from the monotone convergence theorem that the second integral converges to zero as $t \uparrow \infty$. We note that $\Phi - \Phi_t = (\Phi - t)^+$, so

$$\int_\Omega \varphi(x)\langle \nu_x^\infty, \Phi_p^\infty(x,\bullet)\rangle \mathrm{d}\lambda(x) = \lim_{t\to\infty}\lim_{j\to\infty}\int_\Omega \varphi(\Phi(\bullet,v_j)-t)^+\mathrm{d}x, \tag{2.9}$$

Suppose that $(\Phi(\bullet,v_j))$ is uniformly integrable, so we can deduce from (2.9) that

$$0 \leqslant \int_\Omega \varphi(\Phi(\bullet,v_j)-t)^+\mathrm{d}x \leqslant \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\int_{\Phi(\bullet,v_j)>t}(\Phi(\bullet,v_j)-t)\mathrm{d}x \leqslant \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\sup_j \int_{\Phi(\bullet,v_j)>t}\Phi(\bullet,v_j)\mathrm{d}x,$$

which converges to zero by assumption. The proof of this implication is complete.

Conversely, suppose that (2.8) holds and aim to show that $(\Phi(\bullet,v_j))$ is uniformly integrable. Let $\varepsilon > 0$ and abbreviate $f_j := \Phi(\bullet,v_j)$. We infer from (2.9) with $\varphi = \mathbf{1}_{\bar\Omega}$ that there exists t_ε such that for $t \geqslant t_\varepsilon$ we have that

$$\lim_{j\to\infty}\int_\Omega (f_j - t)^+\mathrm{d}x \leqslant \varepsilon.$$

In particular, there exists j_ε such that for $j \geqslant j_\varepsilon$ we have

$$\int_\Omega (f_j - t_\varepsilon)^+\mathrm{d}x \leqslant 2\varepsilon \quad \text{so} \quad \sup_{j\geqslant j_\varepsilon}\int_\Omega (f_j - t_\varepsilon)^+\mathrm{d}x \leqslant 2\varepsilon.$$

By increasing t_ε if necessary, we can therefore prove that

$$\sup_j \int_\Omega (f_j - t)^+\mathrm{d}x \to 0 \quad \text{as } t\to\infty. \tag{2.10}$$

It remains to prove that this is sufficient to have that (f_j) is uniformly integrable. First note that

$$\frac{t}{2}\mathscr{L}^n(f_j > t) \leqslant \int_{f_j > t/2}\left(f_j - \frac{t}{2}\right)\mathrm{d}x,$$

so that

$$\int_{f_j>t} f_j\mathrm{d}x \leqslant t\mathscr{L}^n(f_j>t) + \int_{f_j>t}(f_j - t)\mathrm{d}x \leqslant 2\int_{f_j>t/2}\left(f_j - \frac{t}{2}\right)\mathrm{d}x + \int_{f_j>t}(f_j-t)\mathrm{d}x,$$

which converges to zero by (2.10). This proves that $(\Phi(\bullet,v_j))$ is uniformly integrable.

To prove the last claim in the statement, we fix $\varphi \in \mathrm{C}(\bar\Omega)$ and $F \in \mathrm{C}(S_{\mathbb{V}})$. To capture the L^p-concentration without oscillation effects, we define the integrands

$$F_k(z) := \begin{cases} ((|z|-k)^+)^p\, F\left(\frac{z}{|z|}\right) & z \neq 0 \\ 0 & z = 0 \end{cases},$$

and, further, $\Phi_k(x,(z,z')) := \varphi(x)\left(F_k(z) - F_k(z')\right)$, so that $\Phi_k \in \mathbb{E}_p(\Omega,\mathbb{V}^2)$ and

$$(\Phi_k)_p^\infty(x,(z,z')) = \varphi(x)\left(F_0(z) - F_0(z')\right).$$

Since $(v_j,\tilde v_j)$ is bounded in $\mathrm{L}^p(\Omega,\mathbb{V}^2)$, we have from Corollary 2.8 that it generates a Young measure $\boldsymbol{\eta} \in \mathrm{Y}^p(\Omega,\mathbb{V}^2)$. By uniform integrability of $(v_j - \tilde v_j)$, we have that $(\Phi_k(\bullet,(v_j,\tilde v_j)))_j$ is uniformly integrable, so we can apply the first part of the theorem to get

$$\int_\Omega \Phi_k(x,(v_j(x),\tilde v_j(x)))\mathrm{d}x \to \int_\Omega \langle \eta_x, \Phi_k(x,\bullet)\rangle\mathrm{d}x = \int_\Omega \varphi(x)\langle \eta_x, F_k\otimes \mathbf{1}_{\mathbb{V}} - \mathbf{1}_{\mathbb{V}}\otimes F_k\rangle \mathrm{d}x \to 0,$$

where the first convergence is as $j\to\infty$ and the second, which follows from the dominated convergence theorem, is as $k\to\infty$. We can also write

$$\begin{aligned}\int_\Omega \Phi_k(x,(v_j(x),\tilde v_j(x)))\mathrm{d}x \to &\int_\Omega \varphi(x)\langle \nu_x, F_k\rangle\mathrm{d}x + \int_{\bar\Omega}\varphi(x)\langle \nu_x^\infty, (F_k)_p^\infty\rangle \mathrm{d}\lambda_{\boldsymbol{\nu}}(x) \\ &- \int_\Omega \varphi(x)\langle\tilde\nu_x, F_k\rangle\mathrm{d}x - \int_{\bar\Omega}\varphi(x)\langle\tilde\nu_x^\infty,(F_k)_p^\infty\rangle\mathrm{d}\lambda_{\tilde{\boldsymbol{\nu}}}(x) \quad \text{as } j\to\infty.\end{aligned}$$

Letting $k\to\infty$, we obtain that

$$0 = \int_{\bar\Omega}\varphi(x)\langle \nu_x^\infty, F\rangle\mathrm{d}\lambda_{\boldsymbol{\nu}}(x) - \int_{\bar\Omega}\varphi(x)\langle\tilde\nu_x^\infty, F\rangle\mathrm{d}\lambda_{\tilde{\boldsymbol{\nu}}}(x),$$

which is enough to conclude. □

Before moving on to the next section, we prove a decomposition lemma which states that we can decouple oscillation from concentration:

Lemma 2.20 (The unconstrained decomposition lemma)**.** *Let* $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ *generate* $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ *and write* $v(x) \coloneqq \bar{\nu}_x$ *for* $\mathscr{L}^n$*-a.e.* $x \in \Omega$*. Then there exist* (g_j), $(b_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ *such that*

$$\begin{aligned} &v_j = v + g_j + b_j, \\ &g_j,\, b_j \rightharpoonup 0 \textit{ in } \mathrm{L}^p(\Omega, \mathbb{V}), \\ &(g_j) \textit{ is } p\textit{-uniformly integrable}, \\ &b_j \to 0 \textit{ in } \mathscr{L}^n\textit{-measure.} \end{aligned}$$

Therefore, we have that

$$(v + g_j) \textit{ generates } ((\nu_x)_{x\in\Omega},\, 0,\, \mathtt{n/a}) \quad \textit{and} \quad (b_j) \textit{ generates } ((\delta_0)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar{\Omega}})\,.$$

Sketch. We will use the truncation maps

$$\mathcal{T}_\alpha z \coloneqq \begin{cases} z & |z| \leqslant \alpha \\ k\frac{z}{|z|} & |z| > \alpha \end{cases},$$

which we use to define the integrands $\Phi_\alpha(z) \coloneqq |\mathcal{T}_\alpha z|^p$, which are clearly in $\mathbb{E}_p(\Omega, \mathbb{V})$ with null recession function, so that

$$\lim_{\alpha\to\infty} \lim_{j\to\infty} \int_\Omega |\mathcal{T}_\alpha v_j|^p \mathrm{d}x = \lim_{\alpha\to\infty} \int_\Omega \langle \nu_x, |\mathcal{T}_\alpha|^p \rangle \mathrm{d}x = \int_\Omega \langle \nu_x, |\,\boldsymbol{\cdot}\,|^p \rangle \mathrm{d}x.$$

By a diagonalization argument, we establish that there exists a sequence $\alpha_j \to \infty$ as $j \to \infty$ such that

$$\lim_{j\to\infty} \int_\Omega |\mathcal{T}_{\alpha_j} v_j|^p \mathrm{d}x = \int_\Omega \langle \nu_x, |\,\boldsymbol{\cdot}\,|^p \rangle \mathrm{d}x.$$

We define $g_j \coloneqq \mathcal{T}_{\alpha_j} v_j - v$ and $b_j \coloneqq v_j - \mathcal{T}_{\alpha_j} v_j$ and check the desired properties. □

A complete proof will be given in Section 5, when we investigate the case under differential constraints. Lemma 5.3 reduces to the unconstrained case above when $\mathcal{A} \equiv 0$.

3. Weakly sequential (lower semi-)continuity: the unconstrained case

In this section, we will mainly investigate the claim that convexity of an integrand is equivalent with weak sequential[4] lower semi-continuity of this integrand on L^p. We again freeze the **exponent** $1 < p < \infty$ **and domain** $\Omega \subset \mathbb{R}^n$ **to be bounded and open with** $\mathscr{L}^n(\partial\Omega) = 0$. Along the same lines, we will work with *normal integrands* $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$, by which we mean (jointly) Borel measurable, such that $\mathbb{V} \ni z \mapsto F(x, z)$ is lower semi-continuous for $\mathscr{L}^n$-a.e. $x \in \Omega$. If both $\pm F$ are normal, we will say that F is a *Carathéodory integrand*. We will assume that the integrands satisfy a p-growth condition

$$\text{(G-}p\text{)} \qquad |F(x, z)| \leqslant c(1 + |z|)^p \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega \text{ and all } z \in \mathbb{V}.$$

Of course, these classes of integrands are much larger than the classes $\mathbb{E}_p \subsetneq \mathbb{G}_p \subsetneq \mathrm{C}(\Omega \times \mathbb{V})$. Our primary goal in this section will be to show that Young measure techniques enable us to deal with the measurable x-dependence with ease, a natural complication which is quite difficult to circumvent with other real analysis techniques. We emphasize that, as a corollary of the main results, we will essentially be able to look not only at open domains Ω, but general measurable sets.

We will study the lower semi-continuity of the integrals of F, namely

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}) \implies \liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\mathrm{d}x \geqslant \int_\Omega F(x, v(x))\mathrm{d}x. \tag{3.1}$$

We first sketch the necessity of convexity. From the proof of Theorem 2.7, we have that there exist characteristic functions χ_j such that $\chi_j \overset{*}{\rightharpoonup} \theta \in \mathrm{L}^\infty(\Omega)$, where $\theta \in [0, 1]$. We also let v, $\tilde{v} \in \mathrm{L}^\infty(\Omega, \mathbb{V})$ and note that

$$\chi_j v + (1 - \chi_j)\tilde{v} \overset{*}{\rightharpoonup} \theta v + (1 - \theta)\tilde{v} \quad \text{in } \mathrm{L}^\infty(\Omega, \mathbb{V}). \tag{3.2}$$

Plugging this sequence in (3.1), we obtain

$$\begin{aligned}\int_\Omega F\left(x, \theta v(x) + (1 - \theta)\tilde{v}(x)\right) \mathrm{d}x &\leqslant \liminf_{j\to\infty} \int_\Omega \chi_j(x) F\left(x, v(x)\right) + (1 - \chi_j(x)) F\left(x, \tilde{v}(x)\right) \mathrm{d}x \\ &= \int_\Omega \theta F\left(x, v(x)\right) + (1 - \theta) F\left(x, \tilde{v}(x)\right) \mathrm{d}x.\end{aligned}$$

Since this inequality holds for any $(v, \tilde{v}) \in \mathrm{L}^\infty(\Omega, \mathbb{V}^2)$, we have (for instance, from [9, Prop. 6.24]) that

$$F(x, \theta z + (1 - \theta)z') \leqslant \theta F(x, z) + (1 - \theta)F(x, z') \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega \text{ and all } z, z' \in \mathbb{V}.$$

The main result of this section will be to establish the converse:

Theorem 3.1 (L^p LSC, non-negative integrands)**.** *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be bounded and open, and $F\colon \Omega \times \mathbb{V} \to [0, \infty)$ be a normal integrand satisfying* (G-p). *Suppose that $z \mapsto F(x, z)$ is convex for $\mathscr{L}^n$-a.e. $x \in \Omega$. Then F is weakly sequentially lower semi-continuous on* $\mathrm{L}^p(\Omega, \mathbb{V})$*, i.e.,* (3.1) *holds.*

In other words, under very mild conditions, convexity implies lower semi-continuity. Such a statement or inexpensive modifications appears as part of numerous results in analysis. The proof that we will give uses Young measures and will be very robust and will revolve around the idea of suitably approximating the measurable integrands with (continuous) p-admissible integrands.

We list some of the results that are interesting in their own right and will prove as intermediate steps for the proof of Theorem 3.1. We begin with a lower semi-continuity result on the space of Young measures, for positive normal integrands:

Proposition 3.2 (YM LSC, normal integrands)**.** *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be bounded and open, and $F\colon \Omega \times \mathbb{V} \to [0, \infty)$ be a normal integrand of p-growth. Let $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ generate $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\mathrm{d}x \geqslant \int_\Omega \langle \nu_x, F(x, \,\bullet\,)\rangle \mathrm{d}x.$$

The proof of Theorem 3.1 follows from Propostion 3.2, Jensen's inequality, and Lemma 2.10. To prove Proposition 3.2, we employ the following lower semi-continuity result for signed, jointly lower semi-continuous integrands:

[4]We remark briefly that strong sequential lower semi-continuity of an integrand on L^p is equivalent with lower semi-continuity of the integrand in the second variable. This motivates the definition of the *normal integrands* that we work with in this section. We do not pursue this result here, but refer the reader to [9, Sec. 6.4.1].

Proposition 3.3 (YM LSC, jointly LSC integrands). *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be bounded and open, and $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ be a lower semi-continuous integrand that satisfies $F(x,z) \geqslant -c(1+|z|)^p$ for all $x \in \Omega$, $z \in \mathbb{V}$. Let $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ generate $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\mathrm{d}x \geqslant \int_\Omega \langle \nu_x, F(x,\bullet)\rangle \mathrm{d}x + \int_{\bar\Omega} \langle \nu_x^\infty, H_F(x,\bullet)\rangle \mathrm{d}\lambda(x),$$

where $H_F\colon \Omega \times \mathbb{V} \to \mathbb{R} \cup \{\infty\}$ is the lower recession function *of F, defined by*

$$H_F(x,z) \coloneqq \liminf_{(x',z',t)\to(x,z,\infty)} \frac{F(x',tz')}{t^p} \quad \text{for } x \in \bar\Omega,\ z \in \mathbb{V}.$$

It is clear that in the absence of a growth bound from above, a recession function for F cannot be defined. We will use this proposition and a Luzin type result to prove Proposition 3.2:

Proof of Proposition 3.2. We use Lemma B.2 to find an increasing sequence of compact subsets $C_k \subset \Omega$ such that $\mathscr{L}^n(\Omega \setminus C_k) \leqslant 1/k$ and the restriction of F to $C_k \times \mathbb{V}$ is lower semi-continuous. In particular, the integrand $F_k \coloneqq \mathbf{1}_{C_k} F$ satisfies the assumptions of Proposition 3.3 with $H_{F_k} \geqslant 0$, so that

$$\begin{aligned}\liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\mathrm{d}x &\geqslant \liminf_{j\to\infty} \int_\Omega F_k(x, v_j(x))\mathrm{d}x \geqslant \int_{C_k} \langle \nu_x, F(x,\bullet)\rangle \mathrm{d}x + \int_{\bar\Omega} \langle \nu_x, H_{F_k}(x,\bullet)\rangle \mathrm{d}\lambda(x)\\ &\geqslant \int_{C_k} \langle \nu_x, F(x,\bullet)\rangle \mathrm{d}x \to \int_\Omega \langle \nu_x, F(x,\bullet)\rangle \mathrm{d}x \quad \text{as } k\to\infty,\end{aligned}$$

where the last convergence follows by the monotone convergence theorem. The proof is complete. □

It remains to prove Proposition 3.3, which follows from the monotone convergence theorem and the following precise approximation result:

Theorem 3.4 (Approximation of LSC integrands). *Let $1 \leqslant p < \infty$ and $\Omega \subset \mathbb{R}^n$ be bounded and* ***measurable****, and $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ be a lower semi-continuous integrand that satisfies $F(x,z) \geqslant -c(1+|z|)^p$ for all $x \in \Omega$, $z \in \mathbb{V}$. Then there exists a sequence $(\Phi_k) \subset \mathbb{E}_p(\mathbb{R}^n, \mathbb{V})$ such that*

$$\Phi_k(x,z) \uparrow F(x,z) \text{ for } (x,z) \in \Omega \times \mathbb{V} \quad \text{and} \quad (\Phi_k)_p^\infty(x,z) \uparrow H_F(x,z) \text{ for } (x,z) \in \bar\Omega \times \mathbb{V}.$$

Proof. We begin by defining

$$TF(x,\tilde z) \coloneqq (1-|\tilde z|)^p F\left(x, \frac{\tilde z}{1-|\tilde z|}\right) \quad \text{for } x \in \Omega,\ \tilde z \in B_{\mathbb{V}}.$$

This is formally the same map as the transformation T_p defined in Section 2. We suppress the subscript both for simplicity and to keep the definition of T_p consistent throughout the document. Then TF is lower semi-continuous (as a composition of lower semi-continous and continuous functions) and $TF \geqslant -c$ in $\Omega \times B_{\mathbb{V}}$. We can then define

$$G(x,\tilde z) \coloneqq \begin{cases} \liminf_{\Omega \ni x' \to x,\, B_{\mathbb{V}} \ni \tilde z' \to \tilde z} TF(x',\tilde z') & (x,\tilde z) \in \overline{\Omega \times B_{\mathbb{V}}} \\ \infty & (x',\tilde z') \in (\mathbb{R}^n \setminus \bar\Omega) \times \bar B_{\mathbb{V}} \end{cases},$$

so that G is the lower semi-continuous envelope of

$$\tilde G(x,\tilde z) \coloneqq \begin{cases} TF(x,\tilde z) & (x,\tilde z) \in \Omega \times B_{\mathbb{V}} \\ \infty & (x,\tilde z) \in (\mathbb{R}^n \times \bar B_{\mathbb{V}}) \setminus (\Omega \times B_{\mathbb{V}}) \end{cases},$$

meaning that G is the largest lower semi-continuous function that is smaller than $\tilde G$.

It is easy to see that $G = TF$ on $\Omega \times B_{\mathbb{V}}$, so that $G \geqslant -c$ on $\overline{\Omega \times B_{\mathbb{V}}}$. We claim that $G = H_F$ on $\bar\Omega \times S_{\mathbb{V}}$. We let $z \in S_{\mathbb{V}}$ and $x \in \bar\Omega$ and argue sequentially, considering $\Omega \ni x_j \to x$ and $B_{\mathbb{V}} \ni z_j \to z$. First, let $t_j \to \infty$ and look at

$$\liminf_{j\to\infty} \frac{F(x_j, t_j z_j)}{t_j^p} = \liminf_{j\to\infty} \left(\frac{1}{t_j} + |z_j|\right)^p TF\left(x_j, \frac{t_j z_j}{1 + t_j |z_j|}\right) \geqslant G(x,z),$$

so that $H_F \geqslant G$. Conversely, choosing x_j, z_j such that in addition $TF(x_j, z_j) \to G(x,z)$, we choose $t_j = 1/(1-|z_j|) \to \infty$ to get that

$$\liminf_{j\to\infty} \frac{F(x_j, t_j z_j)}{t_j^p} = \liminf_{j\to\infty} TF(x_j, z_j) = G(x,z),$$

so that $H_F \leqslant G$. We conclude that $G = H_F$ on $\bar\Omega \times S_{\mathbb{V}}$.

It remains to find a suitable approximation from below of G by continuous functions. Take

$$G_k(x,\tilde z) \coloneqq \sup\{G(x',\tilde z') - k(|x-x'| + |\tilde z - \tilde z'|) \colon x' \in \Omega, \tilde z' \in B_{\mathbb{V}}\} \quad \text{for } x \in \mathbb{R}^n,\ \tilde z \in \bar B_{\mathbb{V}}.$$

One then checks that G_k is k-Lipschitz and $G_k \uparrow G$ as $k \to \infty$ on $\overline{\Omega \times B_{\mathbb{V}}}$ and defines

$$\Phi_k(x, z) := (1 + |z|)^p G_k\left(x, \frac{z}{1+|z|}\right) = T_p^{-1} G_k(x, z) \quad \text{for } x \in \mathbb{R}^n,\ z \in \mathbb{V},$$

which satisfies $\Phi_k \in \mathbb{E}_p(\mathbb{R}^n, \mathbb{V})$, $\Phi_k \uparrow F$ on $\Omega \times \mathbb{V}$, and $\Phi_k^\infty \uparrow H_F$ on $\bar{\Omega} \times S_{\mathbb{V}}$. □

Roughly speaking, the main difficulty in the proof of Theorem 3.1 was to deal with the concentration effect, which was achieved by requiring the integrand to be bounded from below. The other way to ensure is to rule out this effect by requiring a suitable uniform integrability property. We will view this as a continuity result for Young measures:

Theorem 3.5 (Oscillation YM continuity). *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be bounded and open, $A \subset \Omega$ be measurable, and $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ be a Carathéodory integrand. Let $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ generate a Young measures $\boldsymbol{\nu}$ and be such that $(F(\,\cdot\,, v_j))$ is uniformly integrable on A. Then*

$$\int_A F(x, v_j(x))\mathrm{d}x \to \int_A \langle \nu_x, F(x, \cdot\,)\rangle \mathrm{d}x < \infty. \tag{3.3}$$

Conversely, suppose that $F \geqslant 0$. Then (3.3) *implies that $(F(\,\cdot\,, v_j))$ is uniformly integrable on A.*

One can then conclude from Theorem 3.5 that we have lower semi-continuity on L^p for convex integrands by using Jensen's inequality as in the proof of Theorem 3.1:

Corollary 3.6 (L^p LSC, no concentration). *Let F, Ω, p be as in the statement of Theorem 3.5. Assume in addition that F is convex in the second variable, as in Theorem 3.1. If $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ converges weakly to v and is such that $(F(\,\cdot\,, v_j)^-)$ is uniformly integrable, then*

$$\liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\mathrm{d}x \geqslant \int_\Omega F(x, v(x))\mathrm{d}x.$$

Proof of Theorem 3.5. We first assume that $(F(\,\cdot\,, v_j))$ is uniformly integrable on A. There is no loss in generality in assuming that $A = \Omega$, otherwise replace F by $\mathbf{1}_A F$ and that $F \geqslant 0$, otherwise treat $F^\pm$ separately.

Let $\varepsilon > 0$ so that for $k \geqslant k_\varepsilon$ we have that

$$\sup_j \int_{F(\,\cdot\,, v_j) > k} F(\,\cdot\,, v_j)\mathrm{d}x < \varepsilon.$$

For each k, we also truncate the integrand at level k by setting $F_k := \min\{F, k\}$, so that F_k is clearly Carathéodory, $0 \leqslant F_k \leqslant k$, and

$$\left| \int_\Omega F(\,\cdot\,, v_j) - F_k(\,\cdot\,, v_j)\mathrm{d}x \right| < \varepsilon \quad \text{for } k \geqslant k_\varepsilon.$$

We then apply Theorems B.3 and B.4 successively to obtain a compact set $C_\varepsilon \subset \Omega$ and continuous integrands $G_k \in \mathrm{C}(\mathbb{R}^n \times \mathbb{V})$ such that $\mathscr{L}^n(\Omega \setminus C_\varepsilon) < \varepsilon/k$, $G_k = F_k$ on $C_\varepsilon \times \mathbb{V}$, and $0 \leqslant G_k \leqslant k$. It follows that $G_k \in \mathbb{E}_p(\Omega, \mathbb{V})$ and $(G_k)_p^\infty \equiv 0$, so that

$$\lim_{j\to\infty} \int_\Omega G_k(x, v_j(x))\mathrm{d}x = \int_\Omega \langle \nu_x, G_k(x, \cdot\,)\rangle \mathrm{d}x. \tag{3.4}$$

We look at the right hand side and aim to let $k \to \infty$. We have that

$$\left| \int_\Omega \langle \nu_x, G_k(x, \cdot\,)\rangle - \langle \nu_x, F_k(x, \cdot\,)\rangle \mathrm{d}x \right| \leqslant \int_{\Omega \setminus C_\varepsilon} \langle \nu_x, |G_k - F_k|(x, \cdot\,)\rangle \mathrm{d}x < \varepsilon.$$

By the monotone convergence theorem, we have that

$$\lim_{k\to\infty} \int_\Omega \langle \nu_x, F_k(x, \cdot\,)\rangle \mathrm{d}x = \int_\Omega \langle \nu_x, F(x, \cdot\,)\rangle \mathrm{d}x.$$

Concerning the left hand side of (3.4), we have that for $k \geqslant k_\varepsilon$

$$\begin{aligned}\left| \int_\Omega G_k(x, v_j(x)) - F(x, v_j(x))\mathrm{d}x \right| &\leqslant \int_{\Omega \setminus C_\varepsilon} |G_k(x, v_j(x)) - F_k(x, v_j(x))|\mathrm{d}x \\ &\quad + \int_{F(\,\cdot\,, v_j) > k} (F(x, v_j(x)) - k)\mathrm{d}x < 2\varepsilon.\end{aligned}$$

Collecting the facts above, we obtain the claimed convergence.

Conversely, suppose that (3.3) holds and let $E \subset A$ be a Borel set. By replacing F with $\mathbf{1}_E F$ and $\mathbf{1}_{A\setminus E} F$ viewed as integrands on Ω, we can apply Theorem 3.2 to infer that

$$\liminf_{j\to\infty} \int_E F(x, v_j(x))\mathrm{d}x \geqslant \int_E \langle \nu_x, F(x, \bullet)\rangle \mathrm{d}x, \quad \liminf_{j\to\infty} \int_{A\setminus E} F(x, v_j(x))\mathrm{d}x \geqslant \int_{A\setminus E} \langle \nu_x, F(x, \bullet)\rangle \mathrm{d}x.$$

Using (3.3), we have that

$$\begin{aligned}\int_{A\setminus E} \langle \nu_x, F(x, \bullet)\rangle \mathrm{d}x + \int_E \langle \nu_x, F(x, \bullet)\rangle \mathrm{d}x &\leqslant \liminf_{j\to\infty} \int_{A\setminus E} F(x, v_j(x))\mathrm{d}x + \liminf_{j\to\infty} \int_E F(x, v_j(x))\mathrm{d}x \\ &\leqslant \liminf_{j\to\infty} \int_A F(x, v_j(x))\mathrm{d}x = \int_A \langle \nu_x, F(x, \bullet)\rangle \mathrm{d}x,\end{aligned}$$

which implies that, on a subsequence,

$$\lim_{j\to\infty} \int_E F(x, v_j(x))\mathrm{d}x = \int_E \langle \nu_x, F(x, \bullet)\rangle \mathrm{d}x.$$

Since the limit is independent of the subsequence, the convergence holds along the full sequence. Since the right hand side is finite by assumption, we can conclude by Theorem 1.6. ☐

Finally, we make the simple observation already present in the proof above that most of the results of this section are valid also for variational integrals defined only on measurable sets:

Corollary 3.7. *Let $A \subset \mathbb{R}^n$ be bounded and* ***measurable*** *and $F\colon A \times \mathbb{V} \to \mathbb{R}$ be an integrand that is normal/Carathéodory/convex in the second variable integrand. Also consider an immaterial bounded open set $\Omega \supset A$. Then $\mathbf{1}_A F\colon \Omega\times\mathbb{V} \to \mathbb{R}$ is itself a normal/Carathéodory/convex in the second variable integrand respectively, so that Theorem 3.1, Proposition 3.2, Proposition 3.3, and Corollary 3.6 hold for variational integrals*

$$v \mapsto \int_A F(x, v(x))\mathrm{d}x.$$

We aim to give similar lower semi-continuity results when the weakly convergent sequence satisfies linear partial differential constraints, as we indicated in the rough framework (CC). We will see in Section 5 that the class of integrands for which the variational integrals are lower semi-continuous in this case is much richer than the class of convex integrands. Before we dive into this, we will examine the case of weak sequential continuity, where we will see that the structure is much more rigid and that most integrands with good continuity properties are explicitly computable.

4. Weak sequential continuity: constant rank constraints and quadratic case

We will make an interlude to discuss elementary proofs of weak continuity claims under differential constraints, as introduced in the set up (CC). Specifically, we will look at which nonlinear quantities are sequentially continuous when applied to a weakly convergent sequence in L^p which can be decomposed into a strongly convergent, PDE constrained part, and a part which might exhibit oscillation and concentration effects. Our main question is to identify the (polynomial) autonomous integrands $F\colon \mathbb{V} \to \mathbb{R}$ such that

$$\text{(WC)}\qquad \left.\begin{array}{ll} v_j \rightharpoonup v & \text{in } \mathrm{L}^p(\Omega,\mathbb{V}) \\ \mathscr{A} v_j \to \mathscr{A} v & \text{in } \mathrm{W}^{-\ell,p}(\Omega,\mathbb{W}) \end{array}\right\} \implies F(v_j) \overset{*}{\rightharpoonup} F(v) \text{ in } \mathscr{D}'(\Omega).$$

Here $\mathscr{A}$ is as in (1.3), $\Omega \subset \mathbb{R}^n$ is bounded and open, and $1 < p < \infty$ will be correlated with the growth of the integrand later. We will see that for p large enough, the class of integrands satisfying (WC) is independent of p. We also recall the definition of $\mathrm{W}^{-\ell,p}(\Omega,\mathbb{W})$, as the linear dual of $\mathrm{W}_0^{\ell,p/(p-1)}(\Omega,\mathbb{W})$, the closure of the space $\mathscr{D}(\Omega,\mathbb{W}) = \mathrm{C}_c^\infty(\Omega,\mathbb{W})$ in the respective Sobolev norm. See also Section D.

First, we examine necessary conditions for *lower semi*-continuity, cf. (3.1). Assume for simplicity that F is continuous. Then, we would like to test for convexity with functions similar to the oscillatory sequence defined in (3.2). Since the integrand is autonomous, we can simply test with plane waves of the form

$$v_j(x) := f(jx\cdot\xi), \quad \text{where } f \text{ is the 1-periodic extension of } \chi_{(0,\theta)}z_1 + \chi_{(\theta,1)}z_2$$

for some z_1, $z_2 \in \mathbb{V}$ and $\theta \in (0,1)$. We then know from the Riemann–Lebesgue lemma, Exercise 2.11, that

$$v_j \overset{*}{\rightharpoonup} \theta z_1 + (1-\theta) z_2 \text{ in } \mathrm{L}^p(\Omega,\mathbb{V}) \quad \text{and} \quad \int_\Omega F(v_j)\mathrm{d}x \to \mathscr{L}^n(\Omega)\left[\theta F(z_1) + (1-\theta)F(z_2)\right]$$

We require in addition that $\mathscr{A} v_j = 0$, or equivalently, using (1.4), $\mathscr{A}(\xi)(z_1 - z_2) = 0$. Therefore, assuming lower semi-continuity, we obtain that

$$\theta F(z_1) + (1-\theta)F(z_2) \geqslant F(\theta z_1 + (1-\theta) z_2) \quad \text{whenever } z_1 - z_2 \in \Lambda_{\mathscr{A}}. \tag{4.1}$$

In other words, although lower semi-continuity along *all* weakly convergent sequences implies convexity of the integrand, lower semi-continuity along PDE constrained sequences implies convexity *only* in the directions of the wave cone; in this case, we say that F is *$\Lambda_{\mathscr{A}}$-convex.* It easily follows from Theorem 4.5 that this result is sharp, in particular, one cannot hope for convexity unless $\Lambda_{\mathscr{A}} = \mathbb{V}$, so $\mathscr{A} \equiv 0$, which is the unconstrained case of Section 3.

However, unlike in Section 3, it is **not** the case that the $\Lambda_{\mathscr{A}}$-convexity condition (4.1) is sufficient for lower semi-continuity along PDE constrained sequences (in general). This has been one of the outstanding problems in the calculus of variations for many years. On the other hand, finding explicit and relevant (classes of) examples of directionally convex functionals satisfying lower semi-continuity properties is a widely unexplored topic in the field. Both these avenues lead beyond the scope of this course. We will however formulate a necessary and sufficient condition, which is unwieldy for the study of weak lower semi-continuity but can be made explicit in the study of weak continuity:

Proposition 4.1. *Let Ω be bounded, open, and $F \in \mathrm{C}(\mathbb{V})$, satisfy*

$$v_j \overset{*}{\rightharpoonup} v \text{ in } \mathrm{L}^\infty(\Omega,\mathbb{V}),\ \mathscr{A} v_j = 0 \text{ in } \mathscr{D}'(\Omega,\mathbb{W}) \implies \liminf_{j\to\infty} \int_\Omega F(v_j)\mathrm{d}x \geqslant \int_\Omega F(v)\mathrm{d}x.$$

*Then F is $\mathscr{A}$-*quasiconvex*, i.e.,*

$$\int_Q F(z + v(x))\mathrm{d}x \geqslant F(z) \quad \text{for } z \in \mathbb{V},\ v \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}),\ \mathscr{A} v = 0,\ \int_Q v\mathrm{d}x = 0, \tag{4.2}$$

where $Q = (0,1)^n$ can be identified with the torus $\mathbb{T}^n$.

To prove this, one takes z, v as in (4.2) and defines $v_j(x) := z + v(jx)$ which converges weakly-* to z, so that the Riemann–Lebesgue lemma yields the conclusion. It is then very easy to infer from this result that the weak continuity condition in (WC) implies that $\pm F$ is $\mathscr{A}$-quasiconvex. This motivates the following:

Definition 4.2. *We say that an integrand $F\colon \mathbb{V} \to \mathbb{R}$ is $\mathscr{A}$-*quasiaffine *if and only if*

$$\int_Q F(z + v(x))\mathrm{d}x = F(z) \quad \text{for } z \in \mathbb{V},\ v \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}),\ \mathscr{A} v = 0,\ \int_Q v\mathrm{d}x = 0.$$

The main aim of this section is to prove that for very broad classes of operators we have that quasiaffinity implies (WC). We henceforth assume that $\mathscr{A}$ **satisfies the spanning cone condition** (SC).

Lemma 4.3. *Let F be $\mathscr{A}$-quasiaffine. Then F' is $\mathscr{A}$-quasiaffine, F is a polynomial, and each homogeneous component of F is $\mathscr{A}$-quasiaffine.*

In what follows, we will work only with s-homogeneous quasiaffine polynomials; we will denote the integrands by $F = P$ in this case. To prove the lemma, we will be using the following:

Exercise 4.4. Let F be $\mathscr{A}$-quasiconvex. Then F is $\Lambda_{\mathscr{A}}$-convex, i.e., (4.1) holds.

Proof of Lemma 4.3. By Exercise 4.4, we have that F is $\Lambda_{\mathscr{A}}$-affine. Since $\Lambda_{\mathscr{A}}$ spans $\mathbb{V}$, we can express the coordinates of $z = (z_1, \ldots, z_N) \in \mathbb{V}$ in a basis contained in $\Lambda_{\mathscr{A}}$. Since F is affine in the z_N direction, we can write $F(z_1, \ldots, z_N) = G(z_1, \ldots, z_{N-1})z_N + H(z_1, \ldots, z_{N-1})$. Since the restriction of F to $\{z_N = 0\}$ is also affine in the $z_1, \ldots, z_{N-1}$ directions, we conclude that H is affine in all these directions. So is G, since F restricted to $\{z_N = 1\}$ is also affine in the $z_1, \ldots, z_{N-1}$ directions. We can thus argue by induction.

Since F is locally Lipschitz, the fact that F' is $\mathscr{A}$-quasiaffine follows from an elementary application of the dominated convergence theorem.

Now let $t \in \mathbb{R}$, $z \in \mathbb{V}$, and $v \in \mathrm{C}^\infty(\mathbb{T}^n, \mathbb{V})$ with $\int v = 0$ and $\mathscr{A}v = 0$. We also write $F = \sum_{h=0}^s P_h$, where P_h is a h-homogeneous polynomial. Then we have that

$$F(tz) = \int_{\mathbb{T}^n} F(tz + tv(x))\mathrm{d}x, \quad \text{so} \quad \sum_{h=0}^s t^h P_h(z) = \sum_{h=0}^s t^h \int_{\mathbb{T}^n} P_h(z + v(x))\mathrm{d}x,$$

which is a polynomial in $t \in \mathbb{R}$. Identifying coefficients implies that each P_h is $\mathscr{A}$-quasiaffine. □

We have seen that by Proposition 4.1 and Lemma 4.3 we have that weak continuity (WC) implies that the integrand is an $\mathscr{A}$-quasiaffine polynomial which can be assumed homogeneous. We will now show that the converse is also true in great generality. We begin with the case of quadratic forms:

Theorem 4.5 (Weak continuity, quadratic theorem)**.** *Let P be an $\mathscr{A}$-quasiaffine quadratic form. Then* (WC) *holds with $F = P$ and $p = 2$.*

This result follows from a slightly more general lower semi-continuity statement:

Theorem 4.6 (LSC for quadratic forms)**.** *Let P be a quadratic form. The following are equivalent:*

(a) *P is $\mathscr{A}$-quasiconvex.*
(b) *P is $\Lambda_{\mathscr{A}}$-convex.*
(c) *$P \geqslant 0$ on $\Lambda_{\mathscr{A}}$.*
(d) *For any $\rho \in \mathrm{C}_c(\Omega)^+$ we have that*

$$\left.\begin{array}{lll} v_j \rightharpoonup v & \text{in} & \mathrm{L}^2_{\mathrm{loc}}(\Omega, \mathbb{V}) \\ \mathscr{A}v_j \to \mathscr{A}v & \text{in} & \mathrm{W}^{-\ell,2}_{\mathrm{loc}}(\Omega, \mathbb{W}) \end{array}\right\} \implies \liminf_{j\to\infty} \int_\Omega \rho(x)P(v_j(x))\mathrm{d}x \geqslant \int_\Omega \rho(x)P(v(x))\mathrm{d}x.$$

An immediate consequence of this theorem is that convexity *strictly* implies quasiconvexity in general.

Proof. Some implications we already know, for instance (d) $\implies$ (a) follows from Proposition 4.1 and (a) $\implies$ (b) follows from Exercise 4.4. To see that (b) $\iff$ (c), we let $z \in \Lambda_{\mathscr{A}}$ and $\tilde{z} \in \mathbb{V}$ and note that convexity in the z-direction is equivalent with $0 \leqslant P''(\tilde{z})[z, z] = P(z)$. It remains to prove that (c) $\implies$ (d), which we do in several steps. Assume that $P(z) \geqslant 0$ for all $z \in \Lambda_{\mathbb{V}}$.

Step I. We show that we can assume that $\rho = \mathbf{1}_\Omega$ and $v_j \in \mathrm{C}_c^\infty(\Omega, \mathbb{V})$.

It is clear that since mollifications of continuous functions converge locally uniformly, we can assume $\rho \in \mathrm{C}_c^\infty(\Omega)^+$. Write $\omega := \operatorname{spt} \rho \Subset \Omega$ and $\psi := \sqrt{\rho} \in \mathrm{C}_c^\infty(\Omega)^+$. We aim to replace v_j with ψv_j. To this end, we note that $\psi v_j \to \psi v$ in $\mathrm{L}^2(\omega, \mathbb{V})$ and $\rho P(v_j) = P(\psi v_j)$. We then compute

$$\mathscr{A}(\psi v_j) = \psi \mathscr{A} v_j + \sum_{|\alpha|=\ell} A_\alpha \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\psi \partial^\beta v_j \to \psi \mathscr{A} v + \sum_{|\alpha|=\ell} A_\alpha \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\psi \partial^\beta v = \mathscr{A}(\psi v),$$

where the convergence is in $\mathrm{W}^{-\ell,2}(\mathbb{R}^n, \mathbb{W})$. The reason for this is that, by the compact Sobolev embedding, we have that $v_j \rightharpoonup v$ in $\mathrm{L}^2_{\mathrm{loc}}$ implies $\partial^\beta v_j \to \partial^\beta v$ in $\mathrm{W}^{-\ell,2}_{\mathrm{loc}}$ if $|\beta| < \ell$.

Step II. We show that we can also assume that $v = 0$.

Write $\tilde{v}_j := v_j - v$, so that $\tilde{v}_j \rightharpoonup 0$ in $\mathrm{L}^2(\mathbb{R}^n, \mathbb{V})$, $\mathscr{A}\tilde{v}_j \to 0$ in $\mathrm{W}^{-\ell,2}(\mathbb{R}^n, \mathbb{W})$, and $\operatorname{spt} \tilde{v}_j \subset \Omega$. Also write $P(z) = \langle z, Mz \rangle$ for $z \in \mathbb{V}$, where $M \in \mathrm{SLin}(\mathbb{V}, \mathbb{V})$ is a symmetric linear map on $\mathbb{V}$. We compute

$$\int_\Omega P(v_j)\mathrm{d}x = \int_\Omega P(\tilde{v}_j)\mathrm{d}x + 2\int_\Omega \langle \tilde{v}_j, Mv \rangle \mathrm{d}x + \int_\Omega P(v)\mathrm{d}x,$$

where the middle term tends to zero as $j \to \infty$. It follows that

$$\liminf_{j\to\infty} \int_\Omega P(v_j)\mathrm{d}x \geqslant \int_\Omega P(v)\mathrm{d}x \iff \liminf_{j\to\infty} \int_\Omega P(\tilde{v}_j)\mathrm{d}x \geqslant 0.$$

Step III. We claim that for each $\varepsilon > 0$ there exists $C_\varepsilon < \infty$ such that

$$P(z) \geqslant -\left(\varepsilon|z|^2 + C_\varepsilon \frac{|\mathscr{A}(\xi)z|^2}{|\xi|^{2\ell}}\right) \quad \text{for } z \in \mathbb{V},\ \xi \in \mathbb{R}^n \setminus \{0\}.$$

To prove this, note that by homogeneity it suffices to prove it for $|z| = 1 = |\xi|$. Let $\varepsilon > 0$. First note that, for instance since P is Lipschitz in $S_{\mathbb{V}}$, there exists $\delta > 0$ such that $|z - z'| < \delta$ implies $|P(z) - P(z')| < \varepsilon$. Therefore, if $\operatorname{dist}(z, \Lambda_{\mathscr{A}}) < \delta$, we have that there exists $z' \in \Lambda_{\mathscr{A}}$ such that $|z - z'| < \delta$, so

$$P(z) > P(z') - \varepsilon \geqslant -\varepsilon. \tag{4.3}$$

On the other hand, the map

$$S^{n-1} \times (S_{\mathbb{V}} \cap \{\operatorname{dist}(\bullet, \Lambda_{\mathscr{A}}) \geqslant \delta\}) \ni (\xi, z) \mapsto \mathscr{A}(\xi)z \in \mathbb{W}$$

is continuous and non-zero on a compact set. It follows that

$$\triangle_\varepsilon := \inf\{|\mathscr{A}(\xi)z| \colon \xi \in S^{n-1},\ z \in S_{\mathbb{V}},\ \operatorname{dist}(z, \Lambda_{\mathscr{A}}) \geqslant \delta\} > 0.$$

Writing $\|M\|$ for the operator norm of M, we have that if $\operatorname{dist}(z, \Lambda_{\mathscr{A}}) \geqslant \delta$, then

$$P(z) \geqslant -\|M\| \geqslant -\frac{\|M\|}{\triangle_\varepsilon^2}|\mathscr{A}(\xi)z|^2 =: -C_\varepsilon|\mathscr{A}(\xi)z|^2. \tag{4.4}$$

Putting (4.3) and (4.4) together, we obtain

$$P(z) \geqslant -\left(\varepsilon + C_\varepsilon|\mathscr{A}(\xi)z|^2\right),$$

which completes the proof of this step.

Step IV. Conclusion. We briefly recall the set up we reduced to: we have $v_j \in \mathrm{C}_c^\infty(\Omega, \mathbb{V})$ converging weakly in L^2 to zero and such that $\mathscr{A}v_j$ converges strongly in $\mathrm{W}^{-\ell,2}$ to zero. We can assume that $\rho = 1$.

By Plancherel's theorem, we have that

$$\begin{aligned}\int_\Omega P(v_j)\mathrm{d}x &= \int_{\mathbb{R}^n} \langle \hat{v}_j, M\hat{v}_j\rangle \mathrm{d}\xi = \int_{\mathbb{R}^n} \langle \Re\hat{v}_j, M\Re\hat{v}_j\rangle + \langle \Im\hat{v}_j, M\Im\hat{v}_j\rangle \mathrm{d}\xi - 2\,\mathrm{i}\int_{\mathbb{R}^n} \langle \hat{\Re} v_j, M\hat{\Im}v_j\rangle \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} P(\Re\hat{v}_j) + P(\Im\hat{v}_j)\mathrm{d}\xi \geqslant -\varepsilon\int_{\mathbb{R}^n} |\Re\hat{v}_j|^2 + |\Im\hat{v}_j|^2 \mathrm{d}\xi - C_\varepsilon \int_{\mathbb{R}^n} \frac{|\mathscr{A}(\xi)\Re\hat{v}_j|^2 + |\mathscr{A}(\xi)\Im\hat{v}_j|^2}{|\xi|^{2\ell}}\mathrm{d}\xi \\ &= -\varepsilon\int_{\mathbb{R}^n} |\hat{v}_j|^2\mathrm{d}\xi + C_\varepsilon\int_{\mathbb{R}^n} \frac{|\mathscr{A}(\xi)\hat{v}_j|^2}{|\xi|^{2\ell}}\mathrm{d}\xi = -\varepsilon\|v_j\|^2_{\mathrm{L}^2(\mathbb{R}^n)} - C_\varepsilon\|\mathscr{A}v_j\|^2_{\mathrm{W}^{-\ell,2}(\mathbb{R}^n)}.\end{aligned}$$

The first term can be made arbitrarily small as (v_j) is bounded in L^2, whereas the second term can be made arbitrarily small for j sufficiently large. Therefore

$$\liminf_{j\to\infty} \int_\Omega P(v_j)\mathrm{d}x \geqslant 0,$$

which completes the proof. □

We will next focus on constant rank operators, where we will prove that the conclusion of Theorem 4.5 also holds[5]. To this end, we recall the definition of the constant rank condition, namely that there exists $r \in \mathbb{N}_0$ such that

$$\operatorname{rank}\mathscr{A}(\xi) = r \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{CR}$$

Recall from Lemma 1.20 that in this case there exists a *potential operator* $\mathscr{B}$ (which is also a homogeneous partial differential operator of constant rank) such that the exact relation

$$\ker\mathscr{A}(\xi) = \operatorname{im}\mathscr{B}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{4.5}$$

It will be very convenient to express $\mathscr{B}$ in jet notation

$$\mathscr{B}u = \mathbf{T}(D^k u) \quad \text{for } u \colon \mathbb{R}^n \to \mathbb{U}, \tag{4.6}$$

where k is the order of $\mathscr{B}$ and $\mathbf{T} \in \operatorname{Lin}(\operatorname{SLin}^k(\mathbb{R}^n, \mathbb{U}), \mathbb{V})$ is basically the tensor of coefficients of $\mathscr{B}$.

We begin by showing that the $\mathscr{A}$-quasiconvexity/quasiaffinity notions reduce to the case of gradients.

Lemma 4.7. *Suppose that $\mathscr{A}$ satisfies the spanning cone* (SC) *and the constant rank* (CR) *conditions. Let $\mathscr{B}$ satisfy the exact relation* (4.5) *and be expressed in jet notation as in* (4.6)*. Let $F \colon \mathbb{V} \to \mathbb{R}$.*

Then F is $\mathscr{A}$-quasiconvex if and only if $F \circ \mathbf{T}$ is k-quasiconvex, see Definition 4.8 below.

[5] By this we mean that (WC) holds for s-homogeneous $\mathscr{A}$-quasiaffine polynomials $F = P$ and $p = s$.

Definition 4.8 (k-quasiconvexity/k-quasiaffinity). *An integrand* $f\colon \mathrm{SLin}^k(\mathbb{R}^n,\mathbb{U})\to\mathbb{R}$ *is said to be* k-quasiconvex *if and only if*

$$f(A)\leqslant\int_{(0,1)^n} f(A+D^k u(x))\mathrm{d}x \quad \text{for } A\in\mathrm{SLin}^k(\mathbb{R}^n,\mathbb{U}),\ u\in\mathrm{C}^\infty(\mathbb{T}^n,\mathbb{U}).$$

We say that f *is* k-quasiaffine *if and only if* $\pm f$ *is* k*-quasiconvex.*

Proof of Lemma 4.7. First, note that by Lemma 1.24, we have that F is $\mathscr{A}$-quasiconvex if and only if

$$F(z)\leqslant\int_{(0,1)^n}F(z+\mathscr{B}u(x))\mathrm{d}x \quad \text{for } z\in\mathbb{V},\ u\in\mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}).$$

This implies that for $A\in\mathrm{SLin}^k(\mathbb{R}^n,\mathbb{U})$, $u\in\mathrm{C}^\infty(\mathbb{T}^n,\mathbb{U})$

$$F\circ\mathbf{T}(A)=F(\mathbf{T}A)\leqslant\int_{(0,1)^n}F(\mathbf{T}A+\mathbf{T}D^ku(x))\mathrm{d}x=\int_{(0,1)^n}F\circ\mathbf{T}(A+D^ku(x))\mathrm{d}x,$$

so that $F\circ\mathbf{T}$ is indeed k-quasiaffine.

Conversely, suppose $F\circ\mathbf{T}$ is k-quasiconvex and note that

$$\Lambda_{\mathscr{A}}=\bigcup_{\xi\in S^{n-1}}\operatorname{im}\mathscr{B}(\xi)=\mathbf{T}\{a\otimes b^{\otimes k}\colon a\in\mathbb{U},\ b\in\mathbb{R}^n\},$$

so that $\mathbb{V}=\operatorname{span}\Lambda_{\mathscr{A}}=\mathbf{T}\operatorname{span}\{a\otimes b^{\otimes k}\colon a\in\mathbb{U},\ b\in\mathbb{R}^n\}=\mathbf{T}\mathrm{SLin}^k(\mathbb{R}^n,\mathbb{U})$, so that for each $z\in\mathbb{V}$ there exists $A\in\mathrm{SLin}^k(\mathbb{R}^n,\mathbb{U})$ such that $z=\mathbf{T}A$. It follows that for $u\in\mathrm{C}^\infty(\mathbb{T}^n,\mathbb{U})$

$$F(z)=F\circ\mathbf{T}(A)\leqslant\int_{(0,1)^n}F\circ\mathbf{T}(A+D^ku(x))\mathrm{d}x=\int_{(0,1)^n}F(z+\mathscr{B}u(x))\mathrm{d}x,$$

which completes the proof. □

The relevance of this observation in Lemma 4.7 comes from the fact that the class of k-quasiaffine integrands is known. We cite this powerful algebraic result without proof:

Theorem 4.9. *Let* $f\colon\mathrm{SLin}^k(\mathbb{R}^n,\mathbb{U})\to\mathbb{R}$ *be a* k*-quasiaffine integrand. Then, for* $u\colon\mathbb{R}^n\to\mathbb{U}$, $f(D^ku)$ *is a linear combination of minors of* DU*, where* $U=D^{k-1}u$.

This result, together with Lemma 4.7, will enable us to reduce the problem of weak continuity for constant rank constrained sequences to the case of Jacobian subdeterminants. In particular, this reduces the computation of $\mathscr{A}$-quasiaffine functions to a linear system.

Theorem 4.10 (Weak continuity, constant rank operators). *Suppose that* $\mathscr{A}$ *satisfies the* (SC) *and* (CR) *conditions. Let* $P\colon\mathbb{V}\to\mathbb{R}$ *be an* s*-homogeneous polynomial, for some* $s\geqslant 2$*. Then*

$$\left.\begin{array}{ll} v_j\rightharpoonup v & \text{in } \mathrm{L}^s_{\mathrm{loc}}(\Omega,\mathbb{V})\\ \mathscr{A}v_j\to\mathscr{A}v & \text{in } \mathrm{W}^{-\ell,s}_{\mathrm{loc}}(\Omega,\mathbb{W})\end{array}\right\}\implies P(v_j)\overset{*}{\rightharpoonup}P(v)\ \text{in } \mathscr{D}'(\Omega).$$

Proof. Let $\rho\in\mathrm{C}^\infty_c(\Omega)$. We claim that

$$\lim_{j\to\infty}\int_\Omega\rho P(v_j)\mathrm{d}x=\int_\Omega\rho P(v)\mathrm{d}x. \tag{4.7}$$

As in Step I of the proof of Theorem 4.5, we note that it suffices to consider $\rho\geqslant 0$, and, moreover, e.g., by replacing $\rho P(v_j)$ with $\rho^{1/2}P(\rho^{1/(2s)}v_j)$ and mollification, we can assume that $\rho\in\mathrm{C}^\infty_c(\omega)^+$ and $v_j\in\mathrm{C}^\infty_c(\omega,\mathbb{V})$ for some open set $\omega\Subset\Omega$. We can then identify v_j with their extensions by zero and assume that

$$v_j\rightharpoonup v \text{ in } \mathrm{L}^s(\mathbb{R}^n,\mathbb{V}) \quad\text{and}\quad \mathscr{A}v_j\to\mathscr{A}v \text{ in } \mathrm{W}^{-\ell,s}(\mathbb{R}^n,\mathbb{W}).$$

In this case, we can apply Fourier analysis to decompose v_j effectively into an exactly $\mathscr{A}$-free part and a strongly convergent part which does not change the weak limiting behaviour of $(P(v_j))_j$.

Let $\mathscr{B}$ be a potential operator for $\mathscr{A}$, i.e., satisfy the exact relation (4.5), which we write in jet notation as (4.6). For $\xi\in\mathbb{R}^n\setminus\{0\}$, we write in Fourier space

$$\hat v_j(\xi)=[\mathrm{Proj}_{\ker\mathscr{A}(\xi)}+\mathrm{Proj}_{\operatorname{im}\mathscr{A}^*(\xi)}]\hat v_j(\xi)=\mathscr{B}(\xi)\mathscr{B}^\dagger(\xi)\hat v_j(\xi)+\mathscr{A}^*(\xi)(\mathscr{A}^*)^\dagger(\xi)\hat v_j(\xi)=\widehat{\mathscr{B}u_j}(\xi)+\widehat{\mathscr{A}^*w_j}(\xi),$$

where u_j, w_j are defined by

$$\hat u_j(\xi):=\mathscr{B}^\dagger(\xi)\hat v_j(\xi)\quad\text{and}\quad \hat w_j(\xi):=(\mathscr{A}^*)^\dagger(\xi)\hat v_j(\xi),$$

so that

$$\widehat{D^ku_j}(\xi):=\mathscr{B}^\dagger(\xi)\hat v_j(\xi)\otimes\xi^{\otimes k}\quad\text{and}\quad\widehat{D^\ell w_j}(\xi):=(\mathscr{A}^*)^\dagger(\xi)\hat v_j(\xi)\otimes\xi^{\otimes\ell}.$$

By Theorem D.1, we have that $u_j \in \mathrm{W}^{k,s}(\mathbb{R}^n,\mathbb{U})$ and $w_j \in \mathrm{W}^{\ell,s}(\mathbb{R}^n,\mathbb{W})$ are bounded uniformly by $\sup_j \|v_j\|_{\mathrm{L}^p}$. In particular we have that $D^k u_j \rightharpoonup D^k u$ in L^s for some limiting function u. On the other hand, writing $\tilde{w}_j = w_j - w$ and $\tilde{v}_j = v_j - v$, where w is the weak limit of (w_j), we have that

$$\mathscr{F}(\mathscr{A}^*\tilde{w}_j)(\xi) = \mathrm{Proj}_{\mathrm{im}\,\mathscr{A}^*(\xi)}\mathscr{F}\tilde{v}_j(\xi) = \mathscr{A}^\dagger(\xi)\mathscr{A}(\xi)\mathscr{F}\tilde{v}_j(\xi) = \mathscr{A}^\dagger\left(\frac{\xi}{|\xi|}\right)\frac{\mathscr{F}(\mathscr{A}\tilde{v}_j)(\xi)}{|\xi|^\ell}.$$

It follows, again by Theorem D.1, that since $\mathscr{A}\tilde{v}_j \to 0$ in $\mathrm{W}^{-\ell,s}(\mathbb{R}^n,\mathbb{W})$, then we also have that $\mathscr{A}^*\tilde{w}_j \to 0$ in $\mathrm{L}^s(\mathbb{R}^n,\mathbb{W})$.

Altogether, we have proved that that

$$v_j = \mathscr{B}u_j + \mathscr{A}^* w_j$$

where $\mathscr{B}u_j$ converges weakly in L^s and $\mathscr{A}^*w_j$ converges strongly in L^s. We can then write

$$P(v_j) - P(\mathscr{B}u_j) = \int_0^1 \langle P'(\mathscr{B}u_j + t\mathscr{A}^*w_j), \mathscr{A}^*w_j\rangle \mathrm{d}t \tag{4.8}$$

and note that we have that P' is $(s-1)$-homogeneous and $\mathscr{A}$-quasiaffine (by Lemma 4.3), as well as, for each $t\in[0,1]$, $\mathscr{B}u_j + t\mathscr{A}^*w_j \rightharpoonup \mathscr{B}u + t\mathscr{A}^*w$ in $\mathrm{L}^s(\mathbb{R}^n,\mathbb{V})$ and $\mathscr{A}(\mathscr{B}u_j + t\mathscr{A}^*w_j) = \mathscr{A}v_j \to \mathscr{A}v = \mathscr{A}(\mathscr{B}u + t\mathscr{A}^*w)$ in $\mathrm{W}^{-\ell,s}(\mathbb{R}^n,\mathbb{W})$. By induction since P' is a linear map for $s=2$, we have that $P'(\mathscr{B}u_j + t\mathscr{A}^*w_j) \rightharpoonup P'(\mathscr{B}u + t\mathscr{A}^*w)$ in $\mathscr{D}'(\Omega)$, which is improved to $\mathrm{L}^{s/(s-1)}$ by the elementary bound one can compute. We then notice that, by homogeneity of P' and Hölder's inequality,

$$\begin{aligned}\left|\int_\Omega \rho\langle P'(\mathscr{B}u_j + t\mathscr{A}^*w_j), \mathscr{A}^*w_j\rangle \mathrm{d}x\right| &\leqslant c\|\rho\|_{\mathrm{L}^\infty}\int_\Omega (|\mathscr{B}u_j|^{s-1} + |\mathscr{A}w_j|^{s-1})|\mathscr{A}w_j|\mathrm{d}x \\ &\leqslant c\|\rho\|_{\mathrm{L}^\infty}\left(\|\mathscr{B}u_j\|_{\mathrm{L}^s}^{s-1} + \|\mathscr{A}^*w_j\|_{\mathrm{L}^s}^{s-1}\right)\|\mathscr{A}^*w_j\|_{\mathrm{L}^s} \leqslant c\|v_j\|_{\mathrm{L}^s}^s,\end{aligned}$$

which is thus bounded independently of $t\in[0,1]$. We then have that

$$\begin{aligned}\int_\Omega \rho\left(P(v_j) - P(\mathscr{B}u_j)\right)\mathrm{d}x &= \int_0^1\int_\Omega \langle P'(\mathscr{B}u_j + t\mathscr{A}^*w_j), \rho\mathscr{A}^*w_j\rangle \mathrm{d}x\mathrm{d}t \\ &\to \int_0^1\int_\Omega \langle P'(\mathscr{B}u + t\mathscr{A}^*w), \rho\mathscr{A}^*w\rangle \mathrm{d}x\mathrm{d}t = \int_\Omega \rho\left(P(v) - P(\mathscr{B}u)\right)\mathrm{d}x,\end{aligned}$$

where the first equality follows from Fubini's theorem, the convergence by the dominated convergence theorem together with the inductive hypothesis, and the last equality by the fundamental theorem of calculus. We can conclude that (4.7) is equivalent with

$$\lim_{j\to\infty}\int_\Omega \rho P(\mathscr{B}u_j)\mathrm{d}x = \int_\Omega \rho P(\mathscr{B}u)\mathrm{d}x.$$

By Lemma 4.7, we have that P is $\mathscr{A}$-quasiaffine if and only if $P\circ\mathbf{T}$ is k-quasiaffine. By Theorem 4.9, we can assume that $P\circ\mathbf{T}(D^k u_j)$ is a minor of order s of DU_j, where $U_j := D^{k-1}u_j$. So it suffices to prove that

$$DU_j \rightharpoonup DU \text{ in } \mathrm{L}^s \implies \lim_{j\to\infty}\int_\Omega \rho M(DU_j)\mathrm{d}x = \int_\Omega \rho M(DU)\mathrm{d}x,$$

where $M(DU) = \det(D_{x'}U')$, where $x = (x', x'')$ and $U = (U', U'')$ with x', $U' \in \mathbb{R}^s$. By the Piola identity, we have that

$$M(DU_j) = D_{x'}U'_{j1}\cdot\Sigma_j \quad \text{where } D^*_{x'}\Sigma_j = 0,\ |\Sigma_j| \leqslant c\prod_{i=2}^s |U'_{ji}|.$$

It follows by integration by parts that

$$\int_\Omega \rho M(DU_j)\mathrm{d}x = \int_\Omega D_{x'}(U'_{j1} - (U'_{j1})_\omega)\cdot(\rho\Sigma_j)\mathrm{d}x = -\int_\Omega (U'_{j1} - (U'_{j1})_\omega)D_{x'}\rho\cdot\Sigma_j\mathrm{d}x.$$

We now argue by induction on $s \geqslant 2$, since Σ_j consists of subdeterminants of order $s-1$. If $s=2$, we simply have that $\Sigma_j \rightharpoonup \Sigma$ in L^2. If $s>2$, we obtain that $\Sigma_j \rightharpoonup \Sigma$ in $\mathrm{L}^{s/(s-1)}$ by the inductive hypothesis and the bound on Σ_j. In any case, since $(U'_{j1} - (U'_{j1})_\omega)_j$ converges strongly in L^s by the compact Sobolev embedding, we have that

$$\lim_{j\to\infty}\int_\Omega \rho M(DU_j)\mathrm{d}x = -\int_\Omega (U'_1 - (U'_1)_\omega)D_{x'}\rho\cdot\Sigma\mathrm{d}x = \int_\Omega \rho M(DU)\mathrm{d}x,$$

where we reversed the integration by parts in the last equality. The proof is complete. □

In fact, a more involved variant of Theorem 4.10 can be proved. We state it here for completeness of the exposition, but will not include a proof.

Theorem 4.11. *Suppose that $\mathscr{A}$ satisfies the* (SC) *and* (CR) *conditions and let $F\colon \mathbb{V}\to\mathbb{R}$. The following are equivalent:*

(a) *F is $\mathscr{A}$-quasiaffine.*
(b) *F is an $\mathscr{A}$-*null Lagrangian*, i.e., $\mathscr{B}^*(DF(\mathscr{B}u))=0$ in Ω for all $u\in \mathrm{C}^k(\bar\Omega,\mathbb{U})$. Here $\Omega\subset\mathbb{R}^n$ is a bounded open set and $\mathscr{B}$ is a potential operator for $\mathscr{A}$ such that* (4.5) *and* (4.6) *hold.*
(c) *For each $r\geqslant 2$ and $z_1,\ldots,z_r\in\mathbb{V}$ such that for each i we find linearly dependent $\xi_i\in S^{n-1}$ such that $\mathscr{A}(\xi_i)z_i=0$, we have that $D^rF(\bullet)[z_1,\ldots,z_r]=0$.*
(d) *F is a polynomial of degree $s\leqslant\min\{n,\dim\mathbb{V}\}$ and*

$$\left.\begin{array}{lll} v_j\rightharpoonup v & \text{in} & \mathrm{L}^s_{\mathrm{loc}}(\Omega,\mathbb{V})\\ \mathscr{A}v_j\to\mathscr{A}v & \text{in} & \mathrm{W}^{-\ell,s}_{\mathrm{loc}}(\Omega,\mathbb{W})\end{array}\right\}\implies F(v_j)\overset{*}{\rightharpoonup}F(v)\ \text{in}\ \mathscr{D}'(\Omega).$$

The claim in (b) agrees with the standard terminology in the calculus of variations in the case $\mathscr{B}=D$. The statement is equivalent with the fact that all smooth maps are solutions of the Euler–Lagrange system of the energy functional

$$\mathscr{E}[u]:=\int_\Omega F(\mathscr{B}u(x))\mathrm{d}x.$$

Equivalently, the value of $\mathscr{E}[u]$ depends only on the boundary values of u. On the other hand, the claim in (c) is a generalization of the fact that F is $\Lambda_{\mathscr{A}}$-affine (when $r=2$) to the case of arbitrary laminations.

We will conclude the section with a few examples of weakly continuous nonlinearities. However, before we list these, we make a case of the fact that there are very few nonlinear quantities of interest, as can be seen from the following sufficient criterion for non-existence:

Lemma 4.12. *Let $\mathscr{A}$ satisfy the spanning cone condition* (SC)*. Suppose that if Q is a quadratic form on $\mathbb{V}$, then $Q=0$ on $\Lambda_{\mathscr{A}}$ implies that $Q\equiv 0$. Then there are no non-affine $\mathscr{A}$-quasiconvex polynomials.*

In particular, whenever the wave cone has algebraic dimension at least three, there are no non-trivial $\mathscr{A}$-quasiaffine integrands.

Proof. The proof is simple and follows by contradiction. Let P be an s-homogeneous $\mathscr{A}$-quasiaffine polynomial, $s\geqslant 2$. Then $Q:=D^{s-2}P\not\equiv 0$ is an $\mathscr{A}$-quasiaffine quadratic form by Lemma 4.3. Equivalently, $Q=0$ on $\Lambda_{\mathscr{A}}$, so that $Q\equiv 0$, which is a contradiction. ☐

Example 4.13. We again compare with Example 1.16. If $\mathscr{A}=0$, we know from Section 3 that the only weakly continuous integrands are trivial (affine). If $\mathscr{A}$ is elliptic, then any $F\in\mathbb{E}_p$ is weakly continuous (in this case, the spanning cone condition is violated). We have already seen that the weakly continuous quantities with respect to sequences of gradients are the minors (Theorem 4.9).

Consider now the example $\mathscr{A}=\operatorname{div}$ acting on vector fields $v\colon\mathbb{R}^n\to\mathbb{R}^n$. If $n=2$, div is essentially the same as curl, so that the only homogenoeus, non-affine div-quasiaffine polynomial is $P=\det$. If $n\geqslant 3$, we can infer from the computation of the wave cone in Example 1.16(d) and Lemma 4.12 that there are no non-trivial div-quasiaffine integrands.

In the case of exterior derivatives of differential forms, all suitable exterior products are weakly continuous. From the quadratic Theorem 4.5 and the computation in Example 1.16(e), we see that $P(z,\tilde z)=z\cdot\tilde z$ is weakly continuous for the div-curl operator, thus retrieving the celebrated div-curl lemma. Finally, one can show that there are no non-trivial weakly continuous quantities with respect to sequences of symmetrized gradients. In the case of Example 1.16(i), one can check that $P(z_1,\ldots,z_n)=z_{i_1}\ldots z_{i_s}$ with $1\leqslant i_1<i_2<\ldots<i_s\leqslant n$ is a weakly continuous nonlinearity.

5. Weak sequential lower semi-continuity: constant rank constraints

In this section we will complete the study of weak (lower semi-)continuity from the previous two sections. In particular, we will establish the fact that $\mathscr{A}$-quasiconvexity of an integrand (recall Propostion 4.1) is also sufficient for lower semi-continuity of the corresponding energy functionals along constant rank constrained sequences. Our results will be formulated for rough integrands, in the same fashion as in Section 3.

Throughout this section, let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\mathscr{L}^n(\partial\Omega) = 0$ and $1 < p < \infty$. One of the main results, together with Theorem 5.6 below, will be the following:

Theorem 5.1 (LSC, constant rank operators, non-negative integrands). *Suppose that $\mathscr{A}$ satisfies the* (SC) *and* (CR) *conditions. Let $1 < q < \infty$ and $F\colon \Omega \times \mathbb{V} \to [0, \infty)$ be a normal integrand of p-growth, i.e., satisfying* (G-p). *Suppose that $z \mapsto F(x, z)$ is $\mathscr{A}$-quasiconvex for $\mathscr{L}^n$-a.e. $x \in \Omega$. Then*

$$\text{(LSC)} \qquad \left.\begin{aligned} v_j &\rightharpoonup v && \text{in } \mathrm{L}^p(\Omega, \mathbb{V}) \\ \mathscr{A}v_j &\to \mathscr{A}v && \text{in } \mathrm{W}^{-\ell,q}(\Omega, \mathbb{W}) \end{aligned}\right\} \implies \liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\mathrm{d}x \geqslant \int_\Omega F(x, v(x))\mathrm{d}x.$$

We will employ Proposition 3.2, which then reduces the analysis to understanding the oscillation measure of the p-Young measure generated by the constrained sequence. This motivates the following:

Definition 5.2. *We say that $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ is an $\mathscr{A}$-*free Young measure *if it is generated by a sequence $(v_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ with convergence properties as in the left hand side of* (LSC).

The following lemma, which extends the unconstrained result of Lemma 2.20, will therefore be quintessential for our subsequent analysis:

Lemma 5.3 (Decomposition lemma I, constant rank constraints). *Let $\mathscr{A}$ as in* (1.3) *be a constant rank operator with potential operator $\mathscr{B}$ such that* (4.5) *and* (4.6) *hold. Let $1 < q < \infty$ and*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}) \quad \text{with} \quad \mathscr{A}v_j \to \mathscr{A}v \text{ in } \mathrm{W}^{-\ell,q}(\Omega, \mathbb{W})$$

generate a p-Young measure $\boldsymbol{\nu}$. Then there exist sequences $(u_j) \subset \mathrm{C}_c^\infty(\Omega, \mathbb{U})$ and $(b_j) \subset \mathrm{L}^p(\Omega, \mathbb{V})$ such that

$$\begin{aligned} & v_j = v + \mathscr{B}u_j + b_j, \\ & \mathscr{B}u_j,\, b_j \rightharpoonup 0 \text{ in } \mathrm{L}^p(\Omega, \mathbb{V}), \\ & (D^k u_j) \text{ is } p\text{-uniformly integrable}, \\ & b_j \to 0 \text{ in } \mathscr{L}^n\text{-measure.} \end{aligned}$$

Therefore, we have that, in $\mathrm{Y}^p(\Omega, \mathbb{V})$

$$(v + \mathscr{B}u_j) \text{ generates } ((\nu_x)_{x\in\Omega},\, 0,\, \texttt{n/a}) \quad \text{and} \quad (b_j) \text{ generates } ((\delta_0)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar\Omega}).$$

Proof. Recall the truncation maps

$$\mathcal{T}_\alpha(z) := \begin{cases} z & |z| \leqslant \alpha \\ k\frac{z}{|z|} & |z| > \alpha \end{cases},$$

which can be used to see that

$$\lim_{\alpha\to\infty} \lim_{j\to\infty} \int_\Omega |\mathcal{T}_\alpha v_j|^p = \int_\Omega \langle \nu_x, |\boldsymbol{\cdot}|^p \rangle \mathrm{d}x,$$

so that we can employ a diagonalization argument to see that there exists a sequence $\alpha_j \uparrow \infty$ such that

$$\lim_{j\to\infty} \int_\Omega |\mathcal{T}_{\alpha_j} v_j|^p = \int_\Omega \langle \nu_x, |\boldsymbol{\cdot}|^p \rangle \mathrm{d}x.$$

By use of Theorem 2.18 applied to the integrand $|\boldsymbol{\cdot}|^p$ and the sequence $(\mathcal{T}_{\alpha_j} v_j)$, we see that the sequence is p-uniformly integrable. Since (v_j) converges weakly in L^1, it is uniformly integrable, so that, for $\varepsilon > 0$,

$$\varepsilon \mathscr{L}^n(|v_j - \mathcal{T}_{\alpha_j} v_j| > \varepsilon) \leqslant \int_{|v_j|>\alpha_j} |v_j| \left(1 - \frac{\alpha_j}{|v_j|}\right) \mathrm{d}x \leqslant \int_{|v_j|>\alpha_j} |v_j| \mathrm{d}x \to 0,$$

so that $(v_j - \mathcal{T}_{\alpha_j} v_j)$ converges to zero in measure.

We write $\boldsymbol{\nu}_1 := ((\nu_x)_{x\in\Omega},\, 0,\, \texttt{n/a})$ and $\boldsymbol{\nu}_2 := ((\delta_0)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar\Omega})$ and conclude from Theorems 2.17 and 2.18 that $(\mathcal{T}_{\alpha_j} v_j)$ generates $\boldsymbol{\nu}_1 \in \mathrm{Y}^p(\Omega, \mathbb{V})$ and $(v_j - \mathcal{T}_{\alpha_j} v_j)$ generates $\boldsymbol{\nu}_2 \in \mathrm{Y}^p(\Omega, \mathbb{V})$.

Let now $r > 1$ be a number such that $r < p$ and $r \leqslant q$. We claim that $\mathcal{T}_{\alpha_j} v_j - v_j \to 0$ in $\mathrm{L}^r(\Omega, \mathbb{V})$. To see this, write

$$\|\mathcal{T}_{\alpha_j} v_j - v_j\|_{\mathrm{L}^r(\Omega)}^r \leqslant c \int_{|v_j|\geqslant\alpha_j} |v_j|^r \mathrm{d}x \leqslant c \int_{|v_j|\geqslant\alpha_j} \frac{|v_j|^p}{\alpha_j^{p-r}} \mathrm{d}x \leqslant \frac{c}{\alpha_j^{p-r}} \int_\Omega |v_j|^p \mathrm{d}x \to 0.$$

In particular, $\mathscr{A}\mathcal{T}_{\alpha_j}v_j \to \mathscr{A}v$ in $\mathrm{W}^{-\ell,r}(\Omega,\mathbb{W})$.

We now aim to find a sequence of cut off functions $\rho_j \in \mathrm{C}_c^\infty(\Omega,[0,1])$ such that $\rho_j \uparrow 1$ that makes $\mathscr{A}\left(\rho_j(\mathcal{T}_{\alpha_j}v_j - v)\right)$ well behaved. First, note that for any such sequence we have that $(\rho_j\mathcal{T}_{\alpha_j}v_j)$ is p-uniformly integrable and that $\left((1-\rho_j)(\mathcal{T}_{\alpha_j}v_j - v)\right)$ converges in measure to zero.

$$v_j := \left[\rho_j(\mathcal{T}_{\alpha_j}v_j - v) + v\right] + \left[(v_j - \mathcal{T}_{\alpha_j}v_j) + (1-\rho_j)(\mathcal{T}_{\alpha_j}v_j - v)\right],$$

where the first term converges weakly in L^p to v and generates $\boldsymbol{\nu}_1$ and the second term converges in measure to zero and generates $\boldsymbol{\nu}_2$. It remains to preserve the differential structure, which adds restrictions to (ρ_j). We write $\tilde{v}_j = \mathcal{T}_{\alpha_j}v_j - v$, so that $\mathscr{A}\tilde{v}_j \to 0$ in $\mathrm{W}^{-\ell,r}(\Omega,\mathbb{W})$ and

$$\mathscr{A}(\rho_j\tilde{v}_j) = \rho_j\mathscr{A}\tilde{v}_j + \sum_{|\alpha|=\ell}\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}\rho_j A_\alpha \partial^\beta \tilde{v}_j, \tag{5.1}$$

where $\partial^\beta\tilde{v}_j \to 0$ in $\mathrm{W}^{-\ell,p}(\Omega,\mathbb{W})$ by the compact Sobolev embedding. Choosing the derivatives of (ρ_j) to blow up slow enough near the boundary of Ω, we can obtain that $\mathscr{A}(\rho_j\tilde{v}_j) \to 0$ in $\mathrm{W}^{-\ell,r}(\Omega,\mathbb{W})$. Since ρ_j is compactly supported inside Ω, we can mollify and assume that $\rho_j \equiv 1$ and $\tilde{v}_j \in \mathrm{C}_c^\infty(\Omega,\mathbb{V})$, which we identify with their extension by zero to $\mathbb{R}^n$ without mention. With this new notation, we record that $(v+\tilde{v}_j)$ generates $\boldsymbol{\nu}_1$ and $(v_j - v - \tilde{v}_j)$ generates $\boldsymbol{\nu}_2$.

As in the proof of Theorem 4.10, we can define

$$\hat{u}_j(\xi) := \mathscr{B}^\dagger(\xi)\mathscr{F}\tilde{v}_j(\xi), \quad \text{so that } \widehat{D^k u_j}(\xi) = \mathscr{B}^\dagger(\xi)\mathscr{F}\tilde{v}_j(\xi)\otimes\xi^{\otimes k} =: \mathscr{F}[H\tilde{v}_j](\xi) \quad \text{for } \xi \neq 0.$$

We can then infer that $\tilde{v}_j - \mathscr{B}u_j \to 0$ in $\mathrm{L}^r(\mathbb{R}^n,\mathbb{V})$, so that $(v_j - v - \mathscr{B}u_j)$ generates $\boldsymbol{\nu}_2$. We claim that $(D^k u_j)$ is p-uniformly integrable in Ω. In that case, we retrieve $\mathscr{B}u_j = \mathbf{T}(D^k u_j)$, where the tensor $\mathbf{T}$ is the linear map in (4.6). It will follow that $(v+\mathscr{B}u_j)$ generates $\boldsymbol{\nu}_1$.

To prove this, first note that for $\alpha > 0$ by Theorem D.1

$$\sup_j \int_{\mathbb{R}^n} |H\tilde{v}_j - H\mathcal{T}_\alpha\tilde{v}_j|^p \mathrm{d}x \leqslant c\sup_j \int_{\mathbb{R}^n} |\tilde{v}_j - \mathcal{T}_\alpha v_j|^p \mathrm{d}x \to 0 \quad \text{as } \alpha\to\infty$$

by p-uniform integrability of $(\tilde{v}_j)$. Let $\varepsilon > 0$ and choose $\alpha > 0$ such that the right hand side is less than ε. Let $s > p$ and notice that, again by Theorem D.1, we have

$$\|H\mathcal{T}_\alpha\tilde{v}_j\|_{\mathrm{L}^s(\mathbb{R}^n)} \leqslant c\|\mathcal{T}_\alpha\tilde{v}_j\|_{\mathrm{L}^s(\Omega)} \leqslant c\alpha,$$

so that $(H\mathcal{T}_\alpha\tilde{v}_j)$ is p-uniformly integrable. Then there exists $\delta > 0$ such that $\mathscr{L}^n(E) < \varepsilon$ implies that

$$\int_E |H\mathcal{T}_\alpha\tilde{v}_j|^p\mathrm{d}x \leqslant \varepsilon.$$

We can therefore estimate

$$\int_E |D^k u_j|^p \mathrm{d}x = \int_E |H\tilde{v}_j|^p\mathrm{d}x \leqslant c\int_{\mathbb{R}^n}|H\tilde{v}_j - H\mathcal{T}_\alpha\tilde{v}_j|^p\mathrm{d}x + c\int_E |H\mathcal{T}_\alpha\tilde{v}_j|^p\mathrm{d}x \leqslant c\varepsilon,$$

which concludes the proof of the claim that $(\mathscr{B}u_j)$ is uniformly integrable.

It remains to use cut off functions to prove that we can assume that $\mathscr{B}u_j$ are compactly supported inside Ω. To this end, let $\phi_j \in \mathrm{C}_c^\infty(\Omega,[0,1])$ be such that $\phi_j \uparrow 1$ be such that $\mathscr{B}(\rho_j u_j)$ is well behaved in a sense that we now describe. First, note that since $\partial^\beta u_j \to 0$ in $\mathrm{L}^p(\mathbb{R}^n,\mathrm{SLin}^k(\mathbb{R}^n,\mathbb{V}))$, we have by the compact Sobolev embedding (see also Remark A.4) that $\partial^\beta u_j \to 0$ in $\mathrm{L}^p(\Omega,\mathbb{V})$ for $|\beta| < k$. In particular, by a Leibniz rule computation similar to the one in (5.1), we can choose ϕ_j to be controlled in $\mathrm{C}^k(\bar{\Omega})$ such that $\mathscr{B}(\phi_j u_j) - \phi_j\mathscr{B}u_j \to 0$ in $\mathrm{L}^p(\Omega,\mathbb{W})$. In particular, $(\mathscr{B}(\phi_j u_j))$ is p-uniformly integrable and $\mathscr{B}(\phi_j u_j) - \mathscr{B}u_j \to 0$ in measure in Ω.

It follows that we can assume that $u_j \in \mathrm{C}_c^\infty(\Omega,\mathbb{U})$ and we can set $b_j := v_j - v - \mathscr{B}u_j$ so that all the required properties are satisfied. □

Remark 5.4. From the proof of the Decomposition Lemma 5.3, we can extract the following stronger statement concerning the oscillation measure of an $\mathscr{A}$-free Young measure: Since $(D^k u_j)$ is p-uniformly integrable, it generates an (oscillation) Young measure $\boldsymbol{\mu} = ((\mu_x)_{x\in\Omega}, 0, \mathtt{n/a})$ in $\mathrm{Y}^p(\Omega,\mathrm{SLin}^k(\mathbb{R}^n,\mathbb{V}))$ which satisfies

$$\nu_x = \tau_{v(x)}\mathbf{T}_{\#}\mu_x \quad \text{for } \mathscr{L}^n\text{-a.e. } x\in\Omega,$$

where $\tau_z\colon \mathcal{M}(\mathbb{V}) \to \mathcal{M}(\mathbb{V})$ denotes *translation by* $z \in \mathbb{V}$ *of a measure*, i.e.,

$$\langle \tau_z\eta, \varphi\rangle := \int_{\mathbb{V}} \varphi(\zeta + z)\mathrm{d}\zeta \quad \text{for } \varphi \in \mathrm{C}_0(\mathbb{V}).$$

In other words, $\mathscr{A}$-free oscillation p-Young measures can be retrieved from (k-)gradient Young measures by simple operations. We briefly remark that this is *not* the case for $p = 1$ due to the lack of boundedness of singular integral operators.

This consequence of the Decomposition Lemma 5.3 is the most important tool we will use in the proof of Theorem 5.1 as it essentially enables us to work only with oscillation Young measures generated by gradients and k-quasiconvex functions (recall Lemma 4.7). The other important step will be a blow up argument (so called *localization principle*) to show that each oscillation measure is itself a gradient Young measure. This will enable us to prove a Jensen type inequality for $\mathscr{A}$-quasiconvex functions and $\mathscr{A}$-free Young measures

$$\langle \nu_x, F(x, \bullet)\rangle \geq F(x, \bar{\nu}_x) \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega \tag{5.2}$$

Before we can perform this second step, we require another preliminary result, a general density lemma in $\mathbb{E}_p(\Omega, \mathbb{V})$ which enables us to test with fewer integrands:

Lemma 5.5. *There exists a countable family* $\{\varphi \otimes f \colon \varphi \in \mathrm{Lip}(\Omega),\ f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{V}) \cap \mathbb{E}_p(\Omega, \mathbb{V})\}$ *whose span is dense in* $\mathbb{E}_p(\Omega, \mathbb{V})$. *Moreover,*

$$|f(z_1) - f(z_2)| \leqslant c\|T_p f\|_{\mathrm{Lip}(B_{\mathbb{V}})}|z_1 - z_2|(1 + |z_1| + |z_2|)^{p-1} \quad \text{for } z_1,\, z_2 \in \mathbb{V},$$

where T_p *is the isomorphism from Section 2 and* $\|g\|_{\mathrm{Lip}} := \|g\|_{\mathrm{L}^\infty} + \|Dg\|_{\mathrm{L}^\infty}$.

Proof. By the Stone–Weierstrass Theorem, polynomials are dense in $\mathrm{C}(\bar{\Omega} \times \bar{B}_{\mathbb{V}})$, so that we can find a countable subset $\{\varphi \otimes g\}$ such that φ and g are monomials in Ω and $B_{\mathbb{V}}$, respectively, which has dense span. Setting $f := T_p^{-1} g$, we obtain the conclusion by direct computation. □

Proof of Theorem 5.1. Let $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ be a p-Young measure generated by (v_j). By Proposition 3.2, we have that

$$\liminf_{j\to\infty} \int_\Omega F(x, v_j(x))\,\mathrm{d}x \geqslant \int_\Omega \langle \nu_x, F(x, \bullet)\rangle\,\mathrm{d}x.$$

By Lemma 2.10, it would therefore suffice to prove the Jensen type inequalities in (5.2).

By Remark 5.4 and Lemma 4.7, it suffices to prove this claim if $F(x, \bullet)$ is k-quasiconvex for $\mathscr{L}^n$-a.e. $x \in \Omega$ and p-uniformly integrable $v_j = D^k u_j \rightharpoonup 0$ in L^p, where $u_j \in \mathrm{C}_c^\infty(\Omega, \mathbb{U})$. In this case, of course $(D^k u_j)$ generates $((\nu_x)_{x\in\Omega}, 0, \mathtt{n/a})$ and $\bar{\nu}_x = 0$ almost everywhere.

We claim that almost every oscillation measure ν_{x_0} can be seen as a parametrized oscillation measure, $\boldsymbol{\nu}_0 := \big((\nu_{x_0})_{y\in Q_1(0)}, 0, \mathtt{n/a}\big)$ that is also generated by gradients. To check this, we will only test with p-admissible integrands of the form $\varphi \otimes f$ in the countable family given by Lemma 5.5. In particular, we can choose x_0 from a subset of Ω of full measure to be a Lebesgue point of all maps $x \mapsto \langle \nu_x, f\rangle$.

Then for $r > 0$ small enough but fixed for now, we can employ Theorem 3.5 to see that

$$\lim_{j\to\infty} \int_{Q_r(x_0)} \varphi\left(\frac{x - x_0}{r}\right) f(D^k u_j(x))\,\mathrm{d}x = \int_{Q_r(x_0)} \varphi\left(\frac{x - x_0}{r}\right) \langle \nu_x, f\rangle\,\mathrm{d}x.$$

We want to make the change of variable $y = r^{-1}(x - x_0)$, so we calculate

$$D^k[u_j(x_0 + ry)] = r^k[D^k u_j](x_0 + ry),$$

so we can define $u_j^{x_0,r}(y) := r^{-k} u_j(x_0 + ry)$ to obtain

$$\lim_{j\to\infty} \int_{Q_1(0)} \varphi(y) f(D^k u_j^{x_0,r}(y)) r^n\,\mathrm{d}y = \int_{Q_1(0)} \varphi(y)\langle \nu_{x_0+ry}, f\rangle r^n\,\mathrm{d}y$$

and use the Lebesgue differentiation theorem to infer that

$$\lim_{r\downarrow 0}\lim_{j\to\infty} \int_{Q_1(0)} \varphi(y) f(D^k u_j^{x_0,r}(y))\,\mathrm{d}y = \int_{Q_1(0)} \varphi(y)\,\mathrm{d}y \langle \nu_{x_0}, f\rangle.$$

By a standard diagonalization argument, we find a sequence $r_j \downarrow 0$ such that $(D^k u_j^{x_0,r_j})$ generates $\boldsymbol{\nu}_0$ in $\mathrm{Y}^p(Q_1(0), \mathrm{SLin}^k(\mathbb{R}^n, \mathbb{U}))$. Either by applying Lemma 5.3 or by repeating the last step in its proof (truncating $(u_j^{x_0,r_j})$ near the boundary and using the strong convergence of the lower order derivatives, possibly by removing lower order polynomials, in this case), we can find a sequence

$$\tilde{u}_j \in \mathrm{C}_c^\infty(Q_1(0), \mathbb{U}), \quad \boldsymbol{\varepsilon}_{D^k \tilde{u}_j} \overset{*}{\rightharpoonup} \boldsymbol{\nu}_0 \text{ in } \mathbb{E}_p(Q_1(0), \mathrm{SLin}^k(\mathbb{R}^n, \mathbb{U}))^*,$$

so that, moreover $D^k \tilde{u}_j \rightharpoonup \bar{\nu}_{x_0} = 0$ in L^p. We can thus infer from Theorem 3.5 that

$$\langle \nu_{x_0}, F(x_0, \bullet)\rangle = \lim_{j\to\infty} \int_{Q_1(0)} F(x_0, D^k \tilde{u}_j(y))\,\mathrm{d}y \geqslant F(x_0, 0),$$

where in the last inequality we used the k-quasiconvexity of $F(x_0, \bullet)$. The proof is complete. □

The other main result of this section will be a lower semi-continuity result for signed integrands. In contrast to Theorem 5.1, the concentration at the boundary must be ruled out. Also, whereas in Theorem 5.1, the constraint allows for L^p-concentration effects, as they cannot decrease the energy when the integrand is positive, below such interior effects are handled by using quasiconvexity. Recall that $1 < p < \infty$.

Theorem 5.6 (LSC, constant rank operators, signed integrands)**.** *Suppose that $\mathscr{A}$ satisfies the* (SC) *and* (CR) *conditions. Let $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ be a jointly lower semi-continuous integrand of p-growth, i.e., satisfying* (G-p)*. Suppose that $z \mapsto F(x,z)$ is $\mathscr{A}$-quasiconvex for $\mathscr{L}^n$-a.e. $x \in \Omega$ and that the recession function of F satisfies*

$$\liminf_{(x',z',t)\to(x,z,\infty)} \frac{F(x',tz')}{t^p} = \lim_{t\to\infty} \frac{F(x,tz)}{t^p} \quad \text{for } (x,z) \in \Omega \times \mathbb{V}. \tag{5.3}$$

Let

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega,\mathbb{V}) \quad \text{and} \quad \mathscr{A}v_j \to \mathscr{A}v \text{ in } \mathrm{W}^{-\ell,p}(\Omega,\mathbb{W}).$$

If on a subsequence $|v_j|^p \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}^+(\Omega)$ with $\mu(\partial\Omega)=0$, then

$$\liminf_{j\to\infty} \int_\Omega F(x,v_j(x))\mathrm{d}x \geqslant \int_\Omega F(x,v(x))\mathrm{d}x.$$

In other words, we do not allow the sequence to concentrate at the boundary. In fact, an equivalent formulation is that if λ is the L^p-concentration measure of a Young measure generated by (v_j), then λ does not charge the boundary of Ω[6].

To prove the assertion, we will employ Proposition 3.3 to account for the concentration effects. By taking $q=p$ in the proof of Theorem 5.1, we know that the Jensen inequalities for oscillation (5.2) hold. It remains to prove an inequality of the form

$$\langle \nu_x^\infty, G(x,\bullet)\rangle \geq 0 \quad \text{for } \lambda\text{-a.e. } x \in \Omega, \tag{5.4}$$

where G is a p-homogeneous $\mathscr{A}$-quasiconvex function. In the case of Theorem 5.6, we will take $G = H_F$, the lower recession function of F, as defined in Proposition 3.3. In this case, the assumption (5.3) is crucial to ensure the $\mathscr{A}$-quasiconvexity of H_F. In order to prove this Jensen inequality, we will first refine the Decomposition Lemma 5.3

Lemma 5.7 (Decomposition lemma II, constant rank constraints)**.** *Let $\mathscr{A}$ as in* (1.3) *be a constant rank operator with potential operator $\mathscr{B}$ such that* (4.5) *and* (4.6) *hold. Let*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p(\Omega,\mathbb{V}) \quad \text{with} \quad \mathscr{A}v_j \to \mathscr{A}v \text{ in } \mathrm{W}^{-\ell,p}(\Omega,\mathbb{W})$$

generate a p-Young measure $\boldsymbol{\nu}$. Then there exist sequences (u_j), $(\tilde u_j) \subset \mathrm{C}_c^\infty(\Omega,\mathbb{U})$ and $(\tilde b_j) \subset \mathrm{L}^p(\Omega,\mathbb{V})$ such that

$$\begin{aligned} &v_j = v + \mathscr{B}u_j + \mathscr{B}\tilde u_j + \tilde b_j,\\ &\mathscr{B}u_j,\ \mathscr{B}\tilde u_j,\ \tilde b_j \rightharpoonup 0 \text{ in } \mathrm{L}^p(\Omega,\mathbb{V}),\\ &(D^k u_j) \text{ is } p\text{-uniformly integrable},\\ &D^k\tilde u_j,\ \tilde b_j \to 0 \text{ in } \mathscr{L}^n\text{-measure,} \end{aligned}$$

and, moreover, in $\mathrm{Y}^p(\Omega,\mathbb{V})$,

$$\begin{aligned} &(v+\mathscr{B}u_j) \text{ generates } \left((\nu_x)_{x\in\Omega},\, 0,\, \texttt{n/a}\right),\\ &(\mathscr{B}\tilde u_j) \text{ generates } \left((\delta_0)_{x\in\Omega},\, \lambda \llcorner \Omega,\, (\nu_x^\infty)_{x\in\Omega}\right),\\ &(\tilde b_j) \text{ generates } \left((\delta_0)_{x\in\Omega},\, \lambda \llcorner \partial\Omega,\, (\nu_x^\infty)_{x\in\partial\Omega}\right). \end{aligned}$$

Proof. Using the Decomposition Lemma 5.3 with $p=q$, we can write $v_j = v + \mathscr{B}u_j + b_j$ with (u_j) as required and (b_j) generating $((\delta_0)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\bar\Omega})$. Consequently, we have that $\mathscr{A}b_j \to 0$ in $\mathrm{W}^{-\ell,p}(\Omega,\mathbb{V})$. Selecting cut off (test) functions $0 \leqslant \rho_j \uparrow 1$. Proceeding like in the proof of Lemma 5.3, we can ensure that $\mathcal{A}(\rho_j b_j) \to 0$ in $\mathrm{W}^{-\ell,p}(\Omega,\mathbb{V})$. Therefore, the same is true of $(\mathscr{A}((1-\rho_j)b_j))$, and clearly $(\rho_j b_j)$, $((1-\rho_j)b_j)$ both converge to zero in measure and weakly in L^p. Since all $(1-\rho_j)b_j = b_j$ near $\partial\Omega$, it is easy to see that $((1-\rho_j)b_j)$ generates $((\delta_0)_{x\in\Omega},\, \lambda\llcorner\partial\Omega,\, (\nu_x^\infty)_{x\in\partial\Omega})$.

Next, one can use the Helmholtz decomposition in Theorem 4.10 and Lemma 5.3 to split

$$\rho_j b_j = \mathscr{B}U_j + \mathscr{A}^* w_j,$$

[6] In fact, we already noticed that we have a relation $\lambda = \mu - \langle \nu_x, |\bullet|^p\rangle$.

which, in this case are such that $\mathscr{A}^* w_j \to 0$ in $\mathrm{L}^p(\mathbb{R}^n,\mathbb{V})$. Repeating the cut off function argument at the end of the proof of Lemma 5.3, we can define $\tilde{u}_j := \phi_j U_j$ in a way such that $\mathscr{B}((1-\phi_j)U_j) \to 0$ in $\mathrm{L}^p(\Omega,\mathbb{V})$. We can then conclude that $\tilde{u}_j$ thus defined and $\tilde{b}_j := (1-\rho_j)b_j + \mathscr{A}^* w_j + \mathscr{B}((1-\phi_j)U_j)$ satisfy the conditions of the lemma. □

Proof of Theorem 5.6. As discussed above, it suffices to prove inequality (5.4). First, note that since we impose $\lambda(\partial\Omega) = 0$, it suffices to look at ν_x^∞ only for $x \in \Omega$. To this end, we will use the Decomposition Lemma 5.7 and assume that (slightly relabeling notation), that a sequence $(\mathscr{B}u_j) \subset \mathrm{C}_c^\infty(\Omega,\mathbb{V})$ generates $((\delta_0)_{x\in\Omega},\, \lambda,\, (\nu_x^\infty)_{x\in\Omega})$. Much like in the proof of Theorem 5.1, we will freeze $x_0 \in \Omega$ and perform a blow up argument, which is more complicated in this case.

It suffices to show that for λ-a.e. $x_0 \in \Omega$, we can find a measure $\tau \in \mathcal{M}^+(\bar{Q}_1(0))$ such that

$$\boldsymbol{\nu}_1 := \big((\delta_0)_{y\in Q_1(0)},\, \tau,\, (\nu_{x_0}^\infty)_{y\in Q_1(0)}\big)$$

is generated by a sequence $(\mathscr{B}\tilde{u}_j)$ for $\tilde{u}_j \in \mathrm{C}_c^\infty(Q_1(0),\mathbb{U})$. In fact, if $\lambda = \lambda^a \mathscr{L}^n \llcorner \Omega + \lambda^s$ is a Radon–Nýkodim decomposition of λ, we will show that for $\lambda^a\mathscr{L}^n$-a.e. $x_0 \in \Omega$, we can choose $\tau = \lambda^a(x_0)\mathscr{L}^n \llcorner Q_1(0)$, whereas for λ^s-a.e. $x_0 \in \Omega$ we can choose τ to be a tangent measure of λ^s at x_0[7].

Much like in the proof of Theorem 5.1, we choose a family of integrands of the form $\varphi \otimes f$ with $\varphi \in \mathrm{C}(\bar{\Omega})$ and $f \in \mathbb{E}_p$ satisfying the properties in Lemma 5.5. For $x_0 \in \Omega$ to be determined, we have that for balls $B(x_0,r) \in \Omega$ such that $\lambda(\partial B(x_0,r)) = 0$ we have that

$$\begin{aligned}\lim_{j\to\infty} \int_{Q_1(0)} \varphi(y) f(\mathscr{B}u_j^{x_0,r}(y))\mathrm{d}y = \int_{Q_1(0)} \varphi(y)\mathrm{d}y f(0) &+ \int_{Q_1(0)} \varphi(y)\langle \nu_{x_0+ry}^\infty, f_p^\infty\rangle \lambda^a(x_0+ry)\mathrm{d}y \\ &+ \int_{Q_1(0)} \varphi(y)\langle \nu_{x_0+ry}^\infty, f_p^\infty\rangle \frac{\mathrm{d}T_\#^{x_0,r}\lambda^s(y)}{r^n},\end{aligned} \tag{5.5}$$

where $y = T^{x_0,r}(x) := r^{-1}(x-x_0)$ and $u_j^{x_0,r}(y) := r^{-k}u_j(x_0+ry)$.

By Lebsgue differentiation and the fact that we sample f from a countable set, we know that $\mathscr{L}^n$-a.e. $x_0 \in \Omega$ we have that

$$\lim_{r\downarrow 0} \fint_{Q_r(x_0)} |\langle \nu_x^\infty, f_p^\infty\rangle\lambda^a(x) - \langle \nu_{x_0}^\infty, f_p^\infty\rangle\lambda^a(x_0)|\mathrm{d}x = 0 \quad \text{and} \quad \lim_{r\downarrow 0} \frac{\lambda^s(Q_r(x_0))}{r^n} = 0,$$

so that at such x_0 we have that, taking a subsequence in $r > 0$,

$$\lim_{r\downarrow 0}\lim_{j\to\infty} \int_{Q_1(0)} \varphi(y) f(\mathscr{B}u_j^{x_0,r}(y))\mathrm{d}y = \int_{Q_1(0)} \varphi(y)\mathrm{d}y f(0) + \int_{Q_1(0)} \varphi(y)\mathrm{d}y \langle \nu_{x_0}^\infty, f_p^\infty\rangle \lambda^a(x_0).$$

The need to take a subsequence accounts for the fact that we can only guarantee that λ can charge at most countably many boundaries $\partial B_r(x_0)$. We employ a diagonalization argument to choose suitable $r_j > 0$ so that $(\mathscr{B}u_j^{x_0,r_j})$ generates $\boldsymbol{\nu}_1$ with $\tau = \lambda^a(x_0)\mathscr{L}^n \llcorner Q_1(0)$. It then follows by Lemma 5.7 that $\boldsymbol{\nu}_1$ can be generated by a sequence $(\mathscr{B}\tilde{u}_j)$ with $u_j \in \mathrm{C}_c^\infty(Q_1(0),\mathbb{U})$.

At λ^s-a.e. $x \in \Omega$ (singular points) we require

$$\begin{aligned}&\lim_{r\downarrow 0} \int_{Q_1(0)} |\langle \nu_{x_0+ry}^\infty, f_p^\infty\rangle - \langle \nu_{x_0}^\infty, f_p^\infty\rangle| \frac{\mathrm{d}T_\#^{x_0,r}\lambda^s(y)}{\lambda^s(Q_r(x_0))} = \lim_{r\downarrow 0} \fint_{Q_r(x_0)} |\langle \nu_x^\infty, f_p^\infty\rangle - \langle \nu_{x_0}^\infty, f_p^\infty\rangle|\mathrm{d}\lambda^s(x) = 0, \\ &\lim_{r\downarrow 0} \frac{1}{\lambda^s(Q_r(x_0))}\int_{Q_r(x_0)} \lambda^a(x)\mathrm{d}x = 0,\end{aligned}$$

so that we can rescale $U_j^{x_0,r}(y) := \big(\lambda^s(Q_r(x_0))^{-1}r^n\big)^{1/p} u_j^{x_0,r}(y)$. By (5.5) with f replaced by $f(\theta(r)\,\bullet)$,

$$\begin{aligned}\lim_{j\to\infty} \int_{Q_1(0)} \varphi(y) f(\mathscr{B}U_j^{x_0,r}(y))\mathrm{d}y = \int_{Q_1(0)} \varphi(y)\mathrm{d}y f(0) &+ \frac{r^n}{\lambda^s(Q_r(x_0))}\int_{Q_1(0)} \varphi(y)\langle \nu_{x_0+ry}^\infty, f_p^\infty\rangle \lambda^a(x_0+ry)\mathrm{d}y \\ &+ \int_{Q_1(0)} \varphi(y)\langle \nu_{x_0+ry}^\infty, f_p^\infty\rangle \frac{\mathrm{d}T_\#^{x_0,r}\lambda^s(y)}{\lambda^s(Q_r(x_0))}.\end{aligned}$$

Since $\Big(\lambda^s(Q_r(x_0,r))^{-1}T_\#^{x_0,r}\lambda^s\Big)$ are probability measures on $Q_1(0)$, we can choose a a weakly-* cluster point τ and conclude that, on a subsequence in $r > 0$,

$$\lim_{r\downarrow 0}\lim_{j\to\infty} \int_{Q_1(0)} \varphi(y) f(\mathscr{B}U_j^{x_0,r}(y))\mathrm{d}y = \int_{Q_1(0)} \varphi(y)\mathrm{d}y f(0) + \int_{Q_1(0)} \varphi\mathrm{d}\tau \langle \nu_{x_0}^\infty, f_p^\infty\rangle.$$

The proof is then concluded as in the previous case. □

[7]See, e.g., [2, Sec. 2.7] for details; we will not make use of any properties of tangent measures here.

6. Characterization of $\mathscr{A}$-free Young measures

In the previous section, we proved that quasiconvexity of an integrand is sufficient for the lower semicontinuity of several classes of integral functionals along PDE constrained sequences. In particular, we proved that quasiconvex integrands satisfy Jensen type ineaqualities (5.2), (5.4) with respect to the Young measures generated by such sequences. The main goal of this section is to prove that these inequalities in fact characterize Young measures generated by PDE constrained sequences. We make this precise in the following ample result.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $\mathscr{L}^n(\partial\Omega) = 0$ and $1 < p < \infty$. For a normal integrand $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ satisfying (G-p), we define the *upper recession function* by

$$F^\infty_p(x,z) := \limsup_{(x',z',t)\to(x,z,\infty)} \frac{F(x',tz')}{t} \quad \text{for } (x,z) \in \Omega \times \mathbb{V}.$$

Thus, we extend the definition of the recession function in $\mathbb{E}_p(\Omega,\mathbb{V})$. This is necessary since, in general, (even autonomous) quasiconvex integrands, despite being locally Lipschitz, need not lie in $\mathbb{E}_p(\Omega,\mathbb{V})$.

Theorem 6.1. *Suppose that $\mathscr{A}$ satisfies the* (SC) *and* (CR) *conditions. Let $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega,\mathbb{V})$.*

If $\boldsymbol{\nu} =: ((\nu_x)_{x\in\Omega},\, \lambda,\, (\nu^\infty_x)_{x\in\bar\Omega})$ is generated by a sequence $(v_j) \subset \mathrm{L}^p(\Omega,\mathbb{V})$ such that $(\mathscr{A}v_j)$ is strongly compact in $\mathrm{W}^{-\ell,p}(\Omega,\mathbb{V})$, then

$$(6.1)\quad \begin{aligned} &\langle f,\nu_x\rangle \geqslant f(\bar\nu_x) \quad \text{for } \mathscr{L}^n\text{-a.e. } x\in\Omega,\\ &\langle f^\infty_p,\nu^\infty_x\rangle \geqslant 0 \quad \text{for } \lambda\text{-a.e. } x\in\Omega \end{aligned} \qquad \text{hold for all } \mathscr{A}\text{-quasiconvex } f\colon \mathbb{V}\to\mathbb{R} \text{ satisfying (G-}p\text{)}.$$

Conversely, suppose that $\lambda(\partial\Omega) = 0$ and write $v(x) := \bar\nu_x$. Let $\mathscr{B}$ be a potential operator for $\mathscr{A}$ such that (4.5) *and* (4.6) *hold. Suppose that the inequalities* (6.1) *hold. Then there exist sequences (u_j), $(\tilde u_j) \subset \mathrm{C}^\infty_c(\Omega,\mathbb{V})$ such that:*

$$\begin{aligned} &(v + \mathscr{B}u_j + \mathscr{B}\tilde u_j) \text{ generates } \boldsymbol{\nu},\\ &(D^k u_j) \text{ is } p\text{-uniformly integrable},\\ &D^k\tilde u_j \to 0 \text{ in measure.} \end{aligned}$$

This shows that quasiconvexity is intrinsic to weak convergence of PDE constrained sequences. The fact that for an $\mathscr{A}$-free Young measure we have the Jensen inequalities in (6.1) was proved in the course of proving Theorems 5.1 and 5.6, see also (5.2) and (5.4). To prove the converse, we first consider the case of x-independent Young measures, so called *homogeneous Young measures.* We will then formulate an approximation argument.

To this end, we first restrict our setup to the homogeneous case as follows: Let $Q \subset \mathbb{R}^n$ be a cube and $z \in \mathbb{V}$ and define

$$\mathrm{Y}^p_h(z) := \Big\{(\nu^0,\nu^\infty) \in \mathcal{M}^+_1(\mathbb{V}) \times \mathcal{M}^+(S_\mathbb{V})\colon \text{there exist } u_j \in \mathrm{C}^\infty_c(Q,\mathbb{U}) \text{ s.t. for all } \Phi \in \mathbb{E}_{p,a} \\ \lim_{j\to\infty} \fint_Q \Phi(z + \mathscr{B}u_j(x))\mathrm{d}x = \langle \nu^0,\Phi\rangle + \langle\nu^\infty,\Phi^\infty_p\rangle\Big\},$$

where $\mathbb{E}_{p,a}(\mathbb{V})$ denotes the set of autonomous integrands in $\mathbb{E}_p(\Omega,\mathbb{V})$, i.e.,

$$\mathbb{E}_{p,a}(\mathbb{V}) = \left\{\Phi \in \mathrm{C}(\mathbb{V})\colon \Phi^\infty_p(x,z) := \lim_{t\to\infty}\frac{\Phi(tz)}{t^p} \in \mathbb{R} \text{ locally uniformly for } z \in \mathbb{V}\right\}.$$

It is easy to see that, with the norm induced from $\mathbb{E}_p(\Omega,\mathbb{V})$, we have that

$$\mathrm{Y}^p_h(z) \subset \mathbb{E}_{p,a}(\mathbb{V})^* \simeq \mathcal{M}(\mathbb{V}) \times \mathcal{M}(S_\mathbb{V}),$$

where the isomorphism is given by the map T_p. We record that, since $p > 1$, we have that $\bar\nu^0 = z$ for elements of $\mathrm{Y}^p_h(z)$. Finally, let us mention that in the "inhomogenization" argument we will only look at measures $(\nu^0,\nu^\infty) \in \mathrm{Y}^p_h$ that have $\nu^0 = \delta_0$ or $\nu^\infty \equiv 0$.

We now formulate the homogeneous step of the converse of Theorem 6.1:

Proposition 6.2 (Characterization of homogeneous $\mathscr{A}$-free Young measures)**.** *Suppose that $\mathscr{A}$ satisfies the* (SC) *and* (CR) *conditions. Let $\mathscr{B}$ be a potential operator for $\mathscr{A}$ such that* (4.5) *and* (4.6) *hold. Let $\nu := (\nu^0,\nu^\infty) \in \mathcal{M}^+_1(\mathbb{V}) \times \mathcal{M}^+(S_\mathbb{V})$ and $z \in \mathbb{V}$.*

Then $\nu \in \mathrm{Y}^p_h(z)$ if and only if $\bar\nu^0 = z$ and

$$\langle \nu^0, f\rangle + \langle \nu^\infty, f^\infty_p\rangle \geqslant f(z) \quad \text{for all } \mathscr{A}\text{-quasiconvex } f\colon \mathbb{V}\to\mathbb{R} \text{ satisfying (G-}p\text{)}.$$

Before we can give a proof, we need a few auxiliary statements concerning properties of homogeneous $\mathscr{A}$-free Young measures and of $\mathscr{A}$-quasiconvex envelopes. To this end, we define the homogeneous elementary measures (without a change in notation) by

$$\langle\!\langle \boldsymbol{\varepsilon}_v, \Phi \rangle\!\rangle := \langle \boldsymbol{\varepsilon}_v, \Phi \rangle_{\mathbb{E}^*_{p,a},\mathbb{E}_{p,a}} := \fint_Q \Phi(v(x))\mathrm{d}x \quad \text{for } v \in \mathrm{L}^p(Q,\mathbb{V}),\ \Phi \in \mathbb{E}_{p,a}(\mathbb{V}).$$

We also use the notation $\langle\!\langle\,\bullet\,,\,\bullet\,\rangle\!\rangle$ for the duality pairing between $\mathbb{E}^*_{p,a}$ and $\mathbb{E}_{p,a}$ in general.

Lemma 6.3. *The set $\{\boldsymbol{\varepsilon}_{z+\mathscr{B}u} \colon u \in \mathrm{C}^\infty_c(Q,\mathbb{U})\}$ is weakly-* dense in $\mathrm{Y}^p_h(z)$. Also, $\mathrm{Y}^p_h(z)$ is weakly-* closed and convex.*

Proof. The density claim follows by definition of $\mathrm{Y}^p_h(z)$. Next, let $\nu = (\nu^0, \nu^\infty)$ lie in its weakly-* closure. Consider a dense subset $\{\Phi_j\} \subset \mathbb{E}_{p,a}(\mathbb{V})$ such that $\Phi_0 := |\bullet|^p$. Then for each j, there exists $u_j \in \mathrm{C}^\infty_c(Q,\mathbb{U})$ such that

$$\left| \langle\!\langle \nu, \Phi_i \rangle\!\rangle - \fint_Q \Phi_i(z + \mathscr{B}u_j(x))\mathrm{d}x \right| \leqslant \frac{1}{j} \quad \text{for all } i = 0, \dots, j.$$

We also have that $(z + \mathscr{B}u_j)$ is bounded in L^p, so that $(\boldsymbol{\varepsilon}_{z+\mathscr{B}u_j})$ is bounded in $\mathbb{E}^*_{p,a}$. By Theorem A.2, we have that $\boldsymbol{\varepsilon}_{z+\mathscr{B}u_j} \overset{*}{\rightharpoonup} \tilde{\nu}$ in $\mathbb{E}^*_{p,a}$ on a subsequence. By the inequality above, we have that $\tilde{nu} = \nu$, so that $\boldsymbol{\varepsilon}_{z+\mathscr{B}u_j} \overset{*}{\rightharpoonup} \nu$ on the full sequence. It follows that $\nu \in \mathrm{Y}^p_h(z)$, so the closedness property is proved.

Finally, to prove the convexity, we let ν_0, $\nu_1 \in \mathrm{Y}^p_h(z)$, $t \in (0,1)$, and $\nu_t := t\nu_0 + (1-t)\nu_1$. We claim that $\nu_t \in \mathrm{Y}^p_h(z)$. By the weakly-* closure and density properties showed above, it suffices to prove the claim for

$$\nu_0 = \boldsymbol{\varepsilon}_{z+\mathscr{B}u_0},\ \nu_1 = \boldsymbol{\varepsilon}_{z+\mathscr{B}u_1}, \text{ and } t = (p/q)^n, \quad p,\ q \in \mathbb{N}.$$

Consider a regular mesh of q^n cubes of side length q^{-1} that cover Q. Let $\{x_i\}_{i=1}^{q^n}$ denote the centers of these cubes and define

$$u(x) := \sum_{i=1}^{p^n} q^k u_1(q(x - x_i)) + \sum_{i=p^n+1}^{q^n} u_0(q(x - x_i)) \quad \text{for } x \in Q.$$

It is easy to see that $u \in \mathrm{C}^\infty_c(Q,\mathbb{U})$ and to check explicitly that $\nu_t = \boldsymbol{\varepsilon}_{z+\mathscr{B}u}$. The proof is complete. ☐

Let $f \in \mathrm{C}(\mathbb{V})$. We define its $\mathscr{A}$-quasiconvex envelope by

$$f^{\mathrm{qc}}_{\mathscr{A}}(z) := \inf\left\{ \fint_{\mathbb{T}^n} f(z + v(x))\mathrm{d}x \colon v \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{V}),\ \mathscr{A}v = 0,\ \int_{\mathbb{T}^n} v(x)\mathrm{d}x = 0 \right\} \quad \text{for } z \in \mathbb{V}.$$

It can be shown that $f^{\mathrm{qc}}_{\mathscr{A}}$ is the largest $\mathscr{A}$-quasiconvex function below f, but we will not use this in the sequel. Instead we will use the following:

Lemma 6.4. *Let $f \in \mathrm{C}(\mathbb{V})$. We have that*

$$f^{\mathrm{qc}}_{\mathscr{A}}(z) = \inf\left\{ \fint_Q f(z + \mathscr{B}u(x))\mathrm{d}x \colon u \in \mathrm{C}^\infty_c(Q,\mathbb{U}) \right\} \quad \text{for } z \in \mathbb{V} \tag{6.2}$$

and $f^{\mathrm{qc}}_{\mathscr{A}}$ is $\mathscr{A}$-quasiconvex. Moreover, if f satisfies (G-p) *and $f^{\mathrm{qc}}_{\mathscr{A}}(z_0) > -\infty$ for some $z_0 \in \mathbb{V}$, we have that $f^{\mathrm{qc}}_{\mathscr{A}}$ satisfies* (G-p).

Proof. Let $z \in \mathbb{V}$. By Lemma 1.24, it is clear that

$$f^{\mathrm{qc}}_{\mathscr{A}}(z) = \inf\left\{ \fint_{\mathbb{T}^n} f(z + \mathscr{B}u(x))\mathrm{d}x \colon u \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{U}) \right\}.$$

Let $\varepsilon > 0$. We next show that for each $u \in \mathrm{C}^\infty(\mathbb{T}^n,\mathbb{U})$ we can find $\tilde{u} \in \mathrm{C}^\infty_c(Q,\mathbb{U})$ such that

$$\fint_Q f(z + \mathscr{B}\tilde{u}(x))\mathrm{d}x \leqslant \fint_{\mathbb{T}^n} f(z + \mathscr{B}u(x))\mathrm{d}x + \varepsilon$$

Here we identified the torus $\mathbb{T}^n$ with a cube Q. Define $u_j(x) := j^{-k}u(jx)$, so that

$$\fint_Q f(z + \mathscr{B}u_j(x))\mathrm{d}x = \fint_Q f(z + \mathscr{B}u(x))\mathrm{d}x.$$

We choose a cut off function $\rho_\delta \in \mathrm{C}^\infty_c(Q,[0,1])$, such that $\rho(x) = 1$ at all x such that $\mathrm{dist}(x,\partial Q) > \delta$ and $\|\partial^\alpha \rho_\delta\|_{\mathrm{L}^\infty} \leqslant c_0\delta^{-|\alpha|}$ for $|\alpha| \leqslant k$. Therefore

$$|\mathscr{B}(\rho_\delta u_j)| \leqslant |\rho_\delta \mathscr{B}u_j| + c_1 \sum_{i=1}^k |D^i\rho_\delta||D^{k-i}\mathscr{B}u_j| \leqslant \|\mathscr{B}u\|_{\mathrm{L}^\infty} + c_0c_1 \sum_{i=1}^k (j\delta)^{-i}\|D^{k-i}u\|_{\mathrm{L}^\infty}.$$

We require $j\delta \geqslant 1$. Let $M := \sup f\left(B(0, \|\mathscr{B}u\|_{\mathrm{L}^\infty} + c_0c_1\|u\|_{\mathrm{W}^{k-1,\infty}})\right)$ and $Q^\delta := \{x \in Q\colon \operatorname{dist}(x\partial Q) > \delta\}$. We have

$$\fint_Q f(z + \mathscr{B}(\rho_\delta u_j)(x))\mathrm{d}x \leqslant \int_{Q\setminus Q^\delta} M\mathrm{d}x + \int_{Q^\delta} f(z+\mathscr{B}u_j(x))\mathrm{d}x \leqslant M\mathscr{L}^n(Q^\delta) + \fint_Q f(z+\mathscr{B}u(x))\mathrm{d}x.$$

Requiring $M\mathscr{L}^n(Q^\delta) < \varepsilon$ completes the proof of (6.2).

The remainder of the proof follows by simple modifications of standard arguments in the calculus of variations, see, e.g., [7, Sec. 6.3], and is left as an exercise to the reader. □

Proof of Proposition 6.2. First, assume that $\nu \in \mathrm{Y}^p_h(z)$ and let $f\colon \mathbb{V} \to \mathbb{R}$ be $\mathscr{A}$-quasiconvex. We first construct $\mathscr{A}$-quasiconvex $g := \max\{f, f^\infty_p + \delta|\bullet|^p - \delta^{-1}\}$ for small $\delta > 0$, which satisfies $g = f^\infty_p + \delta|\bullet|^p - \delta^{-1}$ outside a large set depending on δ, so $g \in \mathbb{E}_{p,a}(\mathbb{V})$; moreover, $g \downarrow f$ and $g^\infty_p \downarrow f^\infty_p$ pointwise as $\delta \downarrow 0$. We can therefore write by use of the monotone convergence theorem that

$$\langle \nu^0, f\rangle + \langle \nu^\infty, f^\infty_p\rangle = \lim_{\delta\downarrow 0}\langle \nu^0, g\rangle + \langle \nu^\infty, g^\infty_p\rangle = \lim_{\delta\downarrow 0}\lim_{j\to\infty}\fint_Q g(z+\mathscr{B}u_j(x))\mathrm{d}x \geqslant \lim_{\delta\downarrow 0} g(z) = f(z),$$

which concludes the proof of the implication.

Conversely, we use the Hahn–Banach separation theorem in the form of C.2, which is applicable to $\mathrm{Y}^p_h(z)$ by Lemma 6.3, to write

$$\mathrm{Y}^p_h(z) = \bigcap\{H \subset \mathbb{E}_{p,a}(\mathbb{V})^*\colon \mathrm{Y}^p_h(z) \subset H \text{ halfspace}\}.$$

Let $H = \{\nu \in \mathcal{M}(\mathbb{V})\times\mathcal{M}(S_{\mathbb{V}})\colon \langle\!\langle \nu, \Phi\rangle\!\rangle \geqslant t\}$ for some $\Phi \in \mathbb{E}_{p,a}(\mathbb{V})$ and some $t \in \mathbb{R}$ be a halfspace containing $\mathrm{Y}^p_h(z)$. Therefore, for all $u \in \mathrm{C}^\infty_c(Q, \mathbb{U})$, we have that

$$t \leqslant \langle\!\langle \boldsymbol{\varepsilon}_{z+\mathscr{B}u}, \Phi\rangle\!\rangle = \fint_Q \Phi(z+\mathscr{B}u(x))\mathrm{d}x,$$

so that, by Lemma 6.4, we have that

$$t \leqslant \Phi^{\mathrm{qc}}_{\mathscr{A}}(z) \leqslant \langle\!\langle \nu, \Phi^{\mathrm{qc}}_{\mathscr{A}}\rangle\!\rangle \leqslant \langle\!\langle \nu, \Phi\rangle\!\rangle, \tag{6.3}$$

which implies that $\nu \in H$. This is true for all such H, so it follows that indeed $\nu \in \mathrm{Y}^p_h(z)$. It remains to justify the second and third inequality in (6.3). The second of these follows simply by the assumed Jensen inequality, whereas the third is due to the fact that $\Phi^{\mathrm{qc}}_{\mathscr{A}} \leqslant \Phi$. The proof is complete. □

We can now proceed with the proof of the inhomogeneous result.

Proof of Theorem 6.1. We already explained that we need only prove the converse. To this end, let $\boldsymbol{\nu} \in \mathrm{Y}^p(\Omega, \mathbb{V})$ be such that $\lambda(\partial\Omega) = 0$. By Theorem 2.17 and 2.18, we have that it suffices to show that there exist sequences (u_j), $(\tilde{u}_j) \subset \mathrm{C}^\infty_c(\Omega, \mathbb{U})$ such that

$$(v + \mathscr{B}u_j) \text{ generates } ((\nu_x)_{x\in\Omega}, 0, \mathtt{n/a}) \quad\text{and}\quad (\mathscr{B}\tilde{u}_j) \text{ generates } ((\delta_0)_{x\in\Omega}, \lambda, (\nu^\infty_x)_{x\in\Omega}).$$

Indeed, this is enough since we would have that $(\mathscr{B}u_j)$ is p-uniformly integrable and $\mathscr{B}\tilde{u}_j \to 0$ in measure, while both sequences converge weakly to 0 in $\mathrm{L}^p(\Omega, \mathbb{V})$. In this case, one can apply the Decomposition Lemma 5.7 to refine the two sequences.

We will test with integrands $\varphi \otimes \Phi \in \mathrm{C}(\bar{\Omega}) \times \mathbb{E}_{p,a}(\mathbb{V})$ as given by Lemma 5.5. In particular, we can assume that $\|\varphi\|_{\mathrm{Lip}}, \|T_p\Phi\|_{\mathrm{Lip}} \leqslant 1$, so that

$$|\Phi(z) - \Phi(z')| \leqslant c|z - z'|(1 + |z| + |z'|)^{p-1} \quad \text{for } z, z' \in \mathbb{V}.$$

We are working with $\|\bullet\|_{\mathrm{Lip}} := \|\bullet\|_{\mathrm{L}^\infty} + \|D\bullet\|_{\mathrm{L}^\infty}$. Let $\varepsilon > 0$.

We write $g(x) := \varphi(x)\langle \nu_x, \Phi\rangle$ and $g_0(x) := \langle \nu_x, \Phi_0\rangle$ where $\Phi_0 = (1 + |\bullet|)^p$, so $g, g_0 \in \mathrm{L}^1(\Omega)$ by the moment condition. We apply Lusin's theorem in the following way: There exists a compact set $C \subset \Omega$ such that, with $G = (g, g_0)$,

$$\mathscr{L}^n(\Omega \setminus C) < \varepsilon|\Omega|, \quad \int_{\Omega\setminus C} |G|\mathrm{d}x < \varepsilon|\Omega|, \quad \text{and } G\big|_C \text{ is continuous.}$$

Using Theorem B.4, we can find $\tilde{G} =: (\tilde{g}, \tilde{g}_0) \in \mathrm{C}(\bar{\Omega})$ such that $\tilde{G} = G$ in C and $\|\tilde{G}\|_{\mathrm{L}^\infty(\Omega)} = \|G\|_{\mathrm{L}^\infty(C)}$. Moreover, $\tilde{G}$ is uniformly continuous, so we can find $\delta \in (0, \varepsilon)$ such that $|\tilde{G}(x) - \tilde{G}(x')| < \varepsilon$ whenever $|x - x'| < \delta$. Finally, consider a regular grid of cubes in $\mathbb{R}^n$ of side length $\delta/2$; we write $\mathcal{F}_\delta$ for the family of such cubes that are contained in Ω. Since $\mathscr{L}^n(\partial\Omega) = 0$, it is clear that $|\bigcup \mathcal{F}_\delta| \uparrow |\Omega|$ as $\delta \downarrow 0$. We write

$$\mathcal{F}^o_\delta := \{Q \in \mathcal{F}_\delta\colon Q \cap C \neq \emptyset\}.$$

Then $\mathcal{F}_\delta^o$ covers $C \cap \bigcup \mathcal{F}_\delta$, so that we can assume by taking δ smaller that

$$\int_{\Omega \setminus \bigcup \mathcal{F}_\delta^o} |G| \mathrm{d}x < \varepsilon |\Omega|.$$

For each cube $Q \in \mathcal{F}_\delta^o$, we choose an arbitrary $x_Q \in Q \cap C$ that are Lebesgue points of the barycenter $v(x) = \bar{\nu}_x$ and such that the oscillation Jensen inequality holds at x_Q. We also record that

$$|G(x_Q) - \tilde{G}(x)| < \varepsilon \quad \text{for all } x \in Q.$$

We can also assume that we have a piecewise constant approximation of the barycenter in $\mathrm{L}^p(\Omega, \mathbb{V})$

$$\int_\Omega |v - v^\varepsilon|^p \mathrm{d}x \leqslant \varepsilon |\Omega|, \quad \text{where } v^\varepsilon := \sum_{Q \in \mathcal{F}_\delta^o} v(x_Q) \mathbf{1}_Q.$$

As a consequence of Proposition 6.2 with $\nu^0 = \nu_{x_Q}$, $\nu^\infty = 0$ we can find $u_Q^\varepsilon \in \mathrm{C}_c^\infty(Q, \mathbb{U})$ such that

$$\left| \langle \nu_{x_Q}, \Phi \rangle - \fint_Q \Phi(\bar{\nu}_{x_Q} + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x \right| + \left| \langle \nu_{x_Q}, \Phi_0 \rangle - \fint_Q \Phi_0(\bar{\nu}_{x_Q} + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x \right| < \varepsilon.$$

Recall here that $\Phi_0 = (1 + |\,\bullet\,|)^p$. We can begin to estimate

$$\left| \int_\Omega g \mathrm{d}x - \int_{\bigcup \mathcal{F}_\delta^0} \tilde{g} \mathrm{d}x \right| \leqslant \int_{\Omega \setminus \bigcup \mathcal{F}_\delta^o} |g| \mathrm{d}x + \int_{F_\delta^o} |g - \tilde{g}| \mathrm{d}x \leqslant 2\varepsilon |\Omega|,$$

so that

$$\left| \int_{\bigcup \mathcal{F}_\delta^0} \tilde{g} \mathrm{d}x - \sum_{Q \in \mathcal{F}_\delta^o} |Q| g(x_Q) \right| \leqslant \varepsilon |\Omega|.$$

We can estimate further

$$\left| \sum_{Q \in \mathcal{F}_\delta^o} \left(|Q| g(x_Q) - \varphi(x_Q) \int_Q \Phi(\bar{\nu}_{x_Q} + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x \right) \right| \leqslant \varepsilon |\Omega|.$$

We then have that

$$\sum_{Q \in \mathcal{F}_\delta^o} \varphi(x_Q) \int_Q \Phi(\bar{\nu}_{x_Q} + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x = \sum_{Q \in \mathcal{F}_\delta^o} \int_Q \varphi(x) \Phi(\bar{\nu}_{x_Q} + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x + \mathcal{E}_1,$$

where

$$\begin{aligned} |\mathcal{E}_1| &\leqslant c \sum_{Q \in \mathcal{F}_\delta^o} \int_Q |\varphi(x_Q) - \varphi(x)| \Phi_0(\bar{\nu}_{x_Q} + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x \leqslant c\delta \sum_{Q \in \mathcal{F}_\delta^o} |Q| (\langle \nu_{x_Q}, \Phi_0 \rangle + \varepsilon) \\ &\leqslant c\delta \sum_{Q \in \mathcal{F}_\delta^o} \left(\int_Q \langle \nu_x, \Phi_0 \rangle \mathrm{d}x + 2\varepsilon |Q| \right) \leqslant c\delta \left(\int_\Omega \langle \nu_x, \Phi_0 \rangle \mathrm{d}x + 2\varepsilon |\Omega| \right), \end{aligned} \tag{6.4}$$

where the integral is finite by the moment condition. We make and recall the abbreviations

$$u^\varepsilon := \sum_{Q \in \mathcal{F}_\delta^o} u_Q^\varepsilon \in \mathrm{C}_c^\infty(\Omega, \mathbb{U}) \quad \text{and} \quad v^\varepsilon = \sum_{Q \in \mathcal{F}_\delta^o} v(x_Q) \mathbf{1}_Q \in \mathrm{L}^p(\Omega, \mathbb{V}).$$

We next look at

$$\sum_{Q \in \mathcal{F}_\delta^o} \int_Q \varphi(x) \Phi(v(x_Q) + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x = \sum_{Q \in \mathcal{F}_\delta^o} \int_Q \varphi(x) \Phi(v(x) + \mathscr{B} u_Q^\varepsilon(x)) \mathrm{d}x + \mathcal{E}_2,$$

where, by using $\|\varphi\|_{\mathrm{L}^\infty} \leqslant 1$,

$$\begin{aligned} |\mathcal{E}_2| &\leqslant c \int_\Omega |v - v^\varepsilon| \left(1 + |v + \mathscr{B} u^\varepsilon| + |v^\varepsilon + \mathscr{B} u^\varepsilon|\right)^{p-1} \mathrm{d}x \\ &\leqslant c \|v - v^\varepsilon\|_{\mathrm{L}^p(\Omega)} \|1 + |v + \mathscr{B} u^\varepsilon| + |v^\varepsilon + \mathscr{B} u^\varepsilon|\|_{\mathrm{L}^p(\Omega)}^{p-1} \\ &\leqslant c \|v - v^\varepsilon\|_{\mathrm{L}^p(\Omega)} \left(\left(\int_\Omega \Phi_0(v^\varepsilon + \mathscr{B} u^\varepsilon) \mathrm{d}x \right)^{(p-1)/p} + \|v - v^\varepsilon\|_{\mathrm{L}^p(\Omega)}^{p-1} \right) \end{aligned}$$

Since $\|v - v^\varepsilon\|_{\mathrm{L}^p(\Omega)} \leqslant (\varepsilon |\Omega|)^{1/p}$ and the estimation

$$\int_\Omega \Phi_0(v^\varepsilon + \mathscr{B} u^\varepsilon) \mathrm{d}x \leqslant c \left(\int_\Omega \langle \nu_x, \Phi_0 \rangle \mathrm{d}x + \varepsilon |\Omega| \right)$$

from (6.4), we are very close to conclude. Writing

$$\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^o}\varphi(x)\Phi(v(x)+\mathscr{B}u^\varepsilon(x))\mathrm{d}x\leqslant c\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^o}(1+|v|)^p\mathrm{d}x\leqslant c\varepsilon|\Omega|$$

and collecting estimates, we have that

$$\left|\int_\Omega\varphi(x)\langle\nu_x,\Phi\rangle\mathrm{d}x-\int_\Omega\varphi(x)\Phi(v(x)+\mathscr{B}u^\varepsilon(x))\mathrm{d}x\right|\to 0\quad\text{as }\varepsilon\downarrow 0.$$

We have thus showed that the oscillation measure has the right gradient structure.

We carry on with the concentration part. We now consider the functions $h(x):=\varphi(x)\langle\nu_x^\infty,\Phi_p^\infty\rangle$ and $h_0(x):=\langle\nu_x^\infty,\Phi_{0,p}^\infty\rangle$, where we recall that $\Phi_0=(1+|\,\bullet\,|)^p$. We have that $H:=(h,h_0)\in\mathrm{L}^1(\Omega;\mathrm{d}\lambda)$. As in the previous case, we will apply Lusin's theorem and Theorem B.4 to find a compact set $K\subset\Omega$ and a continuous extension $\tilde H=:(\tilde h,\tilde h_0)\in\mathrm{C}(\bar\Omega)$, such that

$$\lambda(\Omega\setminus K)<\varepsilon\lambda(\Omega),\qquad\int_{\Omega\setminus K}|H|\mathrm{d}\lambda<\varepsilon\lambda(\Omega),\qquad\tilde H\big|_K=H,\qquad\|\tilde H\|_{\mathrm{L}^\infty(\Omega)}=\|H\|_{\mathrm{L}^\infty(K)}.$$

We then choose $\delta\in(0,\varepsilon)$ and a collection $\mathcal{F}_\delta^c$ of cubes in *exact* analogy with the case of the oscillation measures, by replacing $\mathscr{L}^n$ with λ. At this stage we also use the assumption $\lambda(\partial\Omega)=0$. Finally, for each $Q\in\mathcal{F}_\delta^c$, we choose $x_Q\in Q\cap K$ such that the concentration Jensen inequality holds at x_Q.

By Proposition 6.2 with $\nu^0=\delta_0$ and $\nu^\infty=|Q|^{-1}\lambda(Q)\nu_{x_Q}^\infty$ we can find $\tilde u_Q^\varepsilon\in\mathrm{C}_c^\infty(Q,\mathbb{U})$ such that

$$\left|\Phi(0)+\frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty,\Phi_p^\infty\rangle-\fint_Q\Phi(\mathscr{B}\tilde u_Q^\varepsilon(x))\mathrm{d}x\right|+\left|\Phi_0(0)+\frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty,\Phi_{0,p}^\infty\rangle-\fint_Q\Phi_0(\mathscr{B}\tilde u_Q^\varepsilon(x))\mathrm{d}x\right|<\varepsilon.$$

We can then estimate

$$\int_\Omega\varphi\mathrm{d}x\Phi(0)+\int_\Omega h\mathrm{d}\lambda=\int_{\Omega\setminus\bigcup\mathcal{F}_\delta^c}\varphi\mathrm{d}x\Phi(0)+\sum_{Q\in\mathcal{F}_\delta^c}|Q|\varphi(x_Q)\left(\Phi(0)+\frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty,\Phi_p^\infty\rangle\right)+\mathcal{E}_3,$$

where

$$|\mathcal{E}_3|\leqslant\sum_{Q\in\mathcal{F}_\delta^c}\int_Q|\varphi-\varphi(x_Q)|\mathrm{d}x|\Phi(0)|+2\varepsilon\lambda(\Omega)+\int_Q|\varphi||\tilde h-h(x_Q)|\mathrm{d}\lambda\leqslant\delta|\Omega|+3\varepsilon\lambda(\Omega).$$

We next focus on

$$\left|\sum_{Q\in\mathcal{F}_\delta^c}\left(|Q|\varphi(x_Q)\left(\Phi(0)+\frac{\lambda(Q)}{|Q|}\langle\nu_{x_Q}^\infty,\Phi_p^\infty\rangle\right)-\varphi(x_Q)\int_Q\Phi(\mathscr{B}\tilde u_Q^\varepsilon(x))\mathrm{d}x\right)\right|\leqslant\varepsilon|\Omega|.$$

Defining $\tilde u^\varepsilon:=\tilde u_Q^\varepsilon$ on each $Q\in\mathcal{F}_\delta^c$ and extending by zero to the rest of Ω, we obtain $\tilde u^\varepsilon\in\mathrm{C}_c^\infty(\Omega,\mathbb{U})$. Further, we have

$$\sum_{Q\in\mathcal{F}_\delta^c}\varphi(x_Q)\int_Q\Phi(\mathscr{B}\tilde u_Q^\varepsilon(x))\mathrm{d}x=\int_{\bigcup\mathcal{F}_\delta^c}\varphi(x)\Phi(\mathscr{B}\tilde u_Q^\varepsilon(x))\mathrm{d}x+\mathcal{E}_4,$$

where

$$\begin{aligned}|\mathcal{E}_4|&\leqslant c\delta\sum_{Q\in\mathcal{F}_\delta^c}\int_Q\Phi_0(\mathscr{B}\tilde u_Q^\varepsilon(x))\mathrm{d}x\leqslant c\delta\left(\varepsilon|\Omega|+\sum_{Q\in\mathcal{F}_\delta^c}|Q|\Phi_0(0)+\lambda(Q)h_0(x_Q)\right)\\&\leqslant c\delta\left(\varepsilon|\Omega|+\int_{\bigcup\mathcal{F}_\delta^c}|h_0|\mathrm{d}\lambda\right)\leqslant c\delta\left(\varepsilon|\Omega|+\lambda(\Omega)\right)\end{aligned}$$

where the last integral can be computed explicitly. Collecting, we proved that

$$\left|\int_\Omega\varphi\mathrm{d}x\Phi(0)+\int_\Omega\langle\nu_x^\infty,\Phi_p^\infty\rangle\mathrm{d}\lambda(x)-\int_\Omega\varphi(x)\Phi(\mathscr{B}\tilde u_Q^\varepsilon(x))\mathrm{d}x\right|\to 0\quad\text{as }\varepsilon\downarrow 0.$$

The proof is complete. □

We conclude these notes with a few remarks. The proofs of this section also apply to the unconstrained case $\mathscr{A}\equiv 0$, when one can choose $\mathscr{B}=\mathrm{Id}_{\mathbb{V}}$ and $\mathscr{A}$-quasiconvexity is simply convexity. This has the advantage that the autonomous convex integrands of p-growth always lie in $\mathbb{E}_{p,a}(\mathbb{V})$, so lower semi-continuity results for signed functionals are free of technical conditions (cf., Theorem 5.6). Also, the inequalities in (6.1) follow from the actual Jensen inequality. Therefore, we can obtain an alternative proof of the fact of Theorem B.6, that each parametrized measure in $\mathrm{Y}^p(\Omega,\mathbb{V})$ can be generated by a sequence that converges weakly in L^p.

Finally, we briefly discuss the case $p = 1$, (when we require the PDE constraint to hold in $\mathrm{W}^{-\ell,q}$ for some $q > 1$) which is obviously different since the barycenter of the Young measure retains information from the concentration part. Technically, this difference is seen since if $\Phi(x, z) = \varphi(x)z$, we have that $\Phi_p^\infty \equiv 0$ for $p > 1$, but $\Phi_1^\infty = \Phi$. However, only the first part of the proof is needed to prove the same result in the case $p = 1$. This is, of course, an artifact of weak convergence in L^1 and of the Dunford–Pettis criterion, which implies $\lambda \equiv 0$. If we allow Young measures in $\mathbf{Y}^1(\Omega, \mathbb{V})$ to be generated by sequences that converge weakly-* in $\mathcal{M}(\Omega, \mathbb{V})$, the Jensen type inequalities become

$$\langle f, \nu_x\rangle + \lambda^a(x)\langle f_1^\infty, \nu_x\rangle \geqslant f(\bar{\nu}_x + \lambda^a(x)\bar{\nu}_x^\infty) \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega,$$
$$\langle f_1^\infty, \nu_x^\infty\rangle \geqslant f_1^\infty(\bar{\nu}_x^\infty) \quad \text{for } \lambda^s\text{-a.e. } x \in \Omega$$

for all $\mathscr{A}$-quasiconvex $f\colon \mathbb{V} \to \mathbb{R}$ satisfying $|f| \leqslant c(1 + |\,\bullet\,|)$. Here we wrote $\lambda = \lambda\mathscr{L}^n \llcorner \bar{\Omega} + \lambda^s$ for the Radon–Nýkodim decomposition of λ with respect to Lebesgue measure. For completeness, we mention that then $v = (\bar{\nu}_x + \lambda^a(x)\bar{\nu}_x^\infty)\mathscr{L}^n \llcorner \Omega + \bar{\nu}_x^\infty \lambda^s$ is the decomposition of the barycenter.

An minor adaptation of the proof yields the same result if the barycenter v is an L^1 function, i.e., $\bar{\nu}_x^\infty = 0$ for λ^s-a.e. $x \in \Omega$[8]. The characterization result also holds in the case when the barycenter is a general measure v with $\mathscr{A}v \in \mathrm{W}^{-\ell,q}(\Omega, \mathbb{W})$, but the proof is substantially more involved.

Another important peculiarity of the (general) $p = 1$ case is that the singular singular Jensen type inequality is not necessary for the characterization. This remarkable fact can be proved using very recent powerful results and is very far from being true for $p > 1$.

[8] Of course, in this case, one need not have $\lambda^s \equiv 0$.

Bibliographical remarks

As the original text introducing (oscillation) Young measures, we refer the reader to the L.C. Young's book [19], see also [3, 18, 13, 14, 16]. In the context of the compensated compactness framework (CC), L. Tartar used Young measures in the context of conservation laws [17]. Oscillation Young measures were also used in [4, 6] to describe variational continuum models for microstructure. The first characterizations of oscillation Young measures generated by sequences of gradients by duality with quasiconvex functions are due to D. Kinderlehrer and P. Pedregal [11, 12]. The idea to use a compactification to describe directions of concentration first appeared in the work [8], due to R. DiPerna and A. Majda, concerning equations describing incompressible fluids. A characterization of oscillation and concentration effects in sequences of gradients appeared in [10]. There, unlike in our notes, varifolds are used to capture the concentration behaviour. Our set up is slightly different and follows that of J.J. Alibert and G. Bouchitté for $p = 1$ [1], which also differs from the DiPerna–Majda measures. Generalized Young measures were also used more recently for problems of linear growth in the calculus of variations, see [15].

Appendix A. Functional analysis

Theorem A.1 (Ascoli–Arzelà). *Let (M, d) be a separable metric space. If $\mathcal{F} \subset \mathrm{C}(M, \mathbb{R}^N)$ is a* pointwise bounded[9] *and* equi-continuous[10] *family of functions, then any sequence (f_j) from $\mathcal{F}$ admits a subsequence that converges uniformly on all compact subsets of M.*

The Ascoli–Arzelà theorem implies:

Theorem A.2 (Sequential Banach–Alaoglu). *Let X be a* ***separable*** *Banach space and $(x_j^*) \subset X^*$ be a sequence of bounded linear functionals on X with $\sup_j \|x_j^*\|_{X^*} < \infty$. Then there exists a subsequence $(x_{j_i}^*)$ and $x^* \in X^*$ such that*

$$x_{j_i}^* \overset{*}{\rightharpoonup} x^* \text{ in } X^*,$$

meaning that $\langle x_{j_i}^*, x\rangle \to \langle x^*, x\rangle$ for all $x \in X$. Moreover, $\|x^*\|_{X^*} \leqslant \liminf_{i\to\infty} \|x_{j_i}\|_{X^*}$.

We recall briefly that the weak and weakly-* topologies on infinite dimensional spaces cannot be described by convergence of sequences. To see this, consider the example $X = \ell^2$ and the set $A = \{\sqrt{j} e_j\}_{j\in\mathbb{N}}$, where (e_j) is an orthonormal basis. Then 0 lies in the weak closure of A, but no sequence in A converges weakly to zero. In particular, it is quite important to emphasize that in this course we study weak(ly-*) *sequential* lower semi-continuity.

We also have the L^p variant of the Ascoli–Arzelà theorem:

Theorem A.3 (Riesz–Kolmogorov). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leqslant p < \infty$. A family $\mathcal{F} \subset \mathrm{L}^p(\Omega, \mathbb{V})$ is relatively strongly compact if and only if both of the following two conditions hold:*

- (a) $\sup_{v\in\mathcal{F}} \|v\|_{\mathrm{L}^p(\Omega)} < \infty$.
- (b) $\sup_{v\in\mathcal{F}} \int_{\Omega\cap(\Omega-h)} |v(x+h) - v(x)|^p \,\mathrm{d}x \to 0$ *as* $|h| \to 0$.

This clearly is a much more stronger requirement on the family of L^p functions than the weak compactness criterion in Theorem 1.12(a). The property in (b) can be viewed as an "almost differentiability" requirement; in fact we can formulate the following:

Remark A.4 (Nikolskiĭ conditions and strong L^p-compactness). The condition in (b) can be rephrased as follows: There exists a continuous, increasing modulus of continuity $\omega\colon [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ such that

$$\int_{\Omega\cap(\Omega-h)} |v(x+h) - v(x)|^p \,\mathrm{d}x \leqslant \omega(|h|) \quad \text{for } |h| \leqslant 1,\ v \in \mathcal{F}.$$

In the case when $\omega(t) = t^{\alpha p}$ for some $0 < \alpha \leqslant 1$, we have that elements of $\mathcal{F}$ are uniformly bounded in the Nikolskiĭ space $\mathrm{B}^\alpha_{p,\infty}$. These are spaces in which the fractional Sobolev spaces $\mathrm{W}^{\alpha+\epsilon,p}$ embed locally. In particular, we see that fractional differentiability and/or embeddings into fractional Sobolev spaces are very natural sufficient conditions for strong compactness of embeddings into L^p spaces.

Appendix B. Measure theory

Proposition B.1 (On convergence in measure). *Let $\Omega \subset \mathbb{R}^n$ be a Borel set of finite Lebesgue measure and v_j, $v \in \mathrm{L}^0(\Omega, \mathbb{V})$, the space of equivalence classes of measurable maps from Ω into $\mathbb{V}$. Recall that we say that $v_j \to v$ in ($\mathscr{L}^n$-)measure if and only if for any $\varepsilon > 0$*

$$\mathscr{L}^n\left(\{x \in \Omega\colon |v_j(x) - v(x)| < \varepsilon\}\right) \to 0 \quad \text{as } j \to \infty.$$

If $v_j(x) \to v(x)$ for $\mathscr{L}^n$-a.e. $x \in \Omega$, then $v_j \to v$ in measure. If $v_j \to v$ in measure, then there exists a subsequence such that $v_{j_i}(x) \to v(x)$ for $\mathscr{L}^n$-a.e. $x \in \Omega$.

If we define

$$\mathrm{d}_0(v, \tilde{v}) := \int_\Omega \frac{|v(x) - \tilde{v}(x)|}{1 + |v(x) - \tilde{v}(x)|} \,\mathrm{d}x \quad \text{for } v,\ \tilde{v} \in \mathrm{L}^0(\Omega, \mathbb{V}),$$

then d_0 is a complete metric on $\mathrm{L}^0(\Omega, \mathbb{V})$, such that $v_j \to v$ in measure if and only if $\mathrm{d}_0(v_j, v) \to 0$.

On the other hand, there is no *metric/topology that corresponds to convergence $\mathscr{L}^n$-a.e..*

Lemma B.2 (Luzin for normal integrands). *Let $\Omega \subset \mathbb{R}^n$ be bounded, open, and have $\mathscr{L}^n(\Omega) = 0$. Assume that $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ is a normal integrand. Then for each $\varepsilon > 0$, there exists a compact subset $C_\varepsilon \subset \Omega$ such that $\mathscr{L}^n(\Omega \setminus C_\varepsilon) < \varepsilon$ and the restriction $F\colon C_\varepsilon \times \mathbb{V} \to \mathbb{R}$ is jointly lower semi-continuous.*

This can be seen from Step 1 of the proof of [9, Thm. 6.28]. As a consequence, we have:

[9] meaning $\sup_{f\in\mathcal{F}} |f(x)| < \infty$ for all $x \in M$.

[10] meaning that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $\mathrm{d}(x, y) < \delta$ and $f \in \mathcal{F}$; for example, a uniformly Lipschitz sequence.

Theorem B.3 (Scorza Dragoni). *Let $\Omega \subset \mathbb{R}^n$ be bounded, open, and have $\mathscr{L}^n(\Omega) = 0$. Assume that $F\colon \Omega \times \mathbb{V} \to \mathbb{R}$ is a Carathéodory integrand. Then for each $\varepsilon > 0$, there exists a compact set $C_\varepsilon \subset \Omega$ such that $\mathscr{L}^n(\Omega \setminus C_\varepsilon) < \varepsilon$ and the restriction $F\colon C_\varepsilon \times \mathbb{V} \to \mathbb{R}$ is jointly continuous.*

Theorem B.4 (Tietze–Urysohn–Brouwer). *Let M be a metric space, $A \subset M$ be a closed set and $F\colon A \to \mathbb{R}$ be a continuous function. Then there exists continuous function $\tilde{F}\colon M \to \mathbb{R}$ such that $\tilde{F}(x) = F(x)$ for all $x \in A$ and $\sup_{x\in A} |F(x)| = \sup_{x\in M} |\tilde{F}(x)|$.*

Definition B.5 (Parametrized measures). *Let M, N be locally compact, separable metric spaces and $\lambda \in \mathcal{M}^+(M)$. A measure valued map $\boldsymbol{\nu}\colon M \to \mathcal{M}^+(N)$ is said to be (weakly-*) λ-*measurable *if and only if the map $x \mapsto \langle \boldsymbol{\nu}(x), \varphi\rangle$ is λ-measurable for each $\varphi \in \mathrm{C}_0(N)$ (see also Theorem B.11 below for notation and consistency). We will identify the map $\boldsymbol{\nu}$ with a* parametrized measure *$(\nu_x)_{x\in M}$, where $\nu_x = \boldsymbol{\nu}(x)$.*

Pertaining to parametrized measures, we will discuss the following disintegration theorem:

Theorem B.6 (Disintegration). *Let E, F be open or closed subsets of (possibly different) Euclidean spaces and let $\mu \in \mathcal{M}^+(E\times F)$. Denote by $\gamma \in \mathcal{M}^+(E)$ the projection of μ, i.e., $\gamma(A) := \mu(A\times F)$ for all $A \in \mathbb{B}(E)$. Then there exists a γ-essentially unique, γ-measurable parametrized measure $(\eta_x)_{x\in E} \subset \mathcal{M}^+_1(F)$ such that $\mu = \gamma \otimes \eta_x$, i.e.,*

$$\mu(A) = (\gamma \otimes \eta_x)(A) := \int_E \int_F \mathbf{1}_A(x, z)\mathrm{d}\eta_x(z)\mathrm{d}\gamma(x) \quad \text{for } A \in \mathbb{B}(E \times F),$$

or, equivalently,

$$\int_{E\times F} f(x, z)\mathrm{d}\mu(x, z) = \int_E \int_F f(x, z)\mathrm{d}\eta_x(z)\mathrm{d}\gamma(x) \quad \text{for } f \in \mathrm{L}^1(E \times F; \mathrm{d}\mu).$$

Here we denoted by $\mathbb{B}(E)$ the Borel σ-algebra on E. This result plays a crucial role in the back end of the construction of the Young measures, see Theorem 2.7. A proof can be found in [2, Thm. 2.28].

We will also sparsely use the concept of the push-forward of a measure:

Definition B.7 (Push-forward of a measure). *Let (X_i, Σ_i), $i = 1, 2$, be measure spaces and μ be a measure on X_1, and $f\colon X_1 \to X_2$ be a measurable mapping. The* push-forward *of μ by f is the measure $f_\#\mu$ on X_2 defined by*

$$f_\#\mu(A) := \mu(f^{-1}(A)) \quad \text{for } A \in \Sigma_2,$$

or equivalently

$$\int_{X_2} g\mathrm{d}f_\#\mu = \int_{X_1} g(f(x))\mathrm{d}\mu(x) \quad \text{whenever } g \circ f \in \mathrm{L}^1(X_1; \mathrm{d}\mu).$$

We will now recall the identifications of the duals of important subspaces of $\mathrm{B}(\Omega)$, the space of bounded measurable functions on a measure space Ω. This is a Banach space under the supremum norm. We follow [9, Sec. 1.3], where proofs and more detail (in particular, very general assumptions on the underlined measure spaces) can be found.

For the remainder of this section, we let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set. We write $\mathrm{ba}(\Omega)$ for the space of finitely additive signed measures on Ω which have bounded total variation, the definition of which we recall here

$$\|\mu\|_{\mathrm{TV}} := \sup \left\{ \sum_j |\mu(A_j)|\colon \{A_j\} \subset \mathbb{B}(\Omega) \text{ finite partition of } \Omega \right\}.$$

Theorem B.8 (Riesz representation theorem in B). *Each bounded linear functional L on $\mathrm{B}(\Omega)$ can be identified with a unique measure $\mu \in \mathrm{ba}(\Omega)$ via*

$$L(f) = \int_\Omega f\mathrm{d}\mu \quad \text{for } f \in \mathrm{B}(\Omega), \quad \text{and} \quad \|L\|_{\mathrm{L}^\infty(\Omega)^*} = \|\mu\|_{\mathrm{TV}}.$$

Conversely, all such measures define appropriate functionals.

The above result holds for any measure space (not just topological spaces with the Borel σ-algebra).

Theorem B.9 (Riesz representation theorem in C_b). *Each bounded linear functional L on $\mathrm{C}_b(\Omega)$ can be identified with a unique measure $\mu \in \mathrm{rba}(\Omega)$ via*

$$L(f) = \int_\Omega f\mathrm{d}\mu \quad \text{for } f \in \mathrm{C}_b(\Omega) \quad \text{and} \quad \|L\|_{\mathrm{C}_b(\Omega)^*} = \|\mu\|_{\mathrm{TV}}.$$

Conversely, all such measures define appropriate functionals.

Here we wrote $\mathrm{C}_b(\Omega)$ for the space of continuos and bounded functions on Ω, which is a closed subspace of $\mathrm{L}^\infty(\Omega)$, and $\mathrm{rba}(\Omega)$ for the subspace of (inner and outer) regular measures in $\mathrm{ba}(\Omega)$. Recall that we say that $\mu \in \mathrm{ba}(\Omega)$ is *regular* if

$$\mu^\pm(A) = \inf\{\mu^\pm(O) \colon O \supset A,\, O \text{ open}\} \quad \text{and} \quad \mu^\pm(A) = \sup\{\mu^\pm(C) \colon C \subset A,\, C \text{ compact}\},$$

where $\mu = \mu^+ - \mu^-$ is a Hahn–Jordan decomposition of the signed measure μ.

The result of Theorem B.9, holds more generally for normal Hausdorff spaces Ω (with the Borel σ-algebra). It is however of little use to us in this form, as we lack good (weakly-*) compactness properties unless $\mathrm{C}_b(\Omega)$ is separable, which is the case if and only if Ω is compact and metrizable [9, Thm. 1.194]. For completeness of this presentation, we mention in passing the result of Alexandrov that, if Ω is compact, then measures in $\mathrm{rba}(\Omega)$ are in fact *countably* additive [9, Thm. 1.191].

If $\Omega \subset \mathbb{R}^n$ is compact, it is easy to see that we can identify $\mathrm{C}_b(\Omega)$ with $\mathrm{C}(\bar{\Omega})$, in which case we have:

Theorem B.10 (Riesz representation theorem in C)**.** *Let $K \subset \mathbb{R}^n$ be compact. Then each bounded linear functional L on $\mathrm{C}(K)$ can be identified with a unique measure $\mu \in \mathcal{M}(K)$ via*

$$L(f) = \int_K f \,\mathrm{d}\mu \quad \text{for } f \in \mathrm{C}(K) \quad \textit{and} \quad \|L\|_{\mathrm{C}(K)^*} = \|\mu\|_{\mathrm{TV}}.$$

Conversely, all such measures define appropriate functionals.

This version holds generally for compact and Hausdorff topological spaces K. Finally, we record the locally compact version, where we write $\mathrm{C}_0(\Omega)$ for the uniform closure of compactly supported continuous functions, $\mathrm{C}_c(\Omega)$.

Theorem B.11 (Riesz representation theorem in C_0)**.** *Each bounded linear functional L on $\mathrm{C}_0(\Omega)$ can be identified with a unique measure $\mu \in \mathcal{M}(\Omega)$ via*

$$L(f) = \int_\Omega f \,\mathrm{d}\mu \quad \text{for } f \in \mathrm{C}_0(\Omega) \quad \textit{and} \quad \|L\|_{\mathrm{C}_0(\Omega)^*} = \|\mu\|_{\mathrm{TV}}.$$

Conversely, all such measures define appropriate functionals.

In principle, we will use Theorem B.10 for $K = \bar{\Omega}$ being the closure of a bounded open set $\Omega \subset \mathbb{R}^n$, in which case $\mathrm{C}(\bar{\Omega})^* \simeq \mathcal{M}(\bar{\Omega})$ and use Theorem B.11 for $\Omega \subset \mathbb{R}^n$ being an open set, in which case $C_0(\Omega)$ is the space of continuous functions vanishing on $\partial\Omega$ and/or at infinity and $\mathrm{C}_0(\Omega)^* \simeq \mathcal{M}(\Omega)$. We also mention that the result of Theorem B.11 holds for locally compact Hausdorff spaces Ω.

Appendix C. Convex analysis

Throughout this section, X denotes a locally convex topological vector space, though the definition of convexity and of the convex hull does not depend on the topology. Recall that a set $S \subset X$ is *convex* if and only if $tx + (1-t)y \in S$ whenever $x, y \in S$ and $t \in [0,1]$. The *convex hull* of a set $S \subset X$ is the collection of all convex combinations of elements in S and can be described as the smallest convex set that contains S:

$$\mathrm{co}(S) := \left\{ \sum_{i=1}^k \lambda_i x_i \colon k \in \mathbb{N},\, x_i \in S,\, \lambda_i \geqslant 0,\, \sum_{i=1}^k \lambda_i = 1 \right\} = \bigcap \{C \colon S \subset C \subset X,\, C \text{ convex}\}.$$

We will use the notation $\overline{\mathrm{co}}(S)$ to denote the closure of $\mathrm{co}(S)$. A *closed halfspace* H is a set of the form

$$H = \{x \in X \colon \langle x^*, x \rangle \geqslant t\},$$

where $x^* \in X^*$ is a continuous linear functional on X (for *open halfspaces*, take strict inequality). Recall:

Theorem C.1 (Hahn–Banach separation theorem)**.** *Let X be a locally convex topological vector space and $A, B \subset X$ be disjoint, non-empty, convex sets. Then:*

(a) *if A is open, there exist $x^* \in X^*$ and $t \in \mathbb{R}$ such that*

$$\langle x^*, x \rangle < t \leqslant \langle x^*, y \rangle \quad \textit{for all } x \in A,\, y \in B.$$

(b) *if A is compact and B is closed, there exist $x^* \in X^*$ and $s, t \in \mathbb{R}$ such that*

$$\langle x^*, x \rangle \leqslant s < t \leqslant \langle x^*, y \rangle \quad \textit{for all } x \in A,\, y \in B.$$

As a consequence, we have:

Lemma C.2. *Let X be a locally convex topological vector space and $S \subset X$. Then*

$$\overline{\mathrm{co}}(S) = \bigcap \{H \colon S \subset H \subset X,\, H \textit{ closed halfspace}\}.$$

Appendix D. Harmonic analysis

Let $1 < p < \infty$, $p' := p/(p-1)$, $\ell \in \mathbb{N}$.

Theorem D.1 (Hörmander–Mikhlin multiplier theorem)**.** *Let $m \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\})$ be zero-homogeneous. Then*

$$\left\| \mathscr{F}^{-1}\left(m\hat{f}\right)\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant \|f\|_{\mathrm{L}^p(\mathbb{R}^n)} \quad \textit{for } f \in \mathrm{L}^p(\mathbb{R}^n).$$

Here, the inverse Fourier transform is defined as a principal value integral.

We will also define the (homogeneous) negative Sobolev space $\mathrm{W}^{-\ell,p}(\mathbb{R}^n)$. We consider the space of . For such f we define the norm

The Sobolev space is then defined as the closure of this space in the norm above.

We define the (homogeneous) Sobolev space $\mathrm{W}^{\ell,p}(\mathbb{R}^n) = \{f \in \mathscr{D}'(\mathbb{R}^n) \colon D^\ell f \in \mathrm{L}^p(\mathbb{R}^n)\}/\mathrm{P}_{\ell-1}(\mathbb{R}^n)$, where we quotiented out the polynomials of degree $\ell - 1$. For an open set $\Omega \subset \mathbb{R}^n$, we define the space $\mathrm{W}^{\ell,p}_0(\Omega)$, the closure of the test functions $\mathrm{C}^\infty_c(\Omega)$ in the semi-norm $\|D^\ell \,\boldsymbol{\cdot}\, \|_{\mathrm{L}^p(\Omega)}$. If Ω is bounded, taking the full Sobolev norm (with lower order derivatives), gives an equivalent definition, by Poincaré's inequality. On the other hand, in full space, we have the identification $\mathrm{W}^{\ell,p}(\mathbb{R}^n) = \mathrm{W}^{\ell,p}_0(\mathbb{R}^n)$[11].

Irrespective of the boundedness of Ω, we will define the negative Sobolev space $\mathrm{W}^{-\ell,p}(\Omega)$ as the dual space $\mathrm{W}^{\ell,p'}_0(\Omega)^*$. We have the following standard result:

Proposition D.2. *The space $\mathrm{W}^{-\ell,p}(\mathbb{R}^n)$ can be identified with the closure of the space of Schwartz functions $f \in \mathscr{S}(\mathbb{R}^n)$ such that $\hat{f} = 0$ near 0 in the norm*

$$\|f\|_{\mathrm{W}^{-\ell,p}(\mathbb{R}^n)} := \left\| \mathscr{F}^{-1}\left(\frac{\hat{f}(\xi)}{|\xi|^\ell}\right)\right\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

In other words, we have $\| \,\boldsymbol{\cdot}\, \|_{\mathrm{W}^{\ell,p'}(\mathbb{R}^n)^*} \sim \| \,\boldsymbol{\cdot}\, \|_{\mathrm{W}^{-\ell,p}(\mathbb{R}^n)}$.

[11]Technically, here we defined the homogeneous Sobolev space $\dot{\mathrm{W}}^{\ell,p}(\mathbb{R}^n)$. Since we will not use the inhomogeneous space, we will not make any distinction.

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