Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Reversal modes in magnetic nanowires

by

Katharina Kühn

Doctoral Thesis no.: 1 2007

Reversal Modes in Magnetic Nanowires

Der Fakultät für Mathematik und Informatik
der Universität Leipzig
eingereichte

Dissertation

zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

im Fachgebiet

Mathematik

vorgelegt von

Diplommathematikerin Katharina Kühn

geboren am 25.07.1978 in München

Leipzig, den 09.07.2007.

Contents

Chapter 1

Introduction

The aim of this thesis is to explain phenomena in the reversal of magnetic nanowires by analysing the underlying fundamental equations. In numerical simulations two different reversal modes have been found, one occuring for very thin, the other for thicker wires. We study the two modes analytically and investigate the reasons why they occur.

1.1 Reversal modes of magnetic nanowires in experiments and simulations

In the last years several groups have succeeded in the production and investigation of magnetic wires with less than 100 nm diameter, e.g. [34, 33, 36]. Arrays of such nanowires are in consideration as future high density storage devices [3]. The time necessary to change the magnetisation of a nanowire is directly related to the writing and reading speed of such a device. Therefore it is important to understand their reversal process. It is known that the reversal of the magnetisation starts at one end of the wire and then a domain wall separating the already reversed part from the not yet reversed part is propagating through the wire. (See Figure 1.1 for a sketch of the setting.) However, because of technical difficulties related to the small size of the wires, there are few experimental results about the speed of the wall, e.g. [2, 4, 23, 32], and there are no experimental results about the shape of the wall.

In numerical simulations of the reversal process, several groups, e.g. [16, 22, 40], have observed two different reversal modes. These modes depend on the wire thickness and correspond to very different switching speeds. For thin wires the transverse mode is observed: the magnetisation is constant on each cross section, rotating and moving along the wire (Figure 1.2). For thick wires the vortex mode is observed: the magnetisation is approximately

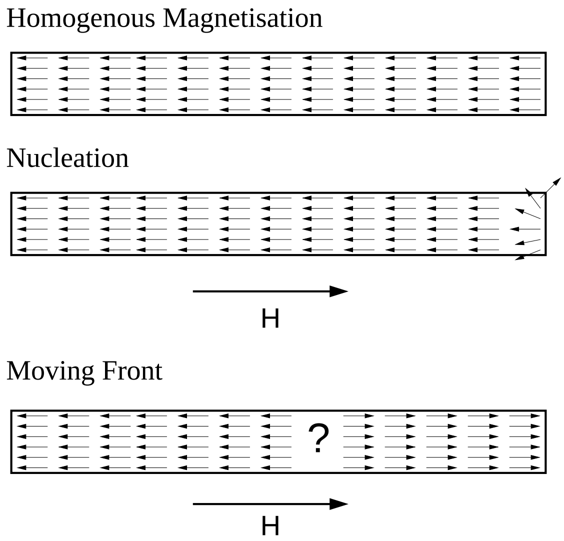

Figure 1.1: The setting

tangential to the boundary and forms a vortex which moves along the wire. (Figure 1.3). In some simulations one can see additional effects like the periodic creation and annihilation of singularities [22]. The vortex mode is much faster than the transverse mode.

For nickel the transition from the transverse mode to the vortex mode occurs at a radius of about 25 nm.

It is the purpose of this thesis to explain the different behaviours by analysing the underlying equations.

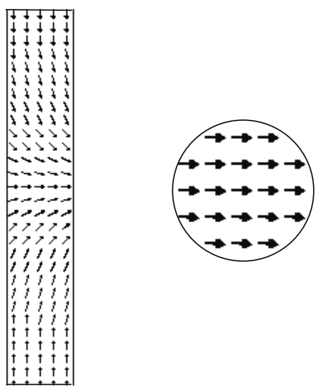

Figure 1.2: Transverse Mode: longitudinal section and cross section

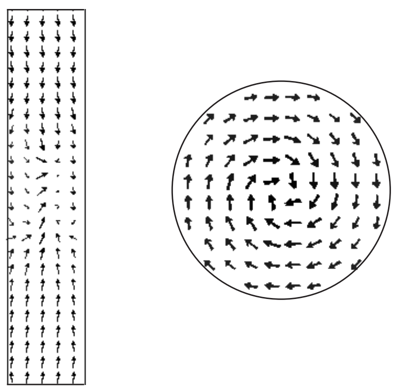

Figure 1.3: Vortex Mode: longitudinal section and cross section

1.2 The micromagnetic model

We work in the framework of micromagnetism. This is a mesoscopic continuum theory that assigns a nonlocal nonconvex energy to each magnetisation m from the domain Σ to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. If necessary, m is extended by zero outside Σ. Experimentally observed ground states of the magnetisation correspond to minimisers of the micromagnetic energy functional

$$E(m) \;:=\; \underbrace{\int_\Sigma A_{\text{ex}}\, |\nabla m|^2}_{\text{exchange energy}} + \underbrace{\int_{\mathbb{R}^3} K_{\text{d}}\, |\nabla u|^2}_{\text{stray field energy}} + \underbrace{E_{\text{an}}(m)}_{\text{anisotropy energy}} - \underbrace{\int_\Sigma J_s\, h \cdot m.}_{\text{external field energy}} \tag{1.1}$$

Here h is an external magnetic field and u is the weak solution of $\Delta u = \operatorname{div} m$ in $\mathbb{R}^3$, i.e., $\nabla u = H(m)$ is the projection of m on gradient fields. A_{ex}, K_d and J_s are material constants. We refer to [13, 24] for a general discussion of the micromagnetic model. Throughout this thesis we will assume that the material is magnetically soft, i.e., we consider the micromagnetic energy without the anisotropy term.

In the micromagnetic model the evolution of the magnetisation is described by the Landau-Lifshitz-Gilbert equation:

$$\partial_t m \;=\; -\gamma m \times H_{\text{eff}} + \alpha m \times (m \times H_{\text{eff}}) \quad \text{where } H_{\text{eff}} = \delta_m E. \tag{1.2}$$

The first term describes the precession of the magnetisation around the effective field, the second term describes a change of the magnetisation in direction of the effective field. The number γ is the gyromagnetic ratio while α is a phenomenological damping constant.

We can simplify (1.2) by considering the overdamped limit. This corresponds to setting $\gamma := 0$ and is exactly the gradient flow of the energy under the conditon $|m| = 1$.

$$\partial_t m \;=\; \alpha m \times (m \times H_{\text{eff}}) = -\alpha H_{\text{eff}} + (\alpha H_{\text{eff}} \cdot m) m \quad \text{where } H_{\text{eff}} = \delta_m E. \tag{1.3}$$

In some parts of the thesis we simplify (1.3) further by keeping on the right hand side only the terms that are highest order with respect to the derivatives. Then we have

$$\partial_t m \;=\; -\alpha A_{\text{ex}}(\Delta m - (\Delta m \cdot m) m). \tag{1.4}$$

This equation is referred to as the harmonic map heat flow equation. For singularities, the highest order terms with respect to the derivatives are dominant. Thus the harmonic map heat flow equation is used to study the role of singularities in the magnetisation.

1.3 Outline of the thesis

A more detailed overview over the results as well as references to related literature can be found in the introduction sections of the individual chapters.

Chapter 2 investigates static 180 degree domain walls. We establish a crossover between two scaling regimes for the energy as a function of the radius R. For small radii the optimal scaling can be realized by a transverse wall for which the magnetisation is constant on each cross section. For large radii a vortex wall yields the optimal scaling. Moreover, we show that for $R \to 0$ the energy minimisation problem Γ-converges to a local, one dimensional problem where the energy is given by $\pi \|\partial_x m\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2}\|m_y\|^2_{L^2(\mathbb{R})}$. We also prove an upper bound on the rate of convergence of the minimal energies.

In Chapter 3 we study the transverse mode via a perturbation argument from the static case. We show that for thin wires and weak external magnetic field there exist travelling wave solutions of the gradient flow equation with respect to the micromagnetic energy including the nonlocal stray field energy. The arguments rely on the good regularity properties of static domain walls in thin wires. A large part of the chapter is devoted to proving these regularity results using the bounds on the rate of convergence established in Chapter 2.

In Chapter 4 we model the evolution of the magnetisation in an infinite cylinder by harmonic map heat flow under an additional external field. Using variational methods, we show the existence of corotationally symmetric travelling wave solutions with a moving vortex. We moreover show that for weak and strong fields the travelling waves connect the original state anti-parallel to the external magnetic field with the fully reversed state in direction of the external field.

In Chapter 5 we interpret our results and compare them to the results of numerical simulations. We also make suggestions for further research.

1.4 Definitions and notation

The chapters are to a large extent self contained. Whenever we are using notation from a different chapter, we are pointing this out explicitely. Some very basic notation that is used throughout this thesis is listed below.

We will use the following conventions. The letter p denotes a point in $\mathbb{R}^3$ and has the components $p = (x, y_1, y_2) = (x, y)$. A map g with values in $\mathbb{R}^3$ has the components $g = (g_x, g_{y_1}, g_{y_2})$. We write f_y for $(0, f_{y_1}, f_{y_2})$, i.e., we view f_y as a map to $\{0\} \times \mathbb{R}^2$.

For a, $b \in \mathbb{R}^n$, $n \in \mathbb{N}$ we denote the scalar product by $a \cdot b$. The characteristic function of a set Ω is denoted by $\mathbb{1}_\Omega$. For $\Omega \subset \mathbb{R}^3$ and $f, g\colon \Omega \to \mathbb{R}^n$, $n \in \mathbb{N}$, we define

$$\begin{aligned} \langle f \rangle_\Omega &:= \frac{1}{|\Omega|} \int_\Omega f(p)\, dp, \\ \langle f, g \rangle_\Omega &:= \int_\Omega f(p) \cdot g(p)\, dp, \end{aligned}$$

whenever the integrals on the right hand side are defined, Moreover we set

$$\begin{aligned} D_R(p) &:= \{q \in \mathbb{R}^2 \ : \ |p-q| < R\}, \qquad & D_R &:= D_R(0), \\ B_R(p) &:= \{q \in \mathbb{R}^3 \ : \ |p-q| < R\}, \qquad & B_R &:= B_R(0), \\ \Sigma &:= \Sigma(R) := \mathbb{R} \times D_R. \end{aligned}$$

For $m\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ let $H(m)$ be the projection of m on gradient fields, that is

$$H(m) = \nabla u, \qquad \text{where } \Delta u = \operatorname{div} m.$$

Chapter 2

Scaling laws of domain walls in magnetic nanowires

2.1 Introduction

In this chapter we investigate the static energy functional. We identify two regimes of scaling, one for very thin wires and the other for thicker wires. We show that, as the radius tends to zero, the energy Γ-converges to a reduced energy.

We use methods that are similar to those used in the study of magnetic films. There, different scalings of the energy are known, corresponding to different regimes of thickness and to different types of domain walls. For an overview see [13].

2.1.1 The model

We study the micromagnetic energy functional for soft magnetic materials without external magnetic field

$$E(m) \;:=\; \underbrace{\int_{\Sigma} A_{\mathrm{ex}}\,|\nabla m|^2}_{E_{ex}(m)} + \underbrace{\int_{\mathbb{R}^3} K_{\mathrm{d}}\,|\nabla u|^2}_{E_H(m)}.$$

Here u is the weak solution of $\Delta u = \operatorname{div} m$ in $\mathbb{R}^3$, i.e., $\nabla u = H(m)$ is the projection of m on gradient fields.

To motivate why transverse walls are energetically favorable in thin wires and vortex walls are energetically favorable in thick wires we rescale and set $m_k(x,y) := m(kx, ky)$. Then

$$E(m_k) \;=\; \int_{\Sigma(kR)} A_{\mathrm{ex}}|\nabla m_k|^2 + \int_{\mathbb{R}^3} K_{\mathrm{d}}|H(m_k)|^2 \;=\; kE_{\mathrm{ex}}(m) + k^3 E_H(m).$$

This calculation suggests that for small radii the biggest contribution to the energy is the exchange energy E_{ex} whereas for big radii the magnetostatic energy becomes important. In order to reduce the magnetostatic energy, it is favorable to avoid surface charges like in the vortex wall. In order to reduce exchange energy, it is favorable to have constant magnetisation on a cross section like in the transverse wall. In this chapter we will investigate this idea in more detail.

To simplify the calculations, we measure distances in multiples of the characteristic length $\sqrt{\frac{A_{\mathrm{ex}}}{K_{\mathrm{d}}}}$, also called the exchange length or the Bloch line width, and energies as multiples of $\sqrt{\frac{A_{\mathrm{ex}}^3}{K_{\mathrm{d}}}}$. In these units the energy E is

$$E(m) := E(m,R) := E_{\mathrm{ex}}(m) + E_H(m) := \int_\Sigma |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla u|^2, \quad (2.1)$$

where u is again the weak solution of $\Delta u = \operatorname{div} m$. We define the admissible set

$$\mathcal{M} := \mathcal{M}(R) := \{m \colon \Sigma(R) \to \mathbb{S}^2 \mid E(m) < \infty\}.$$

We are interested in magnetisations with a 180 degree domain wall, so we would like to consider a subset $\mathcal{M}_l$ of $\mathcal{M}$ with $\lim_{x\to-\infty} m(x,\cdot) = -\vec{e}_x$ and $\lim_{x\to\infty} m(x,\cdot) = \vec{e}_x$. Initially it is not clear in which sense the limits should to be understood. However, in Section 2.4 the set $\mathcal{M}$ will be characterised in the following way.

Theorem 2.1. *Set*

$$\chi \colon \mathbb{R} \to \mathbb{R}^3, \quad x \mapsto \begin{cases} x\vec{e}_x & \text{if } |x| < 1 \\ \operatorname{sign}(x)\vec{e}_x & \text{otherwise.} \end{cases}$$

A function $m \colon \Sigma \to \mathbb{S}^2$ is in $\mathcal{M}$ if and only if one of the four maps $m \pm \vec{e}_x$, $m \pm \chi$ is in $H^1(\Sigma)$.

This motivates the definition

$$\mathcal{M}_l := \mathcal{M}_l(R) := \{m \in \mathcal{M}(R) \mid m - \chi \in H^1(\Sigma)\}. \quad (2.2)$$

To study transverse walls and vortex walls, we consider the following restricted classes of admissible maps

$$\mathcal{T} := \mathcal{T}(R) := \{m \in \mathcal{M}(R) \mid m \text{ is constant on each cross section}\},$$
$$\mathcal{V} := \mathcal{V}(R) := \left\{ m \in \mathcal{M}(R) \;\middle|\; \begin{array}{l} m_y(x,y_1,y_2) \text{ is parallel to } (-y_2,y_1), \\ |m_y| \text{ depends only on } x \text{ and } |y| \end{array} \right\},$$

$$\mathcal{T}_l := \mathcal{T}_l(R) := \mathcal{T}(R) \cap \mathcal{M}_l(R), \qquad \mathcal{V}_l := \mathcal{V}_l(R) := \mathcal{V}(R) \cap \mathcal{M}_l(R),$$

and the infima of the energies

$$E_{\mathcal{M}_l}(R) := \inf_{m\in\mathcal{M}_l(R)} E(m),$$

$$E_{\mathcal{T}_l}(R) := \inf_{m\in\mathcal{T}_l(R)} E(m),$$

$$E_{\mathcal{V}_l}(R) := \inf_{m\in\mathcal{V}_l(R)} E(m).$$

2.1.2 The main results

We discuss the question of existence of optimal wall profiles, the scaling of the energy and the shape of the optimal wall profile.

Theorem 2.2 **(Existence).** *For each radius $R > 0$ there exist minimisers of the energy E in $\mathcal{M}_l(R)$, $\mathcal{T}_l(R)$ and $\mathcal{V}_l(R)$.*

The energy of the optimal wall profile scales like $E_{\mathcal{T}_l}$ when the radius goes to zero and scales like $E_{\mathcal{V}_l}$ for radius to infinity.

Theorem 2.3 **(Energy scaling).** *There exist constants c, C such that*

$$\begin{aligned} &\text{for } R \le 2: & cR^2 &\le E_{\mathcal{M}_l}(R) \le E_{\mathcal{T}_l}(R) \le CR^2,\\ &\text{for } R > 2: & cR^2\sqrt{\ln(R)} &\le E_{\mathcal{M}_l}(R) \le E_{\mathcal{V}_l}(R) \le CR^2\sqrt{\ln(R)}. \end{aligned}$$

Neither $E_{\mathcal{T}_l}$ nor $E_{\mathcal{V}_l}$ has the optimal scaling in the opposite regime: There exists a constant $\tilde{c}$ such that for all $R \in \mathbb{R}^+$ we have

$$E_{\mathcal{T}_l}(R) \ge \tilde{c}R^{\frac{8}{3}} \quad \text{and} \quad E_{\mathcal{V}_l}(R) \ge \tilde{c}R.$$

This shows that the transverse wall is energetically favourable for small radii and the vortex wall is energetically favorable for big radii. However, the constants are not sharp enough to get good estimates for the critical radius where the crossover occurs.

To capture the essence of the energy minimising problem for small radii, we use the notion of Γ-convergence as described in [12].

Theorem 2.4 **(Γ-convergence).** *After rescaling the energy E by a factor of $\frac{1}{R^2}$, the energy minimising problem Γ-converges to a reduced, one dimensional problem. The admissible functions for the reduced problem are maps from $\mathbb{R}$ to $\mathbb{S}^2$, and the energy simplifies to*

$$E_{\mathrm{red}}(m) = \pi\|\partial_x m\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2}\|m_y\|^2_{L^2(\mathbb{R})}.$$

The minimiser m^{red} of the reduced problem exists and is unique up to translation and rotation. Its energy is $\sqrt{8}\pi$ and its profile is that of a Bloch wall, i.e.,

$$m^{\text{red}} = \left(\tanh\left(\frac{x}{\sqrt{2}}\right), \frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)}, 0 \right).$$

Since Γ-convergence implies the convergence of minimisers as well as the convergence of the minimal energies, we can conclude that for small radii minimisers of E are almost constant on the cross section and have a profile that resembles a Bloch wall.

We also prove a bound on the rate of convergence of the minimal energies. This will be an important tool for regularity estimates in the next chapter.

Theorem 2.5 (Rate of convergence). *There exists a constant C such that $\left|E_{\mathcal{M}_l}(R) - \sqrt{8}\pi R^2\right| \leq CR^4|\ln R|$ for all R small enough.*

For $R \gg 1$ we do not know the shape of minimiser. However, we have example functions in $\mathcal{V}_l$ whose energies have the optimal scaling. They have a square root type singularity and the width of their transition regions scales like $R^2\sqrt{\ln(R)}$. The latter is in contrast to the regime $R \ll 1$ where the thickness of the transition region of the optimal walls is of order 1.

In this chapter we often exploit the close connection between the full problem, the reduced one-dimensional problem and the full problem restricted to the set $\mathcal{T}$, of functions that are constant on each cross section. Therefore we do not prove the results in the same order as they are stated above.

2.1.3 Outline of the chapter

We start by collecting some general results about the stray field energy in Section 2.2.

In Section 2.3, we study the restricted class of transverse walls. We establish a lower bound for $E_{\mathcal{T}_l}$ and show that the transverse component of the magnetisation is bounded by the energy. As a corollary we obtain a characterisation of the set $\mathcal{T}$ similar to Theorem 2.1. In this section we use two different representations of the stray field energy: via a Fourier multiplier and via a convolution. The calculations regarding the Fourier multiplier can be found in Section 2.9, the calculations regarding the convolution kernel can be found in Section 2.10.

In Section 2.4, we use the characterisation theorem for transverse walls to prove Theorem 2.1 and the existence of minimisers of the energy in $\mathcal{M}_l$, $\mathcal{T}_l$ and $\mathcal{V}_l$.

In Section 2.5, we investigate the case of small radii. We establish the energy scaling of $E_{\mathcal{M}_l} \sim E_{\mathcal{T}_l} \sim R^2$ and find the Γ-limit for $R \to 0$.

In Section 2.7, we prove the lower bound $E_{\mathcal{M}_l} \geq cR^2\sqrt{\ln(R)}$ for all R that are large enough.

In Section 2.6, we show an upper bound on the rate of convergence of the minimal energies for $R \to 0$.

In Section 2.8, we calculate upper and lower bounds for $E_{\mathcal{V}_l}$ with elementary methods. In particular we get the estimates $E_{\mathcal{V}_l} \leq CR^2\sqrt{\ln(R)}$ for all $R \geq 2$, and $E_{\mathcal{V}_l} \geq CR$ for all $R \in \mathbb{R}^+$. Combining the first estimate with the result of Section 2.7, we see that $E_{\mathcal{M}_l}(R)$ scales like $E_{\mathcal{V}_l}(R) \sim R^2\sqrt{\ln(R)}$ for $R \to \infty$.

2.1.4 Definitions and notation

We will use the following conventions. Generalising the definition in Theorem 2.1, we define the functions

$$\chi_{c^-}^{c^+} : \mathbb{R} \to \mathbb{R}^3, \quad x \mapsto c^{\mathrm{sign}(x)} \min(1, |x|)\, \vec{e}_x, \qquad \chi := \chi_{-1}^{1}.$$

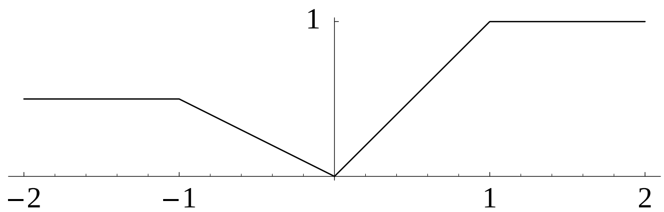

Figure 2.1: The first component of the function $\chi_{c^-}^{c^+}$ for $c^- = \frac{1}{2}$ and $c^+ = 1$

Now let m be a function $\Sigma \to \mathbb{R}^3$. The divergence of m consists of two parts: the *body charges* ρ in the interior of the cylinder and the *surface charges* σ, the divergence from the normal component of the magnetisation on the surface.

$$\rho(p) = \begin{cases} -\operatorname{div} m(p) & \text{if } p \in \Sigma \\ 0 & \text{otherwise} \end{cases} \qquad \sigma(p) = m \cdot \vec{e}_\nu \text{ for all } p \in \partial\Sigma$$

The map u is by definition a weak solution of

$$\Delta u = \operatorname{div} m \qquad \text{in } \mathbb{R}^3, \tag{2.3}$$

if and only if $\nabla u \in L^2(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \eta = \int_\Sigma \rho\eta \; + \int_{\partial\Sigma} \sigma\eta, \qquad \text{for all } \eta \in C_c^\infty(\mathbb{R}^3). \tag{2.4}$$

This defines u only up to a constant. We can remove this ambiguity by requiring $u \in L^6(\mathbb{R}^3)$. Note that there is a constant C such that for all functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ the inequality $\|f - c\|_{L^6(\mathbb{R}^3)} \le C\|\nabla f\|_{L^2(\mathbb{R}^3)}$ holds for some $c \in \mathbb{R}$ if the right hand side exists.

We can decompose u and define u_ρ, u_σ as those maps in $L^6(\mathbb{R}^3)$ that satisfy

$$\int_{\mathbb{R}^3} \nabla u_\rho \cdot \nabla \eta = \int_\Sigma \rho\eta \qquad \text{for all } \eta \in C_c^\infty(\mathbb{R}^3), \tag{2.5}$$

$$\int_{\mathbb{R}^3} \nabla u_\sigma \cdot \nabla \eta = \int_{\partial\Sigma} \sigma\eta \qquad \text{for all } \eta \in C_c^\infty(\mathbb{R}^3). \tag{2.6}$$

Finally we set

$$E_{\rho\rho}(m) := \int_{\mathbb{R}^3} |\nabla u_\rho|^2, \quad E_{\sigma\sigma}(m) := \int_{\mathbb{R}^3} |\nabla u_\sigma|^2, \quad E_{\rho\sigma}(m) := \int_{\mathbb{R}^3} \nabla u_\rho \cdot \nabla u_\sigma.$$

Then we have $E_H(m) = E_{\rho\rho}(m) + E_{\sigma\sigma}(m) + 2E_{\rho\sigma}(m)$.

A special case are functions $m\colon \Sigma \to \mathbb{R}^3$ that are constant on each cross section. To simplify notation, we will often describe such functions by maps $m_\mathbb{R}\colon \mathbb{R} \to \mathbb{R}^3$. For a map $f\colon \mathbb{R} \to \mathbb{R}^3$ we therefore define $f_\Sigma\colon \Sigma \to \mathbb{R}^3$, $(x, y) \mapsto f(x)$ and

$$E(f) := E(f_\Sigma), \qquad E_{\rho\rho}(f) := E_{\rho\rho}(f_\Sigma),$$
$$E_{\sigma\sigma}(f) := E_{\sigma\sigma}(f_\Sigma), \qquad E_{\rho\sigma}(f) := E_{\rho\sigma}(f_\Sigma).$$

We decompose functions $m : \Sigma \to \mathbb{R}^3$ into two parts and set

$$\overline{m}(x) := \frac{1}{|D_R|} \int_{D_R} m(x, y)\, dy, \qquad \tilde{m}(x, y) := m(x, y) - \overline{m}(x).$$

2.2 The stray field energy

Since we are not working on a finite domain, it is initially not clear under which conditions the solution u of the equation $\Delta u = \operatorname{div} m$ exists and ∇u has a finite L^2-norm. An particularly simple case is the case when $m \in L^2(\Sigma)$ since then $\|\nabla u\|_{L^2(\mathbb{R}^3)} \le \|m\|_{L^2(\Sigma)}$. We will reduce the general case to this situation. We define the set

$$\mathcal{D} := \left\{ m\colon \Sigma \to \mathbb{R}^3 \mid \exists c^-, c^+ \in \mathbb{R} \text{ such that } m - \chi_{c^-}^{c^+} \in H^1(\Sigma) \right\}.$$

Lemma 2.6 below states that, for all $m \in \mathcal{D}$, Equation (2.3) has a weak solution and that for such m the stray field energy $E_H(m)$ is finite. Later we will show that $\mathcal{M}$ is a subset of $\mathcal{D}$.

We define the maps G and $K_i\colon \mathbb{R}^3 \to \mathbb{R}$, $i \in \{1, 2, 3\}$, by setting

$$G(p) := \frac{1}{4\pi|p|}, \qquad K_i(p) := \partial_i G(p) = -\frac{1}{4\pi} \frac{p_i}{|p|^3}. \tag{2.7}$$

Lemma 2.6. *For $m \in \mathcal{D}$ define the maps $u, u_\rho, u_\sigma \colon \mathbb{R}^3 \to \mathbb{R}$ by*

$$\begin{aligned} u_\rho(p) &:= \int_\Sigma G(p-p')\rho(p')\,dp', \\ u_\sigma(p) &:= \int_{\partial\Sigma} G(p-p')\sigma(p')\,dp', \\ u(p) &:= u_\rho(p) + u_\sigma(p). \end{aligned}$$

Then the following statements hold.
(i) The map u is a weak solution of (2.3)*, ∇u is in $L^2(\mathbb{R}^3)$ and we have*

$$\begin{aligned} \nabla u_\rho(p) &= \sum_{i=1}^{3} \int_\Sigma K_i(p-p')\rho(p')\vec{e}_i\,dp', \\ \nabla u_\sigma(p) &= \sum_{i=1}^{3} \int_{\partial\Sigma} K_i(p-p')\sigma(p')\vec{e}_i\,dp'. \end{aligned}$$

(ii) The map u_ρ is continuous, contained in $H^2_{\mathrm{loc}}(\mathbb{R}^3)$ and a strong solution of

$$\Delta u_\rho = \rho. \tag{2.8}$$

The map u_σ is also continuous and its restriction to $\mathbb{R}^3 \setminus \partial\Sigma$ is arbitrarily often differentiable. For $r := |y|$ we have

$$\Delta u_\sigma = 0 \qquad \text{in } \mathbb{R}^3 \setminus \partial\Sigma, \tag{2.9}$$

$$\lim_{\substack{y\to y_0\\ y\in\Sigma}} \partial_r u_\sigma - \lim_{\substack{y\to y_0\\ y\notin\Sigma}} \partial_r u_\sigma = -\sigma(y_0) \qquad \text{for } y_0 \in \partial\Sigma. \tag{2.10}$$

(iii) The map u is in $L^2(\Sigma)$ and in $L^2(\partial\Sigma)$. We have

$$\int_{\mathbb{R}^3} |\nabla u(p)|^2\,dp = \int_\Sigma u(p)\rho(p)\,dp + \int_{\partial\Sigma} u(p)\sigma(p). \tag{2.11}$$

Proof. (i) The map u satisfies (2.4) [27, Thm 6.21]. For each $p \in \mathbb{R}^3$, the maps $p' \mapsto K_i(p-p')$ are integrable in Σ and on $\partial\Sigma$, so we can interchange integration and differentiation.

$$\begin{aligned} \nabla u_\rho(p) &= \nabla \int_\Sigma \frac{\rho(p')}{|p'-p|}\,dp' = \int_\Sigma -\frac{\rho(p')\,(p-p')}{|p'-p|^3}\,dp' \\ \nabla u_\sigma(p) &= \nabla \int_{\partial\Sigma} \frac{\sigma(p')}{|p'-p|}\,dp' = \int_{\partial\Sigma} -\frac{\sigma(p')\,(p-p')}{|p'-p|^3}\,dp' \end{aligned}$$

To show $\nabla u \in L^2(\mathbb{R}^3)$ we write m as the sum of $\tilde{\chi} := \chi_{c^-}^{c^+}$ and $g := m - \tilde{\chi} \in L^2(\Sigma)$. Let u_g, $u_{\tilde{\chi}}$ be the weak solutions of (2.3) for $m = g$, $m = \tilde{\chi}$

respectively. We have

$$\begin{aligned}
\nabla u_{\tilde\chi}(p) &= \int_{-1}^{0}\int_{D_1} \frac{-c^-\,(p-p')}{|p'-p|^3}\,dp' + \int_{0}^{1}\int_{D_1} \frac{c^+\,(p-p')}{|p'-p|^3}\,dp' \\
|\nabla u_{\tilde\chi}(p)| &\le \begin{cases} \frac{|c^+|+|c^-|}{|p-R+1|^2} & \text{if } |p| > R+2 \\ 4\pi R^2(|c^+|+|c^-|) & \text{otherwise} \end{cases}
\end{aligned}$$

Thus $\|\nabla u\|^2_{L^2(\mathbb{R}^3)} \le 2\|\nabla u_{\tilde\chi}\|^2_{L^2(\mathbb{R}^3)} + 2\|g\|^2_{L^2(\mathbb{R}^3)}$ is finite.

(ii) The statements can be shown by direct calculation.

(iii) First we show $u \in L^2(\Sigma)\cap L^2(\partial\Sigma)$. Again we can interchange integration and differentiation and calculate

$$\begin{aligned}
u_g(p) &= \int_\Sigma G(p-p')\rho_g(p')\,dp' + \int_{\partial\Sigma} G(p-p')\sigma_g(p')\,dp' \\
&= \operatorname{div}\int_{p-\Sigma} G(p')g(p-p')\,dp' \;=\; \operatorname{div}\int_\Sigma G(p-p')g(p')\,dp' \\
&= \sum_{i=1}^{3}\int_\Sigma K_i(p-p')g_i(p')\,dp'.
\end{aligned}$$

Applying the inequality

$$\left|\int_\Sigma a(p')b(p')\,dp'\right| \le \|a\|_{L^2(\Sigma)}\|b\|_{L^2(\Sigma)}$$

to $a(p') := \sqrt{|K_i(p-p')|}$ and $b(p') := \sqrt{|K_i(p-p')|}\,g_i(p')$ yields

$$\begin{aligned}
\|u_g\|^2_{L^2(\Sigma)} &= \sum_{i=1}^{3}\int_\Sigma\left(\int_\Sigma K_i(p-p')g_i(p')\,dp'\right)^2 dp \\
&\le \sum_{i=1}^{3}\int_\Sigma\left(\int_\Sigma |K_i(p-p')|\,dp'\right)\left(\int_\Sigma |K_i(p-p')|g_i(p')^2\,dp'\right) dp \\
&\le \sum_{i=1}^{3}\|K_i\|_{L^1(\Sigma)}\int_\Sigma\int_\Sigma |K_i(p-p')|\,dp\; g_i(p')^2\,dp' \\
&\le \sum_{i=1}^{3}\|K_i\|^2_{L^1(\Sigma)}\;\|g_i\|^2_{L^2(\Sigma)}. \qquad (2.12)
\end{aligned}$$

Using the estimate for the trace $\|u(x,\cdot)\|_{L^2\partial D_R} \le C\|u(x,\cdot)\|_{H^1(\Sigma)}$ for almost all $x \in \mathbb{R}$ we get a bound on $\|u_g\|_{L^2(\partial\Sigma)}$. For $u_{\tilde\chi}$ we have the following estimate

$$|u_{\tilde\chi}(p)| \le \begin{cases} \frac{|c^+|+|c^-|}{|p|-(R+1)} & \text{if } |p| > R+2 \\ \|\operatorname{div}\tilde\chi\|_{L^2(\mathbb{R}^3)}\|G\|_{L^2(\Sigma)} & \text{otherwise.} \end{cases}$$

Thus $u_{\tilde{\chi}} \in L^2(\Sigma) \cap L^2(\partial\Sigma)$ and therefore $u = u_{\tilde{\chi}} + u_g \in L^2(\Sigma) \cap L^2(\partial\Sigma)$. Let B_d be the ball with radius d as defined in Section 1.4. In the distributional sense

$$\int_{\partial B_d} u\nabla u \, d\vec{\nu} \;=\; \int_{B_d} \operatorname{div}(u\nabla u) = \int_{B_d} |\nabla u|^2 + u\Delta u.$$

Now for $d > 2R+2$ we have

$$\begin{aligned}\left|\int_{\partial B_d} u\nabla u \, d\vec{\nu}\right| \;&\le\; (\|u_g\|_{L^2(\partial B_d)} + \|u_{\tilde{\chi}}\|_{L^2(\partial B_d)})\|\nabla u\|_{L^2(\partial B_d)} \\ &\le\; \left(\|u_g\|_{L^2(\partial B_d)} + 4\sqrt{\pi}\sqrt{|c^+| + |c^-|}\right)\|\nabla u\|_{L^2(\partial B_d)}.\end{aligned}$$

Analogously to (2.12) we have

$$\|u_g\|^2_{L^2(\Sigma)} \;\le\; \sum_{i=1}^{3} \|K_i\|_{L^1(B_d)} \, \|K_i\|_{L^1(\Sigma)} \, \|g_i\|^2_{L^2(\Sigma)} \;\le\; Cd$$

for some constant C. Thus there is a a sequence of radii d_n with $\lim_{n\to\infty} d_n = \infty$ such that $\|u_g\|_{L^2(\partial B_{d_n})} \le 2C$ and $\lim_{n\to\infty} \|\nabla u_g\|_{L^2(\partial B_{d_n})} = 0$. We can conclude

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u|^2 \;&=\; \lim_{n\to\infty} \int_{B_{d_n}} |\nabla u|^2 \;=\; \lim_{n\to\infty} \int_{\partial B_d} u\nabla u \, d\vec{\nu} - \int_{B_{d_n}} u\Delta u \\ &=\; -\int_{\mathbb{R}^3} u\Delta u \;=\; \int_\Sigma u\rho + \int_{\partial\Sigma} u\sigma.\end{aligned}$$

□

For certain functions $m \in \mathcal{D}$ the following lemma gives a bound on $E_{\rho\rho}(m)$. Note that for all $m \in \mathcal{T}$ and all $m \in \mathcal{V}$ the condition "$\operatorname{div} m_y = 0$ in Σ" is satisfied.

Lemma 2.7. *If $m \in \mathcal{D}$ with $\partial_x m_x \ge 0$ and $\operatorname{div} m_y = 0$ in Σ then*

$$E_{\rho\rho}(m) \;\le\; \frac{\pi}{2}(c^+ - c^-)^2 R^3.$$

Proof. In this case $\rho = \partial_x m_x$ and we calculate

$$\begin{aligned}E_{\rho\rho}(m) \;&=\; \int_{D_R}\int_{\mathbb{R}} u_\rho(x,y)\,\partial_x m_x(x,y)\,dx\,dy \\ &=\; \frac{1}{4\pi}\int_{D_R}\int_{D_R}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\partial_x m_x(x,y)\,\partial_x m_x(x',y')}{\sqrt{(x-x')^2 + |y-y'|^2}}\,dx'\,dx\,dy'\,dy \\ &\le\; \int_{D_R}\int_{D_R} \frac{(c^+ - c^-)^2}{4\pi|y-y'|}\,dy'dy \;\le\; |D_R|\int_{D_R} \frac{(c^+ - c^-)^2}{4\pi|y'|}\,dy' \\ &=\; \frac{\pi}{2}(c^+ - c^-)^2 R^3\end{aligned}$$

□

The following lemma concerns convergence in $L^2_{\mathrm{loc}}(\overline{\Sigma})$, which, by definition, coincides with convergence in all $L^2([-l,l]\times D_R)$, $l\in\mathbb{N}$.

Lemma 2.8. *For f, $g_n \in \mathcal{D}\cap L^\infty(\Sigma)$ $(n\in\mathbb{N})$ let u_f, u_{g_n} be the weak solutions of (2.3) for $m=f$, $m=g_n$, respectively. We assume that u_f exists, $\nabla u_f \in L^2(\mathbb{R}^3)$ and that $E(g_n)$, $\|(g_n)_y\|^2_{L^2(\partial\Sigma)}$ as well as $\|g_n\|_{L^\infty(\Sigma)}$ are uniformly bounded by some constant M.*

(i) If $(g_n)_{n\in\mathbb{N}}$ converges to zero in $L^2_{\mathrm{loc}}(\overline{\Sigma})$ then $\int_{\mathbb{R}^3}\nabla u_f\nabla u_{g_n}$ converges to zero, too.

(ii) If $(g_n)_{n\in\mathbb{N}}$ converges in $L^2_{\mathrm{loc}}(\overline{\Sigma})$ to $g_0\in\mathcal{D}\cap L^\infty(\Sigma)$ then

$$\lim_{n\to\infty}\left(E_H(g_n)-E_H(g_n-g_0)\right) \;=\; E_H(g_0).$$

Proof. (i) Set $g:=g_n$ for some arbitrary $n\in\mathbb{N}$. First we assume

$$\begin{aligned}\operatorname{supp}(\rho_f)\cup\operatorname{supp}(\sigma_f) &\;\subset\;]-\infty,l]\times\overline{D}_R,\\ \operatorname{supp}(\rho_g)\cup\operatorname{supp}(\sigma_g) &\;\subset\;[l+k,\infty[\times\overline{D}_R.\end{aligned}$$

According to the definition of distributional divergence we have

$$\int_{\mathbb{R}^3}\nabla u_f\nabla\eta=\int_\Sigma f\nabla\eta \qquad \text{for all } \eta\in C_c^\infty(\mathbb{R}^3).$$

Using a density argument we can transfer the equation to all maps η with $\nabla\eta\in L^1(]-\infty,l]\times D_R)\cap L^2(\mathbb{R}^3)$. To prove $\nabla u_g\in L^1(]-\infty,l]\times D_R)$ we calculate for all $p\in(]-\infty,l]\times D_R)$

$$\begin{aligned}&|\nabla u_g(p)|\\ &=\left|\int_{[l+k,\infty]\times D_R}\frac{\rho_g(p')(p-p')}{|p-p'|^3}\,dp'+\int_{[l+k,\infty]\times\partial D_R}\frac{\sigma_g(p')(p-p')}{|p-p'|^3}\,dp'\,dp\right|\\ &\le\left(\|\rho_g\|_{L^2(\Sigma)}+\|\sigma_g\|_{L^2(\partial\Sigma)}\right)\sqrt{\pi R^2\frac{1}{3(l+k-x)^3}},\end{aligned}$$

thus

$$\|\nabla u_g\|_{L^1(]-\infty,l]\times D_R)}\;\le\;\frac{6R}{\sqrt{k}}\left(\|\rho_g\|_{L^2(\Sigma)}+\|\sigma_g\|_{L^2(\partial\Sigma)}\right).$$

This implies

$$\int_{\mathbb{R}^3}\nabla u_f\cdot\nabla u_g\;=\;\int_\Sigma f\cdot\nabla u_g\;\le\;\frac{6R^3}{\sqrt{k}}\|f\|_{L^\infty(\Sigma)}\left(\|\rho_g\|_{L^2(\Sigma)}+\|\sigma_g\|_{L^2(\partial\Sigma)}\right).$$

Now consider the general case: Let c^+, c^- be such that $f-\tilde{\chi}:=f-\chi_{c-}^{c+}\in H^1(\Sigma)$. For $l>1$ we define

$$\begin{aligned}f^a &:=(f-\tilde{\chi})\,\mathbb{1}_{[-l,l]\times D_R}, \qquad & f^b &:=(f-\tilde{\chi})\,\mathbb{1}_{(\mathbb{R}\setminus[-l,l])\times D_R}\\ g^a &:=g\,\mathbb{1}_{[-2l,2l]\times D_R}, & g^b &:=g\,\mathbb{1}_{(\mathbb{R}\setminus[-2l,2l])\times D_R}.\end{aligned}$$

Then

$$|\nabla u_f \cdot \nabla u_g| = |\nabla u_f \cdot \nabla u_{g^a} + (\nabla u_{f^a} + \nabla u_{\tilde{\chi}}) \cdot \nabla u_{g^b} + \nabla u_{f^b} \cdot (\nabla u_g - \nabla u_{g^a})|.$$

The first summand can be estimated by $\sqrt{E_H(f)}\ \|g\|_{L^2([-l,l]\times D_R)}$, the last one by $\|f - \tilde{\chi}\|_{L^2((\mathbb{R}\setminus[-l,l])\times D_R)} \left(\sqrt{E_H(g)} + \|g\|_{L^2([-2l,2l]\times D_R)}\right)$. To get an upper bound for the second summand we use the calculations above. However, we have to consider additionally the surface charges of g at $\{\pm 2l\} \times D_R$. Their field is

$$\begin{aligned}
\nabla u_{\text{surf.}}(p) \;&=\; \frac{-1}{4\pi}\int_{D_R} g(-2l,y')\frac{p-(-2l,y')}{|p-(-2l,y')|^3} \;+ g(2l,y')\frac{p-(2l,y')}{|p-(2l,y')|^3}\,dy', \\
|\nabla u_{\text{surf.}}(p)| \;&\le\; R^2\|g\|_{L^\infty(\Sigma)}\left(\frac{1}{|x-2l|^2} + \frac{1}{|x+2l|^2}\right).
\end{aligned}$$

We get

$$\begin{aligned}
&|(\nabla u_{f^a} + \nabla u_{\tilde{\chi}}) \cdot \nabla u_{g^b}| \\
&\quad\le\; \frac{12R^3}{\sqrt{l}}\|f\|_{L^\infty(\Sigma)}\left(\|\rho_g\|_{L^2(\Sigma)} + \|\sigma_g\|_{L^2(\partial\Sigma)}\right) \\
&\qquad +\; R^2\|g\|_{L^\infty(\Sigma)}\int_{[-l,l]\times D_R}|f(p)|\;\left(\frac{1}{|x-2l|^2} + \frac{1}{|x+2l|^2}\right)\,dp \\
&\quad\le\; \frac{12R^3}{\sqrt{l}}\|f\|_{L^\infty(\Sigma)}\left(\|\rho_g\|_{L^2(\Sigma)} + \|\sigma_g\|_{L^2(\partial\Sigma)}\right) + \frac{2\pi R^4}{l}\|f\|_{L^\infty(\Sigma)}\|g\|_{L^\infty(\Sigma)}
\end{aligned}$$

and thus we have

$$\begin{aligned}
&\lim_{n\to\infty}|\nabla u_f \nabla u_{g_n}| \\
&\quad\le\; \lim_{n\to\infty}\Bigg(\sqrt{E_H(f)}\ \|g_n\|_{L^2([-2l,2l]\times D_R)} \\
&\qquad +\; \|\nabla u_f\|_{L^2((\mathbb{R}\setminus[-l,l])\times D_R)}\left(\sqrt{E_H(g_n)} + \|g_n\|_{L^2([-2l,2l]\times D_R)}\right) \\
&\qquad +\; \frac{12R^3}{\sqrt{l}}\|f\|_{L^\infty(\Sigma)}\left(\|\rho_{g_n}\|_{L^2(\Sigma)} + \|\sigma_{g_n}\|_{L^2(\partial\Sigma)}\right) \\
&\qquad +\; \frac{2\pi R^4}{l}\|f\|_{L^\infty(\Sigma)}\|g_n\|_{L^\infty(\Sigma)}\Bigg) \\
&\quad=\; \|\nabla u_f\|_{L^2((\mathbb{R}\setminus[-l,l])\times D_R)}\sqrt{M} + \frac{12R^3}{\sqrt{l}}\|f\|_{L^\infty(\Sigma)}2M + \frac{2\pi R^4}{l}\|f\|_{L^\infty(\Sigma)}M,
\end{aligned}$$

which becomes arbitrarily small for large l.

(ii) The second statement is an immediate consequence of (i): Since

$$\begin{aligned} E_H(g_n - g_0) + E_H(g_0) &= E_H(g_n) - \int_{\mathbb{R}^3} 2\nabla u_{g_0} \cdot \nabla u_{g_n} + 2E_H(g_0) \\ &= E_H(g_n) + \underbrace{\int_{\mathbb{R}^3} 2\nabla u_{g_0} \cdot (\nabla u_{g_0} - \nabla u_{g_n})}_{(*)} \end{aligned}$$

and since $(*)$ converges to 0 we have

$$\lim_{n\to\infty} (E_H(g_n) - E_H(g_n - g_0)) = E_H(g_0).$$

□

The proof of the last statement of this section is a two line argument. However, since the estimate will be used frequantly we formulate it as a lemma.

Lemma 2.9. *Let $f, g : \Sigma \to \mathbb{R}^3$ such that $E_H(f), E_H(g) < \infty$. Then*

$$|E_H(f) - E_H(g)| \le E_H(f-g) + 2\sqrt{E_H(f)E_H(f-g)} \tag{2.13}$$

In particular, if $\|f-g\|_{L^2(\Sigma)} < \infty$

$$|E_H(f) - E_H(g)| \le \|f-g\|^2_{L^2(\Sigma)} + 2\|f-g\|_{L^2(\Sigma)}\sqrt{E_H(f)} \tag{2.14}$$

Proof. Let u_f, u_g be the weak solutions of $\Delta u_f = \operatorname{div} f$, $\Delta u_g = \operatorname{div} g$. Then

$$\begin{aligned} |E_H(f) - E_H(g)| &= \left| \int_{\mathbb{R}^3} |\nabla u_g|^2 - |\nabla u_f|^2 \right| \\ &= \int_{\mathbb{R}^3} |\nabla u_g - \nabla u_f|^2 + 2|\nabla u_g \cdot \nabla u_f| \\ &\le E_H(f-g) + 2\sqrt{E_H(f)E_H(f-g)} \end{aligned}$$

The bound $E_H(f-g) \le \|f-g\|^2_{L^2(\Sigma)}$ implies (2.14). □

2.3 Transverse walls

In this section we investigate functions that are constant on the cross section. To simplify notation, we describe such functions by maps from $\mathbb{R}$ to $\mathbb{R}^3$. In particular we will view the functionals E, $E_{\rho\rho}$, $E_{\sigma\sigma}$ and $E_{\rho\sigma}$ also as functionals on $\{f : \mathbb{R} \to \mathbb{R}^3\}$ as described in Subsection 2.1.4. The following lemma simplifies the calculation of E_H.

Lemma 2.10. *If $m : \Sigma \to \mathbb{R}^3$ is constant on each cross section and $E(m) < \infty$ then the following equalities hold:*
(i) $E_{\rho\sigma}(m) = 0$,
(ii) $E_{\sigma\sigma}(m) = E_{\sigma\sigma}(m_{y_1}\vec{e}_{y_1}) + E_{\sigma\sigma}(m_{y_2}\vec{e}_{y_2})$.

Proof. (i) Since ρ is independent of y, the map u_ρ is rotationally symmetric and since $\sigma(x,y) = -\sigma(x,-y)$ we have $u_\sigma(x,y) = -u_\sigma(x,-y)$. So

$$\begin{aligned} \nabla_y u_\rho(x,y) &= -\nabla_y u_\rho(x,-y), & \partial_x u_\rho(x,y) &= \partial_x u_\rho(x,-y), \\ \nabla_y u_\sigma(x,y) &= \nabla_y u_\sigma(x,-y), & \partial_x u_\sigma(x,y) &= -\partial_x u_\rho(x,-y), \end{aligned}$$

and therefore

$$E_{\rho\sigma}(m) = \int_{\mathbb{R}^3} \nabla u_\rho(x,y)\cdot\nabla u_\sigma(x,y) + \nabla u_\rho(x,-y)\cdot\nabla u_\sigma(x,-y)\, dy\, dx = 0.$$

(ii) Let $u_i\colon \mathbb{R}^3 \to \mathbb{R}$, $(i \in \{1,2\})$ be such that $\int_{\mathbb{R}^3} \nabla u_i \cdot \nabla \eta = \int_{\partial\Sigma} m_{y_i}\cdot\eta$ for all $\eta \in C_c^\infty(\mathbb{R}^3)$ and set $\overline{y} := (y_1, -y_2)$. Then

$$\begin{aligned} \partial_{y_1} u_1(x,y) &= \partial_{y_1} u_1(x,\overline{y}), & \partial_{y_1} u_2(x,y) &= -\partial_{y_1} u_2(x,\overline{y}), \\ \partial_{y_2} u_1(x,y) &= -\partial_{y_2} u_1(x,\overline{y}), & \partial_{y_2} u_2(x,y) &= \partial_{y_2} u_2(x,\overline{y}), \\ \partial_x u_1(x,y) &= \partial_x u_1(x,\overline{y}), & \partial_x u_2(x,y) &= -\partial_x u_2(x,\overline{y}), \end{aligned}$$

which yields

$$\int_{\mathbb{R}^3} \nabla u_1(x,y)\cdot\nabla u_2(x,y)\, dy\, dx = 0.$$

So we have

$$E_{\sigma\sigma}(m) = \int_{\mathbb{R}^3} |\nabla u_1|^2 + |\nabla u_2|^2 = E_{\sigma\sigma}(m_{y_1}\vec{e}_{y_1}) + E_{\sigma\sigma}(m_{y_2}\vec{e}_{y_2}).$$

□

So the energy of a map $m\colon \mathbb{R} \to \mathbb{R}^3$ is given by

$$E(m) = \pi R^2 \|\partial_x m\|^2_{L^2(\mathbb{R})} + E_{\sigma\sigma}(m) + E_{\rho\rho}(m).$$

In this section we establish for $E_{\mathcal{T}_l}$ an upper bound and two different lower bounds. First, we show that there exists a constant $C(R)$ such that for all $m \in \mathcal{T}_l$ the energy $E(m)$ is bounded from below by $C(R)\|m_y\|^2_{L^2(\Sigma)}$. This implies the characterisation theorem for $\mathcal{T}$. Second, we combine this first lower bound with an estimate for $E_{\rho\rho}$ to get a lower bound for $E_{\mathcal{T}_l}$.

To get the estimates, we use the representation of the stray field energy via a Fourier multiplier. The derivation of the Fourier multiplier can be found in Section 2.9.

All Fourier transforms in this paper refer only to the first argument, we will still denote them by $\hat{f} := \mathcal{F}(f)$. Moreover, we choose the constants in a way that $\|f(\cdot,a)\|_{L^2} = \|\hat{f}(\cdot,a)\|_{L^2}$.

When we apply the Fourier transform to the defining partial differential equations for u_ρ and u_σ, for every $\xi \in \mathbb{R}$ we get ordinary differential equations that can be solved explicitly. Of course, this only works when m

is constant on the cross section. Using the explicit representation of the Fourier transforms of $\hat{u}_\rho$ and $\hat{u}_\sigma$, we get the Fourier multipliers. The following lemma summarises their properties. The Fourier multipliers involve the modified Bessel functions I_1 and K_1. For a definition and for properties of these functions see Section 2.9.

Theorem 2.11 (Estimates via Fourier multipliers).
(i) For $m_y \in L^2(\mathbb{R}, \{0\} \times \mathbb{R}^2)$ we have

$$\begin{aligned} E_{\sigma\sigma}(m_y) &= R^2 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 g_F(\xi R)\, d\xi \\ &:= R^2 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2\ \pi K_1(|\xi R|) I_1(|\xi R|)\, d\xi. \end{aligned}$$

In particular, g_F is a smooth function, monotonously decreasing in $|t|$ with $g_F(0) = \frac{\pi}{2}$. Moreover, we have the inequalities

$$1 \le \quad g_F(t) \quad \le \frac{\pi}{2} \qquad \text{for } |t| \le 1, \tag{2.15}$$

$$\frac{1}{t} \le \quad g_F(t) \quad \le \frac{\pi}{2t} \qquad \text{for } |t| \ge 1, \tag{2.16}$$

$$\frac{\pi}{4} t^2 |\ln(t)| \le \frac{\pi}{2} - g_F(t) \le \frac{\pi}{2} t^2 |\ln(t)| \qquad \text{for } |t| \le \frac{1}{2}. \tag{2.17}$$

(ii) Let $m_x \colon \mathbb{R} \to \mathbb{R}$ be a map such that $\rho := \partial_x m_x$ is in $L^2(\Sigma)$. We have

$$\begin{aligned} E_{\rho\rho}(m_x \vec{e}_x) &= R^4 \int_{\mathbb{R}} |\hat{\rho}(\xi)|^2 h_F(\xi R)\, d\xi \\ &:= R^4 \int_{\mathbb{R}} |\hat{\rho}(\xi)|^2 \frac{\pi}{|\xi R|^2} \left(1 - 2 I_1(|\xi R|) K_1(|\xi R|)\right)\, d\xi. \end{aligned}$$

In particular, h_F is smooth on $\mathbb{R} \setminus \{0\}$ with

$$\frac{\pi}{2} \le h_F(t) \le \pi |\ln(|t|)| \qquad \text{for } |t| \le \frac{1}{2}, \tag{2.18}$$

$$\frac{\pi}{2} |\ln(|t|)| \le h_F(t) \qquad \text{for } |t| \le 1, \tag{2.19}$$

$$\frac{0.4}{t^2} \le h_F(t) \le \frac{\pi}{t^2} \qquad \text{for } |t| \ge \frac{1}{2}. \tag{2.20}$$

As a Corollary of (i) we directly get an upper bound on $E_{\mathcal{T}_l}$.

Corollary 2.12. *We have $E_{\mathcal{T}_l} \le \sqrt{8}\pi\, R^2 + \frac{\pi}{2} R^3$.*

Proof. Let m^{red} be the minimiser of

$$E_{\mathrm{red}} : \quad m \mapsto \pi \left(\|\partial_x m\|^2_{L^2(\mathbb{R})} + \frac{1}{2} \|m_y\|^2_{L^2(\mathbb{R})} \right) \qquad \text{in } \mathcal{T}_l.$$

This minimiser can be calculated explicitely (cf. Lemma 2.26 below). It is monotonously increasing and we have $E_{\text{red}}(m^{\text{red}}) = \sqrt{8}\pi$. So the combination of Theorem 2.11 (i) and Lemma 2.7 yields the estimate. $\square$

We use Theorem 2.11 now to bound the L^2-norm of m_y from below.

Lemma 2.13. *Let $m_y \colon \mathbb{R} \to \{0\} \times \mathbb{R}^2$ be a map with $|m_y| \leq 1$ for which $E_{\sigma\sigma}(m) + \|\partial_x m_y\|^2_{L^2(\Sigma)}$ is finite. Then $\|m_y\|_{L^2(\mathbb{R})}$ is finite and*

$$E_{\sigma\sigma}(m) + \|\partial_x m_y\|^2_{L^2(\Sigma)} \;\geq\; \|m_y\|^2_{L^2(\mathbb{R})} \cdot \min\{R^2,\ 2\,R^{\frac{4}{3}}\}. \tag{2.21}$$

Proof. First, we assume that $\|m_y\|_{L^2(\Sigma)}$ is finite. In this case we can apply the estimate from Theorem 2.11 (i) and the equality $\|\xi \hat{m}_y\|_{L^2(\Sigma)} = \|\partial_x m_y\|_{L^2(\Sigma)}$. Since $\min_{\xi\in\mathbb{R}}\left(\frac{R}{|\xi|} + \pi R^2\xi^2\right) \geq 2\,R^{\frac{4}{3}}$ we have

$$\begin{aligned}
&E_{\sigma\sigma}(m) + \|\partial_x m_y\|^2_{L^2(\Sigma)} \\
&\quad\geq \int_{-\frac{1}{R}}^{1/R} R^2\,|\hat{m}_y(\xi)|^2\,d\xi + \int\limits_{\mathbb{R}\setminus[-\frac{1}{R},\frac{1}{R}]} \left(\frac{R}{|\xi|} + \pi R^2\xi^2\right)|\hat{m}_y(\xi)|^2\,d\xi \\
&\quad\geq \int_{-\frac{1}{R}}^{1/R} R^2|\hat{m}_y(\xi)|^2\,d\xi + \int\limits_{\mathbb{R}\setminus[-\frac{1}{R},\frac{1}{R}]} 2\,R^{\frac{4}{3}}|\hat{m}_y(\xi)|^2\,d\xi \\
&\quad\geq \|m_y\|^2_{L^2(\mathbb{R})} \cdot \min\left\{R^2,\ 2\,R^{\frac{4}{3}}\right\}.
\end{aligned}$$

In order to treat the general case, we decompose $\mathbb{R}$ in three subsets.

$$I_1 := [-k,k],\quad I_2 := ([-k-1,-k] \cup [k,k+1]),\quad I_3 := (]-\infty,k-1] \cup [k+1,\infty[)$$

For $i \in \{1,2,3\}$ set $m_y^i := m_y \mathbb{1}_{I_i}$ and define u_i as weak solution of $\Delta u_i = \operatorname{div} m_y^i$. Then $E_{\sigma\sigma}(m) = \|\nabla u_1 + \nabla u_2 + \nabla u_3\|^2_{L^2(\mathbb{R}^3)}$. Since the functions m_y^1 and m_y^2 have finite support we can use Lemma 2.6 to calculate ∇u_1 and ∇u_2. According to the definition of distributional divergence the equation $\int_{\mathbb{R}^3} \nabla u_3 \nabla \eta = \int_{\mathbb{R}^3} m_3 \nabla \eta$ holds for all test functions $\eta \in C_c^\infty(\mathbb{R}^3)$. Using a density argument, we can transfer the equation to all maps η with $\nabla\eta \in L^1(I_3 \times D_R) \cap L^2(\mathbb{R}^3)$. In particular the equation holds for $\eta := u_1$ since for $x \notin I_1$ we have

$$\begin{aligned}
|\nabla u_1(p)| &= \left|\int_{I_1\times\partial D_R} \frac{\sigma(y')|y-y'|}{\sqrt{(x-x')^2 + |y-y'|^2}^3}\,dp'\right| \\
&\leq \int_{|x|-k}^{\infty} 4\pi R^2 \frac{1}{|x'|^3}\,dx' \;=\; \frac{2\pi R^2}{(|x|-k)^2}.
\end{aligned}$$

Therefore

$$\|\nabla u_1\|_{L^1(I_3\times D_R)} \;\leq\; 2\pi R^2 \int_{k+1}^{\infty} \frac{2\pi R^2}{(x-k)^2}\, dx \;=\; 4\pi^2 R^4$$

and we can calculate

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\mathbb{R}^3)} \;&=\; \sum_{i=1}^{3}\sum_{j=1}^{3} \int_{\mathbb{R}^3} \nabla u_i(p)\cdot \nabla u_j(p)\, dp \\
&\geq\; \|\nabla u_1\|^2_{L^2(\mathbb{R}^3)} + 2\int_{\mathbb{R}^3} (\nabla u_1 \cdot \nabla u_2 + \nabla u_1 \cdot \nabla u_3) \\
&\geq\; \|\nabla u_1\|^2_{L^2(\mathbb{R}^3)} - 2\|\nabla u_1\|_{L^2(\mathbb{R}^3)} \|m_y\|_{L^2(I_2)} - 2\|\nabla u_1\|_{L^1(I_3\times D_R)} \\
&\geq \frac{1}{2}\|\nabla u_1\|^2_{L^2(\mathbb{R}^3)} - 8 - 8\pi^2 R^4.
\end{aligned}$$

Finally we get

$$\begin{aligned}
&E_{\sigma\sigma}(m) + \|\partial_x m_y\|^2_{L^2(\Sigma)} \\
&\quad\geq\; \frac{1}{2}\|\partial_x m_y\|^2_{L^2(I_1)} + \frac{1}{2}\|\nabla u_1\|^2_{L^2(\mathbb{R}^3)} - 8 - 8\pi^2 R^4 \\
&\quad\geq\; \frac{1}{2}\|m_y\|^2_{L^2([-k,k])} \min\{R^2,\ 2\,R^{\frac{4}{3}}\} - 8 - 8\pi^2 R^4.
\end{aligned}$$

Since k was arbitrary, $E(m)$ is only finite if $\|m_y\|_{L^2(\mathbb{R})}$ is finite. □

Corollary 2.14. *A map $m\colon \Sigma \to \mathbb{S}^2$ is in $\mathcal{T}$ if and only if m is constant on each cross section and one of the four functions $m \pm \vec{e}_x$, $m \pm \chi$ is in $H^1(\Sigma)$.*

Proof. If one of the four functions $m \pm \vec{e}_x$, $m \pm \chi$ is in $H^1(\Sigma)$, then m is in $\mathcal{D}$ as defined in Section 2.2. So, according to Lemma 2.6 (i), $E_H(m)$ is finite, and thus $E(m)$ is finite.

To show the other implication we assume that $m \in \mathcal{T}$. Then $E_{\text{ex}}(m)$ is finite, therefore ∇m is in $L^2(\Sigma)$. Moreoverm we have

$$\|1 - |m_x|\|^2_{L^2(\mathbb{R})} \;\leq\; \|1 - m_x^2\|_{L^1(\mathbb{R})} \;=\; \|m_y^2\|_{L^1(\mathbb{R})} \;=\; \|m_y\|^2_{L^2(\mathbb{R})}.$$

So either one of the four functions $m \pm \vec{e}_x$, $m \pm \chi$ is in $H^1(\Sigma)$ or m_x oscillates infinitely often between $+1$ and -1. In the latter case we have an infinite sequence of disjoint intervals $(I_n)_{n\in\mathbb{N}}$ such that $m_x(I_n) = [-\frac{1}{2}, \frac{1}{2}]$ for all

$n \in \mathbb{N}$. But by assumption both

$$\begin{aligned} E(m) &\geq E_{\text{ex}}(m) \;\geq\; \sum_{n=1}^{\infty} \frac{1}{|I_n|} \qquad \text{and} \\ E(m) &\geq E_{\sigma\sigma}(m) + \|\partial_x m_y\|^2_{L^2(\Sigma)} \\ &\geq \|m_y\|^2_{L^2(\mathbb{R})} \min\left\{R^2,\; 2\,R^{\frac{4}{3}}\right\} \\ &\geq \min\left\{R^2,\; 2\,R^{\frac{4}{3}}\right\} \sum_{n=1}^{\infty} \frac{1}{4}|I_n| \end{aligned}$$

have to be finite. This is impossible. □

The following lemma concerns monotonously increasing rearrangement as defined in [1]. One result in that article is the decrease of a certain energy functional under monotonously increasing rearrangement. We apply this result to the functional $E_{\rho\rho}$.

Lemma 2.15. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a map such that $f - \chi \in H^1(\mathbb{R})$ and let f_{mon} be the monotonously increasing rearrangement of f as defined in [1]. Then $E_{\rho\rho}(f\vec{e}_x) \geq E_{\rho\rho}(f_{\text{mon}}\vec{e}_x)$.*

Proof. Let u be the weak solution of $\Delta u = (\partial_x f)\mathbb{1}_{D_R}$ and set $G\colon \mathbb{R}^3 \to \mathbb{R}$, $p \mapsto \frac{1}{4\pi|p|}$, as before. Then

$$E_{\rho\rho}(f\vec{e}_x) \;=\; \int_\Sigma f\partial_x u \;=\; \int_\Sigma f(x)\,\partial_x\left(\int_\Sigma G(p-p')\,\partial_{x'} f(x')\,dp'\right)\,dp.$$

First, assume $f - \chi \in C_c^\infty(\mathbb{R})$ and integrate by parts carefully. For details of the calculation see Section 2.10. Then

$$E_{\rho\rho}(f\vec{e}_x) \;=\; \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \big(f(x) - f(x')\big)^2\, h(x-x')\,dx'\,dx,$$

where

$$h(x) \;:=\; \int_{D_R}\int_{D_R} \partial_{xx} G(x, y-y')\,dy'\,dy$$

is a positive, integrable function. Thus we are in a situation where we can apply [1, Theorem 2.11] and get $E_{\rho\rho}(f\vec{e}_x) \geq E_{\rho\rho}(f_{\text{mon}}\vec{e}_x)$.

When $f - \chi \in H^1(\mathbb{R})$ we use an approximation argument. In [11] it is shown that symmetric rearrangement is continuous in $H^1(\mathbb{R})$. This result can be easily generalised to the case of monotonously increasing rearrangement. □

We use the previous lemma to estimate $E_{\rho\rho}$ from below.

Lemma 2.16. *Let $m_x : \mathbb{R} \to [-1,1]$ be a function such that one of the four functions $m_x \pm 1$, $m_x \pm \chi$ is in $H^1(\mathbb{R})$. Then*

$$E_{\rho\rho}(m_x\vec{e}_x) \;\geq\; \min\left\{\frac{4}{3}\,\frac{R^4}{\|1-m_x^2\|_{L^1(\mathbb{R})}},\; \frac{1}{3}\,R^3\right\}.$$

In particular, for $m \in \mathcal{T}$ we have

$$E_{\rho\rho}(m) \;\geq\; \min\left\{\frac{4}{3}\,\frac{R^4}{\|m_y\|^2_{L^2(\mathbb{R})}},\; \frac{1}{3}\,R^3\right\}.$$

Proof. We assume that m_x is monotonously increasing, since monotonously increasing rearrangement of m_x decreases $E_{\rho\rho}(m_x\vec{e}_x)$. The estimates for the Fourier multiplier in Theorem 2.11 (ii) yield

$$E_{\rho\rho}(m_x\vec{e}_x) \;\geq\; \frac{\pi}{2}R^4\int_{-\frac{1}{2R}}^{\frac{1}{2R}} \hat{\rho}^2(\xi)\,d\xi.$$

We set $c := \min\left\{\frac{\hat{\rho}(0)}{\|\partial_\xi\hat{\rho}\|_{L^\infty(\mathbb{R})}},\, \frac{1}{2R}\right\}$. Then

$$\begin{aligned}
E_{\rho\rho}(m_x\vec{e}_x) \;&\geq\; \frac{\pi}{2}R^4\int_{-c}^{c}\left(\hat{\rho}(0) - \|\partial_\xi\hat{\rho}\|_{L^\infty(\mathbb{R})}|\xi|\right)^2\,d\xi \\
&\geq\; \pi R^4\int_0^c\left(\hat{\rho}(0) - \frac{\hat{\rho}(0)}{c}\,\xi\right)^2\,d\xi \\
&=\; \frac{\pi}{3}R^4\hat{\rho}(0)^2 c \;=\; \min\left\{\frac{\pi}{3}R^4\frac{\hat{\rho}(0)^3}{\|\partial_\xi\hat{\rho}\|_{L^\infty(\mathbb{R})}},\; \frac{\pi}{6}R^3\,\hat{\rho}(0)^2\right\}.
\end{aligned}$$

We calculate

$$\hat{\rho}(0) \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\rho \;=\; \frac{2}{\sqrt{2\pi}}.$$

For all $t > 0$ we have

$$\begin{aligned}
\int_{-t}^{t}|\rho(x)x|\,dx \;&=\; t\,m_x(t) - t\,m_x(-t)) - \int_{-t}^{t}|m_x(x)|\,dx \\
&\leq\; \int_{-t}^{t}1 - |m_x(x)|\,dx \;\leq\; \int_{-t}^{t}1 - m_x(x)^2\,dx.
\end{aligned}$$

In the limit $t \to \infty$ we get

$$\|\partial_\xi\hat{\rho}\|_{L^\infty(\mathbb{R})} \;\leq\; \frac{1}{\sqrt{2\pi}}\|x\rho\|_{L^1(\mathbb{R})} \;\leq\; \frac{1}{\sqrt{2\pi}}\|1-m_x^2\|_{L^1(\mathbb{R})},$$

thus

$$E_{\rho\rho}(m_x\vec{e}_x) \;\geq\; \min\left\{\frac{4}{3}R^4\frac{1}{\|1-|m_x|^2\|_{L^1(\mathbb{R})}},\; \frac{1}{3}R^3\right\}$$

□

Combining Lemma 2.13 and Lemma 2.16 we get a lower bound for $E_{\mathcal{T}_l}$.

Theorem 2.17. *We have* $E_{\mathcal{T}_l} \geq \min\left\{\frac{1}{3}R^3, 3\,R^{\frac{8}{3}}\right\}$.

Proof. For all $m \in \mathcal{T}_l$ we have

$$\begin{aligned} E(m) &= \pi\|\partial_x m\|^2_{L^2(\mathbb{R})} + E_{\sigma\sigma}(m) + E_{\rho\rho}(m) \\ &\geq \|m_y\|^2_{L^2(\mathbb{R})} \min\left\{R^2,\, 2R^{\frac{4}{3}}\right\} + \min\left\{\frac{4}{3}\frac{R^4}{\|m_y\|_{L^2(\mathbb{R})}},\, \frac{1}{3}R^3\right\} \\ &\geq \min\left\{2\sqrt{\frac{4}{3}}R^3,\, 2\sqrt{\frac{8}{3}}R^{\frac{8}{3}},\, \frac{1}{3}R^3\right\} \geq \min\left\{3R^{\frac{8}{3}},\, \frac{1}{3}R^3\right\}. \end{aligned}$$

□

2.4 The characterisation theorem for $\mathcal{M}_l$ and the existence of minimisers

In this section we decompose m as in Subsection 2.1.4.

Lemma 2.18. *Let* $m : \Sigma \to \mathbb{R}^3$ *we have*

$$E_{ex}(\tilde{m}) + E_{ex}(\overline{m}) = E_{ex}(m), \tag{2.22}$$

$$\int_{D_R} |\overline{m}(x)|^2 + |\tilde{m}(x,y)|^2\, dy = \int_{D_R} |m(x,y)|^2\, dy \quad \text{for all } x \in \mathbb{R}, \tag{2.23}$$

$$16R^2\, \|\nabla_y \tilde{m}(x,\cdot)\|^2_{L^2(D_R)} \geq \|\tilde{m}(x,\cdot)\|^2_{L^2(D_R)} \quad \text{for all } x \in \mathbb{R}. \tag{2.24}$$

Proof. The first two equations can be shown by direct calculation. Since $\int_{D_R} \tilde{m}(\cdot, y)\, dy \equiv 0$ we have

$$\begin{aligned} E_{ex}(m) &= \int_\Sigma |\partial_x \overline{m} + \partial_x \tilde{m}|^2 + |\nabla_y \tilde{m}|^2 \\ &= E_{ex}(\overline{m}) + E_{ex}(\tilde{m}) + 2\int_{\mathbb{R}} \partial_x \overline{m}\, \partial_x \int_{D_R} \tilde{m} \\ &= E_{ex}(\overline{m}) + E_{ex}(\tilde{m}). \end{aligned}$$

Analogously, for all $x \in \mathbb{R}$ we can calculate

$$\begin{aligned} \int_{D_R} |m(x,y)|^2\, dy &= \int_{D_R} |\overline{m}(x) + \tilde{m}(x,y)|^2\, dy \\ &= \int_{D_R} |\overline{m}(x)|^2 + |\tilde{m}(x,y)|^2\, dy + 2\overline{m}(x)\int_{D_R} \tilde{m}(x,y)\, dy \\ &= \int_{D_R} |\overline{m}(x)|^2 + |\tilde{m}(x,y)|^2\, dy. \end{aligned}$$

The last equation is an instance of the Poincaré inequality [18, p. 164]. □

We show an estimate for $\|m_y\|_{L^2(\Sigma)}$. We use the estimates from Section 2.3 to bound $\|\overline{m}_y\|_{L^2(\Sigma)}$ and use the exchange energy to bound $\|\tilde{m}_y\|_{L^2(\Sigma)}$.

Lemma 2.19. *There exist constants C_1, C_2 that depend only on R such that*

$$\|m_y\|^2_{H^1(\Sigma)} \leq C_1 E(m), \qquad \|\sigma\|^2_{L^2(\partial\Sigma)} \leq C_2 E(m)$$

Proof. Integration of (2.24) yields

$$E_H(\tilde{m}) \leq \|\tilde{m}\|^2_{L^2(\Sigma)} \leq 16R^2 \|\nabla_y m\|^2_{L^2(\Sigma)} \leq 16R^2 E(m), \tag{2.25}$$

thus

$$E(\overline{m}) \leq E_{\text{ex}}(m) + 2E_H(\tilde{m}) + 2E_H(m) \leq (32R^2 + 2)E(m) < \infty.$$

Using (2.23), (2.25) and Lemma 2.13 we get the estimate

$$\begin{aligned} \|m_y\|^2_{L^2(\Sigma)} &= \left(\|\tilde{m}_y\|^2_{L^2(\Sigma)} + \|\overline{m}_y\|^2_{L^2(\Sigma)}\right) \\ &\leq 16R^2 E(m) + \frac{E(\overline{m})}{c_1} \leq \left(16R^2 + \frac{32R^2 + 2}{c_1}\right) E(m) \end{aligned}$$

where $c_1 = \min\{R^2,\ 2\,R^{\frac{4}{3}}\}$ which implies the first statement. The second statement is a consequence of the trace estimate for Sobolev spaces. □

Like in Corollary 2.14, this estimate implies directly the characterisation of maps $m\colon \Sigma \to \mathbb{S}^2$ with finite energy.

Theorem 2.20. *A map $m\colon \Sigma \to \mathbb{S}^2$ is in $\mathcal{M}$ if and only if one of the four functions $m \pm \vec{e}_x$, $m \pm \chi$ is in $H^1(\Sigma)$.*

We use this result to show the existence of minimisers.

Theorem 2.21. *For every $R > 0$ there exist minimisers of E in $\mathcal{M}_l$, $\mathcal{T}_l$ and $\mathcal{V}_l$.*

Proof. We use the direct method to find a minimiser in $\mathcal{M}_l$. Let $(m^n)_{n\in\mathbb{N}}$ be a minimising sequence in $\mathcal{M}_l$. Since the problem is invariant under translations we can choose the functions m^n in a way that $\overline{m}^n_x(0) = 0$ and $\overline{m}^n_x(x) \leq 0$ for $x \leq 0$. The energy $E(m^n)$ is bounded, therefore $\|\nabla m^n\|_{L^2(\Sigma)}$ is bounded. So there is a map $m^{\lim}\colon \Sigma \to \mathbb{S}^2$ and a subsequence, denoted with $(m^n)_{n\in\mathbb{N}}$ as well, such that ∇m^n converges weakly to $\nabla m^{\lim}$ in $L^2(\Sigma)$ and m^n converges strongly to $m^{\lim}$ in $L^2_{\text{loc}}(\overline{\Sigma})$. Then in particular $\overline{m}^{\lim}_x(0) = 0$. The functional E_{ex} is lower semicontinous with respect to weak L^2 convergence of $(\nabla m^n)_{n\in\mathbb{N}}$, and the functional E_H is lower semicontinous with respect to convergence in $L^2_{\text{loc}}(\overline{\Sigma})$ (Lemma 2.8). Thus $E(m^{\lim}) \leq \liminf_{n\to\infty} E(m^n)$

and we only have to show $m^{\lim} \in \mathcal{M}_l$. Since $E(m^{\lim})$ is finite and $\overline{m}_x^{\lim} \leq 0$ for $x \leq 0$ we have that either $m^{\lim} \in \mathcal{M}_l$ or $m^{\lim} + \vec{e}_x \in H^1(\Sigma)$.

Clearly $E(m^{\lim}) > 0$. Indeed, if $E(m^{\lim})$ is zero, $m^{\lim}$ has to be constant on Σ with $m_y^{\lim} \equiv 0$ (Lemma 2.19). Thus $m^{\lim} \equiv \vec{e}_x$ or $m^{\lim} \equiv -\vec{e}_x$. This is in contradiction to $\overline{m}_x^{\lim}(0)$ being zero.

We now assume $m^{\lim} + e_x \in H^1(\Sigma)$ in order to show by contradiction $m^{\lim} \in \mathcal{M}_l$. The proof will be in the spirit of concentration compactness: If the sequence $(m^n)_{n\in\mathbb{N}}$ converges to a map $m^{\lim} \notin \mathcal{M}_l$ the maps m^n "split" into two parts. We show that the sum of the energies of the parts is strictly greater than the energy that can be obtained when the splitting does not occur.

For $m + \vec{e}_x \in H^1(\Sigma)$ we can construct sequences of maps $(g^n)_{n\in\mathbb{N}}$ and $(h^n)_{n\in\mathbb{N}}$ with the following properties:

(1) $g^n \in \mathcal{M}_l$, $(g^n)_{n\in\mathbb{N}}$ converges to $-\vec{e}_x$ in $L^2_{\text{loc}}(\overline{\Sigma})$, $E(g_n)$ is uniformly bounded.

(2) $(h^n)_{n\in\mathbb{N}}$ converges to $m^{\lim}$ in $L^2(\Sigma)$.

(3) $\lim_{n\to\infty} \int_\Sigma \nabla g_n \cdot \nabla h_n = 0$.

(4) $m^n = g^n + h^n + \vec{e}_x$.

We give an explicit construction of $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ below, as the last part of the proof.

Let u_{g^n}, u_{h^n} be weak solutions of (2.3) for $m = g^n$, $m = h^n$ respectively. Then $\nabla u_{g^n} = \nabla u_{g_n + \vec{e}_x}$ and, using Lemma 2.8, we get

$$\left| \int_{\mathbb{R}^3} \nabla u_{g^n} \cdot \nabla u_{h^n} \right| \leq \left| \int_{\mathbb{R}^3} m^{\lim} \nabla u_{g^n} \right| + \left\| h^n - m^{\lim} \right\|_{L^2(\Sigma)} \sqrt{E_H(g^n)} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore

$$\lim_{n\to\infty} E(m^n) = \lim_{n\to\infty} \left(E(h^n) + E(g^n)\right) \geq E(m^{\lim}) + E_{\mathcal{M}_l} > E_{\mathcal{M}_l}.$$

This is a contradiction to $(m^n)_{n\in\mathbb{N}}$ being a minimising sequence in $\mathcal{M}_l$.

The limit of a sequence whose elements are all in $\mathcal{T}$, $\mathcal{V}$, respectively, is in that class, too. Therefore we can find minimisers in $\mathcal{T}_l$ and $\mathcal{V}_l$ in exactly the same way as we have found minimisers in $\mathcal{M}_l$.

Construction of $(g^n)_{n\in\mathbb{N}}$ *and* $(h^n)_{n\in\mathbb{N}}$. Since $\|\nabla m^n\|_{L^2(\Sigma)}$ is uniformly bounded and $(m^n + \vec{e}_x)_{n\in\mathbb{N}}$ converges in $L^2_{\text{loc}}(\overline{\Sigma})$ to a map $m^{\lim} + \vec{e}_x \in H^1(\Sigma)$ there exists a sequence $l_n \to \infty$, such that

$$\lim_{n\to\infty} \left(\|m^n(-l_n, \cdot) + \vec{e}_x\|_{H^1(D_R)} + \|m^n(l_n, \cdot) + \vec{e}_x\|_{H^1(D_R)} \right) = 0,$$

$$\lim_{n\to\infty}\Big(\|m^n - m^{\lim}\|_{L^2([-l_n,l_n]\times D_R)}\Big) \;=\; 0.$$

Then the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ implies

$$\lim_{n\to\infty}\big(\|m^n(-l_n,\cdot)+\vec{e}_x\|_{L^\infty(D_R)} + \|m^n(l_n,\cdot)+\vec{e}_x\|_{L^\infty(D_R)}\big) = 0.$$

We set

$$g^n(x,y) := \begin{cases} m^n(x,y) & \text{if } x\in\mathbb{R}\setminus[-l_n,l_n],\\ -\vec{e}_x & \text{if } x\in[-l_n+1,l_n-1],\\ \frac{\alpha^n(x)}{|\alpha^n(x)|} & \text{if } x\in[-l_n,-l_n+1],\\ \frac{\beta^n(x)}{|\beta^n(x)|} & \text{if } x\in[l_n-1,l_n], \end{cases} \qquad h^n := m^n - g^n - \vec{e}_x,$$

where

$$\begin{aligned} \alpha^n(x,y) \;&:=\; (l_n+x)\,(-\vec{e}_x) \;+\; (-l_n-x+1)\,m^{\lim}(-l_n,y),\\ \beta^n(x,y) \;&:=\; (1+x-l_n)\,(-\vec{e}_x) \;+\; (-x+l_n)\,m^{\lim}(l_n,y). \end{aligned}$$

See Figure 2.2 for a sketch of m^n_x, g^n_x and h^n_x.

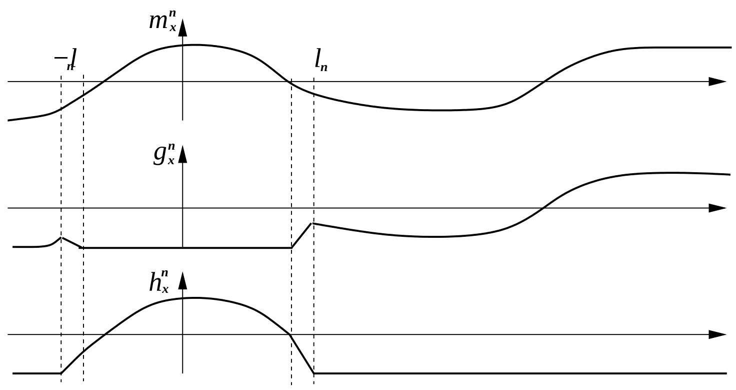

Figure 2.2: The functions m^n_x, g^n_x and h^n_x

Then (1) and (4) are surely satisfied and

$$\lim_{n\to\infty}\|g_n+\vec{e}_x\|_{H^1(([-l_n,-l_n+1]\cup[l_n-1,l_n])\times D_R)} \;=\; 0.$$

On $(]-\infty,-l_n]\cup[-l_n+1,l_n-1]\cup[l_n,\infty[)\times D_R$ either $\nabla g_n\equiv 0$ or $\nabla h_n\equiv 0$. Since $\|\nabla m_n\|_{L^2(\Sigma)}$ is uniformly bounded we have

$$\begin{aligned}\lim_{n\to\infty}\int_\Sigma \nabla g_n\cdot\nabla h_n \;&=\; \lim_{n\to\infty}\int_{([-l_n,-l_n+1]\cup[l_n-1,l_n])\times D_R}\nabla g_n\cdot(\nabla m_n-\nabla g_n)\\ &=\; 0.\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\lim_{n\to\infty} \left\| h^n - m^{\lim} \right\|^2_{L^2(\Sigma)} \\
&\leq \lim_{n\to\infty} \left(\left\| m^n - m^{\lim} \right\|^2_{L^2([-l_n+1,l_n-1]\times D_R)} + \left\| m^{\lim} + \vec{e}_x \right\|^2_{L^2(\mathbb{R}\setminus[-l_n,l_n]\times D_R)} \right. \\
&\quad \left. + \left\| m^n - g^n - \vec{e}_x - m^{\lim} \right\|^2_{L^2(([-l_n,-l_n+1]\cup[l_n-1,l_n]\times D_R)} \right) \\
&\leq \lim_{n\to\infty} \left(\left\| m^n - m^{\lim} \right\|^2_{L^2([-l_n,l_n]\times D_R)} + \left\| m^{\lim} + \vec{e}_x \right\|^2_{L^2((\mathbb{R}\setminus[-l_n+1,l_n-1])\times D_R)} \right. \\
&\quad \left. + \| g^n + \vec{e}_x \|_{L^2(([-l_n,-l_n+1]\cup[l_n-1,l_n]\times D_R)} \right) \\
&= 0.
\end{aligned}
$$

Thus the maps g^n and h^n maps have the required properties (1) to (4). □

2.5 Energy scaling and Γ-convergence for $R \to 0$

In this section we look at sequences of radii that converge to zero. We prove that $\frac{1}{R^2}E(m)$ Γ-converges to a reduced, one dimensional problem whose minimiser can be calculated explicitly. In particular, Γ-convergence implies convergence of the minimal energies. Therefore we do not only get the estimate $cR^2 \leq E_{\mathcal{M}_l} \leq CR^2$ for all $R \leq R_0$ and some fixed $c, C, R_0 > 0$ but we also know that for $R_0 \to 0$ the constants c, C both converge to the minimal energy of the reduced problem. In our case this energy is $\sqrt{8}\pi$.

In this section we make the implicit dependencies on the radius R explicit. Instead of Σ we write $\Sigma(R)$, instead of $E(m)$ we write $E(m, R)$, etc.

Definition 2.22. (i) The admissible set for the *full variational problem for* $R \in \mathbb{R}^+$ is

$$\mathcal{M}(R) = \left\{ m\colon \Sigma(R) \to \mathbb{S}^2 | \; E(m, R) < \infty \right\}.$$

For each admissible function $m \in \mathcal{M}(R)$ we set

$$\acute{m}\colon \Sigma(1) \to \mathbb{S}^2, \qquad \acute{m}\left(x, \frac{y}{R}\right) := m(x, y).$$

After rescaling, the energy functional of the full variational problem is

$$\frac{1}{R^2}E(m, R) = \int_{\Sigma(1)} \left(|\partial_x \acute{m}(p)|^2 + \frac{1}{R^2}|\nabla_y \acute{m}(p)|^2 \right) dp + \frac{1}{R^2}E_H(m, R).$$

(ii) The energy functional for the *reduced variational problem* is

$$E_{\text{red}}(m) := \pi \|\partial_x m\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2}\|m_y\|^2_{L^2(\mathbb{R})}.$$

The admissible set is

$$\mathcal{M}(0) \;=\; \{m\colon \mathbb{R} \to \mathbb{S}^2 |\; E_{\mathrm{red}}(m) < \infty\}.$$

(iii) We use the following notion of convergence: Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers that converges to zero, let $m^n \in \mathcal{M}(R_n)$ and let $m^0 \in \mathcal{M}(0)$. We say the sequence $(m^n)_{n\in\mathbb{N}}$ converges to m^0 if

- $\nabla_y \acute{m}^n$ converges to 0 strongly in $L^2(\Sigma(1))$ and
- $\partial_x \acute{m}^n$ converges to $\partial_x m^0$ weakly in $L^2(\Sigma(1))$ and
- $\acute{m}^n$ converges to m^0 strongly in $L^2_{\mathrm{loc}}(\overline{\Sigma})$.

Theorem 2.23. *The reduced variational problem (Definition 2.22 (ii)) is the Γ-limit of the full variational problem (Definition 2.22 (i)) with respect to the convergence stated in Definition 2.22 (iii). This means*

- Compactness: *Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to zero, let $m^n \in \mathcal{M}(R_n)$ and let $(\frac{1}{R_n^2}E(m^n, R_n))_{n\in\mathbb{N}}$ be bounded. Then there exists a subsequence of $(m^n)_{n\in\mathbb{N}}$ which converges in the sense of Definition 2.22 (iii) to some $m^0 \in \mathcal{M}(0)$.*

- Lower semicontinuity: *For every convergent sequence $(m^n)_{n\in\mathbb{N}}$, $m^n \in \mathcal{M}(R_n)$ with limit $m^0 \in \mathcal{M}(0)$ we have*

$$E_{\mathrm{red}}(m^0) \;\le\; \liminf_{n\to\infty} \frac{1}{R_n^2} E(m^n, R_n).$$

- Construction: *For each $m^0 \in \mathcal{M}(0)$ and each sequence $(R_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero there is a sequence $(m^n)_{n\in\mathbb{N}}$ with $m^n \in \mathcal{M}(R_n)$ such that*

$$E_{\mathrm{red}}(m^0) \;=\; \lim_{n\to\infty} \frac{1}{R_n^2} E(m^n, R_n).$$

To show Theorem 2.23 for all $m \in \mathcal{T}(R)$ we need

$$\lim_{R\to 0} \frac{1}{R^2} E_{\sigma\sigma}(m, R) \;=\; \frac{\pi}{2} \|m_y\|^2_{L^2(\mathbb{R})}$$
$$\lim_{R\to 0} \frac{1}{R^2} E_{\rho\rho}(m, R) \;=\; 0.$$

However, later it will be useful to have the following stronger results.

Lemma 2.24. *(i) Let $m_y \in H^1(\mathbb{R}, \{0\} \times \mathbb{R}^2)$ be a map and let R be so small that $-\ln(R) \geq 1$. Then*

$$0 \;\leq\; R^2 \frac{\pi}{2} \|m_y\|^2_{L^2(\mathbb{R})} - E_{\sigma\sigma}(m_y, R) \;\leq\; 3\pi R^4 \, |\ln(R)| \, \|m_y\|^2_{H^1(\mathbb{R})}.$$

(ii) Let $m_x : \mathbb{R} \to \mathbb{R}$ be a map such that $\rho := \partial_x m_x \in L^2(\mathbb{R})$ and let R be so small that $-\ln(R) \geq 1$. Then

$$0 \;\leq\; E_{\rho\rho}(m_x \vec{e}_x, R) \;\leq\; 5\pi R^4 \, |\ln(R)| \, E(m_x \vec{e}_x, 1).$$

Proof. (i) Theorem 2.11 directly implies the lower bound $R^2 \frac{\pi}{2} \|m_y\|_{L^2(\mathbb{R})} - E_{\sigma\sigma}(m_y, R) > 0$. For the upper bound, Theorem 2.11 implies the relation

$$\begin{aligned} & R^2 \frac{\pi}{2} \|m_y\|^2_{L^2(\mathbb{R})} - E_{\sigma\sigma}(m_y, R) \\ & \leq \underbrace{-\frac{\pi}{2} R^2 \int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{m}_y(\xi)|^2 \, \ln(|\xi R|) \xi^2 R^2 \, d\xi}_{A} + \underbrace{\frac{\pi}{2} R^2 \int\limits_{\mathbb{R} \setminus [-\frac{1}{2R}, \frac{1}{2R}]} |\hat{m}_y(\xi)|^2 \, d\xi}_{B} . \end{aligned}$$

Using the identity $-\ln(|\xi R|) = -\ln(|\xi|) - \ln(R)$ and the fact that $-\ln(|\xi|)$ is negative for $|\xi| > 1$ we have

$$\begin{aligned} A \;&=\; -\frac{\pi}{2} R^2 \int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{m}_y(\xi)|^2 \ln(|\xi|) \xi^2 R^2 \, d\xi - \frac{\pi}{2} R^2 \int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{m}_y(\xi)|^2 \ln(R) \xi^2 R^2 \, d\xi \\ &\leq\; \frac{\pi}{2} R^4 \int_{-1}^{1} |\hat{m}_y(\xi)|^2 \, |\ln(|\xi|)| \, \xi^2 \, d\xi + \frac{\pi}{2} R^4 \, |\ln(R)| \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 \xi^2 \, d\xi \\ &\leq\; \frac{\pi}{2} R^4 \|\hat{m}_y\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2} R^4 \, |\ln(R)| \, \|\partial_x m_y\|^2_{L^2(\mathbb{R})} \\ &\leq\; \frac{\pi}{2} R^4 \, |\ln(R)| \, \|m_y\|^2_{H^1(\mathbb{R})}. \end{aligned}$$

The second summand can be bounded by

$$B \;\leq\; \frac{\pi}{2} R^2 \int_{\mathbb{R} \setminus [-\frac{1}{2R}, \frac{1}{2R}]} |\hat{m}_y(\xi)|^2 (2\xi R)^2 \, d\xi \;\leq\; 2\pi R^4 \|\partial_x m_y\|^2_{L^2(\mathbb{R})},$$

so we have

$$0 \;\leq\; R^2 \frac{\pi}{2} \|m_y\|^2_{L^2(\mathbb{R})} - E_{\sigma\sigma}(m_y, R) \;\leq\; 3\pi R^4 \, |\ln(R)| \, \|m_y\|^2_{H^1(\mathbb{R})}.$$

(ii) To show the second statement we note that (2.19) implies

$$\pi \int_{-1}^{1} |\hat{\rho}(\xi)|^2 \, |\ln(\xi)| \, d\xi \;\leq\; 2E_{\rho\rho}(m_x \vec{e}_x, 1).$$

Like above we calculate

$$
\begin{aligned}
&E_{\rho\rho}(m_x\vec{e}_x, R)\\
&\quad\le \underbrace{\pi R^4 \int_{-\frac{1}{2R}}^{\frac{1}{2R}} -|\hat{\rho}(\xi)|^2 \ln(|\xi R|)\, d\xi}_{A} + \underbrace{\pi R^4 \int_{\mathbb{R}\setminus[-\frac{1}{2R},\frac{1}{2R}]} |\hat{\rho}(\xi)|^2 \frac{1}{|\xi R|^2}\, d\xi}_{B}
\end{aligned}
$$

$$
\begin{aligned}
A &\le \pi R^4 |\ln(R)| \int_{-\frac{1}{2R}}^{\frac{1}{2R}} |\hat{\rho}(\xi)|^2\, d\xi + \pi R^4 \int_{-1}^{1} |\hat{\rho}(\xi)|^2\, |\ln(|\xi|)|\, d\xi\\
&\le \pi R^4 |\ln(R)| \|\partial_x m_x\|^2_{L^2(\mathbb{R})} + 2R^4 E_{\rho\rho}(m_x\vec{e}_x, 1)\\
B &\le \pi R^4 \int_{\mathbb{R}} 4|\hat{\rho}(\xi)|^2\, d\xi \;=\; 4\pi R^4 \|\partial_x m_x\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Thus $E_{\rho\rho}(m_x\vec{e}_x, R) \le 5\pi R^4\, |\ln(R)|\, E(m_x\vec{e}_x, 1)$. □

Lemma 2.25. *Let $m^n \in \mathcal{M}(R_n)$ with $\lim_{n\to\infty} R_n = 0$ and assume that $\frac{1}{R_n^2}E(m^n, R_n)$ is bounded by some number C. If $(m^n)_{n\in\mathbb{N}}$ converges in the sense of Definition 2.22 to $m^{\lim} \in \mathcal{M}(0)$ then we have*

$$
\lim_{n\to\infty} \frac{1}{R_n^2}\left(\|\nabla m^n\|^2_{L^2(\Sigma(R_n))} - \|\nabla m^n - \nabla m^{\lim}\|^2_{L^2(\Sigma(R_n))}\right) \;=\; \pi \|\partial_x m^{\lim}\|^2_{L^2(\mathbb{R})},
$$

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{R_n^2} E_H(m^n, R_n) - \frac{\pi}{2}\left\|\overline{m}_y^n - m_y^{\lim}\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad= \lim_{n\to\infty}\left(\frac{1}{R_n^2} E_{\sigma\sigma}(\overline{m}^n, R_n) - \frac{\pi}{2}\left\|\overline{m}_y^n - m_y^{\lim}\right\|^2_{L^2(\mathbb{R})}\right) \;=\; \frac{\pi}{2}\left\|m_y^{\lim}\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Proof. Since $m^{\lim}$ is constant on each cross section we have

$$
\begin{aligned}
&\frac{1}{R_n^2}\|\nabla m^n\|^2_{L^2(\Sigma(R_n))} - \frac{1}{R_n^2}\|\nabla m^n - \nabla m^{\lim}\|^2_{L^2(\Sigma(R_n))}\\
&\quad= \frac{2}{R_n^2}\left(\int_{\Sigma(R_n)} \nabla m^n \cdot \nabla m^{\lim}\right) - \frac{1}{R_n^2}\|\nabla m^{\lim}\|^2_{L^2(\Sigma(R_n))}\\
&\quad= \underbrace{2\left(\int_{\Sigma(1)} (\partial_x \acute{m}^n - \partial_x \acute{m}^{\lim})\partial_x m^{\lim}\right)}_{(*)} + \pi \|\partial_x m^{\lim}\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

By assumption, $(*)$ converges to 0, so we obtain

$$
\lim_{n\to\infty} \frac{1}{R_n^2}\left(\|\nabla m^n\|^2_{L^2(\Sigma(R_n))} - \|\nabla m^n - \nabla m^{\lim}\|^2_{L^2(\Sigma(R_n))}\right) \;=\; \pi \|\partial_x m^{\lim}\|^2_{L^2(\mathbb{R})}.
$$

We now show the second equation. The Poincaré inequality (2.25) yields

$$
E_H(\tilde{m}) \;\le\; \|\tilde{m}^n\|^2_{L^2(\Sigma(R_n))} \;\le\; 16R_n^2 \|\nabla_y m\|^2_{L^2(\Sigma(R_n))} \;\le\; 16R_n^4 C.
$$

Using this estimate and Lemma 2.9 we can calculate

$$\begin{aligned}
&\left| \frac{1}{R_n^2} E_H(m^n, R_n) - \frac{1}{R_n^2} E_{\sigma\sigma}(\overline{m}^n, R_n) \right| \\
&\quad \le \frac{1}{R_n^2} \left(E_H(\tilde{m}^n, R_n) + 2\sqrt{E_H(\overline{m}^n, R_n)}\, \sqrt{E_H(\tilde{m}^n, R_n)} \right) \\
&\quad \le \frac{1}{R_n^2} \left(16CR_n^4 + 2\sqrt{2E_H(m^n, R_n) + 2E_H(\tilde{m}^n, R_n)}\, \sqrt{16CR_n^4} \right) \\
&\quad \le 16CR_n^2 + 2\sqrt{2CR_n^2 + 32CR_n^4}\, 4\sqrt{C}.
\end{aligned}$$

So we have

$$\lim_{n\to\infty} \left(\frac{1}{R_n^2} E_H(m^n, R_n) - \frac{1}{R_n^2} E_{\sigma\sigma}(\overline{m}^n, R_n) \right) = 0. \tag{2.26}$$

To calculate $\lim_{n\to\infty} E_{\sigma\sigma}(\overline{m}_n, R_n)$ we use the Fourier multiplier of Theorem 2.11. We have

$$\begin{aligned}
E_{\sigma\sigma}(\overline{m}_n, R_n) &= \int_{\mathbb{R}} g_F(R_n\xi)\, |\hat{\overline{m}}_y^n(\xi)|^2 \, d\xi \\
&= \underbrace{\int_{\mathbb{R}} g_F(R_n\xi) \left(|\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2 \right) \, d\xi}_{a_n} \\
&\quad + \underbrace{2\int_{\mathbb{R}} g_F(R_n\xi)\, \hat{m}_y^{\lim}(\xi) \left(\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi) \right) \, d\xi}_{b_n} .
\end{aligned}$$

Here g_F is a continous function with $\frac{\pi}{2} = g_F(0) \ge g_F(t)$ for all $t \in \mathbb{R}$. Considering the first summand a_n we have, for every $t > 0$, the relation

$$\begin{aligned}
0 &\le \frac{\pi}{2} \|\hat{m}^{\lim}\|_{L^2(\mathbb{R})}^2 + \liminf_{n\to\infty} \left(\frac{\pi}{2} \|\hat{\overline{m}}_y^n - \hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2 - a_n \right) \\
&\le \frac{\pi}{2} \|\hat{m}^{\lim}\|_{L^2(\mathbb{R})}^2 + \limsup_{n\to\infty} \left(\frac{\pi}{2} \|\hat{\overline{m}}_y^n - \hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2 - a_n \right) \\
&\le \limsup_{n\to\infty} \underbrace{\int_{-t}^{t} (g_F(0) - g_F(R_n\xi)) \left(|\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2 \right) \, d\xi}_{a_{n,1}} \\
&\quad + \limsup_{n\to\infty} \underbrace{\frac{\pi}{2} \int_{\mathbb{R}\setminus[-t,t]} |\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2 \, d\xi}_{a_{n,2}} .
\end{aligned}$$

Because of (2.26) and the relation $\|\partial_x \overline{m}_y^n\|_{L^2(\Sigma(R_n))}^2 \le CR_n^2$ the term

$$\frac{1}{R_n^2} \left(E_{\sigma\sigma}(\overline{m}_y^n) + \|\partial_x \overline{m}_y^n\|_{L^2(\Sigma(R_n))}^2 \right)$$

is uniformly bounded for small R. Therefore, with Lemma 2.13, $\|\overline{m}_y^n\|_{L^2(\mathbb{R})}$ is bounded. This implies for every $t > 0$

$$\begin{aligned}\limsup_{n\to\infty} a_{n,1} &\leq \limsup_{n\to\infty} \left(g_F(0) - g_F(R_n t)\right)\left(3\|\hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2 + 2\|\hat{\overline{m}}_y^n\|_{L^2(\mathbb{R})}^2\right) \\ &= 0.\end{aligned}$$

Regarding $a_{n,2}$, we have the relation

$$\begin{aligned} a_{n,2} &\leq \frac{\pi}{2}\int_{\mathbb{R}\setminus[-t,t]} \frac{\xi^2}{t^2}\left(|\hat{m}_y^{\lim}(\xi)|^2 + |\hat{\overline{m}}_y^n(\xi) - \hat{m}_y^{\lim}(\xi)|^2\right) \\ &\leq \frac{\pi}{2}\int_{\mathbb{R}}\left(\frac{3\xi^2}{t^2}|\hat{m}^{\lim}(\xi)|^2 + \frac{2\xi^2}{t^2}|\hat{\overline{m}}^n(\xi)|^2\right) \\ &= \frac{3\pi}{2t^2}\|\partial_x m^{\lim}\|_{L^2(\mathbb{R})}^2 + \frac{\pi}{t^2}\|\partial_x \overline{m}^n\|_{L^2(\mathbb{R})}^2 \leq \frac{3}{t^2}C. \end{aligned}$$

Since t was arbitrary we obtain

$$\lim_{n\to\infty}\left(a_n - \frac{\pi}{2}\|\hat{\overline{m}}_y^n - \hat{m}_y^{\lim}\|_{L^2(\mathbb{R})}^2\right) = \frac{\pi}{2}\|\hat{m}^{\lim}\|_{L^2(\mathbb{R})}^2.$$

We now show that the summand b_n converges to zero. Since $\|\overline{m}_y^n\|_{L^2(\mathbb{R})}$ is bounded, the sequence $(f_n)_{n\in\mathbb{N}}$,

$$f_n \colon \mathbb{R} \to \mathbb{R}, \qquad \xi \mapsto g_F(\xi R_n)(\hat{\overline{m}}^n(\xi) - \hat{m}^{\lim}(\xi))$$

converges weakly, up to a subsequence. Since $\lim_{n\to\infty} \overline{m}_y^n - m^{\lim} = 0$ in $L^2_{\mathrm{loc}}(\mathbb{R})$, the only possible limit of the sequence $(f_n)_{n\in\mathbb{N}}$ is 0. In particular, this implies $\lim_{n\to\infty} b_n = 0$. Therefore we have

$$\begin{aligned} &\lim_{n\to\infty}\left(E_{\sigma\sigma}(\overline{m}_n, R_n) - \left\|\overline{m}_y^n - m_y^{\lim}\right\|_{L^2(\mathbb{R})}^2\right) \\ &= \lim_{n\to\infty}\left(a_n - \left\|\overline{m}_y^n - m_y^{\lim}\right\|_{L^2(\mathbb{R})}^2\right) = \frac{\pi}{2}\left\|m_y^{\lim}\right\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

as claimed.

□

Proof of Theorem 2.23. We have to check the three properties of Γ-convergence: compactness, lower semicontinuity and construction.

First we show compactness. If $\frac{1}{R_n^2}E(m^n, R_n)$ is bounded, then

$$\lim_{n\to\infty}\|\nabla_y \acute{m}^n\|_{L^2(\Sigma(1))} = \lim_{n\to\infty}\|\nabla_y m^n\|_{L^2(\Sigma(R_n))} = 0$$

and $\|\partial_x \acute{m}^n\|_{L^2(\Sigma(1))} = \frac{1}{R_n}\|\partial_x m^n\|_{L^2(\Sigma(R_n))}$ is bounded, so $(\partial_x \acute{m}^n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence. Since for every $n \in \mathbb{N}$ the embedding

$H^1([-n,n] \times D^R) \hookrightarrow L^2([-n,n] \times D^R)$ is compact, we can again pick a subsequence, denoted with $(m^n)_{n\in\mathbb{N}}$ as well, such that $(\acute{m}^n)_{n\in\mathbb{N}}$ converges in $L^2_{\text{loc}}(\overline{\Sigma}(1))$, too. This subsequence converges in the sense of Definition 2.22 (iii) to a map $m^0 \colon \mathbb{R} \to \mathbb{S}^2$.

Lower semicontinuity is a direct consequence of Lemma 2.25. Moreover Lemma 2.24 ensures that, for a sequence $(R_n)_{n\in\mathbb{N}}$ converging to zero and $m^0 \in \mathcal{M}(0)$, we can construct a recovery sequence by setting $m^n \colon \Sigma(R_n) \to \mathbb{S}^2$, $(x,y) \mapsto m^0(x)$. □

We now determine the minimiser of the reduced problem.

Lemma 2.26. *The minimiser of E_{red} in*

$$\mathcal{M}_l(0) := \{m \in \mathcal{M}(0) : m - \chi \in H^1(\mathbb{R})\}$$

is unique up to translation and rotation. It is given by

$$m^{\text{red}} \colon \mathbb{R} \to \mathbb{S}^2, \quad x \mapsto \left(\tanh\left(\tfrac{x}{\sqrt{2}}\right), \frac{1}{\cosh\left(\tfrac{x}{\sqrt{2}}\right)}, 0 \right),$$

and its energy is $\sqrt{8}\pi$.

Proof. To find minimisers of E_{red} we parametrise m by the angle $\theta \colon \Sigma \to [0,1]$ and set $m_x = -\cos(\pi\theta)$. Using the inequality $a^2 + b^2 \geq 2ab$ and the fact that $\pi \sin(\pi\theta)\, \partial_x\theta = \partial_x \cos(\pi\theta)$ we get

$$E_{\text{red}}(m) \;\geq\; \pi \int_{\mathbb{R}} (\pi\partial_x\theta)^2 + \frac{1}{2}\sin^2(\pi\theta) \;\geq\; \pi \int_{\mathbb{R}} \sqrt{2}\pi \,|\sin(\pi\theta)\,\partial_x\theta| \;\geq\; \sqrt{8}\pi.$$

Assume that $|\nabla m| = \pi|\nabla\theta|$, i.e., that the direction m_y does not change. Then the first inequality is an equality. Assume moreover that θ is a monotonously increasing solution of

$$\partial_x\theta \;=\; \frac{1}{\sqrt{2}\pi}\sin(\pi\theta), \tag{2.27}$$

then the second inequality is an equality. Such a map is unique up to translation and rotation and we have

$$\theta(x) \;=\; \frac{2}{\pi}\arctan\left(e^{\frac{x}{\sqrt{2}}}\right) \;=\; \frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{1+e^{\sqrt{2}x}}}\right),$$

$$m^{\text{red}}(x) \;=\; (-\cos(\pi\theta), \sin(\pi\theta), 0) \;=\; \left(\tanh\left(\frac{x}{\sqrt{2}}\right), \frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)}, 0\right).$$

□

Theorem 2.27. *(i) Let m^{red} be as in Lemma 2.26. For each positive sequence $(R_n)_{n\in\mathbb{N}}$ converging to zero and each sequence of minimisers $m^n \in \mathcal{M}_l(R_n)$, the rescaled energy $\frac{1}{R_n^2}E(m^n, R_n)$ converges to $E_{\text{red}}(m^{\text{red}}) = \sqrt{8}\pi$. Moreover, there is a sequence of translations T^n such that a subsequence of $(T^n(m_n))_{n\in\mathbb{N}}$ converges, up to a rotation, to m^{red} in the sense of Definition 2.22 (iii).*

(ii) We have

$$\lim_{n\to\infty} \frac{1}{R_n^2} E_{ex}(m^n, R^n) \;=\; \pi\|\partial_x m^{\text{red}}\|^2_{L^2(\mathbb{R})} \;=\; \sqrt{2}\pi \tag{2.28}$$

$$\lim_{n\to\infty} \frac{1}{R_n^2} E_H(m^n, R^n) \;=\; \frac{\pi}{2}\|m_y^{\text{red}}\|^2_{L^2(\mathbb{R})} \;=\; \sqrt{2}\pi \tag{2.29}$$

Proof. (i) For Γ-limits, the following statement holds [12, Corollary 7.17, p. 78]: Let $(r_n)_{n\in\mathbb{N}}$ be a sequence converging to zero and let m^n be the minimiser of the full problem for R_n. Then every accumulation point m of $(m^n)_{n\in\mathbb{N}}$ is a minimiser of the reduced problem. Moreover, the full energy of m_n converges to the reduced energy of m. This statement implies the first part of the theorem directly.

(ii) This is a direct consequence of $\lim_{n\to\infty} \frac{1}{R_n^2} E(m^n, R_n) = E_{\text{red}}(m^{\text{red}})$ and Lemma 2.25. □

Using the fact that we are considering minimisers, we can prove an even stronger convergence result.

Lemma 2.28. *Let m^{red} be as in Lemma 2.26, let $(R_n)_{n\in\mathbb{N}}$ be a sequence converging to zero and let $m^n \in \mathcal{M}_l(R_n)$ be a sequence of minimisers converging to m^{red} in the sense of Definition 2.22 (iii). Then*

$$\lim_{n\to\infty} \frac{1}{R_n}\|m^n - m^{\text{red}}\|_{H^1(\Sigma(R))} \;=\; 0.$$

Proof. In view of Lemma 2.25 and Theorem 2.27 (ii) we have

$$\lim_{n\to\infty} \frac{1}{R_n}\|\overline{m}_y^n - m_y^{\text{red}}\|_{L^2(\Sigma(R_n))} \;=\; 0,$$

$$\lim_{n\to\infty} \frac{1}{R_n}\|\nabla m^n - \nabla m^{\text{red}}\|_{L^2(\Sigma(R_n))} \;=\; 0. \tag{2.30}$$

Combining (2.30) and the Poincaré inequality (2.25) implies

$$\lim_{n\to\infty} \frac{1}{R_n}\|\tilde{m}^n\|_{L^2(\Sigma(R_n))} \;=\; 0.$$

Fix x_0 such that $|m_x^{\text{red}}(x)| > \frac{1}{2}$ for $x \in \mathbb{R}\setminus[-x_0, x_0]$. By assumption we have

$$\lim_{n\to\infty} \frac{1}{R_n}\|m^n - m^{\text{red}}\|_{L^2([-x_0,x_0]\times D_{R_n})} \;=\; 0.$$

Thus it remains to show

$$\lim_{n\to\infty} \frac{1}{R_n} \|\overline{m}_x^n - m_x^{\mathrm{red}}\|_{L^2(\Sigma(R_n)\setminus[-x_0,x_0]\times D_{R_n})} \;=\; 0.$$

Since

$$\begin{aligned}
\lim_{n\to\infty} & \frac{1}{R_n^2} \int_{\Sigma(R_n)} \left| |m_y^n|^2 - |m_y^{\mathrm{red}}|^2 \right| \\
&\leq \lim_{n\to\infty} \frac{1}{R_n^2} \left\| m_y^n - m_y^{\mathrm{red}} \right\|_{L^2(\Sigma(R_n))} \left(\left\| m_y^n - m_y^{\mathrm{red}} \right\|_{L^2(\Sigma)} + 2 \left\| m_y^{\mathrm{red}} \right\|_{L^2(\Sigma(R_n))} \right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\left| |m_y^n|^2 - |m_y^{\mathrm{red}}|^2 \right| &= \left| 1 - |m_x^n|^2 - (1 - |m_x^{\mathrm{red}}|^2) \right| \\
&= \left| -|\tilde{m}_x^n|^2 - |\overline{m}_x^n|^2 + |m_x^{\mathrm{red}}|^2 \right| \\
&\geq -|\tilde{m}_x^n|^2 + \left| (|\overline{m}_x^n| + |m_x^{\mathrm{red}}|)(|\overline{m}_x^n| - |m_x^{\mathrm{red}}|) \right| \\
&\geq -|\tilde{m}_x^n|^2 + (|\overline{m}_x^n| - |m_x^{\mathrm{red}}|)^2,
\end{aligned}$$

the functions $|\overline{m}_x^n|$ converge in $L^2(\mathbb{R})$ to $|m_x^{\mathrm{red}}|$.

Now (2.30) implies that they converge also in $H^1(\mathbb{R})$, and with the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ they converge in $L^\infty(\mathbb{R})$. In particular, there is n_0 such that for all $n > n_0$ we have $\left| |\overline{m}_x^n| - |m_x^{\mathrm{red}}| \right| < \frac{1}{2}$. This implies that the functions $\mathrm{sign}(\overline{m}_x^n)|_{]-\infty,-x_0]}$ and $\mathrm{sign}(\overline{m}_x^n)|_{[x_0,\infty[}$ are constant. Now the fact that m^n converges in the sense of Definition 2.22 (iii) to m^{red} implies

$$\mathrm{sign}(\overline{m}_x^n)|_{\mathbb{R}\setminus]-x_0,x_0[} \;=\; \mathrm{sign}(m_x^{\mathrm{red}})|_{\mathbb{R}\setminus]-x_0,x_0[}$$

and we have

$$\begin{aligned}
\lim_{n\to\infty} & \frac{1}{R_n} \|\overline{m}_x^n - m_x^{\mathrm{red}}\|_{L^2(\Sigma(R_n)\setminus[-x_0,x_0]\times D_{R_n})} \\
&= \lim_{n\to\infty} \frac{1}{R_n} \| |\overline{m}_x^n| - |m_x^{\mathrm{red}}| \|_{L^2(\Sigma(R_n)\setminus[-x_0,x_0]\times D_{R_n})} \;=\; 0.
\end{aligned}$$

□

2.6 The rate of convergence of the minimal energies $E_{\mathcal{M}_l}$

Theorem 2.27 implies that

$$\lim_{R\to 0} \frac{1}{R^2} E_{\mathcal{M}_l}(R) - \sqrt{8}\,\pi \;=\; 0.$$

In this section we prove an upper bound on the difference $|\frac{1}{R^2}E_{\mathcal{M}_l}(R)-\sqrt{8}\pi|$ in terms of R. Let m^R be a minimiser of $E(m,R)$ in $\mathcal{M}_R$. In the first lemma we consider $\|\tilde{m}^R\|_{L^2(\Sigma)}$. We already know from Lemma 2.28 that

$$\lim_{R\to 0}\frac{1}{R^2}\|\tilde{m}^R\|_{L^2(\Sigma(R))} \;\le\; \lim_{R\to 0}\frac{4}{R}\|\nabla_y\tilde{m}^R\|_{L^2(\Sigma(R))} \;=\; 0.$$

Using the fact that m^R is a minimiser, we can improve this estimate.

Lemma 2.29. *There exist positive constants R_0, C such that for all $R\le R_0$*

$$\left\|\frac{\overline{m}^R}{|\overline{m}^R|}-\overline{m}^R\right\|_{L^2(\Sigma)} \;\le\; \left\|\tilde{m}^R\right\|_{L^2(\Sigma)}, \tag{2.31}$$

$$\left\|\partial_x\frac{\overline{m}^R}{|\overline{m}^R|}\right\|^2_{L^2(\Sigma)}-\left\|\partial_x\overline{m}^R\right\|^2_{L^2(\Sigma)} \;\le\; \frac{1}{4}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma)}, \tag{2.32}$$

$$E\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right)-E(m^R) \;\le\; 64R^2\left\|\nabla\tilde{m}^R\right\|_{L^2(\Sigma)}-\frac{1}{2}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma)}, \tag{2.33}$$

$$\left\|\tilde{m}^R\right\|_{L^2(\Sigma)} \;\le\; CR^3, \qquad \left\|\nabla\tilde{m}^R\right\|_{L^2(\Sigma)} \;\le\; CR^2. \tag{2.34}$$

Proof. Let R_0 be so small that for all $R\le R_0$ the following inequalities hold:

$$\inf_{x\in\mathbb{R}}|\overline{m}^R(x)| \;\ge\; \frac{1}{2}, \quad \frac{48}{R}E(m^R) \;\le\; \frac{1}{4}, \quad 64R^2 \;\le\; \frac{1}{4}, \quad E(m^R) \;\le\; 16R^2.$$

Estimate (2.31) is the result of the following calculation. In the last step we use (2.23).

$$\left\|\frac{\overline{m}^R}{|\overline{m}^R|}-\overline{m}^R\right\|^2_{L^2(\Sigma)} \;=\; \int_\Sigma\left(1-|\overline{m}^R|\right)^2 \;\le\; \int_\Sigma 1-\left|\overline{m}^R\right|^2 \;=\; \left\|\tilde{m}^R\right\|^2_{L^2(\Sigma)}.$$

To prove (2.32) we first show that $1-|\overline{m}^R|^2$ is small, using (2.23) and the Poincaré inequality (2.24).

$$\begin{aligned}
1-\left|\overline{m}^R\right|^2 \;&=\; \frac{1}{|D_R|}\int_{D_R}\left|\tilde{m}^R\right|^2 \;\le\; \frac{1}{|D_R|}\int_{\mathbb{R}}\left|\partial_x\int_{D_R}\left|\tilde{m}^R\right|^2\right|\\
&\le\; \frac{2}{|D_R|}\left\|\tilde{m}^R\right\|_{L^2(\Sigma)}\left\|\partial_x\tilde{m}^R\right\|_{L^2(\Sigma)}\\
&\le\; \frac{8R}{|D_R|}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma)} \;\le\; \frac{3}{R}\left\|\nabla\tilde{m}^R\right\|^2_{L^2(\Sigma)}
\end{aligned} \tag{2.35}$$

Moreover we have

$$\partial_x\overline{m}^R \;=\; \partial_x\left(|\overline{m}^R|\,\frac{\overline{m}^R}{|\overline{m}^R|}\right) \;=\; |\overline{m}^R|\;\partial_x\frac{\overline{m}^R}{|\overline{m}^R|}+\left(\partial_x|\overline{m}^R|\right)\,\frac{\overline{m}^R}{|\overline{m}^R|},$$

and since $\partial_x \frac{\overline{m}^R}{|\overline{m}^R|}$ is perpendicular to $\frac{\overline{m}^R}{|\overline{m}^R|}$ this yields

$$\left\| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right\|^2_{L^2(\Sigma)} - \|\partial_x \overline{m}^R\|^2_{L^2(\Sigma)} \;\leq\; \int_\Sigma (1 - |\overline{m}^R|^2) \left| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right|^2$$

$$= \int_\Sigma (1 - |\overline{m}^R|^2) \left| \frac{\partial_x \overline{m}^R}{|\overline{m}^R|} - \frac{\partial_x |\overline{m}^R| \, \overline{m}^R}{|\overline{m}^R|^2} \right|^2 \;\leq\; 4 \int_\Sigma (1 - |\overline{m}^R|^2) \left| \frac{\partial_x \overline{m}^R}{|\overline{m}^R|} \right|^2 .$$

Using (2.35), the assumption $|\overline{m}| \geq \frac{1}{2}$, and the bound $\frac{48}{R} E(m^R) \leq \frac{1}{4}$, we get

$$\left\| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right\|^2_{L^2(\Sigma)} - \|\partial_x \overline{m}^R\|^2_{L^2(\Sigma)} \;\leq\; \frac{48}{R} \|\nabla \tilde{m}^R\|^2_{L^2(\Sigma)} \int_\Sigma |\partial_x \overline{m}^R|^2$$

$$\leq\; \frac{48}{R} \|\nabla \tilde{m}^R\|^2_{L^2(\Sigma)} E(m^R) \;\leq\; \frac{1}{4} \|\nabla \tilde{m}^R\|^2_{L^2(\Sigma)}.$$

We consider (2.33). With (2.22) we have

$$E\left(\frac{\overline{m}^R}{|\overline{m}^R|}, R \right) - E(m^R, R)$$

$$= \underbrace{\left\| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right\|^2_{L^2(\Sigma)} - \|\partial_x \overline{m}^R\|^2_{L^2(\Sigma)} - E_{ex}(\tilde{m}^R)}_{A} + \underbrace{E_H\left(\frac{\overline{m}^R}{|\overline{m}^R|} \right) - E_H(m^R)}_{B} .$$

For the first summand we have

$$A \;\leq\; -\frac{3}{4} \|\nabla \tilde{m}\|^2_{L^2(\Sigma)}.$$

For the second summand we use (2.14), (2.31), (2.24) and the assumptions $64R^2 \leq \frac{1}{4}$, $E(m^R) \leq 16R^2$. We calculate

$$B \;\leq\; \left\| \frac{\overline{m}^R}{|\overline{m}^R|} - m^R \right\|^2_{L^2(\Sigma)} + 2\sqrt{E_H(m^R)} \left\| \frac{\overline{m}^R}{|\overline{m}^R|} - m^R \right\|_{L^2(\Sigma)}$$

$$\leq\; 4 \left\| \tilde{m}^R \right\|^2_{L^2(\Sigma)} + 4\sqrt{E_H(m^R)} \left\| \tilde{m}^R \right\|_{L^2(\Sigma)}$$

$$\leq\; 64R^2 \left\| \nabla \tilde{m}^R \right\|^2_{L^2(\Sigma)} + 16R\sqrt{E(m^R)} \left\| \nabla \tilde{m}^R \right\|_{L^2(\Sigma)}$$

$$\leq\; \frac{1}{4} \left\| \nabla \tilde{m}^R \right\|^2_{L^2(\Sigma)} + 64R^2 \left\| \nabla \tilde{m}^R \right\|_{L^2(\Sigma)} .$$

Adding the summands yields (2.33).

Now the fact that m^R is the minimiser of $E(m^R)$ in $\mathcal{M}_l(R)$ implies (2.34): We have

$$0 \;\leq\; E\left(\frac{\overline{m}^R}{|\overline{m}^R|} \right) - E(m^R),$$

and thus $\|\nabla \tilde{m}^R\|_{L^2(\Sigma)} \leq 128R^2$. The bound on $\|\tilde{m}^R\|_{L^2(\Sigma)}$ follows from the Poincaré inequality (2.24). □

We now determine an upper bound on the rate of convergence of the minimal energies $E_{\mathcal{M}_l}$.

Theorem 2.30. *There exists $C, R_0 > 0$ such that for all $R \le R_0$ we have*

$$\left| \frac{1}{R^2} E_{\mathcal{M}_l}(R) - \sqrt{8}\,\pi \right| \;\le\; CR^2 |\ln(R)|.$$

Proof. Let m^{red} be as in Lemma 2.26 and let $R \le R_0$ where R_0 as in Lemma 2.29. On the one hand, since by definition $E\left(m^R, R\right) \le E\left(m, R\right)$ for all $m \in \mathcal{M}_l(R)$ and since $E_{\text{red}}\left(m^{\text{red}}\right) = \sqrt{8}\,\pi$ (Lemma 2.26) we have

$$\begin{aligned} \frac{1}{R^2} E\left(m^R, R\right) - \sqrt{8}\,\pi \;&\le\; \frac{1}{R^2} E\left(m^{\text{red}}, R\right) - E_{\text{red}}\left(m^{\text{red}}\right) \\ &=\; \frac{1}{R^2} E_{\rho\rho}\left(m^{\text{red}}, R\right) + \frac{1}{R^2} E_{\sigma\sigma}\left(m^{\text{red}}, R\right) - \frac{\pi}{2} \|m^{\text{red}}\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

So Lemma 2.24 implies

$$\frac{1}{R^2} E\left(m^R, R\right) - \sqrt{8}\,\pi \;\le\; \frac{1}{R^2} E_{\rho\rho}\left(m^{\text{red}}, R\right) \;\le\; 5\pi R^2 \;|\ln(|R|)|\; E_{\rho\rho}(m^{\text{red}}, 1).$$

On the other hand we can use (2.33) and Lemma 2.24 (i) to get the estimate

$$\begin{aligned} &\sqrt{8}\,\pi - \frac{1}{R^2} E\left(m^R, R\right) \\ &\quad\le\; E_{\text{red}}\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right) - \frac{1}{R^2} E\left(m^R, R\right) \\ &\quad=\; E_{\text{red}}\left(\frac{\overline{m}^R}{|\overline{m}^R|}\right) - \frac{1}{R^2} E\left(\frac{\overline{m}^R}{|\overline{m}^R|}, R\right) + \frac{1}{R^2} E\left(\frac{\overline{m}^R}{|\overline{m}^R|}, R\right) - \frac{1}{R^2} E\left(m^R, R\right) \\ &\quad\le\; \frac{\pi}{2} \left\| \frac{\overline{m}^R}{|\overline{m}^R|} \right\|^2_{L^2(\mathbb{R})} - \frac{1}{R^2} E_{\sigma\sigma}\left(\frac{\overline{m}^R}{|\overline{m}^R|}, R\right) + 64 \|\nabla \tilde{m}^R\|_{L^2(\Sigma(R))} \\ &\quad\le\; 3\pi R^2 \;|\ln(R)| \left\| \frac{\overline{m}^R_y}{|\overline{m}^R|} \right\|^2_{H^1(\mathbb{R})} + 64 \left\| \nabla \tilde{m}^R \right\|_{L^2(\Sigma(R))}. \end{aligned}$$

Since (2.31) and (2.32) imply that $\left\| \frac{\overline{m}^R_y}{|\overline{m}^R|} \right\|_{H^1(\mathbb{R})}$ is uniformly bounded, with (2.34) we see that there exists $C > 0$ such that

$$\sqrt{8}\,\pi - \frac{1}{R^2} E\left(m^R, R\right) \;\le\; CR^2 \;|\ln(R)|.$$

□

2.7 Lower bound for the energy scaling for $R \to \infty$

In this section we look at the scaling of $E_{\mathcal{M}_l}$ for big radii. We find a lower bound, which will be complemented by an upper bound on $E_{\mathcal{V}_l}$ in the next section. To simplify the calculations, instead of the functional E we consider the functional

$$I\colon \mathcal{M}(1) \times \mathbb{R}^+ \to \mathbb{R}, \qquad I(m,R) \;:=\; E_{\mathrm{ex}}(m) + R^2 E_H(m).$$

Then we have for all $m \in \mathcal{M}(1)$ the relation $E(m^R, R) = R\, I(m,R)$, where $m^R := m(Rx, Ry)$.

Theorem 2.31. *There are constants C, $R_0 \in \mathbb{R}^+$ such that for all $R \geq R_0$*

$$\inf_{m \in \mathcal{M}_l(1)} I(m,R) \;\geq\; CR\sqrt{\ln(R)}, \quad \textit{i.e.,} \quad E_{\mathcal{M}_l} \;\geq\; CR^2\sqrt{\ln(R)}.$$

Proof. Let $R \geq R_0 := 2e$ and let m be a minimiser of $I(\cdot, R)$. We define

$$\overline{\overline{m}}\colon \mathbb{R} \to \mathbb{R}^3, \quad x \mapsto \langle \overline{m} \rangle_{[x-\frac{R}{2}, x+\frac{R}{2}]}$$

$$\begin{aligned} a \;&:=\; \sup\left\{ x \in \mathbb{R} : \, \overline{\overline{m}}_x(x) \leq -\frac{1}{\sqrt{2}} \right\}, \\ b \;&:=\; \inf\left\{ x > a : \, \overline{\overline{m}}_x(x) \geq \frac{1}{\sqrt{2}} \right\}, \end{aligned}$$

and set $d := b - a$. We have $d \geq \frac{R}{2}$ since otherwise

$$\begin{aligned} \overline{\overline{m}}_x(b) - \overline{\overline{m}}_x(a) \;&=\; \left(\frac{1}{R} \int_b^{b+\frac{R}{2}} \overline{m}_x \right) - \left(\frac{1}{R} \int_{a-\frac{R}{2}}^{a} \overline{m}_x \right) \\ &\leq\; \frac{1}{2} + \frac{1}{2} < \frac{2}{\sqrt{2}} \;=\; \overline{\overline{m}}_x(b) - \overline{\overline{m}}_x(a). \end{aligned}$$

We distinguish three different cases.

Case 1: $E_{\mathrm{ex}} \geq \frac{d}{200}$. We use a test function to show $E_H \sim \frac{\ln(d)}{d}$ and complement the estimate with the lower bound on the exchange energy. We define

$$\begin{aligned} \phi_1' &: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \frac{1}{R} \mathbb{1}_{[a-\frac{R}{2}, a+\frac{R}{2}]}(x) - \frac{1}{R} \mathbb{1}_{[b-\frac{R}{2}, b+\frac{R}{2}]}(x), \\ \phi_1 &: \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \int_{-\infty}^{x} \phi_2'(t)\, dt, \\ \psi_1 &: \mathbb{R}^2 \to \mathbb{R}, \qquad y \mapsto \mathbb{1}_{D_1}(y) + \mathbb{1}_{D_d \setminus D_1}(y) \frac{\ln\left(d/|y|\right)}{\ln(d)}. \end{aligned}$$

Then

$$\|\phi_1\|^2_{L^2(\mathbb{R})} \le d, \qquad \|\phi_1'\|^2_{L^2(\mathbb{R})} \le \frac{2}{R}$$

$$\|\nabla_y \psi_1\|^2_{L^2(\mathbb{R}^2)} = \int_1^d \frac{2\pi r}{\ln(d)^2\, r^2}\, dr = \frac{2\pi}{\ln(d)},$$

$$\|\psi_1\|^2_{L^2(\mathbb{R}^2)} = \int_1^d 2\pi r \left(\frac{\ln(\frac{1}{r}d)}{\ln(d)}\right)^2 dr = \frac{2\pi d^2}{\ln(d)^2}\int_{\frac{1}{d}}^1 r\ln(r)^2\, dr \le \frac{\pi d^2}{2\ln(d)^2}.$$

Let u be the weak solution of $\Delta u = \operatorname{div} m$. Then we have for all differentiable functions $f : \mathbb{R}^3 \to \mathbb{R}$ the equality $\int_\Sigma m \cdot \nabla f = \int_{\mathbb{R}^3} \nabla u \cdot \nabla f$. So we can calculate

$$\begin{aligned}
\sqrt{2}\pi &= \pi\left(\overline{\overline{m}}_x(a) - \overline{\overline{m}}_x(b)\right) = \int_\Sigma m_x\, \phi_1'\, \psi_1 \\
&= \int_\Sigma m_x\, \phi_1'\, \psi_1 + \phi_1\, (\nabla_y \psi_1)\cdot m_y = \int_\Sigma \nabla(\phi_1\, \psi_1)\cdot \nabla u \\
&\le \sqrt{\|\phi_1'\|^2_{L^2(\mathbb{R})}\, \|\psi_1\|^2_{L^2(\mathbb{R}^2)} + \|\phi_1\|^2_{L^2(\mathbb{R})}\, \|\nabla_y\psi_1\|^2_{L^2(\mathbb{R}^2)}}\ \|\nabla u\|_{L^2(\mathbb{R}^3)} \\
&\le \sqrt{\frac{\pi}{R}\frac{d^2}{\ln(d)^2} + \frac{2\pi d}{\ln(d)}}\ \|\nabla u\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Since $d \ge \frac{1}{2}R \ge e$ and $E_{\text{ex}} \ge \frac{d}{200}$ we have

$$\begin{aligned}
&I(m,R) \\
&\ge \inf_{d\ge\frac{R}{2}}\left(\frac{d}{200} + R^2\left(\frac{1}{2\pi R}\,\frac{d^2}{\ln(d)^2} + \frac{d}{\pi\ln(d)}\right)^{-1}\right) \\
&\ge \frac{1}{2}\min\left\{\inf_{d\ge\frac{R}{2}}\left(\frac{d}{200} + 2\pi R^3\frac{\ln(d)^2}{d^2}\right),\ \inf_{d\ge\frac{R}{2}}\left(\frac{d}{200} + \pi R^2\frac{\ln(d)}{d}\right)\right\} \\
&\ge \frac{1}{2}\min\left\{\inf_{d\ge\frac{R}{2}}\left(\frac{d}{200} + 2\pi R^3\frac{\ln\left(\frac{R}{2}\right)^{\frac{3}{2}}}{d^2}\right),\ \inf_{d\ge\frac{R}{2}}\left(\frac{d}{200} + \pi R^2\,\frac{\ln\left(\frac{R}{2}\right)}{d}\right)\right\} \\
&= \frac{1}{2}\min\left\{\frac{3}{20}\left(\frac{\pi}{10}\right)^{\frac{1}{3}} R\sqrt{\ln\left(\frac{R}{2}\right)},\ 2\sqrt{\frac{\pi}{200}}R\sqrt{\ln\left(\frac{R}{2}\right)}\right\} \\
&\ge \frac{1}{40}R\sqrt{\ln(R)}.
\end{aligned}$$

Case 2: $E_{\text{ex}} < \frac{d}{200}$ *and* $\|\hat{\overline{m}}_y\|^2_{L^2(\Sigma)} \ge \frac{1}{5}d$. We prove a bound on $\|\overline{m}_y\|_{L^2([-1,1])}$ and use Theorem 2.11 (i). For the Fourier transform $\hat{\overline{m}}_y$ and the Fourier transform of the derivative $\xi\,\hat{\overline{m}}_y$ we have the inequality

$$\|\xi\,\hat{\overline{m}}_y\|^2_{L^2(\Sigma)} \le \frac{1}{200}d \le \frac{1}{40}\|\hat{\overline{m}}_y\|^2_{L^2(\Sigma)},$$

so $\|\hat{\overline{m}}_y(\xi)\|^2_{L^2[-1,1]} \geq \left(1 - \frac{1}{40}\right) \|\overline{m}_y\|^2_{L^2(\mathbb{R})}$. Using Theorem 2.11 (i) we get

$$E_{\sigma\sigma}(\overline{m}_y) \;\geq\; \|\hat{\overline{m}}_y(\xi)\|^2_{L^2[-1,1]} \;\geq\; 0.195\, d.$$

The Poincaré inequality yields $\|\tilde{m}\|^2_{L^2(\Sigma)} \leq \frac{16}{200} d$, thus

$$\begin{aligned} I(m,R) \;&\geq\; R^2 E_H(m) \;\geq\; R^2 \left(\sqrt{E_H(\overline{m})} - \|\tilde{m}\|_{L^2(\Sigma)}\right)^2 \\ &\geq\; R^2 d(\sqrt{0.195} - \sqrt{0.08})^2 \;\geq\; \frac{1}{40} R^2 d \;\geq\; \frac{1}{80} R^3. \end{aligned}$$

Case 3: $E_{ex} < \frac{d}{200}$ *and* $\|\overline{m}_y\|^2_{L^2(\Sigma)} \leq \frac{1}{5} d$. First we show that $\int_{a-\frac{R}{2}}^{b+\frac{R}{2}} \overline{m}_x^2$ is large. For all $x \in \mathbb{R}$ we have

$$\begin{aligned} 1 \;&=\; \frac{1}{\pi} \int_{D_1} |\overline{m}(x) + \tilde{m}(x,y)|^2 \, dy \\ &=\; |\overline{m}(x)|^2 + \frac{1}{\pi} \int_{D_1} 2\overline{m}(x) \cdot \tilde{m}(x,y) + |\tilde{m}(x,y)|^2 \, dy \\ &=\; |\overline{m}(x)|^2 + \frac{1}{\pi} \|\tilde{m}(x,\cdot)\|^2_{L^2(D_1)}. \end{aligned}$$

Since $\|\tilde{m}\|^2_{L^2(\Sigma)} \leq \frac{16}{200} d$, in particular

$$\begin{aligned} \int_{a-\frac{R}{2}}^{b+\frac{R}{2}} \overline{m}_x^2 \;&\geq\; R + d - \frac{1}{\pi} \|\overline{m}_y\|^2_{L^2(\Sigma)} - \frac{1}{\pi} \|\tilde{m}\|^2_{L^2(\Sigma)} \\ &\geq\; R + d - \frac{d}{5\pi} - \frac{16d}{200\pi} \;\geq\; R + 0.9d. \end{aligned}$$

Thus there is at least one $x_0 \in [a,b]$ such that $\int_{x_0-\frac{R}{2}}^{x_0+\frac{R}{2}} \overline{m}_x^2 \geq 0.9R$. We proceed similar to Case 1. We set

$$\begin{aligned} \phi_2' : \mathbb{R} \to \mathbb{R}, \qquad & x \mapsto \frac{1}{R} \mathbb{1}_{[x_0 - tfracR2,\, x_0 + \frac{R}{2}]}(x) \left(\overline{m}_x(x) - \overline{\overline{m}}_x(x_0)\right), \\ \phi_2 : \mathbb{R} \to \mathbb{R}, \qquad & x \mapsto \int_{-\infty}^{x} \phi_2'(t) \, dt, \\ \psi_2 : \mathbb{R}^2 \to \mathbb{R}, \qquad & y \mapsto \mathbb{1}_{D_1}(y) + \mathbb{1}_{D_R \setminus D_1}(y) \frac{\ln(R/|y|)}{\ln(R)}. \end{aligned}$$

Then

$$|\phi_2'| \leq \frac{2}{R}, \qquad \phi_2 \equiv 0 \text{ on } \mathbb{R} \setminus \left[x_0 - \tfrac{R}{2}, x_0 + \tfrac{R}{2}\right], \qquad |\phi_2| \leq 1.$$

$$\|\nabla_y \psi_1\|^2_{L^2(\mathbb{R}^2)} \;=\; \frac{2\pi}{\ln(R)}, \qquad \|\psi_1\|^2_{L^2(\mathbb{R}^2)} \;\leq\; \frac{\pi R^2}{2\ln(R)^2}.$$

We have

$$
\begin{aligned}
\pi\,(0.9-0.5) \;&\leq\; \frac{\pi}{R}\int_{x_0-\frac{R}{2}}^{x_0+\frac{R}{2}} \overline{m}_x(x)^2 - \overline{\overline{m}}_x(x_0)^2\,dx\\
&= \frac{\pi}{R}\int_{x_0-\frac{R}{2}}^{x_0+\frac{R}{2}} \overline{m}_x(x)\,(\overline{m}_x(x) - \overline{\overline{m}}_x(x_0))\,dx\\
&= \int_\Sigma \overline{m}_x\phi_2'\,\psi_2 \;=\; \int_\Sigma m\cdot\nabla(\phi_2\psi_2) \;=\; \int_{\mathbb{R}^3}\nabla u\cdot\nabla(\phi_2\psi_2)\\
&\leq \sqrt{\|\phi_2'\|^2_{L^2(\mathbb{R})}\,\|\psi\|^2_{L^2(\mathbb{R}^2)} + \|\phi_2\|^2_{L^2(\mathbb{R})}\,\|\nabla_y\psi\|^2_{L^2(\mathbb{R}^2)}}\;\|\nabla u\|_{L^2(\mathbb{R}^3)}\\
&\leq \sqrt{\frac{2\pi R}{\ln(R)^2}+\frac{2\pi R}{\ln(R)}}\;\|\nabla u\|_{L^2(\mathbb{R}^3)} \;\leq\; \sqrt{4\pi\,\frac{R}{\ln(R)}}\;\|\nabla u\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Thus

$$
I(m,R) \;\geq\; R^2\|\nabla u\|^2_{L^2(\mathbb{R}^3)} \;\geq\; \frac{\pi 0.4^2}{4}R\,\ln(R) \;\geq\; \frac{1}{10}R\,\ln(R).
$$

In all three cases we have the inequality $I(m,R) \geq \frac{1}{40}\,R\,\sqrt{\ln(R)}$ for $R \geq 2e$. □

2.8 The vortex wall - an example of a set of functions with low energy for big radii

In this section we show upper and lower bounds for the energy $E_{\mathcal{V}_l}$. We see that for large radii the upper bound has the same scaling as the lower bound for $E_{\mathcal{M}_l}$. This shows that this scaling is indeed optimal and that for large radii the energy of the minimisers in $\mathcal{V}_l$ is at most a constant factor larger than the energy of the minimisers in $\mathcal{M}_l$. For small radii $E_{\mathcal{V}_l}$ scales like R and is thus much larger than $E_{\mathcal{M}_l} \sim E_{\mathcal{T}_l} \sim R^2$.

Because of the symmetry of the functions $m \in \mathcal{V}$, we use spherical coordinates x, r, ϕ in the domain and polar coordinates $\theta : \Sigma \to [0,1]$, $\gamma : [0,\pi]$ in the image,

$$
p = \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ r\,\cos\phi \\ r\,\sin\phi \end{pmatrix}, \qquad m = \begin{pmatrix} m_x \\ m_{y_1} \\ m_{y_2} \end{pmatrix} = \begin{pmatrix} -\cos(\pi\theta) \\ \sin(\pi\theta)\,\cos\gamma \\ \sin(\pi\theta)\,\sin\gamma \end{pmatrix}.
$$

For $m \in \mathcal{V}$ we have $\gamma = \phi + \frac{\pi}{2}$, so m is uniquely determined by the angle θ. The exchange energy is

$$
\|\nabla m\|^2_{L^2(\Sigma)} = \int_\Sigma \frac{1}{|y|^2}\sin^2(\pi\theta(p)) + (\pi\,\partial_x\theta(p))^2 + (\pi\,\nabla_y\theta(p))^2\,dp.
$$

2.8.1 Bounds on $E_{\mathcal{V}_l}$ for small radii

The following theorem gives upper and lower bounds for $E_{\mathcal{V}_l}$ that are valid for all $R > 0$. However, they are reasonably sharp only for small R.

Theorem 2.32. *We have*

$$8\pi\, R \;\le\; E_{\mathcal{V}_l} \;\le\; 12\pi\, R + \frac{\pi}{2}\, R^3.$$

Proof. Set

$$E_1(\theta) \;:=\; \int_\Sigma \pi^2(\partial_x\theta)^2 + \frac{1}{r^2}\sin^2(\pi\theta),$$

Using the Modica Mortola trick, we find for all functions $\theta : \Sigma \to [0,1]$ with $\lim_{x\to-\infty}\theta(x,y) = 0$ and $\lim_{x\to\infty}\theta(x,y) = 1$

$$E_1(\theta) \;\ge\; \int_\Sigma \frac{2\pi}{r}|\sin(\pi\theta)\,\partial_x\theta| \;\ge\; \int_0^R\int_{-\infty}^{\infty} 4\pi\,\partial_x(\cos\pi\theta)\;dx\;dr \;=\; 8\pi\,R. \tag{2.36}$$

In particular we have $E(m) \ge 8\pi R$ for all $m \in \mathcal{V}_l$.

A function θ satisfies equation (2.36) with equality if and only if it is a monotonously increasing solution of

$$\partial_x\theta \;=\; \frac{1}{\pi r}\sin(\pi\theta).$$

This solution is unique up to translation. It is given by

$$\theta_1(x,r) \;:=\; \frac{2}{\pi}\arctan(e^{\frac{x}{r}}).$$

Let $m^1 \in \mathcal{V}_l$ be the magnetisation corresponding to θ_1. We calculate $E(m_1)$. Since

$$|\partial_r\theta_1(x,r)| \;=\; \frac{2x\;e^{-\frac{x}{r}}}{\pi r^2\;(1+e^{-\frac{2x}{r}})} \;\le\; \frac{2x}{\pi r^2}\,e^{\frac{-|x|}{r}},$$

we have

$$\int_\Sigma \pi^2(\partial_r\theta_1)^2 \;\le\; \int_0^R\int_{-\infty}^{\infty} 2\pi^3 r\left(\frac{2x\;e^{\frac{-|x|}{r}}}{\pi r^2}\right)^2 \;=\; 4\pi\,R.$$

Finally, using Lemma 2.7, we get

$$E_{\mathcal{V}_l} \;\le\; E(m_1) \;\le\; E_1(\theta_1) + \int_\Sigma \pi^2(\partial_r\theta_1)^2 + E_{\rho\rho}(m) \;\le\; 12\pi R + \frac{\pi}{2}R^3.$$

□

2.8.2 An upper bound on $E_{\mathcal{V}_l}$ for big radii

In this subsection we use a family of maps $m_\alpha^R \in \mathcal{V}_l(R)$ and show that for an appropriate choice of α and large R we have the estimate $E(m_\alpha^R) \leq C\, R^2 \sqrt{\ln(R)}$.

First, we estimate an integral over a cylindrical surface.

Lemma 2.33. *For $r, l \in \mathbb{R}^+$, with $l \geq r$, and $p := (x, y)$ with $|y| \geq r$ and*

$$Z_{r,l} \;:=\; \left\{(x', y') \in [-l, l] \times \partial D_r\right\}$$

we have

$$\int_{Z_{r,l}} \frac{1}{|p - p'|} dp' \;\leq\; 2\pi^2 r \left(1 + \ln\left(\frac{1}{r} l\right)\right).$$

Proof. We show the lemma by direct calculation. Set

$$A(p) \;:=\; \int_{Z_{r,l}} \frac{1}{|p - p'|} dp' \;=\; \int_{-l}^{l} \int_{\partial D_r} \frac{1}{\sqrt{(x - x')^2 + |y - y'|^2}}\, dy'\, dx'.$$

Then $A(p) \;\leq\; A(|p| \vec{e}_{y_1}) \;\leq\; A(r \vec{e}_{y_1})$, so

$$A(p) \;\leq\; \int_{-l}^{l} \int_{\partial D_r} \frac{1}{\sqrt{x'^2 + (y - r\, \vec{e}_{y_1})^2}}\, dy'\, dx'.$$

We unroll the cylinder surface. This stretches $|p - p'|$ by a factor less or equal to $\frac{\pi}{2}$.

$$\begin{aligned} A \;&\leq\; \int_{-l}^{l} \int_{-\pi r}^{\pi r} \frac{\pi}{2} \frac{1}{\sqrt{x'^2 + t^2}}\, dt\, dx' \\ &=\; \int_{-r}^{r} \int_{-\pi r}^{\pi r} \frac{\pi}{2} \frac{1}{\sqrt{x'^2 + t^2}}\, dt\, dx' + 2 \int_{r}^{l} \int_{-\pi r}^{\pi r} \frac{\pi}{2} \frac{1}{\sqrt{x'^2 + t^2}}\, dt\, dx' \end{aligned}$$

In the first summand we replace the rectangle by a disc with the same area.

$$A \;\leq\; \int_{D_{2r}} \frac{\pi}{2} \frac{1}{|p'|}\, dp' + 2 \int_{r}^{l} \frac{\pi^2 r}{x}\, dx \;=\; 2\pi^2 r \left(1 + \ln\left(\frac{1}{r} l\right)\right)$$

□

We now set

$$\theta_\alpha \colon \mathbb{R} \times D_1 \to \mathbb{R} \qquad \theta_\alpha(x, r) \;:=\; \begin{cases} 0 & \text{if } x < -\alpha\sqrt{r} \\ 0.5 + \frac{x}{2\alpha\sqrt{r}} & \text{if } -\alpha\sqrt{r} \leq x \leq \alpha\sqrt{r} \\ 1 & \text{if } x > \alpha\sqrt{r}, \end{cases} \tag{2.37}$$

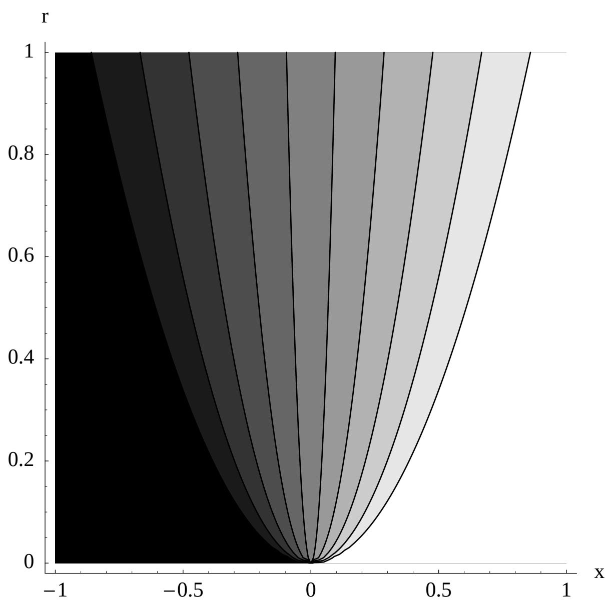

Figure 2.3: Contour plot of the function θ_1

let $m_\alpha \in \mathcal{V}_l(1)$ be the rotationally symmetric magnetisation corresponding to θ_α and define $m_\alpha^R : \Sigma(R) \to \mathbb{S}^2,\ (x,y) \mapsto m_\alpha\left(\frac{x}{R}, \frac{y}{R}\right)$. See Figure 2.3 for a plot of θ_1. Note that m_α^R has a square root type singularity and that the size of the transition region is $R\alpha$.

Theorem 2.34. *For $R \geq e$ and $\alpha_0 := R\sqrt{\ln(R)}$ we have*

$$E_{\mathcal{V}_l} \ \leq\ E(m_{\alpha_0}^R) \ \leq\ 38R^2\sqrt{\ln(R)}.$$

Proof. Let I be as in Section 2.7,

$$I\colon \mathcal{M}(1) \times \mathbb{R}^+ \to \mathbb{R}, \qquad I(m,R) \ :=\ E_{\text{ex}}(m) + R^2 E_H(m).$$

Then $E(m_\alpha^R, R) = R\, I(m_\alpha, R)$. First, we estimate $I(m_\alpha)$ for general $\alpha \geq 1$

and then choose a suitable α_0. According to Lemma 2.6 we have

$$
\begin{aligned}
E_{\rho\rho}(m_\alpha) &= \int_\Sigma u_\alpha(p)\rho_\alpha(p)\,dp \;=\; \int_\Sigma\int_\Sigma \frac{\rho_\alpha(p)\rho_\alpha(p')}{4\pi|p-p'|}\,dp'\,dp \\
&= \int_{\{(p,p')\in\Sigma\times\Sigma\,:\,|y'|<|y|\}} \frac{\rho_\alpha(p)\rho_\alpha(p')}{4\pi|p-p'|}\,dp'\,dp \\
&\quad+ \int_{\{(p,p')\in\Sigma\times\Sigma\,:\,|y|<|y'|\}} \frac{\rho_\alpha(p)\rho_\alpha(p')}{4\pi|p-p'|}\,dp'\,dp \\
&= 2\int_\Sigma\int_{\mathbb{R}\times D_{|y|}} \frac{\rho_\alpha(p)\rho_\alpha(p')}{4\pi|p-p'|}\,dp'\,dp \\
&= \frac{1}{2\pi}\int_\Sigma\int_0^{|y|}\int_{\partial D_t}\int_{\mathbb{R}} \frac{\rho_\alpha(x',y')\;\rho_\alpha(p)}{|(x',y')-p|}\,dx'\,dy'\,dt\,dp.
\end{aligned}
$$

Since $\rho_\alpha = \operatorname{div} m_\alpha = \partial_x (m_\alpha)_x$, we have

$$
\rho_\alpha(x',r') \;=\; \begin{cases} \frac{\pi}{2\alpha\sqrt{|y'|}}\,\sin\left(\frac{\pi x'}{2\alpha\sqrt{|y'|}}\right) \;\le\; \frac{\pi}{2\alpha\sqrt{|y'|}} & \text{if } |x'|<\alpha\sqrt{|y'|} \\ 0 & \text{otherwise.} \end{cases}
$$

Using this estimate and applying Lemma 2.33 with $l=\alpha\sqrt{r'}$ yields

$$
\begin{aligned}
E_{\rho\rho}(m_\alpha) &\le \frac{1}{2\pi}\int_\Sigma \rho_\alpha(p)\int_0^{|y|}\int_{\partial D_{r'}}\int_{-\alpha\sqrt{r'}}^{\alpha\sqrt{r'}} \frac{\pi}{2\alpha\sqrt{r'}}\;\frac{1}{|(x',y')-p|}\,dx'\,dy'\,dr'\,dp \\
&\le \frac{1}{2\pi}\int_\Sigma \rho_\alpha(p)\int_0^{|y|} \frac{\pi}{2\alpha\sqrt{r'}}\;2\pi^2 r'\left(1+\ln\left(\frac{\alpha\sqrt{r'}}{r'}\right)\right)\,dr'\,dp \\
&= \frac{\pi^2}{2\alpha}\int_\Sigma \rho_\alpha(p)\int_0^{|y|} \sqrt{r'}\left(1+\ln(\alpha)-\ln(\sqrt{r'})\right)\,dr'\,dp \\
&\le \frac{\pi^2}{2\alpha}\int_\Sigma \rho_\alpha(p)(2+\ln(\alpha))\,dp \;\le\; \frac{\pi^3}{\alpha}(2+\ln(\alpha)).
\end{aligned}
$$

The exchange energy is

$$
\begin{aligned}
E_{\mathrm{ex}}(m_\alpha) &= \int_\Sigma \pi^2(\partial_x\theta_\alpha)^2+\pi^2(\nabla_y\theta_\alpha)^2+\frac{1}{r^2}\sin^2(\pi\theta_\alpha)\,dp \\
&= \int_0^1 2\pi r\int_{-\alpha\sqrt{r}}^{\alpha\sqrt{r}} \frac{\pi^2}{4\alpha^2 r}+\pi^2\left(\frac{x}{4\alpha}\,r^{-\frac{3}{2}}\right)^2+\frac{1}{r^2}\sin^2\left(\frac{\pi x}{2\alpha\sqrt{r}}\right)\,dx\,dr \\
&= \int_0^1 \frac{\pi^3}{\alpha}\sqrt{r}+\frac{\pi^3}{8\alpha^2}\,r^{-2}\,\frac{2\alpha^3\sqrt{r}^3}{3}+\frac{2\pi\alpha\sqrt{r}}{r}\,dr \\
&= \frac{2\pi^3}{3\alpha}+\frac{\pi^3\alpha}{6}+4\pi\alpha.
\end{aligned}
$$

For $R > e$ we choose $\alpha := R\sqrt{\ln R}$. Then $\ln(\alpha) \le 2\ln(R)$ and

$$\begin{aligned}
E_{\rho\rho}(m_\alpha) \;&\le\; \frac{2\pi^3}{R\sqrt{\ln(R)}}(2+2\ln(R)) \;\le\; \frac{8\pi^3}{e^2}R\sqrt{\ln(R)} \;\le\; 34R\sqrt{\ln(R)},\\
E_{\mathrm{ex}}(m_\alpha) \;&\le\; \frac{2\pi^3}{3R\sqrt{\ln(R)}} + \left(\frac{\pi^3}{6}+4\pi\right)R\sqrt{\ln(R)}\\
&\le\; \left(\frac{2\pi^3}{e^2}+\frac{\pi^3}{6}+4\pi\right)R\sqrt{\ln(R)} \;\le\; 21R\sqrt{\ln(R)}.
\end{aligned}$$

Thus

$$I(m_\alpha, R) \;\le\; 38R\sqrt{\ln(R)}, \qquad E(m_\alpha^R) \;\le\; 38R^2\sqrt{\ln(R)}.$$

□

2.9 The Fourier multiplier

For functions that are constant on the cross section, we use a partial Fourier transform to find estimates for $E_{\sigma\sigma}$ and $E_{\rho\rho}$. As in Section 2.3, we view E, $E_{\rho\rho}$ and $E_{\sigma\sigma}$ not only as functionals on $\mathcal{M}$ but also on $\{f\colon \mathbb{R} \to \mathbb{R}^3\}$.

As before we apply the Fourier transformation only to the first argument of a function. For all functions $f\colon \mathbb{R}\times A \to \mathbb{R}^n$ for which $f(\cdot, a) \in L^1(\mathbb{R})$ for all $a \in A$, we define

$$\hat{f}(\xi, a) \;:=\; \mathcal{F}(f)(\xi, a) \;:=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x, a)e^{-i\xi x}\,dx.$$

We generalise this formula in the usual way to functions f with $f(\cdot, a) \in L^2(\mathbb{R})$. With the above normalisation we have $\|f(\cdot, a)\|_{L^2} = \|\hat{f}(\cdot, a)\|_{L^2}$. We denote the inverse Fourier transform of a map $g\colon \mathbb{R}\times A \to \mathbb{R}^n$ by $\mathcal{F}^{-1}(g) = \check{g}$.

We now formally Fourier transform the defining equations for u. In cylindrical coordinates, the formal Fourier transform of equation (2.9), (2.10) and (2.8) is

$$-\xi^2 v_\sigma + \partial_{rr}v_\sigma + \frac{1}{r}\partial_r v_\sigma + \frac{1}{r^2}\partial_{\phi\phi}v_\sigma \;=\; 0 \qquad \text{if } r \ne R, \tag{2.38}$$

$$\lim_{r\searrow R}\partial_r v_\sigma(\xi, r, \phi) - \lim_{r\nearrow R}\partial_r v_\sigma(\xi, r, \phi) \;=\; -\hat{\sigma}(\xi, R, \phi), \tag{2.39}$$

$$-\xi^2 v + \partial_{rr}v_\rho + \frac{1}{r}\partial_r v_\rho + \frac{1}{r^2}\partial_{\phi\phi}v_\rho \;=\; \hat{\rho}. \tag{2.40}$$

The next lemma clarifies in which sense v_σ and v_ρ are the Fourier transforms of u_σ and u_ρ, and allows us to calculate $E_{\rho\rho}$ and $E_{\sigma\sigma}$ in Fourier space.

Lemma 2.35. *Let* $f\colon \{(\xi, y) \in \mathbb{R} \times \mathbb{R}^2\} \to \mathbb{R}$ *be a function such that for all* $y \in \mathbb{R}^2$ *the map* $f(\cdot, y)$ *is in* $L^2(\mathbb{R})$*, and let* $g\colon \{(x, y) \in \mathbb{R} \times \mathbb{R}^2\} \to \mathbb{R}$ *be such that for all* $y \in \mathbb{R}^2$ *we have* $g(\cdot, y) = \mathcal{F}^{-1}(f(\cdot, y))$*.*

If $\xi f \in L^2(\mathbb{R}^3)$ *and* $\nabla_y f \in L^2(\mathbb{R}^3)$ *then, in the sense of equality in* $L^2(\mathbb{R}^3)$*,* $\partial_x g = \mathcal{F}^{-1}(i\xi\, f)$ *and* $\nabla_y g = \mathcal{F}^{-1}(\nabla_y f)$*. In particular we have*

$$\|\nabla g\|^2_{L^2(\mathbb{R}^3)} \;=\; \|\xi f\|^2_{L^2(\mathbb{R}^3)} + \|\nabla_y f\|^2_{L^2(\mathbb{R}^3)}.$$

If $\xi f \notin L^2(\mathbb{R}^3)$ *or* $\nabla_y f \notin L^2(\mathbb{R}^3)$ *then* $\nabla g \notin L^2(\mathbb{R}^3)$*.*

Proof. If $i\xi\, f$ is in $L^2(\mathbb{R}^3)$ then $i\xi\, f(\cdot, y)$ is almost everywhere in $L^2(\mathbb{R})$, and we have almost everywhere

$$F_\xi^{-1}(i\xi\, f(\cdot, y)) = \partial_x g(\cdot, y), \qquad \|\partial_x g(\cdot, y)\|_{L^2(\mathbb{R})} = \|\xi g(\cdot, y)\|_{L^2(\mathbb{R})}$$

(c.f. [39, Satz V.2.14]). Thus we have in the sense of equality in $L^2(\mathbb{R}^3)$ the equation $F_\xi^{-1}(i\xi\, f) = \partial_x g$. Conversely, the finiteness of $\|\xi g(\cdot, y)\|_{L^2(\mathbb{R})}$ implies the finiteness of $\|\partial_x g(\cdot, y)\|_{L^2(\mathbb{R})}$.

Let $\nabla_y f \in L^2(\mathbb{R}^3)$, and let $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^2)$ be a test function. Then

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{F}^{-1}(\nabla_y f) \cdot \overline{\phi} \;&=\; \int_{\mathbb{R}^3} \nabla_y f \cdot \overline{\mathcal{F}(\phi)} \;=\; -\int_{\mathbb{R}^3} f \;\operatorname{div}_y \overline{\mathcal{F}(\phi)} \\ &=\; -\int_{\mathbb{R}^3} f\; \overline{\mathcal{F}(\operatorname{div}_y \phi)} \;=\; -\int_{\mathbb{R}^3} \mathcal{F}^{-1}(f)\; \overline{\operatorname{div}_y \phi}. \end{aligned}$$

But the first term being equal to the last for all test functions ϕ means precisely that $\mathcal{F}^{-1}(\nabla_y f)$ is the weak y-gradient of g. Therefore we have $\|\nabla_y f\|_{L^2(\mathbb{R}^3)} = \|\nabla_y g\|_{L^2(\mathbb{R}^3)}$. Conversely, the finiteness of $\|\nabla_y g\|_{L^2(\mathbb{R}^3)}$ implies the finiteness of $\|\nabla_y f\|_{L^2(\mathbb{R}^3)}$. ☐

In the next lemma and the subsequent corollary we will determine the Fourier transform of u_σ. We use Bessel functions to solve the differential equations (2.38) and (2.39). Let I_k and K_k denote the modified k^{th} Bessel functions

$$I_k(x) \;=\; e^{\frac{-\pi k i}{2}} J_k(ix), \qquad K_k(x) \;=\; \frac{\pi i}{2} e^{\frac{\pi i}{2}} H_k^{(1)}(ix),$$

where J_k is the k^{th} Bessel function of first kind and $H_k^{(1)}$ is the k^{th} Hankel function of first kind. In particular, both I_k and K_k are solutions of the differential equation

$$r^2 \partial_{rr} u + r \partial_r u - (r^2 + k^2) u \;=\; 0.$$

The function I_k is continuous in zero and K_k vanishes at infinity. For an overview over the different kinds of Bessel functions and their properties see [26].

Lemma 2.36. *Let $m_y\colon \mathbb{R} \to \{0\} \times \mathbb{R}^2$ be a map, set $m_\kappa := m_y \cdot \vec{e}_\kappa$ where κ is the angle between the unit vector $\vec{e}_\kappa$ and $\vec{e}_{y_1}$, and set*

$$v_\kappa\colon \mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi[\to \mathbb{R},$$

$$(\xi, r, \phi) \mapsto \begin{cases} \hat{m}_\kappa(\xi)\, R\, K_1(|\xi|R)\, I_1(|\xi|r)\, \cos(\phi - \kappa) & \text{if } r \leq R \\ \hat{m}_\kappa(\xi)\, R\, I_1(|\xi|R)\, K_1(|\xi|r)\, \cos(\phi - \kappa) & \text{if } r > R \end{cases} \tag{2.41}$$

If m_y is square integrable with $m_y = m_\kappa \vec{e}_\kappa$, then for all $\xi \in \mathbb{R}$ the map $v_\kappa(\xi, \cdot)$ is continuous in $\mathbb{R}^2$, differentiable in $\mathbb{R}^2 \setminus \partial D_R$ and satisfies the equations (2.38) *and* (2.39)*.*

Proof. A simple calculation shows that v_κ satisfies (2.38) for $r \neq R$ and is continous at $r = R$. Using the differention rules for Bessel functions and the identity

$$K_1(t)(I_0(t) + I_2(t)) + I_1(t)(K_0(t) + K_2(t)) \;=\; \frac{2}{t} \tag{2.42}$$

we see that (2.39) is satisfied. ☐

Corollary 2.37. *Let the notation be as in Lemma 2.36 and set $v := v_0 + v_\pi$. For all $m_y \in L^2(\mathbb{R}, \{0\} \times \mathbb{R}^2)$, the map $v(\xi, \cdot)$ is continuous in $\mathbb{R}^2$ and differentiable in $\mathbb{R}^2 \setminus \partial D_R$. The map v satisfies the equations* (2.38) *and* (2.39)*. Moreover, $v(\cdot, y)$ is in $L^2(\mathbb{R})$ for all $y \in \mathbb{R}^2$, and both ξv and $\nabla_y v$ are in $L^2(\mathbb{R}^3)$. The map $u := \check{v}$ satisfies* (2.6) *and we have*

$$E_{\sigma\sigma}(m_y) = \int_{\mathbb{R}^3} |\nabla_y v_0|^2 + \xi^2 |v_0|^2 + \int_{\mathbb{R}^3} |\nabla_y v_\pi|^2 + \xi^2 |v_\pi|^2. \tag{2.43}$$

Proof. Note that $E_{\sigma\sigma}(v) = E_{\sigma\sigma}(v_0) + E_{\sigma\sigma}(v_\pi)$ (Lemma 2.10 (ii)). The statements now follow from Lemma 2.36 and direct calculation. ☐

We now use the explicit representation of $\hat{u}_\sigma$ to find upper and lower bounds on the Fourier multiplier of $E_{\sigma\sigma}$.

Proof of Theorem 2.11 (i). Using the notation of Lemma 2.36 we define

$$\begin{aligned}
G_{\text{in}}(\kappa,\xi) \;&:=\; \int_0^R \int_0^{2\pi} r|\xi v_\kappa|^2 + r|\partial_r v_\kappa|^2 + \frac{1}{r}|\partial_\phi v_\kappa|^2 \, d\phi \, dr \\
&= |\hat m_\kappa(\xi)|^2 R^2 \xi^2 K_1(|\xi|R)^2 \int_0^R \int_0^{2\pi} \cos^2(\phi-\kappa)\, r \\
&\quad \left(\frac{1}{4}(I_0(|\xi|r)+I_2(|\xi|r))^2 + I_1(|\xi|r)^2\right) + \sin^2(\phi-\kappa)\,\frac{1}{r\xi^2} I_1(|\xi|r)^2 \, d\phi\, dr \\
&= |\hat m_\kappa(\xi)|^2 R^2 \xi^2 K_1(|\xi|R)^2 \int_0^R \pi r \left(\frac{1}{2} I_0(|\xi|r)^2 + I_1(|\xi|r)^2 + \frac{1}{2} I_2(|\xi|r)^2\right) dr \\
&= R^2 |\hat m_\kappa(\xi)|^2 \frac{\pi}{2} |\xi| R\, K_1(|\xi|R)^2 \left(I_0(|\xi|R) I_1(|\xi|R) + I_1(|\xi|R) I_2(|\xi|R)\right) \\
&= R^2 |\hat m_\kappa(\xi)|^2 \frac{\pi}{2} |\xi| R\, I_1(|\xi|R) K_1(|\xi|R)^2 \left(I_0(|\xi|R) + I_2(|\xi|R)\right).
\end{aligned}$$

Here we have used the recurrence relations for Bessel functions and the equalities

$$\begin{aligned}
\partial_t(tI_0(t)I_1(t)) &= I_0(t)\, I_1(t) + \frac{t}{2} I_0(t)\,(I_0(t)+I_2(t)) + tI_1(t)^2 \\
&= \frac{t}{2} I_0(t)\,(I_0(t)-I_2(t)) + \frac{t}{2} I_0(t)\,(I_0(t)+I_2(t)) + tI_1(t)^2 \\
&= tI_0(t)^2 + tI_1(t)^2 \\
\partial_t(tI_1(t)I_2(t)) &= \partial_t(2I_1(t)^2 + tI_1(t)I_0(t)) \\
&= t(I_2(t)-I_0(t))(I_2(t)+I_0(t)) + tI_0(t)^2 + tI_1(t)^2 \\
&= tI_1(t)^2 + tI_2(t)^2.
\end{aligned}$$

Similarly we can prove

$$\begin{aligned}
G_{\text{out}}(\kappa,\xi) :=& \int_R^\infty \int_0^{2\pi} r|\xi v_\kappa|^2 + r|\partial_r v_\kappa|^2 + \frac{1}{r}|\partial_\phi v_\kappa|^2 \, d\phi\, dr \\
=& \; R^2 |\hat m_\kappa(\xi)|^2 \frac{\pi}{2} |\xi| R\, I_1(|\xi|R)^2 K_1(|\xi|R)^1 \left(K_0(|\xi|R) + K_2(|\xi|R)\right).
\end{aligned}$$

Thus we have $E_{\sigma\sigma}(m_y) = R^2 \int_{\mathbb{R}} |\hat m_y(\xi)|^2 g_F(\xi R)\, d\xi$ with

$$g_F(t) := \frac{\pi}{2}|t| K_1(|t|) I_1(|t|)\ \left(K_1(|t|)(I_0(|t|)+I_2(|t|)) + I_1(|t|)(K_0(|t|)+K_2(|t|)\right).$$

Now (2.42) yields $g_F(t) = \pi K_1(|t|) I_1(|t|)$. □

We now consider $E_{\rho\rho}$. Again, we first give an explicit representation of $\hat u_\rho$ and then find estimates for the Fourier multiplier.

Lemma 2.38. *Let $m_x\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\rho := \partial_x m_x$ is in $L^2(\Sigma)$ and set*

$$v\colon \mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi[\to \mathbb{R},$$

$$(\xi, r, \phi) \mapsto \begin{cases} \frac{\hat{\rho}}{\xi^2} \left(|\xi R| \, K_1(|\xi|R) \, I_0(|\xi|r) - 1 \right) & \text{if } r \leq R \\ \frac{\hat{\rho}}{\xi^2} \, |\xi R| \, I_1(|\xi|R) \, K_0(|\xi|r) & \text{if } r > R \end{cases} \tag{2.44}$$

Then $v(\xi, \cdot)$ is continuously differentiable and v is a solution of (2.40)*. The map $v(\cdot, y)$ is in $L^2(\mathbb{R})$ for all $y \in \mathbb{R}^2$. If both ξv and $\nabla_y v$ are in $L^2(\mathbb{R}^3)$, the map $u := \check{v}$ is a solution of* (2.8) *and we have $E_{\rho\rho}(m_x \, \vec{e}_x) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \xi^2 |v|^2 + |\nabla_y v|^2 \, dy \, d\xi$. Otherwise $E_{\rho\rho}(m_x)$ is infinite.*

Proof. Simple calculation shows that v is is a solution of (2.40) for $r \neq R$ and continuous at $r = R$. To see that $\partial_r v$ is continuous at $r = R$ we use the differentiation rules for Bessel functions and the identity $I_0(t)K_1(t) + K_1(t)I_0(t) = \frac{1}{t}$.

□

Proof of Theorem 2.11 (ii). Using the notation of Lemma 2.36 we define

$$\begin{aligned}
H_{\text{in}}(\xi) &:= \int_0^R 2\pi r \left(|\partial_r v(\xi, r)|^2 + |\xi v(\xi, r)|^2 \right) \, dr \\
&= R^2 \frac{\hat{\rho}(\xi)^2}{\xi^2} \, 2\pi \int_0^R r \left(\xi^2 K_1(|\xi|R)^2 I_1(|\xi r|)^2 + \left(\xi K_1(\xi R) I_0(|\xi r|) - \frac{1}{R} \right)^2 \right) \, dr \\
&= R^4 \hat{\rho}(\xi)^2 \, \frac{2\pi}{(\xi R)^2} \left(|\xi| R K_1(|\xi|R)^2 I_1(|\xi|R) I_0(|\xi|R) - 2K_1(\xi R) I_1(|\xi|R) + \frac{1}{2} \right) \\
H_{\text{out}}(\xi) &:= \int_R^\infty 2\pi r \left(|\partial_r v(\xi, r)|^2 + |\xi v(\xi, r)|^2 \right) \, dr \\
&= R^2 \hat{\rho}^2 \, 2\pi I_1(|\xi R|)^2 \int_R^\infty \left(r K_1(|\xi r|)^2 + r K_0(|\xi r|)^2 \right) dr \\
&= R^4 \hat{\rho}^2 \, \frac{2\pi}{|\xi| R} \, I_1(|\xi|R)^2 K_0(|\xi|R) K_1(|\xi|R).
\end{aligned}$$

Thus we have $E_{\rho\rho}(m_x \vec{e}_x) = R^4 \int_{\mathbb{R}} |\hat{m}_y(\xi)|^2 h_F(\xi R) \, d\xi$, with

$$\begin{aligned}
h_F(t) &:= 2\pi \left(\frac{1}{2t^2} + K_1(|t|) I_1(|t|) \left(\frac{1}{|t|} I_0(|t|) K_1(|t|) + \frac{1}{|t|} I_1(|t|) K_0(|t|) - \frac{2}{t^2} \right) \right) \\
&= \frac{\pi}{t^2} \left(1 - 2K_1(|t|) I_1(|t|) \right).
\end{aligned}$$

□

2.10 Calculations for the monotonously increasing rearrangement

Let G as in (2.7) and set

$$h(x) := \int_{D_R} \int_{D_R} \partial_{xx} G(x, y - y') \, dy' \, dy$$

In this section we will present the calculations that show that for functions $f\colon \mathbb{R} \to \mathbb{R}$

$$E_{\rho\rho}(f\vec{e}_x) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}(f(x)-f(x'))^2 h(x-x')\,dx'\,dx$$

and that h has the properties stated in Lemma 2.15

Lemma 2.39. *For $g \in C_c^\infty(\mathbb{R})$ set $f := g + \chi$. Define $u\colon \mathbb{R}^3 \to \mathbb{R}$ to be the unique solution of $\Delta u = \partial_x f(x) \cdot \mathbb{1}_{D_R}$ in $L^2(\mathbb{R}^3)$. Then there is a constant C that depends on g but not on ϵ and a map $\kappa\colon \Sigma\times]0,1[\to \mathbb{R}$ such that $\|\kappa\|_{L^1(\Sigma)} \leq C\epsilon$ and for all $\epsilon > 0$ we have*

$$\partial_x u(x,y) = \int\limits_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]}\int_{D_R}(f(x)-f(x'))\,\partial_{xx}\,G(p-p')\,dp' \;+\kappa(p,\epsilon).$$

Remark. At the first glance it looks like we could send ϵ to zero and write

$$\partial_x u(x,y) = f(x) - \int_\Sigma f(x')\partial_{xx}\,G(p-p')\,dp',$$

but since $\partial_{xx}G$ is not integrable, it is important how we approach the singularity.

Proof. We proof the lemma analogously to [15, Theorem 1, p.23]. The main tool is partial integration. We split the integral

$$\partial_x u(p) = \partial_x \int_\Sigma -G(p-p')\,\partial'_x f(x')\,dp'$$

into two parts. Away from the singularity we have

$$\begin{aligned}
A(p,\epsilon) &= \partial_x \int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]}\int_{D_R} -G(p-p')\,\partial_{x'} f(x')\,dp' \\
&= \int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]}\int_{D_R} -\partial_x G(p-p')\,\partial_{x'} f(x')\,dp' \\
&= \int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]}\int_{D_R} -\partial_{xx} G(p-p')\,f(x')\,dp' \\
&\quad - \int_{D_R} f(x-\epsilon)\,\partial_x G(-\epsilon,y-y')\,dy' + \int_{D_R} f(x+\epsilon)\,\partial_x G(\epsilon,y-y')\,dy' \\
&= \int_{D_R}\int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]} \big(f(x)-f(x')\big)\;\partial_{xx}G(p-p')\,dx'\,dy' + \alpha(p,\epsilon),
\end{aligned}$$

where

$$\alpha(p,\epsilon) = \int_{D_R}\partial_x G(\epsilon,y-y')\,(f(x+\epsilon)+f(x-\epsilon)-2f(x))\,dy'.$$

We can estimate

$$\begin{aligned}
\|\alpha(\cdot,\epsilon)\|_{L^1(\Sigma)} &\leq 2\pi R^2 \, \|f(x+\epsilon)-f(x)\|_{L^1(\mathbb{R})} \max_{y\in D_R} \int_{D_R} |\partial_x G(\epsilon, y-y')| \, dy' \\
&= 2\pi R^2 \, \|f(x+\epsilon)-f(x)\|_{L^1(\mathbb{R})} \int_{D_R} 2\pi r \frac{\epsilon}{\sqrt{r^2+\epsilon^2}^3} \, dr \\
&= 8\pi^2 R^2 \, \|f(x+\epsilon)-f(x)\|_{L^1(\mathbb{R})} \left(1 - \frac{\epsilon}{\sqrt{R^2+\epsilon^2}}\right).
\end{aligned}$$

Since $|f(x+\epsilon)-f(x)) \leq \epsilon \|\partial_x f\|_{L^\infty(\mathbb{R})}$ and $f(x+\epsilon)-f(x) = 0$ outside a finite region of size $|\operatorname{supp}(g)| + 2 + 2\epsilon$, we have $\|\alpha(p,\epsilon)\|_{L^1(\Sigma)} \leq C_1\epsilon$.

The intgral close to the singularity is small:

$$\begin{aligned}
\int_\Sigma |B(p,\epsilon)| \, dp &= \int_\Sigma \left| \partial_x \int_{x-\epsilon}^{x+\epsilon} \int_{D_R} -G(p-p') \, \partial_{x'} f(x') \, dp' \right| \, dp \\
&= \int_\Sigma \left| \int_{x-\epsilon}^{x+\epsilon} \int_{D_R} \partial_{x'} G(p-p') \, \partial_{x'} f(x') \, dp' \right| \, dp \\
&\leq \int_\Sigma \left| \int_{x-\epsilon}^{x+\epsilon} \int_{D_R} G(t, y-y') \, \partial_{x'x'} f(x') \, dx' \, dy' \right| \, dp \\
&\quad + \int_\Sigma \left| \int_{D_R} G(\epsilon, y') \, |f'(x+\epsilon) - f'(x-\epsilon)| \, dy' \right| \, dp \\
&\leq 4\epsilon \int_\Sigma \|\partial_{x'x'} f\|_{L^\infty([x-\epsilon,x+\epsilon])} \int_{D_R} \frac{1}{|y'|} \, dy' \, dp
\end{aligned}$$

So again, since the support of f is bounded, there is some constant C_2 such that $\|B(\cdot,\epsilon)\|_{L^1(\Sigma)}\| \leq C_2\epsilon$. □

Lemma 2.40. *For $x \neq 0$ we define*

$$h(x) := \int_{D_R} \int_{D_R} \partial_{xx} G(x, y-y') \, dy' \, dy.$$

Then h is integrable, positive, symmetric and monotonously decreasing in $|x|$.

Proof. We have

$$\partial_{xx} G(p) = \frac{1}{4\pi} \left(\frac{3x^2}{|p|^5} - \frac{1}{|p|^3} \right).$$

Thus

$$h(x) = \int_{D_R} \int_{D_R} \frac{1}{4\pi} \left(\frac{3x^2}{\sqrt{x^2+(y-y')^2}^5} - \frac{1}{\sqrt{x^2+(y-y')^2}^3} \right) dy' \, dy.$$

Obviously the map h is well defined for all $x \neq 0$ and symmetric. To show that h is integrable and monotonously decreasing in $|x|$, we parameterise the disc $\{y' \in D_R\}$ in dependence of y by the angle α and the distance to y (c. f. Figure 2.4). The integral $\int_{D_R} \cdot \, dy'$ then transforms to the integral $\int_0^{2\pi} \int_0^{R(\alpha,y)} r \cdot \, dr \, d\alpha$.

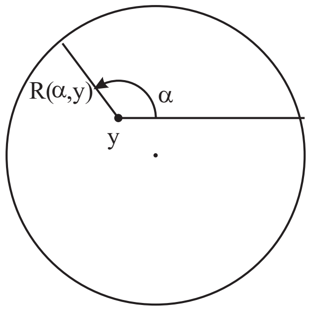

Figure 2.4: Parameterisation of the disc

$$
\begin{aligned}
h(x) &= \int_{D_R} \int_0^{2\pi} \int_0^{R(\alpha,y)} \frac{r}{4\pi} \left(\frac{3x^2}{\sqrt{x^2+r^2}^5} - \frac{1}{\sqrt{x^2+r^2}^3} \right) dr \, d\alpha \, dy \\
&= \frac{1}{4\pi} \int_{D_R} \int_0^{2\pi} -\frac{x^2}{\sqrt{x^2+R(\alpha,y)^2}^3} + \frac{1}{\sqrt{x^2+R(\alpha,y)^2}} \, d\alpha \, dy \\
&= \frac{1}{4\pi} \int_{D_R} \int_0^{2\pi} \frac{R(\alpha,y)^2}{\sqrt{x^2+R(\alpha,y)^2}^3} \, d\alpha \, dy
\end{aligned}
$$

So we see that h is decreasing in $|x|$. Moreover, $h(x) \le \frac{2\pi R^4}{|x|^3}$, therefore the integral $\int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} h(x) \, dx$ is finite for all $\epsilon > 0$. To show that $\int_{-\epsilon}^{\epsilon} h(x) \, dx$ is finite, we decompose D_R into an inner disc with radius $R - \sqrt{x}$ and an outer ring. In the inner disc we estimate the integrand by $\max_{\{y \in D_{R-\sqrt{x}},\, \alpha \in [0,2\pi]\}} \left(\frac{1}{R(\alpha,y)} \right) = \frac{1}{\sqrt{x}}$, and in the outer ring we estimate the integrand by $\max_{\{R \in \mathbb{R}\}} \left(\frac{R^2}{\sqrt{x^2+R^2}^3} \right) \le \frac{1}{|x|}$. So

$$
\begin{aligned}
h(x) &\le \int_{D_{R-\sqrt{x}}} \frac{1}{2R\sqrt{x}} \, dy + \int_{D_R \setminus D_{R-\sqrt{x}}} \frac{1}{2x} \, dy \\
&\le \frac{\pi}{2} R^2 \frac{1}{\sqrt{x}} + \pi R \frac{1}{\sqrt{x}},
\end{aligned}
$$

and thus h is integrable. □

Theorem 2.41. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a map such that $g := f - \chi \in C_c^\infty(\mathbb{R})$. Then*

$$
E_{\rho\rho}(f \, \vec{e}_x) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x) - f(x'))^2 \, h(x - x') \, dx' \, dx
$$

where h is defined as in Lemma 2.40 and thus a positive, integrable, and symmetric function and monotonously decreasing in $|x|$.

Proof. Let the notation be as in Lemma 2.39 and Lemma 2.40. Then

$$\begin{aligned}
&E_{\rho\rho}(f\,\vec{e}_x) \\
&= \int_\Sigma f(x) \int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]} \int_{D_R} (f(x) - f(x'))\partial_{xx}\, G(p-p')\; dy'\; dx' \;+ \kappa(\epsilon, p)\; dp \\
&= \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]} \frac{1}{2}(f(x) - f(x'))^2 h(x-x')\; dx'\; dx}_{T_\epsilon} + \underbrace{\int_\Sigma f(x)\kappa(p,\epsilon)\; dp.}_{D_\epsilon}
\end{aligned}$$

Because h is integrable, f is bounded, and $\|\kappa(\cdot,\epsilon)\|_{L^1(\Sigma)} \leq C\epsilon$, we can send ϵ to zero and get

$$E_{\rho\rho}(f\,\vec{e}_x) \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2}(f(x) - f(x'))^2 h(x-x')\; dx'\; dx.$$

□

Chapter 3

The transverse mode

3.1 Introduction

In this chapter we investigate the transverse mode via a perturbation argument for the static case. We show the existence of travelling wave solutions for the gradient flow of the micromagnetic energy (1.3), including the non-local stray field energy.

We are perturbing domain walls in very thin wires, where they are almost constant on the cross section by adding a weak additional magnetic field h to the equation. Then the solutions of the perturbed equation are almost constant on the cross section, too.

The pertrurbetion argument relies crucially on the fact that the wires are thin, since we need strong regularity of the static domain wall. We will proof strong regularity in the case of thin wires, and we cannot expect it for thick wires where the examples of low energy configurations are vortex walls which have a singurlarity and are not even continuous.

Various models for the transverse mode have been analysed previously.

Thiaville and Nakatani [38] study a one dimensional model for the transverse mode and compare it with numerical simulations. We discuss the relation between our results and their results in Chapter 5. Carbou and Labbé [10] consider a similar model. They prove that one dimensional domain walls are asymptotically stable.

Sanchez [35] considers the limit of the Landau-Lifshitz equation when the diameter of the domain and the exchange coefficient in the equation simultaneously tend to zero and performs an asymptotic expansion.

3.1.1 The model

Let E_h be the micromagnetic energy of (2.1) with an additional external magnetic field in direction of the axis of the wire, that is,

$$E_h(m) = \int_\Sigma |\nabla m|^2 + |H(m)|^2 - hm_x. \tag{3.1}$$

We assume that the evolution of the magnetisation can be described by gradient flow of the energy under the condition $|m| \equiv 1$ with Neumann boundary conditions, that is,

$$\partial_t m = -\delta_m E_h(m) + (\delta_m E_h(m) \cdot m)m \quad \text{in } \Sigma, \qquad \partial_\nu m = 0 \quad \text{on } \partial\Sigma, \tag{3.2}$$

where

$$\delta_m E_h(m) = -2\Delta m + 2H(m) - h\vec{e}_x. \tag{3.3}$$

We are interested in travelling wave solutions of this equation. Because of the rotational symmetry of the cylinder we have to take into account that the solutions may rotate around the axis of the cylinder. Set

$$R_\phi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}, \qquad \tilde{R}_\phi := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}.$$

and note that $\partial_t R_{\omega t} = \omega \tilde{R}_{\omega t + \frac{\pi}{2}}$. Rotating travelling waves with speed c and angular velocity ω satisfy

$$m(t, x, y) = R_{\omega t}\, m(0, R_{-\omega t}(x - ct, y)).$$

Setting

$$\Phi(m) := \begin{pmatrix} m_x \\ m_{y_2} \\ -m_{y_1} \end{pmatrix} + \begin{pmatrix} 0 & & \\ & \partial_{y_1} m_{y_1} & \partial_{y_2} m_{y_1} \\ & \partial_{y_1} m_{y_2} & \partial_{y_2} m_{y_2} \end{pmatrix} \begin{pmatrix} 0 \\ y_2 \\ -y_1 \end{pmatrix},$$

we have

$$\begin{aligned}
&\partial_t m(t, x, y) \\
&\quad = \omega \tilde{R}_{\omega t + \frac{\pi}{2}} m\,(0, R_{-\omega t}(x - ct, y)) - cR_{\omega t}\, \partial_x m\,(0, R_{-\omega t}(x - ct, y)) \\
&\qquad - R_{\omega t} \nabla_y m(0, R_{-\omega t}(x - ct, y))\, \omega \tilde{R}_{-\omega t + \frac{\pi}{2}} \vec{y} \\
&\quad = -c\partial_x m(t, x, y) + \omega \tilde{R}_{\frac{\pi}{2}}\, m(t, x, y) - \omega \nabla_y m(t, x, y) \tilde{R}_{\frac{\pi}{2}} \vec{y} \\
&\quad = -c\partial_x m(t, x, y) - \omega \Phi(m(t, x, y)).
\end{aligned}$$

In particular, rotating travelling waves that are a solution of (3.2) satisfy the stationary equation

$$\begin{aligned} -\delta_m E_h(m) + (\delta_m E_h(m) \cdot m)m + c\partial_x m + \omega \Phi(m) &= 0 \quad \text{in } \Sigma, \\ \partial_\nu m &= 0 \quad \text{on } \partial\Sigma. \end{aligned} \tag{3.4}$$

Set

$$F(m,h,c,\omega) := -\delta_m E_h(m) + (\delta_m E_h(m)\cdot m)m + c\partial_x m + \omega\Phi(m).$$

To find solutions of (3.4) we consider first the case $h = 0$ and then use a perturbation argument. For this we have to work in a function space that is large enough to contain the solutions and small enough that F is differentiable in this function space. In this space we have to restrict the search to solutions with $|m| \equiv 1$. To specify the conditions at $\pm\infty$ we define a function $\chi : \mathbb{R} \to \mathbb{R}^3$ with $\lim_{x\to\pm\infty} \chi(x) = \pm\vec{e}_x$. Contrary to the definition of χ in Chapter 2, we need χ to be smooth, so we set

$$\chi : \mathbb{R} \to \mathbb{R}^3, \qquad x \mapsto \tanh(x)\vec{e}_x. \tag{3.5}$$

Moreover, we have to include further conditions in the set of admissible solutions to break the translation invariance and the rotation invariance of the problem. We proceed a follows.

(1.) We find a solution $m = m_{\mathrm{sol}} \in H^2(\Sigma,\mathbb{R}^3)+\chi$ of (3.4) for $c = 0$, $\omega = 0$, $h = 0$.

(2.) Depending on m_{sol} we define the set of admissible functions $\mathcal{S}$ and show that $\mathcal{S}$ is a Banach submanifold of $H^2(\Sigma,\mathbb{R}^3)+\chi$.

(3.) We find a continuously differentiable function

$$N\colon \mathcal{S}\times L^2(\Sigma,\mathbb{R})\times\mathbb{R}^3 \to L^2(\Sigma,\mathbb{R}^3)\times\mathbb{R}$$

such that (m,c,ω,h) is a solution of (3.4) if and only if there exists $\alpha \in L^2(\Sigma,\mathbb{R})$ that satisfies $N(m,\alpha,c,\omega,h) = (0,h)$.

(4.) We show that the derivative DN of N in $(m_{\mathrm{sol}},0,0,0,0)$ is invertible.

(5.) Then, according to the inverse function theorem [41, Theorem 73.B, p.552], there exists a neighbourhood U of $(m_{\mathrm{sol}},0,0,0,0)$ and a neighbourhood V of $(0,0)$ such that $N|_U \to V$ is bijective. In particular, there exists $h_0 > 0$, such that for all $|h| < h_0$ there is $m_h, \alpha_h, c_h, \omega_h$ with $N(m_h,\alpha_h,c_h,\omega_h,h) = 0$. In other words, for all $|h| < h_0$ there exists a solution of (3.4).

In Chapter 2 we have shown that for every $R > 0$ there exists a minimiser $m^R \in \mathcal{M}_l(R)$ of the energy $E = E_0$, where $\mathcal{M}_l(R)$ is as in (2.2). For $c = 0$, $\omega = 0$, $h = 0$, (3.4) simplifies to

$$0 = -\delta_m E(m) + (\delta_m E(m)\cdot m)m \quad \text{in } \Sigma, \qquad \partial_\nu m|_{\partial\Sigma} = 0 \quad \text{on } \partial\Sigma. \tag{3.6}$$

This is the Euler-Lagrange equation to this minimisation problem with the constraint $|m| = 1$, so we can take $m_{\mathrm{sol}} := m^R$. However, the regularity that we get from the considerations in Chapter 2 is too weak for our purposes, e.g., we do not know that $m^R \in H^2(\Sigma, \mathbb{R}^3) + \chi$. In general, we cannot even expect H^2-regularity since our examples for domain walls with low energy in the case of large radii are vortex walls that are not even continuous (cf. Section 2.8). So the arguments regarding regularity rely crucially on the fact that we are considering wires with small radii.

The regularity of minimisers of the micromagnetic energy has been studied independently by Carbou [8] and Hardt and Kinderlehrer [19]. Carbou investigates critical points of the micromagnetic energy in two and three dimensions using the Euler-Lagrange equation. He finds that critical points in $H^1(D_1, \mathbb{S}^2)$ are smooth, while critical points in $H^1(B_1, \mathbb{S}^2)$ are smooth away from a set of one dimensional Hausdorff measure zero.

Hardt and Kinderlehrer use the fact that on small scales the exchange energy is the dominant part of the micromagnetic energy. Using the notion of *almost-minimisers* they show how the stray field energy can be treated as a lower order perturbation. They find that minimisers of the micromagnetic energy functional on bounded, sufficiently regular domains are smooth away from a discrete set Z.

Their results moreover imply that, if the exchange energy of m is small enough, the set Z is empty. Here "small enough" depends on the domain. Using their results, it is not much work to prove that, for small $R > 0$, the functions m^R are smooth, but we do not get uniform bounds.

However, we do need uniform bounds to show that that $\|m^R - m^{\mathrm{red}}\|_{C^1(\Sigma(R))}$ converges to 0 and that $DN(m_{\mathrm{sol}}, 0, 0, 0, 0)$ is invertible. Using ideas from [19], the Morrey-Campanato approach to regularity, and the bounds on the rate of convergence of the minimal energies from Section 2.6, we obtain uniform bounds in H^2 and $C^{1,\beta}$ for $\beta < \frac{1}{8}$.

3.1.2 Outline of the chapter

In the beginning of Section 3.2 we state the main regularity and convergence result (Theorem 3.1). This result is proved later, in Section 3.4–3.6. Using Theorem 3.1 we go through the steps (1.)–(5.) discussed above to show the existence of travelling wave solutions for small radii and small external magnetic field.

The argument of Section 3.2 uses the invertibility of an operator representing the "interesting" part of $DN(m^R, 0, 0, 0, 0)$. This invertibility is shown in Section 3.3 and relies again on the fact that m^R is close to m^{red}.

Section 3.4–3.6 are devoted to regularity. We work in the framework of Morrey-Campanato theory, that is, we replace $C^{k,p}$-estimates by scaled L^2 estimates.

In Section 3.4, we give a short introduction to the ideas and definitions we are using to prove regularity results.

In Section 3.5, we show decay estimates for almost-minimisers and prove that, for R small enough, they imply uniform $C^{0,\alpha}$-regularity. This part is included for the convenience of the reader and presents, with some modifications, the arguments of Hardt and Kinderlehrer [19].

In Section 3.6 we use the Euler-Lagrange equations for m^R and apply the Morrey-Campanato approach to regularity (cf. [17]) to prove uniform $C^{1,\beta}$ regularity. To get a bound on $\|m^R\|_{C^{1,\beta}(\Sigma(R))}$ that is uniform in R, we need uniform bounds on integrals of the form $\frac{1}{R^\gamma}\int_{B_R(a)}|\nabla m|^2$. We show such bounds in Section 3.6. The proof relies on the bound for the rate of convergence of the minimal energies of Section 2.6.

3.1.3 Notation

The notation introduced in Section 2.1 is kept, replacing only the definition of χ by (3.5).

For $m \in \mathcal{M}(R)$ the micromagnetic energy without external magnetic field is denoted by $E(m) = E(m, R)$ and the micormagnetic energy incuding the external magnetic field is denoted by $E_h(m) = E_h(m, R)$.

As in Chapter 2, we will not distinguish between functions $f\colon \Sigma \to \mathbb{R}^n$ that are constant on the cross section and functions $f\colon \mathbb{R} \to \mathbb{R}^n$. We set

$$m^{\mathrm{red}}\colon \mathbb{R} \to \mathbb{S}^2, \qquad x \mapsto \left(\tanh(\tfrac{x}{\sqrt{2}}), \frac{1}{\cosh(\frac{x}{\sqrt{2}})}, 0\right)$$

and, as above, let $m^R\colon \Sigma(R) \to \mathbb{S}^2$ be a minimiser of E in $\mathcal{M}_l(R)$. To have convergence of m^R to m^{red} in the sense of Definition 2.22 without having to pass to a subsequence , we require

$$\|m^R - m^{\mathrm{red}}\|_{L^2(\Sigma)} \leq \|v - m^{\mathrm{red}}\|_{L^2(\Sigma)} \quad \text{for all other minimizers } v \in \mathcal{M}_l(R).$$

For a set $A \subset L^2(\mathbb{R}^3)$, we denote the closure of A in $L^2(\mathbb{R}^3)$ by A_{L^2}.

3.2 The perturbation argument

We start our pertubation argument quoting the regularity results from Section 3.6 (cf. Theorem 3.33). The first part shows that we have a solution of (3.6) with sufficiently good regularity. The second part bounds the difference between m^R and m^{red}.

Theorem 3.1. *(i) For R small enough, $m^R \in H^2(\Sigma(R)) \cap C^1(\Sigma(R))$.*
(ii) We have

$$\begin{aligned}\lim_{R\to 0} \frac{1}{R}\|m^R - m^{\mathrm{red}}\|_{H^1(\Sigma(R))} &= 0,\\ \lim_{R\to 0} \frac{1}{R}\|\Phi(m^R) - \Phi(m^{\mathrm{red}})\|_{L^2(\Sigma(R))} &= 0,\\ \lim_{R\to 0} \|m^R - m^{\mathrm{red}}\|_{C^1(\Sigma(R))} &= 0,\\ \lim_{R\to 0} \|\Phi(m^R) - \Phi(m^{\mathrm{red}})\|_{C^0(\Sigma(R))} &= 0.\end{aligned}$$

As described previously, the next step in the perturbation argument is to show that we are working on a sufficiently smooth manifold. Set

$$\begin{aligned}\mathcal{S}^R &:= \left\{ f \in H^2(\Sigma(R), \mathbb{R}^3) + \chi \;\middle|\; \begin{array}{ll} |f| \equiv 1, & \partial_\nu f = 0 \text{ on } \partial\Sigma, \\ \langle \partial_x m^R, f\rangle_\Sigma = 0, & \langle \Phi(m^R), f\rangle_\Sigma = 0. \end{array}\right\},\\ T\mathcal{S}^R &:= \left\{ f \in H^2(\Sigma(R), \mathbb{R}^3) \;\middle|\; \begin{array}{ll} f\cdot m^R \equiv 0, & \partial_\nu f = 0 \text{ on } \partial\Sigma, \\ \langle \partial_x m^R, f\rangle_\Sigma = 0, & \langle \Phi(m^R), f\rangle_\Sigma = 0. \end{array}\right\}.\end{aligned}$$

Lemma 3.2. *There exists $R_0 > 0$, such that for all $R \le R_0$ the set $\mathcal{S}^R$ is a submanifold of $H^2(\Sigma, \mathbb{R}^3) + \chi$. The tangent space of $\mathcal{S}^R$ in m^R is $T\mathcal{S}^R$.*

Proof. We show the Lemma in two steps. We define

$$\mathcal{W}^R := \left\{ m \in H^2(\Sigma, \mathbb{R}^3) \,\middle|\, \partial_\nu m|_{\partial\Sigma} = 0,\ \langle m, \partial_x m^R\rangle_\Sigma = 0,\ \langle m, \Phi(m^R)\rangle_\Sigma = 0 \right\}.$$

First, since $\partial_x m^R, \Phi(m^R) \in L^2(\Sigma)$ and since the trace of a function in $H^2(\Sigma)$ is in $H^1(\partial\Sigma)$, the set $\mathcal{W} + \chi$ is a closed affine subspace of $H^2(\Sigma, \mathbb{R}^3) + \chi$. Second, we show that $\mathcal{S}^R$ is a submanifold of $\mathcal{W}^R + \chi$. Set

$$\phi\colon \mathcal{W}^R + \chi \to \left\{ f \in H^2(\Sigma, \mathbb{R}) \;\middle|\; \partial_\nu f|_{\partial\Sigma} = 0 \right\}, \qquad m \mapsto |m| - 1,$$

then $\mathcal{S}^R = \phi^{-1}(0)$. On $\{m \in \mathcal{W}^R : |\phi(m)| < 1\}$ the function ϕ is continuously differentiable and the derivative in m is

$$D\phi(m)\colon \mathcal{W}^R \to \left\{ f \in H^2(\Sigma, \mathbb{R}) \;\middle|\; \partial_\nu f|_{\partial\Sigma} = 0 \right\}, \qquad g \mapsto \frac{g\cdot m}{|m|}. \tag{3.7}$$

If R is small enough, for every $m \in \mathcal{S}^R$ the differential $D\phi(m)$ is surjective: Since the equality $\partial_x m^{\mathrm{red}} \cdot \Phi(m^{\mathrm{red}}) = 0$ implies

$$\begin{aligned}&\det \begin{pmatrix} \langle \partial_x m_R^{\mathrm{red}}, \partial_x m_R^{\mathrm{red}}\rangle_\Sigma & \langle \partial_x m^R, \Phi(m_R^{\mathrm{red}})\rangle_\Sigma \\ \langle \partial_x m_R^{\mathrm{red}}, \Phi(m_R^{\mathrm{red}})\rangle_\Sigma & \langle \Phi(m_R^{\mathrm{red}}), \Phi(m_R^{\mathrm{red}})\rangle_\Sigma \end{pmatrix}\\ &= \pi R^2 \left(\|\partial_x m^{\mathrm{red}}\|^2_{L^2(\mathbb{R})} + \|\Phi(m_R^{\mathrm{red}})\|_{L^2(\mathbb{R})} \right),\end{aligned}$$

so with Theorem 3.1 (ii) there exists R_0 such that for all $R \le R_0$ we have

$$\det \begin{pmatrix} \langle \partial_x m^R, \partial_x m^R \rangle_\Sigma & \langle \partial_x m^R, \Phi(m^R) \rangle_\Sigma \\ \langle \partial_x m^R, \Phi(m^R) \rangle_\Sigma & \langle \Phi(m^R), \Phi(m^R) \rangle_\Sigma \end{pmatrix} \;>\; 0.$$

Therefore, for every $f \in H^2(\Sigma, \mathbb{R})$ with $\partial_\nu f|_{\partial\Sigma} = 0$ we can find unique numbers b_1, b_2 such that

$$\begin{aligned} \langle fm + b_1 \partial_x m^R + b_2 \Phi(m^R),\ \partial_x m^R \rangle_\Sigma &= 0, \\ \langle fm + b_1 \partial_x m^R + b_2 \Phi(m^R),\ \Phi(m^R) \rangle_\Sigma &= 0, \end{aligned}$$

and $fm + b_1\partial_x m^R + b_2\Phi(m^R)$ is a pre-image of f in $\mathcal{W}^R$. Moreover, since in a Hilbert space every subspace splits, in particular $D\phi^{-1}(0)$ splits. Thus 0 is a regular value of ϕ and we can apply [41, Thm. 73C, p.556] to conclude that $\mathcal{S}^R$ is a submanifold of $\mathcal{W}^R + \chi$. Because of (3.7) the space $T\mathcal{S}^R$ is the tangent space of $\mathcal{S}^R$ in m^R. □

We consider the map

$$s \colon \mathcal{S}^R \;\to\; L^2(\Sigma, \mathbb{R}^3), \quad m \mapsto -\delta_m E_h(m) + (\delta_m E_h(m) \cdot m)m,$$

that is, with (3.3),

$$s(m) = \underbrace{2(\Delta m - (\Delta m \cdot m)m - H(m) + (H(m) \cdot m)m)}_{s_1} + \underbrace{h\vec{e}_x - (h\vec{e}_x \cdot m)m}_{s_2}.$$

The space $H^2(\Sigma, \mathbb{R}) + \chi$ embeds into $C^0(\Sigma, \mathbb{R})$, and functions $m \mapsto \Delta m$, and $m \mapsto H(m)$ are continuous linear maps from $\mathcal{S}^R$ to $L^2(\Sigma, \mathbb{R}^3)$. For the last statement see Lemma 2.6. Thus $s_1 \colon \mathcal{S}^R \to L^2(\Sigma, \mathbb{R}^3)$ is well defined and continuously differentiable.

Moreover, we have

$$|h\vec{e}_x - (h\vec{e}_x \cdot m)m| \;=\; h\left|(1 - m_x^2)\vec{e}_x + m_x m_y\right| \;\le\; 2h|m_y|,$$

so $s_2 \colon \mathcal{S}^R \to L^2(\Sigma, \mathbb{R}^3)$ is well defined and continuously differentiable, too.

Thus we can define the continuously differentable map

$$\begin{aligned} &N^R \colon \mathcal{S}^R \times L^2(\Sigma, \mathbb{R}) \times \mathbb{R}^3 \;\to\; L^2(\Sigma, \mathbb{R}^3) \times \mathbb{R}, \\ &(m, \alpha, c, \omega, h) \;\mapsto \\ &\quad (-\delta_m E_h(m, R) + (\delta_m E_h(m, R) \cdot m)m + c\partial_x m + \omega\Phi(m) + \alpha m,\ h). \end{aligned}$$

Since $(-\delta_m E_h(m, R) + (\delta_m E_h(m, R) \cdot m)m + c\partial_x m + \omega\Phi(m)) \perp m$ for all $m \in \mathcal{S}^R$ we have $N^R(m, \alpha, c, \omega, h) = (0, h)$ if and only if m is a solution of (3.4) and $\alpha = 0$.

The differential of N^R in $(m^R, 0_{L^2(\Sigma,\mathbb{R})}, 0_{\mathbb{R}^3})$ is

$$\begin{aligned} DN^R(m^R,0,0)\colon T\mathcal{S}^R \times L^2(\Sigma, R^3)\times\mathbb{R}^3 \;&\to\; L^2(\Sigma,\mathbb{R}^3)\times\mathbb{R} \\ (g,\alpha,c,\omega,h) \;&\mapsto\; (-L^R(g) + c\partial_x m^R + \omega\Phi(m_0) + \alpha m^R,\; h), \end{aligned}$$

where

$$\begin{aligned} L^R\colon H^2(\Sigma,\mathbb{R}^3) &\to L^2(\Sigma,\mathbb{R}^3) \\ g \;\;\mapsto\;\; & \delta_m E(g,R) - (\delta_m E(g,R)\cdot m^R)m^R \\ & - (\delta_m E(m^R,R)\cdot g)m^R - (\delta_m E(m^R,R)\cdot m^R)g. \end{aligned} \tag{3.8}$$

We will consider the restrictions of L^R to different subspaces of $H^2(\Sigma,\mathbb{R}^3)$. We will call this restrictions L^R as well, but name always the domain and the range.

Lemma 3.3. *For all $R > 0$ and all $g, f \in T\mathcal{S}^R$ we have*

$$L^R(g) \;=\; \delta_m E(g,R) - (\delta_m E(g,R)\cdot m^R)m^R - (\delta_m E(m^R,R)\cdot m^R)g, \tag{3.9}$$

$$L^R(g)\cdot f \;=\; \delta_m E(g,R)\cdot f - (\delta_m E(m^R,R)\cdot m^R)g\cdot f. \tag{3.10}$$

Moreover $L^R(T\mathcal{S}^R) \subseteq (T\mathcal{S}^R)_{L^2}$ and the operator $L^R\colon T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ is symmetric.

Proof. Since m^R is a solution of (3.6), $\delta_m E(m^R,R)$ is pointwise parallel to m^R. The elements of $T\mathcal{S}^R$ are pointwise orthogonal to m^R. This implies (3.9) and (3.10). Since the elements of $T\mathcal{S}^R$ satisfy Neumann boundary conditions, for all $g, f \in T\mathcal{S}^R$ we have $\langle L^R f, g\rangle_\Sigma = \langle f, L^R g\rangle_\Sigma$.
It remains to show that $L^R(T\mathcal{S}^R) \subseteq (T\mathcal{S}^R)_{L^2}$. We have

$$(T\mathcal{S}^R)_{L^2} \;:=\; \left\{ f\in L^2(\Sigma(R),\mathbb{R}^3) \;\middle|\; \begin{array}{l} f\cdot m^R \equiv 0, \\ \langle \partial_x m^R, f\rangle_\Sigma = 0, \quad \langle \Phi(m^R), f\rangle_\Sigma = 0 \end{array} \right\}.$$

Looking at (3.9), we see that $L^R(g) \perp m^R$. Set $v(t,x,y) := m^R(x+t,y)$. Then $v(t,\cdot)$ satisfies for all $t\in\mathbb{R}$ the equation

$$0 \;=\; \delta_m E(v(t,\cdot),R) - (\delta_m E(v(t,\cdot),R)\cdot v(t,\cdot))\, v(t,\cdot),$$

therefore we have for all $g \in T\mathcal{S}^R$

$$\begin{aligned} 0 \;&=\; \partial_t \left\langle \delta_m E(v(t,\cdot),R) - (\delta_m E(v(t,\cdot),R)\cdot v(t,\cdot))\, v(t,\cdot),\; g\right\rangle_\Sigma \Big|_{t=0} \\ &=\; \langle L(\partial_x m^R), g\rangle_\Sigma \;=\; \langle L(g), \partial_x m^R\rangle_\Sigma\,. \end{aligned}$$

Analogously, we have for $w(\phi,x,y) := R_\phi(m^R(R_{-\phi}(x,y))$ the equation

$$0 \;=\; \delta_m E(w(\phi,\cdot),R) - (\delta_m E(w(\phi,\cdot),R)\cdot v(\phi,\cdot))\, w(\phi,\cdot)$$

and thus for all $g \in T\mathcal{S}^R$

$$\begin{aligned} 0 &= \partial_\phi \left\langle \delta_m E(w(\phi,\cdot),R) - (\delta_m E(w(\phi,\cdot),R)\cdot v(\phi,\cdot))\, w(\phi,\cdot),\ g\right\rangle_\Sigma|_{\phi=0} \\ &= \left\langle L(\Phi(m^R)), g\right\rangle_\Sigma \;=\; \left\langle L(g), \Phi(m^R)\right\rangle_\Sigma . \end{aligned}$$

□

Note that $DN^R(m^R,0,0)$ is bijective if and only if

(a) $\partial_x m^R$ and $\Phi(m^R)$ are linearly independent,

(b) $L^R\colon T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ bijective.

Since $\lim_{R\to 0}\|m^R - m^{\mathrm{red}}\|_{C^1(\Sigma(R))} = 0$ and since $\partial_x m^{\mathrm{red}}$ and $\Phi(m^{\mathrm{red}})$ are linearly independent, (a) is satisfied if R is small enough. In Section 3.3 we will show that (b) is satisfied for small R, too. Altogether, we have the following theorem.

Theorem 3.4. *(m,c,ω) is a solution of (3.4) if and only if there exists $\alpha \in L^2(\Sigma,\mathbb{R})$ such that $N^R(m,\alpha,c,\omega,h) = (0,h)$.*

The function N^R is continuously differentiable and, if R is small enough, $DN^R(m^R,0,0)$ is bijective.

If N^R is continuously differentiable and $DN^R(m^R)$ is invertible, according to the inverse function theorem [41, Theorem 73.B, p. 552] there exists a neighbourhood U of $(m^R, 0_{L^2(\Sigma,\mathbb{R})}, 0_{\mathbb{R}^3})$ and a neighbourhood V of $(0_{L^2(\Sigma,\mathbb{R}^3)}, 0_{\mathbb{R}})$ such that $N^R|_U \to V$ is bijective. So for every h small enough, we can find $m_h, \alpha_h, c_h, \omega_h$ such that $N^R(m_h,\alpha_h,c_h,\omega_h,h) = 0$. That is, we have proved our main theorem.

Theorem 3.5. *For all $R > 0$ small enough there exists $h_R > 0$ such that for all h with $h < h_R$ there is exists a solution (m_h, c_h, ω_h) of (3.4).*

3.3 Invertibility of L^R

The goal of this section is to prove the following theorem.

Theorem 3.6. *For R small enough, the operator $L^R\colon T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$, as defined in (3.8), is invertible, and its inverse is continuous.*

We proceed in two steps. First, we define a map L_0^R and show that for functions m in a certain space $T\mathcal{S}_0^R$ we have $\left\langle L_0^R(m), m\right\rangle_\Sigma \geq \frac{1}{4}\|m\|^2_{L^2(\Sigma)}$. Then we prove that, for small R, the operator L^R is similar to L_0^R and the space $T\mathcal{S}^R$ is similar to $T\mathcal{S}_0^R$. The map L_0^R is related to the reduced energy functional of Chapter 2 (Definition 2.22),

$$E_{\text{red}} \colon \{m \in H^1(\mathbb{R}) + \chi : |m| = 1\} \to \mathbb{R}, \qquad m \mapsto \pi \int_{\mathbb{R}} |\partial_x m|^2 + \frac{1}{2}|m_y|^2.$$

We introduce, similar to E_{red}, the functional E^0, setting

$$E^0(\cdot, R) \colon \mathcal{M}(R) \to \mathbb{R}, \qquad m \mapsto \int_{\Sigma} |\partial_x m|^2 + \frac{1}{2}|m_y|^2 + 20R^2|\nabla_y m|^2.$$

Then we have

$$\delta_m E^0(m, R) \;=\; -2\partial_{xx} m + (0, m_{y_1}, m_{y_2}) - 40R^2 \Delta_y m.$$

Lemma 3.7. *The minimiser of E^0 in $\mathcal{M}_l(R)$ is unique up to translation and rotation. It is given by*

$$m_R^{\text{red}} \colon \quad (x, y) \mapsto m^{\text{red}}(x) = \left(\tanh\left(\tfrac{x}{\sqrt{2}}\right),\ \frac{1}{\cosh\left(\tfrac{x}{\sqrt{2}}\right)},\ 0 \right),$$

and we have

$$\begin{aligned} |\partial_x m_R^{\text{red}}(x, y)| \;&=\; \frac{1}{\sqrt{2}}|m_y(x, y)| \\ \frac{\partial_x m_R^{\text{red}}(x, y)}{|\partial_x m_R^{\text{red}}(x, y)|} \;&=\; \left(\frac{1}{\cosh\left(\tfrac{x}{\sqrt{2}}\right)}, -\tanh\left(\tfrac{x}{\sqrt{2}}\right), 0 \right), \\ \Phi(m_R^{\text{red}}(x, y)) \;&=\; \left(0, 0, \frac{1}{\cosh\left(\tfrac{x}{\sqrt{2}}\right)} \right). \end{aligned}$$

Proof. For every function $m \in \mathcal{M}_l(R)$ we have

$$E^0(m) \;=\; \int_{D_R} E_{\text{red}}(m(\cdot, y))\, dy + 20R^2 \|\nabla_y m\|^2_{L^2(\Sigma)}.$$

Since m^{red} is the only minimiser of E_{red} in $\{m \in H^1(\mathbb{R}) + \chi : |m| = 1\}$, up to translation and rotation (Lemma 2.26), the function m_R^{red} is the only minimiser of E^0 in $\mathcal{M}_l(R)$, up to translation and rotation.

A direct calculation yields the results for $\partial_x m_R^{\text{red}}$ and $\Phi(m_R^{\text{red}})$. □

Remark. Since the domain of definition will not always be clear from the context, for the rest of this section we will distinguish the functions m^{red} and m_R^{red}.

We now set, in analogy to (3.8),

$$
\begin{aligned}
L_0^R \colon H^2(\Sigma,\mathbb{R}^3) &\to L^2(\Sigma,\mathbb{R}^3),\\
g \;\mapsto\; & \delta_m E^0(g,R) - (\delta_m E^0(g,R)\cdot m_R^{\mathrm{red}})m_R^{\mathrm{red}}\\
&- (\delta_m E^0(m_R^{\mathrm{red}},R)\cdot g)m_R^{\mathrm{red}} - (\delta_m E^0(m_R^{\mathrm{red}},R)\cdot m_R^{\mathrm{red}})g, \qquad (3.11)
\end{aligned}
$$

and define

$$
T\mathcal{S}_0^R \;:=\; \left\{ f\in H^2(\Sigma(R),\mathbb{R}^3) \;\middle|\; \begin{matrix} f\cdot m_R^{\mathrm{red}}\equiv 0, & \partial_\nu f = 0 \text{ on } \partial\Sigma,\\ \left\langle \partial_x m_R^{\mathrm{red}}, f\right\rangle = 0, & \left\langle \Phi(m_R^{\mathrm{red}}), f\right\rangle = 0. \end{matrix} \right\}.
$$

Lemma 3.8. *For all $R>0$ and all $g,f\in T\mathcal{S}_0^R$ we have*

$$
L_0^R(g) \;=\; \delta_m E^0(g,R) - (\delta_m E^0(g,R)\cdot m_R^{\mathrm{red}})m_R^{\mathrm{red}} - (\delta_m E^0(m_R^{\mathrm{red}},R)\cdot m_R^{\mathrm{red}})g, \qquad (3.12)
$$

$$
L_0^R(g)\cdot f \;=\; \delta_m E^0(g,R)\cdot f - (\delta_m E^0(m_R^{\mathrm{red}},R)\cdot m_R^{\mathrm{red}})g\cdot f. \qquad (3.13)
$$

Moreover, $L_0^R(T\mathcal{S}_0^R)\subseteq (T\mathcal{S}_0^R)_{L^2}$, and the operator $L_0^R\colon T\mathcal{S}_0^R\to (T\mathcal{S}_0^R)_{L^2}$ is symmetric.

Proof. We can argue exactly as in Lemma 3.3. □

Theorem 3.9. *For all $R>0$ and all $m\in T\mathcal{S}_0^R$ we have*

$$
\left\langle L_0^R(m), m\right\rangle_\Sigma \;\geq\; \frac{1}{4}\|m\|^2_{L^2(\Sigma)}
$$

Proof. The relations $|\partial_x m^{\mathrm{red}}| = \frac{|m_y^{\mathrm{red}}|}{\sqrt{2}}$ and

$$
\partial_{xx} m^{\mathrm{red}}\cdot m^{\mathrm{red}} + |\partial_x m^{\mathrm{red}}|^2 \;=\; \partial_x(\partial_x m^{\mathrm{red}}\cdot m^{\mathrm{red}}) \;=\; 0
$$

imply

$$
\delta_m E^0(m_R^{\mathrm{red}},R)\cdot m_R^{\mathrm{red}} \;=\; -2\partial_{xx} m_R^{\mathrm{red}}\cdot m_R^{\mathrm{red}} + |(m_R^{\mathrm{red}})_y|^2 \;=\; 2|(m_R^{\mathrm{red}})_y|^2.
$$

Thus, with Lemma 3.8, for all $g,h\in T\mathcal{S}_0^R$ we have

$$
\begin{aligned}
L_0^R(g)\cdot h \;&=\; \delta_m E^0(g,R)\cdot h - (\delta_m E^0(m_R^{\mathrm{red}},R)\cdot m_R^{\mathrm{red}})g\cdot h\\
&=\; \left(\delta_m E_0(g,R) - 2|(m_R^{\mathrm{red}})_y|^2 g\right)\cdot h.
\end{aligned}
$$

We define the vector $\vec{e}_s$ to be the unit vector in direction of $\partial_x m^{\mathrm{red}}$, i.e.,

$$
\vec{e}_s(x) \;:=\; \frac{\partial_x m^{\mathrm{red}}(x)}{|\partial_x m^{\mathrm{red}}(x)|} \;=\; \left(m_{y_1}^{\mathrm{red}}(x), m_x^{\mathrm{red}}(x),\; 0\right),
$$

and the sets

$$\begin{aligned}
\mathcal{W}_1 &:= \left\{ m \in T\mathcal{S}_0^R \;:\; \int_{D_R} m(x,y)\, dy \equiv 0 \right\}, \\
\mathcal{W}_2 &:= \left\{ m \in T\mathcal{S}_0^R \;:\; m(x,y) \;=\; \alpha(x)\, \vec{e}_{y_2} \text{ for some } \alpha \in H^2(\mathbb{R},\mathbb{R}) \right\}, \\
\mathcal{W}_3 &:= \left\{ m \in T\mathcal{S}_0^R \;:\; m(x,y) \;=\; \alpha(x)\, \vec{e}_s(x) \;\text{ for some } \alpha \in H^2(\mathbb{R},\mathbb{R}) \right\}.
\end{aligned}$$

Then $T\mathcal{S}_0^R$ is the direct sum of $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$, and we have $L_0^R(\mathcal{W}_i) \subset (\mathcal{W}_i)_{L^2}$ for $i \in \{1,2,3\}$.

Assume $m \in \mathcal{W}_1$. Using the Poincaré inequality (2.24) we have

$$\begin{aligned}
&\left\langle L_0^R m, m \right\rangle_\Sigma \\
&= 40R^2 \|\nabla_y m\|^2_{L^2(\Sigma)} + 2\|\partial_x m\|^2_{L^2(\Sigma)} + \|m_y\|^2_{L^2(\Sigma)} - 2\|(m_R^{\text{red}})_y m\|^2_{L^2(\Sigma)} \\
&\geq \frac{40}{16}\|m\|^2_{L^2(\Sigma)} - 2\|m\|^2_{L^2(\Sigma)} \;=\; \frac{1}{2}\|m\|^2_{L^2(\Sigma)}.
\end{aligned}$$

Assume $m \in \mathcal{W}_2$. Then $m(x,y) = \alpha(x)\mathbb{1}_{D_R}(y)\vec{e}_{y_2}$ for some $\alpha \in H^2(\mathbb{R},\mathbb{R})$, we have

$$L_0^R(m)|_{(x,y)} \;=\; \left(-2\,\partial_{xx}\alpha(x) + \alpha(x) - 2|m_y^{\text{red}}(x)|^2\alpha(x)\right) \mathbb{1}_{D_R}(y)\, \vec{e}_{y_2}. \tag{3.14}$$

$$\left\langle L_0^R(m), m \right\rangle_\Sigma \;=\; \pi R^2 \left(2\|\partial_x \alpha\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} (1 - 2|m_y^{\text{red}}|^2)\alpha^2 \right) \tag{3.15}$$

and

$$1 - 2|m_y^{\text{red}}(x)|^2 \;\geq\; \frac{1}{4} \qquad \text{for } |x| \geq 1.6. \tag{3.16}$$

Since $\Phi(m_R^{\text{red}}) \cdot \vec{e}_{y_2}$ is positive (Lemma 3.7), and since $\left\langle \Phi(m_R^{\text{red}}), m \right\rangle = 0$, the function α has to change sign.

First, assume that α changes sign in $[-1.6, 1.6]$. We have

$$\inf_{\{f:[-1.6,1.6]\to\mathbb{R}, f \text{ changes sign}\}} \left(\frac{2\|\partial_x f\|^2_{L^2([-1.6,1.6])}}{\|f\|^2_{L^2([-1.6,1.6])}} \right) \;=\; \frac{2\pi^2}{3.2^2},$$

the infimum is attained and the minimizers are multiples of $x \mapsto \sin\left(\frac{\pi}{3.2}x\right)$. Thus we have

$$\begin{aligned}
&2\|\partial_x \alpha\|^2_{L^2([-1.6,1.6])} + \int_{-1.6}^{1.6} (1 - 2|m_y^{\text{red}}|^2)\alpha^2 \\
&\geq\; 2\|\partial_x \alpha\|^2_{L^2([-1.6,1.6])} - \|\alpha\|^2_{L^2([-1.6,1.6])} \;\geq\; \left(\frac{2\pi^2}{3.2^2} - 1\right) \|\alpha\|^2_{L^2([-1.6,1.6])}.
\end{aligned}$$

and therefore, with (3.16) and (3.15)

$$\begin{aligned}\left\langle L_0^R(m), m\right\rangle_\Sigma &\geq \pi R^2 \left(\left(\frac{2\pi^2}{3.2^2} - 1 \right) \|\alpha\|^2_{[-1.6,1.6]} + \frac{1}{4}\|\alpha\|^2_{L^2([-1.6,1.6])} \right) \\ &\geq \frac{1}{4}\|m\|^2_\Sigma.\end{aligned}$$

Now assume that α does not change sign in $[-1.6, 1.6]$ and let S_- be the set where α has the opposite sign as in $[-1.6, 1.6]$. With Lemma 3.7 we see that $\Phi(m^{\mathrm{red}_R}(x, y)) \cdot \vec{e}_{y_2} \geq 0.5$ for $|x| < 1.6$, and since $\left\langle \Phi(m_R^{\mathrm{red}}) \cdot \vec{e}_{y_2}, \alpha\right\rangle_\mathbb{R} = 0$ we have

$$\begin{aligned}\sqrt{1.6}\|\alpha\|_{L^2[-1.6,1.6]} &\leq \left| \left\langle \Phi(m_R^{\mathrm{red}}) \cdot \vec{e}_{y_2}, |\alpha| \right\rangle_{[-1.6,1.6]} \right| \\ &\leq \left| \left\langle \Phi(m_R^{\mathrm{red}}) \cdot \vec{e}_{y_2}, |\alpha| \right\rangle_{S_-} \right| \\ &\leq \int_{S_-} 2e^{-\frac{|x|}{\sqrt{2}}}|\alpha| \;\leq\; \int_{S_-} \sqrt{8}e^{-\frac{|x|}{\sqrt{2}}}|\partial_x\alpha| \\ &\leq \left\| \sqrt{8}e^{-\frac{|x|}{\sqrt{2}}} \right\|_{L^2(\mathbb{R}\setminus[-1.6,1.6])} \|\partial_x\alpha\|_{L^2(\mathbb{R}\setminus[-1.6,1.6])} \\ &\leq 4\sqrt{\frac{1}{\sqrt{2}}e^{-1,6\sqrt{2}}}\|\partial_x\alpha\|_{L^2(\mathbb{R}\setminus[-1.6,1.6])} \\ &\leq 1.1\|\partial_x\alpha\|_{L^2(\mathbb{R}\setminus[-1.6,1.6])}.\end{aligned}$$

Thus (3.16) implies

$$\begin{aligned}&\left\langle L_0^R(m), m\right\rangle_\Sigma \\ &\geq \pi R^2 \left(\frac{1}{2}\|\alpha\|^2_{\mathbb{R}\setminus[-1.6,1.6]} + 2\|\partial_x\alpha\|^2_{L^2(\mathbb{R}\setminus[-1.6,1.6])} - \|\alpha\|^2_{L^2[-1.6,1.6]} \right) \\ &\geq \pi R^2 \left(\frac{1}{2}\|\alpha\|^2_{\mathbb{R}\setminus[-1.6,1.6]} + \left(\frac{2\sqrt{1.6}}{1.1} - 1 \right) \|\alpha\|^2_{L^2[-1.6,1.6]} \right) \\ &\geq \frac{1}{2}\|m\|^2_\Sigma.\end{aligned}$$

<u>Assume $m \in \mathcal{W}_3$.</u> Then $m(x, y) = \alpha(x)\mathbb{1}_{D_R}(y)\vec{e}_s(x)$ for some $\alpha \in H^2(\mathbb{R}, \mathbb{R})$. The function $L_0^R(m)$ is pointwise parallel to $\vec{e}_s$, we have $\partial_x\vec{e}_s \cdot \vec{e}_s = 0$ and

$$\begin{aligned}0 &= \partial_x(\partial_x\vec{e}_s \cdot \vec{e}_s) \;=\; |\partial_x\vec{e}_s|^2 + \partial_{xx}\vec{e}_s \cdot \vec{e}_s \\ &= |\partial_x m^{\mathrm{red}}|^2 + \partial_{xx}\vec{e}_s \cdot \vec{e}_s \;=\; \frac{1}{2}|m_y^{\mathrm{red}}|^2 + \partial_{xx}\vec{e}_s \cdot \vec{e}_s.\end{aligned}$$

So $\partial_{xx}(\alpha\vec{e}_s)\cdot\vec{e}_s = \partial_{xx}\alpha + \frac{1}{2}|m_y^{\text{red}}|^2\alpha$. Moreover, we have $\vec{e}_s\cdot\vec{e}_y = m_x^{\text{red}}$ and therefore

$$\begin{aligned} L_0^R(m)\cdot\vec{e}_s &= -2\partial_{xx}\alpha + |m_y^{\text{red}}|^2\alpha + |m_x^{\text{red}}|^2\alpha - 2|m_y^{\text{red}}|^2\alpha \\ &= -2\partial_{xx}\alpha + \left(1 - 2|m_y^{\text{red}}|^2\right)\alpha \end{aligned} \tag{3.17}$$

Comparing (3.17) and (3.14), we can conclude like in the case $m\in\mathcal{W}_2$ that $\left\langle L^R(m), m\right\rangle \geq \frac{1}{4}\|m\|^2_{L^2(\Sigma)}$. □

The next lemma compares the operators L_0^R and L^R on the space $H^2(\Sigma)$.

Lemma 3.10. *For each $\epsilon > 0$ there exists a radius $R_\epsilon > 0$ such that*

$$\left\langle L_0^R(m), m\right\rangle_\Sigma - \left\langle L^R(m), m\right\rangle_\Sigma \;\leq\; \epsilon\|m\|^2_{H^1(\Sigma)}$$

for all $R < R_\epsilon$ and all $m\in H^2(\Sigma)$.

Proof. For $\epsilon\in]0,1]$ we can find $\tilde{R}_\epsilon \leq \min\left(\frac{1}{\sqrt{20}},\epsilon\right)$ such that for all $R<\tilde{R}_\epsilon$ the following inequalities hold (Theorem 3.1):

$$\|m_R^{\text{red}} - m^R\|_{C^1(\Sigma)} \leq \epsilon, \quad \|m_R^{\text{red}} - m^R\|_{L^2(\Sigma)} \leq \epsilon R, \quad \|\nabla_y m^R\|_{L^\infty(\Sigma)} \leq \epsilon.$$

After reducing $\tilde{R}_\epsilon$ we can assume that $E_{\rho\rho}(m_R^{\text{red}}, R) < \epsilon^2R^2$ for all $R\leq\tilde{R}_\epsilon$ and that

$$\frac{1}{2}\|g_y\|^2_{L^2(\Sigma)} - E_{\sigma\sigma}(g,R) \;\leq\; \epsilon^2\|g_y\|^2_{H^1(\Sigma)}$$

for all $R\leq\tilde{R}_\epsilon$ and all $g\colon\Sigma\to\mathbb{R}^3$ that are constant on the cross section and satisfy $\|g_y\|_{L^2(\Sigma)}<\infty$ (Lemma 2.24). Then, for $R<\tilde{R}_\epsilon$ and $m\in T\mathcal{S}^R$ we have

$$\begin{aligned} &\left\langle L_0^R m, m\right\rangle_\Sigma - \left\langle L^R m, m\right\rangle_\Sigma \\ &\quad= \underbrace{\left\langle \delta_m E^0(m,R), m\right\rangle_\Sigma - \left\langle \delta_m E(m,R), m\right\rangle_\Sigma}_{A} \\ &\qquad \underbrace{-\left\langle |(m_R^{\text{red}})_y|^2, |m|^2\right\rangle_\Sigma + 2\left\langle H(m^R)\cdot m^R, |m|^2\right\rangle_\Sigma}_{B} \\ &\qquad + \underbrace{\int_\Sigma \left(-2|\partial_x m_R^{\text{red}}|^2 + 2|\partial_x m^R|^2 + 2|\nabla_y m^R|^2\right) m^2}_{C} \end{aligned}$$

We estimate the summands separately. As introduced in Subsection 2.1.4 we decompose m in $\overline{m}$ and $\tilde{m}$. Since $40R^2 \leq 2$ and since $\|f\|_{L^2(\Sigma)} \geq$

$\|H(f)\|_{L^2(\mathbb{R}^3)}$ for every $f \in L^2(\Sigma, \mathbb{R}^3)$, we get for the first summand

$$\begin{aligned}
A &= \|m_y\|^2_{L^2(\Sigma)} + \left(\frac{40}{R^2} - 2\right) \|\nabla_y m\|_{L^2(\Sigma)} - 2E_H(m, R) \\
&\leq \|m_y\|^2_{L^2(\Sigma)} - 2E_H(m, R) \\
&= \|\overline{m}_y\|^2_{L^2(\Sigma)} - 2E_H(\overline{m}, R) + \|\tilde{m}_y\|^2_{L^2(\Sigma)} - 2E_H(\tilde{m}, R) + 4\int_\Sigma H(m)\,\tilde{m} \\
&\leq \|\overline{m}_y\|^2_{L^2(\Sigma)} - 2E_{\sigma\sigma}(\overline{m}, R) + \|\tilde{m}_y\|^2_{L^2(\Sigma)} + 4\|\overline{m}\|_{L^2(\Sigma)}\|\tilde{m}\|_{L^2(\Sigma)} \\
&\leq 2\epsilon\|\overline{m}_y\|^2_{H^1(\Sigma)} + \|\tilde{m}_y\|^2_{L^2(\Sigma)} + 4\|m\|_{L^2(\Sigma)}\|\tilde{m}\|_{L^2(\Sigma)}.
\end{aligned}$$

Using the Poincaré inequality (2.24) and the assumption $R \leq \epsilon$ we obtain

$$\begin{aligned}
A &\leq \epsilon\|\overline{m}_y\|^2_{H^1(\Sigma)} + 16R^2\|\nabla\tilde{m}\|^2_{L^2(\Sigma)} + 16R\|\nabla\tilde{m}\|_{L^2(\Sigma)}\|\overline{m}\|_{L^2(\Sigma)} \\
&\leq 33\epsilon\|m\|^2_{H^1(\Sigma)}.
\end{aligned}$$

For the second summand we calculate

$$\begin{aligned}
B = &\underbrace{\int_\Sigma \Big((m_R^{\text{red}})_y - 2H(m_R^{\text{red}})\Big) \cdot m_R^{\text{red}}\,|m|^2}_{B_1} + \underbrace{2\int_\Sigma H(m_R^{\text{red}}) \cdot (m_R^{\text{red}} - m^R)\,|m|^2}_{B_2} \\
&+ \underbrace{2\int_\Sigma H(m_R^{\text{red}} - m^R) \cdot m^R\,|m|^2}_{B_3},
\end{aligned}$$

$$\begin{aligned}
|B_1| &\leq \|(m_R^{\text{red}})_y - 2H(m_R^{\text{red}})\|_{L^2(\Sigma)}\|m_R^{\text{red}}\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \\
&\leq \Big(\|2H((m_R^{\text{red}})_x\vec{e}_x)\|_{L^2(\Sigma)} + \|(m_R^{\text{red}})_y - 2H((m_R^{\text{red}})_y)\|_{L^2(\Sigma)}\Big)\|m\|^2_{L^4(\Sigma)} \\
&\leq 2\sqrt{E_{\rho\rho}(m_R^{\text{red}}, R)} + 2\sqrt{\frac{1}{2}\|(m_R^{\text{red}})_y\|^2_{L^2(\Sigma)} - E_{\sigma\sigma}(m_R^{\text{red}}, R)}\;\|m\|^2_{L^4(\Sigma)} \\
&\leq 2\sqrt{\epsilon^2 R^2} + 2\sqrt{\epsilon^2\pi R^2\|m_y^{\text{red}}\|^2_{H^1(\mathbb{R})}}\;\|m\|^2_{L^4(\Sigma)} \leq 6\epsilon R\|m\|^2_{L^4(\Sigma)}, \\
|B_2| &\leq 2\|H(m_R^{\text{red}})\|_{L^2(\Sigma)}\|m_R^{\text{red}} - m^R\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \\
&\overset{(*)}{=} 2\sqrt{E_{\sigma\sigma}(m, R) + E_{\rho\rho}(m_R^{\text{red}}, R)}\;\|m_R^{\text{red}} - m^R\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \\
&\leq 2\sqrt{\frac{\pi}{2}R^2\|(m_R^{\text{red}})_y\|^2_{L^2(\mathbb{R})} + E_{\rho\rho}(m_R^{\text{red}}, R)}\;\epsilon\|m\|^2_{L^4(\Sigma)} \\
&\leq 2\sqrt{\sqrt{2}\pi R^2 + \epsilon^2 R^2}\;\epsilon\|m\|^2_{L^4(\Sigma)} \leq 6\epsilon R\|m\|^2_{L^4(\Sigma)}, \\
|B_3| &\leq 2\|m_R^{\text{red}} - m^R\|_{L^2(\Sigma)}\|m^R\|_{L^\infty(\Sigma)}\|m\|^2_{L^4(\Sigma)} \leq 2R\epsilon\|m\|^2_{L^4(\Sigma)}.
\end{aligned}$$

For $(*)$ we have used Lemma 2.10.

Because of the Sobolev embedding $H^1(\Sigma(1)) \hookrightarrow L^4(\Sigma(1))$ there exists a constant C_{Sobolev} such that

$$\|u\|_{L^4(\Sigma(1))} \;\leq\; C_{\text{Sobolev}} \|u\|_{H^1(\Sigma(1))} \quad \text{for all } u\colon \Sigma(1) \to \mathbb{R}^n.$$

Rescaling implies for all $R \leq 1$

$$\|u\|_{L^4(\Sigma(R))} \;\leq\; \frac{1}{\sqrt{R}} C_{\text{Sobolev}} \|u\|_{H^1(\Sigma(R))} \quad \text{for all } u\colon \Sigma(R) \to \mathbb{R}^n.$$

Thus,

$$|B| \;\leq\; 14 C_{\text{Sobolev}}^2 \epsilon \|m\|_{H^1(\Sigma)}^2.$$

Since $\partial_x m^{\text{red}} = \frac{1}{\sqrt{2}} |m_y^{\text{red}}| \leq \frac{1}{\sqrt{2}}$ (Lemma 3.7) the third summand C can be estimated by

$$\begin{aligned} C &\leq 2\|\partial_x m_R^{\text{red}} - \partial_x m^R\|_{L^\infty(\Sigma)} \|\partial_x m_R^{\text{red}} + \partial_x m^R\|_{L^\infty(\Sigma)} \|m\|_{L^2(\Sigma)}^2 + 2\epsilon \|m\|_{L^2(\Sigma)}^2 \\ &\leq 2\epsilon \left(\frac{2}{\sqrt{2}} + \epsilon \right) \|m\|_{L^2(\Sigma)}^2 + 2\epsilon \|m\|_{L^2(\Sigma)}^2 \;\leq\; 7\epsilon \|m\|_{L^2(\Sigma)}, \end{aligned}$$

and therefore we have for all $R \leq \tilde{R}_\epsilon$

$$\left\langle L_0^R m, m \right\rangle_\Sigma - \left\langle L^R m, m \right\rangle_\Sigma \;\leq\; (40 + 14 C_{\text{Sobolev}}^2) \epsilon \|m\|_{H^1(\Sigma)}^2.$$

□

Using Lemma 3.10, we transfer the result of Lemma 3.8 to the operator L^R.

Lemma 3.11. *For each $0 < \epsilon < \frac{1}{4}$ there exists R_ϵ such that*

$$\left\langle L^R(m), m \right\rangle_\Sigma \;\geq\; \left(\frac{1}{4} - \epsilon \right) \|m\|_{L^2(\Sigma)}^2$$

for all $R < R_\epsilon$ and all $m \in T\mathcal{S}^R$.

Proof. Let $P_0\colon H^2(\Sigma) \to T\mathcal{S}_0^R$ be the L^2-orthogonal projection. Since

$$m_R^{\text{red}} \perp \partial_x m_R^{\text{red}}, \quad m_R^{\text{red}} \perp \Phi(m_R^{\text{red}}), \quad \left\langle \partial_x m_R^{\text{red}}, \Phi(m_R^{\text{red}}) \right\rangle_\Sigma = 0,$$

we have for all $m \in T\mathcal{S}^R$

$$\begin{aligned} &P_0(m) \\ &= m - (m \cdot (m_R^{\text{red}} - m^R)) m_R^{\text{red}} + \left\langle m, \partial_x m^R - \partial_x m_R^{\text{red}} \right\rangle_\Sigma \frac{\partial_x m_R^{\text{red}}}{\|\partial_x m_R^{\text{red}}\|_{L^2(\Sigma)}^2} \\ &\quad + \left\langle m, \Phi(m^R) - \Phi(m_R^{\text{red}}) \right\rangle_\Sigma \frac{\Phi(m_R^{\text{red}})}{\|\Phi(m_R^{\text{red}})\|_{L^2(\Sigma)}^2}, \end{aligned}$$

that is,

$$\begin{aligned}
\|m\|_{L^2(\Sigma)} - \|P_0(m)\|_{L^2(\Sigma)} \\
&\leq \|m\|_{L^2(\Sigma)} \|m^R - m_R^{\mathrm{red}}\|_{L^\infty(\Sigma)} + \|m\|_{L^2(\Sigma)} \frac{\|\partial_x m^R - \partial_x m_R^{\mathrm{red}}\|_{L^2(\Sigma)}}{\|\partial_x m_R^{\mathrm{red}}\|_{L^2(\Sigma)}} \\
&\quad + \|m\|_{L^2(\Sigma)} \frac{\|\Phi(m^R) - \Phi(m_R^{\mathrm{red}})\|_{L^2(\Sigma)}}{\|\Phi(m_R^{\mathrm{red}})\|_{L^2(\Sigma)}}
\end{aligned}$$

Thus, with Theorem 3.1, we can find $\tilde{R}_\epsilon$ such that

$$\|m\|_{L^2(\Sigma)} - \|P_0(m)\|_{L^2(\Sigma)} \ \leq\ \epsilon \|m\|_{L^2(\Sigma)} \qquad \text{for all } R \leq \tilde{R}_\epsilon.\ m \in T\mathcal{S}^R.$$

After reducing R_ϵ we can assume that Lemma 3.10 implies

$$\left\langle L^R m, m \right\rangle_\Sigma \ \geq\ \left\langle L_0^R m, m \right\rangle_\Sigma - \epsilon \|m\|_{H^1(\Sigma)}^2 \qquad \text{for all } R \leq \tilde{R}_\epsilon,\ m \in T\mathcal{S}^R.$$

Then we have

$$\begin{aligned}
\left\langle L^R m, m \right\rangle_\Sigma &\geq (1-\epsilon) \left\langle L_0^R m, m \right\rangle_\Sigma + \epsilon \|\nabla m\|_{L^2(\Sigma)}^2 \\
&\quad - \epsilon \bigg(\int_\Sigma \underbrace{\left(2|\nabla m_R^{\mathrm{red}}|^2 + |(m_R^{\mathrm{red}})_y|^2\right)}_{\leq 2} m^2 \bigg) - \epsilon \|m\|_{H^1(\Sigma)}^2 \\
&\geq (1-\epsilon) \left\langle L_0^R m, m \right\rangle_\Sigma - 3\epsilon \|m\|_{L^2(\Sigma)}^2.
\end{aligned}$$

Since the operator L_0^R is the second variation of the energy E^0 and since m_0^R is a minimiser of the energy, the operator L_0^R is positive semidefinite. Moreover, it is symmetric on the set $\{m \in H^2(\Sigma, \mathbb{R}^3) : \partial_\nu m|_{\partial\Sigma} = 0\}$, so $L_0^R(T\mathcal{S}_0^R) \subset (T\mathcal{S}_0^R)_{L^2}$ implies

$$\begin{aligned}
\left\langle L_0^R m, m \right\rangle_\Sigma &= \left\langle L_0^R(P_0(m)), P_0(m) \right\rangle_\Sigma + \left\langle L_0^R(m - P_0(m)), m - P_0(m) \right\rangle_\Sigma \\
&\geq \left\langle L_0^R(P_0(m)), P_0(m) \right\rangle_\Sigma \ \geq\ \frac{1}{4} \|P_0(m)\|_{L^2(\Sigma)}^2 \\
&\geq \frac{1-\epsilon}{4} \|m\|_{L^2(\Sigma)}^2.
\end{aligned}$$

Thus, $\left\langle L^R m, m \right\rangle_\Sigma \geq \frac{1}{4}\|m\|_{L^2(\Sigma)}^2 - 4\epsilon \|m\|_{L^2(\Sigma)}^2$. □

Lemma 3.12. *There exists $\lambda, C > 0$, $\tilde{R} > 0$ such that*

$$\begin{aligned}
\|L^R(g) + \lambda g\|_{L^2(\Sigma)} &\geq \|g\|_{H^2(\Sigma)} \\
\|L^R(g)\|_{L^2(\Sigma)} &\leq C \|g\|_{H^2(\Sigma)}
\end{aligned}$$

for all $R \leq \tilde{R}$ and all $g \in T\mathcal{S}^R$.

Proof. We split the operator in two parts and set

$$\begin{aligned} L_H^R(g) &:= 2\left(H(g) - (H(g)\cdot m^R)m^R - (H(m^R)\cdot g)m^R - (H(m^R)\cdot m^R)g\right), \\ L_\Delta^R(g) &:= 2\left(-\Delta g + (\Delta g\cdot m^R)m^R + (\Delta m^R\cdot g)m^R + (\Delta m^R\cdot m^R)g\right). \end{aligned}$$

First, we show $H(m^R) \in L^\infty(\Sigma)$ using a result by Carbou and Fabrie [9] for bounded domains. Let $\eta : \Sigma \to [0,1]$ be a smooth function with

$$\eta(p) = 1 \ \text{ for } p \in [-1,1]\times D_R, \qquad \eta(p) = 0 \ \text{ for } p' \in \Sigma \setminus ([-2,2]\times D_R),$$

and set $\eta_x : (x', y') \mapsto \eta(x'-x, y)$.
Then [9, Lemma 2.3] and the Sobolev embedding $W^{1,4}(\Sigma) \hookrightarrow L^\infty(\Sigma)$ imply that there exist constants C, $\tilde{C}$ independent of x such that

$$\begin{aligned} \|H(m\cdot\eta_x)\|_{L^\infty(\Sigma)} &\le \tilde{C}\|H(m\cdot\eta_x)\|_{W^{1,4}(\Sigma)} \\ &\le \|m\cdot\eta_x\|_{W^{1,4}(\Sigma)} \le (2\pi)^{\frac{1}{4}}\|m\|_{C^1(\Sigma)} \end{aligned} \tag{3.18}$$

Moreover, using the representation of H in terms of K_i of Lemma 2.6 we obtain for all $p = (x,y) \in \Sigma$ the estimate

$$\begin{aligned} &H(m\cdot(1-\eta_x))(p) \\ &\quad\le 3\left(\|K_i\|_{L^1(\Sigma\setminus([-1,1]\times D_R))} + \|K_i\|_{L^1(\partial\Sigma\setminus([-1,1]\times\partial D_R))}\right)\|m\|_{C^1(\Sigma)}. \end{aligned} \tag{3.19}$$

The combination of (3.18) and (3.19) implies that $\|H(m^R)\|_{L^\infty(\Sigma)}$ is finite. Since $\|H(g)\|_{L^2(\Sigma)} \le \|g\|_{L^2(\Sigma)}$ and since $\|m^R\|_{L^\infty(\Sigma)} = 1$, we have for all $g \in (T\mathcal{S})_{L^2}$

$$\|L_H^R(g)\|_{L^2(\Sigma)} \le \left(4 + 4\|H(m^R)\|_{L^\infty(\Sigma)}\right)\|g\|_{L^2(\Sigma)}. \tag{3.20}$$

Thus the operator $L_H^R : (T\mathcal{S})_{L^2} \to (T\mathcal{S})_{L^2}$ is continuous.
Since $\partial_i m^R \perp m^R$ ($i \in \{1,2,3\}$) and since $g \perp m^R$ for all $g \in T\mathcal{S}^R$, we have

$$\begin{aligned} 0 &= \Delta(m^R\cdot g) = \Delta m^R\cdot g + 2\nabla m^R\cdot\nabla g + m^R\cdot\Delta g, \\ 0 &= \operatorname{div}(\nabla m^R\cdot m^R) = \Delta m^R\cdot m^R + |\nabla m^R|^2, \end{aligned}$$

and therefore

$$L_\Delta^R(g) = -2\Delta g - 4(\nabla m^R\cdot\nabla g)m^R - 2|\nabla m^R|^2 g. \tag{3.21}$$

Moreover, we have the estimate

$$\begin{aligned} \|-2\Delta g + \lambda g\|_{L^2(\Sigma)}^2 &= \int_\Sigma \left(4|\Delta g|^2 - 4\Delta g\cdot\lambda g + \lambda^2 g^2\right) \\ &= \int_\Sigma \left(4|D^2 g|^2 + 4\lambda|\nabla g|^2 + \lambda^2|g|^2\right) \\ &\ge \frac{1}{3}\left(2\|D^2 g\|_{L^2(\Sigma)} + 2\sqrt{\lambda}\|\nabla g\|_{L^2(\Sigma)} + \lambda\|g\|_{L^2(\Sigma)}\right)^2. \end{aligned}$$

This yields

$$
\begin{aligned}
&\|L^R_\Delta(g) + \lambda g\|_{L^2(\Sigma)} \\
&\geq \|-2\Delta g + \lambda g\|_{L^2(\Sigma)} - 4\|m^R\|_{C^1}\|\nabla g\|_{L^2(\Sigma)} - 2\|m^R\|^2_{C^1(\Sigma)}\|g\|_{L^2(\Sigma)} \\
&\geq \frac{2}{\sqrt{3}}\|D^2 g\|_{L^2(\Sigma)} + \left(\frac{2\sqrt{\lambda}}{\sqrt{3}} - 4\|m^R\|_{C^1(\Sigma)}\right)\|\nabla g\|_{L^2(\Sigma)} \\
&\quad + \left(\frac{\lambda}{\sqrt{3}} - 2\|m^R\|^2_{C^1(\Sigma)}\right)\|g\|_{L^2(\Sigma)},
\end{aligned}
$$

and we can choose λ such that

$$\|L^R_\Delta(g) + \lambda g\|_{L^2(\Sigma)} \geq \|g\|_{H^2(\Sigma)}. \tag{3.22}$$

Combining (3.20) and (3.21), we obtain the second estimate.

□

Using the above estimates, we show that the operator L^R is bijective and has an continuous inverse.

Lemma 3.13. *There exists* $\tilde{R} > 0$ *such that for all* $R < \tilde{R}$
(i) the operator $L^R : T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ *is injective,*
(ii) $L^R(T\mathcal{S}^R)$ *is dense in* $(T\mathcal{S}^R)_{L^2}$*,*
(iii) $L^R(T\mathcal{S}^R)$ *is closed in* $(T\mathcal{S}^R)_{L^2}$*,*
(iv) the operator $L^R : T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ *is bijective,*
(v) the operator $(L^R)^{-1} : (T\mathcal{S}^R)_{L^2} \to T\mathcal{S}^R$ *is bounded.*

Proof. Let $\tilde{R}$ be so small and λ, C so large that for all $R \leq \tilde{R}$ and all $g \in T\mathcal{S}^R$ Lemma 3.11 and Lemma 3.12 imply

$$
\begin{aligned}
\langle L^R(m), m\rangle_\Sigma &\geq \frac{1}{8}\|m\|^2_{L^2(\Sigma)}, && (3.23)\\
\|L^R(g) + \lambda g\|_{L^2(\Sigma)} &\geq \|g\|_{H^2(\Sigma)}, && (3.24)\\
\|L^R(g)\|_{L^2(\Sigma)} &\leq C\|g\|_{H^2(\Sigma)}. && (3.25)
\end{aligned}
$$

(i) This is a direct implication of (3.23).

(ii) We show the statement by contradiction and assume that $L^R(T\mathcal{S}^R)$ is not dense in $(T\mathcal{S}^R)_{L^2}$. Then there exists $v \in (T\mathcal{S}^R)_{L^2}$ that is orthogonal on $L^R(T\mathcal{S}^R)$, and there exists w in $T\mathcal{S}^R$ such that

$$\langle w, g\rangle_\Sigma < \frac{1}{10C}\|w\|_{L^2(\Sigma)}\|g\|_{L^2(\Sigma)} \qquad \text{for all } g \in L^R(T\mathcal{S}^R),\ g \not\equiv 0.$$

Thus with (3.25) we have

$$\left\langle L^R(w), w\right\rangle_\Sigma \;<\; \frac{1}{10C}\|L^R(w)\|_{L^2(\Sigma)}\|w\|_{L^2(\Sigma)} \;\leq\; \frac{1}{10}\|w\|^2_{L^2(\Sigma)}.$$

This is in contradiction to (3.23).

(iii) Let $(L^R(g_n))_{n\in\mathbb{N}}$, $g_n \in T\mathcal{S}^R$ be a sequence that converges to some $h_0 \in (T\mathcal{S}^R)_{L^2}$ in $L^2(\Sigma, \mathbb{R}^3)$. We have to show $h_0 \in L^R(T\mathcal{S}^R)$.

Because of (3.23), the sequence $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Sigma, \mathbb{R}^3)$, so the sequence $(L^R(g_n) + \lambda g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Sigma, \mathbb{R}^3)$, too. Thus the estimate (3.24) implies that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Sigma, \mathbb{R}^3)$ converging to some $g_0 \in T\mathcal{S}^R$. Using (3.25), we obtain

$$L^R(g_0) \;=\; \lim_{n\to\infty} L^R(g_n) \;=\; h_0.$$

(iv) Since $L^R(T\mathcal{S}^R)$ is dense and closed in $(T\mathcal{S}^R)_{L^2}$, we have $L^R(T\mathcal{S}^R) = (T\mathcal{S}^R)_{L^2}$. Thus, with (i), $L^R : T\mathcal{S}^R \to (T\mathcal{S}^R)_{L^2}$ is bijective.

(v) The arguments for (iii) imply: If $L^R(g_0) = \lim_{n\to\infty} L^R(g_n)$ in $L^2(\Sigma)$ then g_n converges to g_0 in $H^2(\Sigma)$. □

Theorem 3.6 summarises the statements of Lemma 3.13.

3.4 Introduction to regularity

In this section we give a short introduction to the ideas and definitions that we are using to prove regularity results. First we extend functions defined on Σ to a larger domain and discuss properties of this extension. Then we show how scaled L^2-estimates can be used to prove regularity results.

In the rest of this chapter we will somtimes denote $\vec{e}_x$, $\vec{e}_{y_1}$, $\vec{e}_{y_2}$ by $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ and the derivatives ∂_x, ∂_{y_1}, ∂_{y_2} by ∂_1, ∂_2, ∂_3. We will do this in order to write sums like $\sum_i \partial_i f \vec{e}_i$ in a compact way. Moreover, Df denotes the derivative of a function f, and $D^n f$ denotes the n^{th} derivative of f.

We will frequently use the following well known estimates.

Lemma 3.14. *(i) For $u \in H^1(B_\rho \to R^n)$, $n \in \mathbb{N}$ we have*

$$\left\|u - \langle u\rangle_{B_\rho}\right\|_{L^2(B_\rho)} \;\leq\; 8\rho\|\nabla u\|_{L^2(B_\rho)}. \tag{3.26}$$

(ii) For $u \in H^1(B_\rho \to R^n)$, $n \in \mathbb{N}$ and $v \in R^n$ we have

$$\int_{B_\rho} \left|u - \langle u\rangle_{B_\rho}\right|^2 \;\leq\; \int_{B_\rho} |u - v|^2 . \tag{3.27}$$

Proof. (i) This is the Poincaré inequality. See [18, p. 164].

(ii) We calculate

$$\begin{aligned}
&\int_{B_\rho} |u-v|^2 \\
&= \int_{B_\rho} \left(\left| u - \langle u \rangle_\rho \right|^2 + \left| \langle u \rangle_\rho - v \right|^2 \right) + 2 \int \left(u - \langle u \rangle_\rho \right) \left(\langle u \rangle_\rho - v \right) \\
&= \int_{B_\rho} \left(\left| u - \langle u \rangle_\rho \right|^2 + \left| \langle u \rangle_\rho - v \right|^2 \right).
\end{aligned}$$

□

3.4.1 Extending functions

We need uniform bounds on m^R not only in the interior of $\Sigma(R)$ but up to the boundary. There are two strategies to get such estimates. Either we argue first in the interior and then locally at the boundary, or we extend the functions globally to a function on a larger domain and work in the interior of the large domain. The first strategy is more flexible since it works in arbitrary sufficiently regular domains. It is applied by Hardt and Kinderlehrer [19]. However, since we are considering only cylinders, it is simpler to extend functions $f\colon \Sigma(R) \to \mathbb{R}$ by reflection on the boundary to functions $f^*\colon \Sigma(\frac{3}{2}R) \to \mathbb{R}$, even if this means that we have to generalise the notion of almost-minimisers (see Definition 3.15 and the remark following it).

For $x, y \in \Sigma(\frac{3}{2}R)$, $\Omega \subset \Sigma(\frac{3}{2}R)$ we set

$$(x,y)^* := \begin{cases} (x,y) & \text{if } |y| \le R \\ \left(x, (2R - |y|)\frac{y}{|y|}\right) & \text{otherwise,} \end{cases} \qquad \Omega^* := \{p^* \mid p \in \Omega\}, \quad (3.28)$$

and for $f\colon \Sigma \to \mathbb{R}^n$ we define

$$f^*\colon \Sigma(\tfrac{3}{2}R) \to \mathbb{R}^n, \quad p \mapsto f(p^*).$$

If f is continuous, f^* is continuous and $f \in W^{1,q}(\Sigma)$ implies $f^* \in W^{1,q}(\frac{3}{2}\Sigma)$ for all $1 \le p \le \infty$. Moreover, if $\partial_\nu f = 0$ on $\partial\Sigma$ and $f \in W^{k,q}(\Sigma)$ for $k \in \{2,3\}, 1 \le p \le \infty$, then $f^* \in W^{k,q}(\frac{3}{2}\Sigma)$.

We now discuss the relation between derivatives and integrals of f and derivatives and integrals of f^*. Setting

$$\kappa\colon \Sigma(\tfrac{3}{2}R) \to \mathbb{R}, \qquad (x,y) \mapsto \frac{|y|}{|y^*|},$$

we have for $\Omega \subset \Sigma(\frac{3}{2}R) \setminus \Sigma(R)$ the relation

$$\int_\Omega \frac{1}{\kappa} f^* = \int_{\Omega^*} f. \tag{3.29}$$

For the cylindrical coordinate system (x, r, ϕ), let $\vec{e}_x$, $\vec{e}_r = \vec{e}_r(p)$, $\vec{e}_\phi = \vec{e}_\phi(p)$ be the canonical unit vectors. We define the matrix-valued function $A^R \colon \Sigma(\frac{3}{2}R) \setminus (\mathbb{R} \times \{0\}) \to \mathbb{R}^{3\times 3}$, setting

$$A^R(p)\vec{e}_x = \vec{e}_x, \quad A^R(p)\vec{e}_r = \vec{e}_r, \quad A^R(p)\vec{e}_\phi = \kappa\vec{e}_\phi.$$

Then A^R is symmetric, and in the cartesian coordinate system we have

$$A^R(p)\, \nabla f^*(p) \;=\; \nabla f(p^*),$$

We derive an elliptic equation in divergence form for f^*. If f^* is regular enough, using (3.4.1) for every $\phi \in C_c^\infty(\Sigma(R))$ we have

$$\begin{aligned} \int\limits_{\Sigma(\frac{3}{2}R)\setminus\Sigma(R)} \operatorname{div}\left(\frac{1}{\kappa}\left(A^R\right)^2 \nabla f^*\right) \phi^* &= - \int\limits_{\Sigma(\frac{3}{2}R)\setminus\Sigma(R)} \frac{1}{\kappa}\left(A^R \nabla f^*\right)\cdot\left(A^R\nabla\phi^*\right) \\ &= -\int_{\Sigma(R)} \nabla f \cdot \nabla \phi \;=\; \int_{\Sigma(R)} \Delta f\, \phi, \end{aligned}$$

so (3.29) yields for $p \in \Sigma(\frac{3}{2}R) \setminus \overline{\Sigma}(R)$ the equation

$$\operatorname{div}\left(\frac{1}{\kappa}\left(A^R(p)\right)^2 \nabla f^*(p)\right) \;=\; \frac{1}{\kappa(p)}\Delta f(p^*). \tag{3.30}$$

In order to keep the notation simple we will write $m^R \colon \Sigma(\frac{3}{2}R) \to \mathbb{S}^2$ for the continuation of m^R whose correct name would be $(m^R)^*$.

The highest order term of the energy E with respect to the derivatives is the exchange energy $\int_\Sigma |\nabla m|^2$. The stray field energy can be considered as a lower order perturbation. We will use this fact to show $C^{0,\alpha}$ regularity using the notion of almost-minimisers.

Definition 3.15. Let $\Omega \subset \mathbb{R}^3$, $c \geq 0$, $0 < \alpha < 1$ and $A \colon \Omega \to \mathbb{R}^{3\times 3}$. Moreover, assume that $A(p)$ is symmetric for all $p \in \Omega$ and that there exist positive constants $\underline{\lambda}$, $\overline{\lambda}$ with $|\underline{\lambda} f|^2 \leq |Af|^2 \leq |\overline{\lambda} f|^2$ for all $f \colon \Omega \to \mathbb{R}^3$.

A function $m \colon \Omega \to \mathbb{S}^2$ is called (c, α)-*almost-minimiser for* $|A\nabla \cdot|^2$ if for every $\Omega' \subset \Omega$ with $|\Omega'| \leq 1$ and every map $g \colon \Omega \to \mathbb{S}^2$ with $g|_{\Omega\setminus\Omega'} = m|_{\Omega\setminus\Omega'}$ we have

$$\int_{\Omega'} |A\nabla m|^2 \;\leq\; \left(\int_{\Omega'} |A\nabla g|^2\right) + c|\Omega'|^{\frac{1+\alpha}{3}}.$$

Remark. Hardt and Kinderlehrer [19] use the notion of (c,α)-almost-minimisers only for (c,α)-almost-minimisers of $|\nabla\cdot|^2$.

Theorem 3.16. *(i) The function $m^R\colon \Sigma(\frac{3}{2}R)\to\mathbb{S}^2$ is a (c,α)-almost-minimiser for $|\frac{1}{\sqrt{\kappa}}A^R\nabla\cdot|^2$ with $c=4\left(1+\sqrt{E(m^R)}\right)$ and $\alpha=\frac{1}{2}$.*

(ii) Let u be a weak solution of $\Delta u=\operatorname{div}(m^R\mathbb{1}_{\Sigma(R)})$ in $\mathbb{R}^3$ and set

$$\zeta\colon \Sigma\left(\tfrac{3}{2}R\right)\to\mathbb{R}^3,\qquad p\mapsto\frac{1}{\kappa(p)}\left(\nabla u(p^*)-\left(\nabla u(p^*)\cdot m^R(p)\right)m^R(p)\right).$$

Then m^R is a weak solution of

$$-\operatorname{div}\left(\frac{1}{\kappa}\left(A^R\right)^2\nabla m^R\right)\;=\;\frac{1}{\kappa}\,|A^R\nabla m^R|^2m^R-\zeta\quad\text{in }\Sigma(\tfrac{3}{2}R).\tag{3.31}$$

Proof. (i) First assume $\Omega\subset\Sigma$, $|\Omega|\le 1$ and let $v\colon\Sigma\to\mathbb{S}^2$ be a map with $v|_{\Sigma\setminus\Omega}=m|_{\Sigma\setminus\Omega}$. Then $E(m^R)\le E(v)$ and with (2.14) we have

$$\begin{aligned}\int_\Sigma|\nabla m^R|^2-\int_\Sigma|\nabla v|^2\;&\le\;E_H(v)-E_H(m^R)\\ &\le\;\|m^R-v\|^2_{L^2(\Omega)}+2\|m^R-v\|_{L^2(\Omega)}\sqrt{E(m^R)}\\ &\le\;4|\Omega|+4\sqrt{E(m^R)}\sqrt{|\Omega|}\\ &\le\;4\left(1+\sqrt{E(m^R)}\right)|\Omega|^{\frac{1+0.5}{3}}.\end{aligned}$$

Now consider arbitrary $\Omega\subset\Sigma(\frac{3}{2}R)$ with $|\Omega|\le 1$. We set

$$v_*\colon\Sigma(R)\setminus\Sigma\left(\tfrac{1}{2}R\right)\to\mathbb{S}^2,\quad (x,y)\mapsto v\left(x,\tfrac{y}{|y|}(2R-|y|)\right).$$

Then, with (3.29) and (3.4.1) we have

$$\int_{\Omega\setminus\Sigma}\frac{1}{\kappa}|A^R\nabla v|^2\;=\;\int_{(\Omega\setminus\Sigma)^*}|\nabla v_*|^2,$$

and therefore

$$\begin{aligned}&\int_\Omega\left|\frac{1}{\sqrt{\kappa}}A^R\nabla m^R\right|^2-\left|\frac{1}{\sqrt{\kappa}}A^R\nabla v\right|^2\\ &=\;\int_{\Omega\cap\Sigma}|\nabla m^R|^2-|\nabla v|^2+\int_{(\Omega\setminus\Sigma)^*}|\nabla m^R|^2-|\nabla v_*|^2\\ &\le\;4\left(1+\sqrt{E(m^R)}\right)\left(|\Omega\cap\Sigma|^{\frac{1+0.5}{3}}+|(\Omega\setminus\Sigma)^*|^{\frac{1+0.5}{3}}\right)\\ &\le\;4\left(1+\sqrt{E(m^R)}\right)\;|\Omega|^{\frac{1+0.5}{3}}.\end{aligned}$$

(ii) The function $m^R\colon \Sigma(R) \to \mathbb{S}^2$ is a local minimiser of E with the constraint $|m| \equiv 1$, so it satisfies the Euler-Lagrange equation

$$\begin{aligned} 0 &= \delta_m E(m^R) - (\delta_m E(m^R) \cdot m^R) m^R \quad \text{in } \Sigma(R), \\ \text{where } \delta_m E(m^R) &= -2\Delta m^R + 2\nabla u. \end{aligned} \tag{3.32}$$

Since $|m^R| = 1$, we have $\partial_i m^R \perp m^R$ for all $i \in \{1,2,3\}$, and therefore

$$0 = \sum_i \partial_i(\partial_i m^R \cdot m^R) = \Delta m^R \cdot m^R + |\nabla m^R|^2, \quad \text{i.e., } -\Delta m^R \cdot m^R = |\nabla m^R|^2.$$

With this identity (3.32) becomes

$$0 = -\Delta m^R + \nabla u - |\nabla m^R|^2 m^R - (\nabla u \cdot m^R) m^R \quad \text{in } \Sigma(R). \tag{3.33}$$

Using (3.4.1) and (3.30), we have for all $p \in \Sigma(\frac{3}{2}R)$ the relation

$$\begin{aligned} &-\operatorname{div}\left(\frac{1}{\kappa(p)} \left(A^R(p)\right)^2 \nabla m^R(p)\right) = -\frac{1}{\kappa(p)} \Delta m(p^*) \\ &= \frac{1}{\kappa(p)} \left(-\nabla u(p^*) + |\nabla m^R(p^*)|^2 m^R(p^*) + (\nabla u(p^*) \cdot m^R(p^*)) m^R(p^*)\right) \\ &= \frac{1}{\kappa(p)} \, |A^R(p) \nabla m^R(p)|^2 m^R(p) - \zeta(p). \end{aligned}$$

Remembering that $\nabla m^R \in L^2(\Sigma(\frac{3}{2}R))$, we have for every test function $\eta \in C_c^\infty(\Sigma(\frac{3}{2}))$ the equality

$$\int_{\Sigma(\frac{3}{2}R)} \frac{1}{\kappa} \left(\left(A^R\right)^2 \nabla m^R\right) \cdot \nabla \eta = \int_{\Sigma(\frac{3}{2}R)} \left(\frac{1}{\kappa} \, |A^R \nabla m^R|^2 m^R - \zeta\right) \eta,$$

that is, $m^R : \Sigma \to \mathbb{S}^2$ is a weak solution of (3.31). □

3.4.2 Preliminary lemmas regarding scaled L^2-estimates

All considerations regarding regularity are in the spirit of Morrey-Campanato theory. The idea is to replace L^p- and $C^{k,\alpha}$-estimates by scaled L^2-estimates. For example, there exists an integral characterisation of Hölder continuous functions [17, Thm 1.2, p. 70]. The bound on the $C^{0,\alpha}$-norm does depend on the domain of the function, but by a simple rescaling argument we get a version of the estimate, that still depends on the shape of the domain but not on the size.

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with piecewise smooth boundary. Let $0 < \eta \leq 1$ and let $f\colon \eta\Omega \to \mathbb{R}^n$, $n \in \mathbb{N}$ such that*

$$\int_{B_r(a) \cap \eta\Omega} |f - \langle f \rangle_{B_r(a) \cap \eta\Omega}|^2 \leq C_f r^{3+\alpha}$$

for all B_r with $r \le \eta \operatorname{diam}(\Omega)$. Then we have for all $p, p' \in \eta\Omega$ the relation

$$|f(p) - f(p')| \;\le\; C_{\mathrm{Camp}} C_f |p - p'|^{\frac{\alpha}{2}},$$

where $C_{\mathrm{Camp}} = C_{\mathrm{Camp}}(\Omega, \alpha)$.

To get the integral estimates needed in Theorem 3.17 we often compare the integral over an arbitrary ball $B_\rho(a)$ with the integral over the ball $B_{\eta\rho}(a)$ that is by a fixed factor η smaller. We have the following Lemma.

Lemma 3.18. *Let $f\colon B_{\rho_0} \to \mathbb{R}$ be a map, let $\gamma > 0$ and assume that there exists $\eta \in]0,1[$ such that for all $\rho \le \rho_0$*

$$\frac{1}{(\eta\rho)^\gamma} \int_{B_{\eta^n\rho}} |f| \;\le\; \frac{1}{2\rho^\gamma} \left(\int_{B_\rho} |f| \right) + C_0. \tag{3.34}$$

Then we have for all $r \le \rho \le \rho_0$ the relation

$$\frac{1}{r^\gamma} \int_{B_r} |f| \;\le\; \frac{1}{(\eta\rho)^\gamma} \left(\int_{B_\rho} |f| \right) + \frac{2C_0}{\eta^\gamma}.$$

Proof. Induction yields for all $n \in \mathbb{N}$, $0 < \rho < \rho_0$ the estimate

$$\frac{1}{(\eta^n\rho)^\gamma} \int_{B_{\eta\rho}} |f| \;\le\; \frac{1}{2^n\rho^\gamma} \left(\int_{B_\rho} |f| \right) + C_0 \sum_{i=0}^{n-1} \frac{1}{2^i} \;\le\; \frac{1}{\rho^\gamma} \left(\int_{B_\rho} |f| \right) + 2C_0.$$

For arbitrary $r \le \rho$ choose k such that $\eta^{k+1}\rho < r \le \eta^k\rho$. Then we have

$$\frac{1}{r^\gamma} \int_{B_r} |f|^2 \;\le\; \frac{1}{(\eta^{k+1}\rho)^\gamma} \int_{B_{\eta^k\rho}} |f| \;\le\; \left(\frac{1}{(\eta\rho)^\gamma} \int_{B_\rho} |f| \right) + \frac{2}{\eta^\gamma} C_0.$$

□

In the following $\Omega \subset \mathbb{R}^3$ is a domain. To show estimates like (3.34), we often write a function u as the sum of the weak solution v of an elliptic equation and the rest w. In Lemma 3.20 below we compare the integral $\int_{B_r} |\nabla v|^2$ over balls of different sizes. The proof of Lemma 3.20 relies on the following standard estimate for elliptic operators, that can be found, for example, in [18, Thm. 8.10].

Theorem 3.19. *Let $A \in C^{k,1}(\Omega, \mathbb{R}^{3\times 3})$, let $f \in H^{k+2}(\Omega, \mathbb{R})$ let $\underline{\lambda} > 0$ and let $u \in H^1(\Omega)$ be a weak solution of $\operatorname{div}(A\nabla u) = f$. Moreover, assume that $q \cdot (Aq) \ge \underline{\lambda}^2 |q|^2$ for all $q \in \mathbb{R}^3$. Then for any subdomain $\Omega' \Subset \Omega$ we have $u \in H^{k+2}(\Omega')$ and*

$$\|u\|_{H^{k+2}(\Omega')} \;\le\; C_{\mathrm{ElEst}} \left(\|u\|_{H^1(\Omega)} + \|f\|_{H^{k+2}(\Omega)} \right)$$

where $C_{\mathrm{ElEst}} = C_{\mathrm{ElEst}}(\underline{\lambda}, k, d, \|A\|_{C^{k,1}})$ and $d = \operatorname{dist}(\Omega, \Omega')$.

Lemma 3.20. *Let $0 < \rho \le 1$, $A\colon B_\rho \to \mathbb{R}^{3\times 3}$ and let $v \in H^1(B_\rho)$ be a weak solution of $Lv = \operatorname{div}(A\nabla v) = 0$ in B_ρ. Moreover, assume that*

$$q \cdot (Aq) \ge \underline{\lambda}^2 |q|^2, \quad |q' \cdot (Aq)| \le \overline{\lambda}^2 |q||q'| \quad \text{for all } q, q' \in \mathbb{R}^3.$$

Then we have for all $0 < \eta \le 1$

$$\frac{1}{(\eta\rho)^3} \int_{B_{\eta\rho}} |\nabla v|^2 \;\le\; \frac{C_{\text{inEst}}}{\rho^3} \int_{B_\rho} |\nabla v|^2, \tag{3.35}$$

$$\frac{1}{(\eta\rho)^5} \int_{B_{\eta\rho}} |\nabla v - \langle \nabla v \rangle_{B_{\eta\rho}}|^2 \;\le\; \frac{C_{\text{inEst}}}{\rho^5} \int_{B_\rho} |\nabla v - \langle \nabla v \rangle_{B_\rho}|^2, \tag{3.36}$$

where $C_{\text{inEst}} = C_{\text{inEst}}(\underline{\lambda}, \overline{\lambda}, K)$ and

$$K \;:=\; \max\left(\|\nabla A\|_{C^0(B_\rho)} \rho,\ \|D^2 A\|_{C^0(B_\rho)} \rho^2,\ \|D^3 A\|_{C^0(B_\rho)} \rho^3 \right).$$

Proof. First we prove (3.35). Set

$$u\colon B_1 \to \mathbb{R}, \quad p \mapsto v(\rho p) - \langle v \rangle_{B_\rho}, \qquad \tilde{A}\colon B_1 \to \mathbb{R}^{3\times 3}, \quad p \mapsto A(\rho p).$$

Then $\operatorname{div}(\tilde{A}u) = 0$ in B_1. Combining Theorem 3.19 with the Sobolev embedding $H^2(B_{\frac12}) \hookrightarrow L^\infty(B_{\frac12})$, and the Poincaré estimate (3.26) yields

$$\begin{aligned}
\|\nabla u\|_{L^\infty(B_{\frac12})} &\le C_{\text{Sobolev}} \|u\|_{H^3(B_{\frac12})} \\
&\le C_{\text{Sobolev}} C_{\text{ElEst}}(\underline{\lambda}, 1, \tfrac12, \|\tilde{A}\|_{C^{1,1}(B_1)})\, \|u\|_{H^1(B_1)} \\
&\le 9 C_{\text{Sobolev}} C_{\text{ElEst}}(\underline{\lambda}, 1, \tfrac12, \|\tilde{A}\|_{C^{1,1}(B_1)})\, \|\nabla u\|_{L^2(B_1)},
\end{aligned}$$

where C_{Sobolev} is the constant of the embedding $H^2(B_{\frac12}) \hookrightarrow L^\infty(B_{\frac12})$. Since

$$\|\tilde{A}\|_{C^{1,1}(B_1)} \;\le\; \overline{\lambda} + \|DA\|_{C^0(B_\rho)} \rho + \|D^2 A\|_{C^0(B_\rho)} \rho^2$$

we can find $C_1(\underline{\lambda}, \overline{\lambda}, K)$ such that

$$C_1 \;\ge\; 9 C_{\text{Sobolev}} C_{\text{ElEst}}(\underline{\lambda}, 1, \tfrac12, \|\tilde{A}\|_{C^{1,1}(B_1)}).$$

Then we have for all $\eta \le \frac12$

$$\begin{aligned}
\frac{1}{(\eta\rho)^3} \int_{B_{\eta\rho}} |\nabla v|^2 &= \frac{1}{\eta^3 \rho^2} \int_{B_\eta} |\nabla u|^2 \;\le\; \frac{1}{\rho^2} \|\nabla u\|^2_{L^\infty(B_{\frac12})} \\
&\le \frac{1}{\rho^2} C_1^2 \|\nabla u\|^2_{L^2(B_1)} \;=\; \frac{C_1^2}{\rho^3} \int_{B_\rho} |\nabla v|^2.
\end{aligned}$$

For $\eta > \frac12$ we have $\frac{1}{(\eta\rho)^3} \int_{B_{\eta\rho}} v^2 \le \frac{8}{\rho^3} \int_{B_\rho} v^2$. Therefore, if

$$C_{\text{inEst}}(\underline{\lambda}, \overline{\lambda}, K) \;\ge\; \max(C_1(\underline{\lambda}, \overline{\lambda}, K)^2,\, 8),$$

then equation (3.35) holds for all $0 < \eta \leq 1$.

Now we consider (3.36). Set

$$u\colon B_1 \to \mathbb{R}, \qquad p \mapsto v(\rho p) - \langle v\rangle_{B_\rho} - \sum_i \langle \partial_i v\rangle_{B_\rho} p_i,$$
$$\tilde{A}\colon B_1 \to \mathbb{R}^{3\times 3}, \qquad p \mapsto A(\rho p).$$

and let C_{Sobolev} be the constant for the Sobolev embedding $H^2(B_{\frac{1}{2}}) \hookrightarrow L^\infty(B_{\frac{1}{2}})$. Then, using the Poncaré inequality (3.26), we have for all $0 < \eta \leq \frac{1}{2}$

$$\begin{aligned}
\frac{1}{\eta^5}\int_{B_\eta} |\nabla u - \langle \nabla u\rangle_{B_\eta}|^2 \;&\leq\; \frac{64}{\eta^3}\int_{B_\eta} |D^2 u|^2 \;\leq\; 64\,\frac{4}{3}\pi\,\|D^2 u\|^2_{L^\infty(B_{\frac{1}{2}})} \\
&\leq\; 320\, C_{\text{Sobolev}} \|u\|^2_{H^4(B_{\frac{1}{2}})} \;\leq\; 320\, C_{\text{Sobolev}} C^2_{\text{ElEst}}(\underline{\lambda}, 2, \tfrac{1}{2}, \|\tilde{A}\|_{C^{2,1}(B_1)}) \|u\|^2_{H^1(B_1)} \\
&\leq\; 320(64+1)\, C_{\text{Sobolev}} C^2_{\text{ElEst}}(\underline{\lambda}, 2, \tfrac{1}{2}, \|\tilde{A}\|_{C^{2,1}(B_1)}) \|\nabla u\|^2_{L^2(B_1)}.
\end{aligned}$$

As before, we can find $C_2(\underline{\lambda}, \overline{\lambda}, K)$ such that

$$C_2 \;\geq\; 320\cdot 65 C_{\text{Sobolev}} C^2_{\text{ElEst}}(\underline{\lambda}, 2, \tfrac{1}{2}, \|\tilde{A}\|_{C^{2,1}(B_1)}).$$

Then we have for all $0 < \eta \leq \frac{1}{2}$,

$$\begin{aligned}
\frac{1}{(\eta\rho)^5}\int_{B_{\eta\rho}} |\nabla v - \langle \nabla v\rangle_{B_{\eta\rho}}|^2 \;&=\; \frac{1}{\eta^5\rho^4}\int_{B_\eta} |\nabla u - \langle \nabla u\rangle_{B_\eta}|^2 \\
\leq\; \frac{C_2}{\rho^4}\|\nabla u\|^2_{L^2(B_1)} \;&=\; \frac{C_2}{\rho^5}\int_{B_\rho} |\nabla v - \langle \nabla v\rangle_{B_\rho}|^2.
\end{aligned}$$

For $\eta \geq \frac{1}{2}$ we have

$$\frac{1}{(\eta\rho)^5}\int_{B_{\eta\rho}} |\nabla v - \langle \nabla v\rangle_{B_{\eta\rho}}|^2 \;\leq\; \frac{2^5}{\rho^5}\int_{B_\rho} |\nabla v - \langle \nabla v\rangle_{B_\rho}|^2,$$

So we can set $C_{\text{inEst}} := \max(C_1^2,\, C_2^2,\, 2^5)$. ☐

One simple example for the method of comparing integrals of balls of different sizes is the following estimate, which we will use in Section 3.6.

Lemma 3.21. *Let u be the solution of $\Delta u = \operatorname{div} m^R \mathbb{1}_{\Sigma(R)}$ in $\mathbb{R}^3$. Assume that $R \leq 1$. Then for every $0 < \gamma < 3$ there exists an absolute constant C_γ such that for all $r \leq 1$, $a \in \mathbb{R}^3$ we have $\int_{B_r(a)} |\nabla u| \leq C_\gamma r^\gamma$.*

Proof. For arbitrary $a \in \mathbb{R}^3$, $0 < \rho \leq 1$ define $v, w\colon B_\rho(a) \to \mathbb{R}^3$ as the solutions of

$$\Delta v \;=\; 0 \quad \text{in } B_\rho(a), \qquad v \;=\; u \quad \text{on } \partial B_\rho(a), \tag{3.37}$$
$$\Delta w \;=\; \operatorname{div} m^R \quad \text{in } B_\rho(a), \qquad w \;=\; 0 \quad \text{on } \partial B_\rho(a). \tag{3.38}$$

Using Lemma 3.20, we find an absolute constant C_1 such that for each $0 < \eta < 1$

$$\|\nabla v\|^2_{L^2(B_{\eta\rho})} \;\le\; C_1\eta^3\|\nabla v\|^2_{L^2(B_\rho)}.$$

Testing (3.38) with w yields

$$\begin{aligned}\|\nabla w\|^2_{L^2(B_\rho(a))} &= -\int_{B_\rho(a)} \operatorname{div} m^R\, w \;=\; \int_{B_\rho(a)} m^R\cdot\nabla w\\ &\le \sqrt{\frac{4}{3}\pi\rho^3}\;\|\nabla w\|_{L^2(B_\rho(a))},\end{aligned}$$

so we have $\|\nabla w\|^2_{L^2(B_\rho(a))} \le \frac{4}{3}\pi\rho^3$. Since v is the minimizer of $\|\nabla f\|^2_{L^2(B_\rho(a)}$ in the set $\{f\in H^1(B_\rho(a)) : f = u \text{ on } \partial B_\rho(a)\}$, we have in particular

$$\int_{B_\rho(a)} |\nabla v|^2 \;\le\; \int_{B_\rho(a)} |\nabla u|^2.$$

Now choose η such that $C_1\eta^{3-\gamma}\le\frac{1}{4}$. Then

$$\begin{aligned}\frac{1}{(\eta\rho)^\gamma}\|\nabla u\|^2_{L^2(B_{\eta\rho}(a))} &\le \frac{2}{(\eta\rho)^\gamma}\Big(\|\nabla v\|^2_{B_{\eta\rho}(a)} + \|\nabla w\|^2_{B_{\eta\rho}(a)}\Big)\\ &\le \frac{1}{2\rho^\gamma}\int_{B_\rho(a)}|\nabla u|^2 + \frac{8}{3\eta^\gamma}\pi\end{aligned}$$

and Lemma 3.18 yields for all $r\le 1$

$$\frac{1}{r^\gamma}\|\nabla u\|^2_{L^2(B_r(a))} \;\le\; \frac{1}{\eta^\gamma}\|\nabla u\|^2_{L^2(B_1(a))} + \frac{16}{3\eta^{2\gamma}}\pi \;\le\; \underbrace{\frac{1}{\eta^\gamma}E(m^R,R) + \frac{16}{3\eta^{2\gamma}}\pi}_{=:C}$$

Thus,

$$\frac{1}{r^\gamma}\|\nabla u\|_{L^1(B_r(a))} \;\le\; \frac{1}{r^\gamma}\|\nabla u\|_{L^2(B_r(a))}\sqrt{|B_r|} \;\le\; C\sqrt{\frac{4}{3}}\pi.$$

□

3.5 A decay estimate for almost-minimisers and $C^{0,\frac{1}{4}}$ regularity of m^R

In this section we compare the integral $\frac{1}{r^{1+\alpha}}\int_{B_r}|\nabla m|^2$ for (c,α)-minimizers m over balls of different sizes. The arguments are essentially the same as the arguments in [19]. First, using a comparison function, we find a hybrid inequality for almost-minimisers.

Lemma 3.22. *Let $\mu \in \mathbb{R}^3$, let $B_\rho(a) \subset \Omega$, and let m be an (c,α)-minimizer for $|A\nabla \cdot|^2$ in Ω with $\underline{\lambda}|f| \le |Af| \le \overline{\lambda}|f|$ for all $f \in \mathbb{R}^3$. Then we have for all $0 < \tau < 1$ the relation*

$$\frac{1}{\frac{1}{2}\rho}\int_{B_{\frac{\rho}{2}}(a)} |\nabla m|^2$$
$$\le \frac{\overline{\lambda}^2}{\underline{\lambda}^2}\frac{\tau}{\rho}\left(\int_{B_\rho(a)} |\nabla m|^2\right) + 6\cdot 10^4 \frac{\overline{\lambda}^2}{\underline{\lambda}^2}\frac{1}{\tau\rho^3}\left(\int_{B_\rho(a)} |m-\mu|^2\right) + c\rho^\alpha. \quad (3.39)$$

Proof. Without loss of generality we can assume $a = 0$. Since

$$\left|\left\{ r \in \left[\frac{\rho}{2},\rho\right] \;\middle|\; \int_{\partial B_r} |\nabla_{\tan} m|^2 \;\ge\; \frac{5}{\rho}\int_{B_\rho} |\nabla m|^2 \right\}\right| \;\le\; \frac{1}{5}\rho,$$
$$\left|\left\{ r \in \left[\frac{\rho}{2},\rho\right] \;\middle|\; \int_{\partial B_r} |m-\mu|^2 \;\ge\; \frac{5}{\rho}\int_{B_\rho} |m-\mu|^2 \right\}\right| \;\le\; \frac{1}{5}\rho,$$

there exists a radius r^* with $\frac{\rho}{2} \le r^* \le \rho$ such that

$$\int_{\partial B_{r^*}} |\nabla_{\tan} m|^2 \;\le\; \frac{5}{\rho}\int_{B_\rho} |\nabla m|^2 \quad (3.40)$$

and

$$\int_{\partial B_{r^*}} |m-\mu|^2 \;\le\; \frac{5}{\rho}\int_{B_\rho} |m-\mu|^2. \quad (3.41)$$

For $0 < \delta < 1$ we define the map $h\colon B_{r^*} \to \mathbb{R}^3$ by setting

$$h(p) \;:=\; \begin{cases} \mu & \text{if } p \in B_{(1-\delta)r^*}, \\ \frac{r^*-|p|}{\delta r^*}\mu + \frac{|p|-(1-\delta)r^*}{\delta r^*}\; m\left(\frac{r^* p}{|p|}\right) & \text{otherwise.} \end{cases}$$

Then $h|_{\partial B_{r^*}} = m|_{\partial B_{r^*}}$ and

$$\int_{B_{r^*}} |\nabla h|^2 \;=\; \int_{B_{r^*}\setminus B_{(1-\delta)r^*}} |\nabla_{\tan} h|^2 + |\partial_r h|^2$$
$$\le\; \delta r^*\left(\int_{\partial B_{r^*}} |\nabla_{\tan} m|^2\right) + \frac{1}{\delta r^*}\left(\int_{\partial B_{r^*}} |m-\mu|^2\right). \quad (3.42)$$

Like in [20, p.556] we can construct a function

$$w\colon B_{r^*} \to \mathbb{S}^2 \quad \text{such that} \quad w|_{\partial B_{r^*}} = h|_{\partial B_{r^*}}, \quad \int_{B_{r^*}} |\nabla w|^2 \;\le\; 48\int_{B_{r^*}} |\nabla h|^2.$$

as follows:

For $b \in B_{\frac{1}{2}}$, we define the projection $P_b(p) := \frac{p-b}{|p-b|}$. Then

$$\begin{aligned}
\int_{B_{\frac{1}{2}}} \int_{B_{r^*}} |\nabla(P_b \circ h(p))|^2 \, dp \, db \;&=\; \int_{B_{\frac{1}{2}}} \int_{B_{r^*}} \left| \nabla \left(\frac{h(p)-b}{|h(p)-b|} \right) \right|^2 dp \, db \\
&= \int_{B_{\frac{1}{2}}} \int_{B_{r^*}} \left| \frac{\nabla h(p) \, |h(p)-b| \;-\; (h(p)-b) \, \nabla |h(p)-b|}{|h(p)-b|^2} \right|^2 dp \, db \\
&\le 2 \int_{B_{r^*}} \int_{B_{\frac{1}{2}}} \frac{|\nabla h(p)|^2}{|h(p)-b|^2} \, db \, dp \;\le\; 2 \int_{B_{r^*}} |\nabla h(p)|^2 \int_{B_{\frac{1}{2}}} \frac{1}{|y|^2} \, dy \, dp \\
&\le 4\pi \int_{B_{r^*}} |\nabla h(p)|^2 \, dp.
\end{aligned}$$

Thus we can choose $b^* \in B_{\frac{1}{2}}$ such that

$$\int_{B_{r^*}} |\nabla(P_{b^*} \circ h(p))|^2 \, dp \;\le\; \frac{4\pi}{|B_{\frac{1}{2}}|} \int_{B_{r^*}} |\nabla h(p)|^2 \, dp \;=\; 12 \int_{B_{r^*}} |\nabla h(p)|^2 \, dp.$$

We define the map $w := (P_{b^*}|_{\mathbb{S}^2})^{-1} \circ P_{b^*} \circ h$. Then $w|_{\partial B_{r^*}} = h|_{\partial B_{r^*}}$ and

$$\left| D\left((P_{b^*}|_{\mathbb{S}^2})^{-1} \right) \right| \;=\; \left(\inf_{p \in \mathbb{S}^2} |DP_{b^*}(p)| \right)^{-1} \;\le\; \frac{3}{2} \;\le\; 2.$$

Thus we have

$$\int_{B_r} |\nabla w|^2 \;\le\; \|\nabla(P_{b^*}|_{\mathbb{S}^2}^{-1})\|^2_{L^\infty(\mathbb{S}^2)} \int_{B_{r^*}} |\nabla(P_{b^*} \circ h)|^2 \;\le\; 48 \int_{B_{r^*}} |\nabla h|^2 \quad (3.43)$$

Using that m is a (c,α)-almost-minimiser for $|A\nabla \cdot|^2$ and combining (3.40)-(3.43) yields

$$\begin{aligned}
\int_{B_{\frac{\rho}{2}}} \underline{\lambda}^2 |\nabla m|^2 \;&\le\; \int_{B_{r^*}} |A\nabla m|^2 \;\le\; \int_{B_{r^*}} |A\nabla w|^2 + cr^{1+\alpha} \\
&\le \int_{B_{r^*}} \overline{\lambda}^2 |\nabla w|^2 + cr^{1+\alpha} \;\le\; \int_{B_{r^*}} 48\overline{\lambda}^2 |\nabla h|^2 + cr^{1+\alpha} \\
&\le 48\overline{\lambda}^2 \delta\rho \left(\int_{\partial B_{r^*}} |\nabla_{\tan} m|^2 \right) + \frac{96\overline{\lambda}^2}{\delta\rho} \left(\int_{\partial B_{r^*}} |m-\mu|^2 \right) + cr^{1+\alpha} \\
&\le 240\overline{\lambda}^2 \delta \left(\int_{B_\rho} |\nabla m|^2 \right) + \frac{480\overline{\lambda}^2}{\delta\rho^2} \left(\int_{B_\rho} |m-\mu|^2 \right) + cr^{1+\alpha}.
\end{aligned}$$

For $0 < \tau < 1$ set $\delta := \frac{\tau}{120}$. Then we have $\frac{480}{\delta} \le \frac{6 \cdot 10^4}{\tau}$, and thus

$$\int_{B_{\frac{\rho}{2}}} |\nabla m|^2 \;\le\; \frac{\overline{\lambda}^2}{\underline{\lambda}^2} 2\tau \left(\int_{B_\rho} |\nabla m|^2 \right) + 6 \cdot 10^4 \frac{\overline{\lambda}^2}{\underline{\lambda}^2} \frac{1}{\tau\rho^2} \left(\int_{B_\rho} |m-\mu|^2 \right) + cr^{1+\alpha},$$

as claimed. □

The following lemma relies on the observation that, when we appropriately rescale almost-minimisers that have small energy on small balls, we get a sequence of functions that converges to the solution of an elliptic equation. For solutions of elliptic equations we have Lemma 3.20, so for elements of the sequence that are close to this solution we have a similar estimate.

Lemma 3.23. *For $n \in \mathbb{N}$ let m_n be an (c, α)-almost-minimiser for $|A_n \nabla \cdot|^2$ in some domain Ω_n and assume that there exist constants C, $\underline{\lambda}$, $\overline{\lambda}$ such that*

$$\|\nabla A_n\|_{C^1(\Omega_i)} \le C, \qquad \underline{\lambda}|v| \le |A_n v| \le \overline{\lambda}|v| \ \text{ for all } n \in \mathbb{N},\ v \in \mathbb{R}^3.$$

Let $B_{r_n}(a_n) \subset \Omega_n$ and set

$$\epsilon_n^2 := \frac{1}{r_n} \int_{B_{r_n}(a_n)} |\nabla m_n|^2.$$

If

$$\lim_{n\to\infty} r_n = 0, \qquad \lim_{n\to\infty} \epsilon_n = 0, \qquad \lim_{n\to\infty} \frac{r_n^\alpha}{\epsilon_n^2} = 0,$$

then there exists $C_1(\underline{\lambda}, \overline{\lambda}) > 0$ such that for each $0 < \eta < 1$ there exists a subsequence $(m_{n_k})_{k\in\mathbb{N}}$ of $(m_n)_{n\in\mathbb{N}}$ such that for all elements of the subsequence and for all $\beta \in [\eta, 1]$

$$\frac{1}{(\beta r_{n_k})^3} \int_{B_{\beta r_{n_k}}(a_{n_k})} \left| m_{n_k} - \langle m_{n_k} \rangle_{B_{\beta r_{n_k}}(a_{n_k})} \right|^2 \ \le\ C_1 \beta^2 \epsilon_{n_k}^2.$$

Proof. Set

$$v_n \colon\ B_1 \to \mathbb{R}^3, \qquad p \mapsto \frac{1}{\epsilon_n}\Big(m_n(a_n + r_n p) - \langle m_n \rangle_{B_{r_n}(a_n)} \Big), \tag{3.44}$$

$$\tilde{A}_n \colon\ B_1 \to \mathbb{R}^{3\times 3}, \qquad p \mapsto A_n(a_n + r_n p). \tag{3.45}$$

Since $\|\nabla v_n\|^2_{L^2(B_1)} = 1$ and $\|\nabla \tilde{A}_n\|_{L^\infty(B_1)} \le C r_n$ for all $n \in \mathbb{N}$ there exist a constant matrix A and a map $v \colon B_1 \to \mathbb{R}^3$ such that $(\tilde{A}_n, v_n)_{n\in\mathbb{N}}$ converges, up to a subsequence, to (A, v). Here the convergence in the first component is strong in $L^\infty(B_1)$, the convergence in the second component is weak in $H^1(B_1)$ and strong in $L^2(B_1)$. We denote the subsequence with $(\tilde{A}_n, v_n)_{n\in\mathbb{N}}$ as well.

We show that v is a weak solution of $\operatorname{div}(A^2 \nabla v) = 0$, i.e., that

$$\int_{B_1} (A\nabla v) : (A \nabla \zeta) = 0 \quad \text{for all } \zeta \in C_c^\infty(B_1, \mathbb{R}^3).$$

Fix $\zeta \in C_c^\infty(B_1, \mathbb{R}^3)$ and define for $0 < t < 1$ the maps

$$m_n^t \colon B_1 \to \mathbb{R}^3, \qquad p \mapsto m_n(a_n + r_n p) - t\epsilon_n \zeta(p),$$

$$w_n^t \colon B_1 \to \mathbb{R}^3, \qquad p \mapsto \frac{1}{\epsilon_n}\left(\frac{m_n^t(p)}{|m_n^t(p)|} - \left\langle \frac{m_n^t}{|m_n^t|} \right\rangle_{B_1} \right).$$

Since m_n is (c,α)-almost-minimising for $|A_n\nabla\cdot|^2$ we have

$$\begin{aligned}
&\int_{B_1} |\tilde{A}_n\nabla w_n^t|^2 - |\tilde{A}_n\nabla v_n|^2 \\
&= \frac{1}{r_n\epsilon_n^2}\int_{B_{r_n}(a_n)} \left| A_n\,\nabla\left(\frac{m_n^t\left(\frac{p-a_n}{r_n}\right)}{\left|m_n^t\left(\frac{p-a_n}{r_n}\right)\right|}\right)\right|^2 - |A_n\nabla m_n(p)|^2\,dp \\
&\geq -\frac{4\pi}{3\epsilon_n^2} r_n^\alpha. \qquad (3.46)
\end{aligned}$$

On the other hand we have for $t < \frac{1}{2}\|\zeta\|_{L^\infty(B_1)}$,

$$\begin{aligned}
\nabla w_n^t &= \frac{1}{\epsilon_n}\frac{\nabla m_n^t}{|m_n^t|} - \frac{1}{\epsilon_n}\frac{\nabla|m_n^t|^2}{2|m_n^t|^3}\,m_n^t \\
&= \frac{1}{\epsilon_n}\frac{\nabla m_n^t}{|m_n^t|} - \frac{1}{\epsilon_n}\frac{(m_n^t)^T\nabla m_n^t}{|m_n^t|^3}\,m_n^t \\
&= \frac{\nabla v_n + t\nabla\zeta}{|m_n^t|} - \underbrace{\frac{t(m_n^0)^T\nabla\zeta + t\epsilon_n\zeta^T\nabla v_n + \epsilon_n t^2\zeta^T\nabla\zeta}{|m_n^t|^3}}_{=:s}(m_n^0 + t\zeta)
\end{aligned}$$

Note that we consider m_n^t, v_n, ζ to be row vectors and gradients to be column vectors. Since

$$\begin{aligned}
(\tilde{A}_n\nabla v_n):(\tilde{A}_n s\, m^0) &= \left(\sum_k (\tilde{A}_n)_{ik}\,\partial_k(v_n)_j\right):\left(\sum_{\tilde{k}}(\tilde{A}_n)_{i\tilde{k}}\;s_{\tilde{k}}\;(m_n^0)_j\right) \\
&= \sum_{ijk\tilde{k}}(\tilde{A}_n)_{ik}\,\partial_k(v_n)_j\,(\tilde{A}_n)_{i\tilde{k}}\;s_{\tilde{k}}\;(m_n^0)_j \\
&= \sum_k (s^T\tilde{A}_n^2)_k \underbrace{\sum_j \partial_k(v_n)_j\,(m_n^0)_j}_{=0} \;=\; 0
\end{aligned}$$

there exists a constant $\tilde{C}$, depending only on $\overline{\lambda}$ and ζ such that

$$|\tilde{A}_n\nabla w_n^t|^2 \leq \frac{|\tilde{A}_n\nabla v_n|^2 + 2t(\tilde{A}_n\nabla v_n):(\tilde{A}_n\nabla\zeta)}{|m_n^t|^2} + \tilde{C}t^2$$

and thus

$$\begin{aligned}
&\int_{B_1}|\tilde{A}_n\nabla w_n^t|^2 - |\tilde{A}_n\nabla v_n|^2 \\
&\leq \int_{B_1}|\tilde{A}_n\nabla v_n|^2\left(\frac{1}{|m_n^t|^2} - 1\right) + \frac{2t(\tilde{A}_n\nabla v_n):(\tilde{A}\nabla\zeta)}{|m_n^t|^2} + \tilde{C}t^2. \qquad (3.47)
\end{aligned}$$

Since $|1-|m_n^t|| \le \epsilon_n t|\zeta|_{L^\infty}$, combining (3.46) with (3.47) and sending n to infinity, we have

$$0 \;\le\; \int_{B_1} 2t\,(A\nabla v) : (A\nabla\zeta) + Ct^2$$

which, for $t \to 0$ yields $0 \le \int_{B_1}(A\nabla v) : (A\nabla\zeta)$. Replacing ζ by $-\zeta$ we have the reverse inequality, and since ζ was arbitrary, we see that v is a weak solution of $\operatorname{div}(A^2\nabla v) = 0$ in B_1.

The Poincaré inequality (3.26) and Lemma 3.20 imply that for each $0 < \beta < 1$

$$\begin{aligned}\frac{1}{\beta^3}\int_{B_\beta}|v-\langle v\rangle_{B_\beta}|^2 \;&\le\; \frac{64}{\beta}\int_{B_\beta}|\nabla v|^2 \;\le\; 64\,C_{\mathrm{inEst}}(\underline{\lambda},\overline{\lambda},0)\,\beta^2\int_{B_1}|\nabla v|^2\\ &\le\; 64\,C_{\mathrm{inEst}}(\underline{\lambda},\overline{\lambda},0)\,\beta^2.\end{aligned}$$

Now for $\eta \in]0,1]$ chosse n_0 so large that

$$\int_{B_1}|v_n - v|^2 \;\le\; C_{\mathrm{inEst}}(\underline{\lambda},\overline{\lambda},0)\,\eta^5 \quad \text{for all } n \ge n_0.$$

Then with (3.27) we have for all $\beta \in [\eta, 1]$ and all $n \ge n_0$ the estimate

$$\begin{aligned}&\frac{1}{(r_n\beta)^3}\int_{B_{\beta r_n}(a_n)}\left|m_n - \langle m_n\rangle_{B_\beta}\right|^2 \;\le\; \frac{\epsilon_n^2}{\beta^3}\int_{B_\beta}\left|v_n - \langle v\rangle_{B_\beta}\right|^2\\ &\le\; \frac{\epsilon_n^2}{\beta^3}\int_{B_\beta}2|v_n-v|^2 + 2|v-\langle v\rangle_{B_\beta}|^2 \;\le\; 130\,C_{\mathrm{inEst}}(\underline{\lambda},\overline{\lambda},0)\,\beta^2\epsilon_n^2.\end{aligned}$$

□

Using Lemmas 3.22 and 3.23 we show by contradiction the decay estimate for almost-minimisers.

Theorem 3.24. *Let m be a (c,α)-almost-minimiser for $|A\nabla\cdot|^2$ in Ω with $\underline{\lambda}|f| \le |Af| \le \overline{\lambda}|f|$ for all $f \in \mathbb{R}^3$. Then there exist positive constants $\epsilon_0, r_0, C_2, C_3$ and $\eta < 1$, all depending only on c, α, $\underline{\lambda}$ and $\overline{\lambda}$, with the following properties: If $r \le \rho \le r_0$ with $B_\rho(a) \subset \Omega$ and $\frac{1}{\rho}\int_{B_\rho(a)}|\nabla m|^2 < \epsilon_0$, we have the estimates*

$$\frac{1}{(\eta\rho)^{1+\alpha}}\int_{B_{\eta\rho}(a)}|\nabla m|^2 \;\le\; \max\left(\frac{1}{2\rho^{1+\alpha}}\left(\int_{B_\rho}|\nabla m|^2\right),\; C_2\right), \tag{3.48}$$

$$\frac{1}{r^{1+\alpha}}\int_{B_r(a)}|\nabla m|^2 \;\le\; C_3\left(\frac{1}{\rho^{1+\alpha}}\left(\int_{B_\rho}|\nabla m|^2\right)+1\right). \tag{3.49}$$

Proof. First we show (3.48). For $C_1 = C_1(\underline{\lambda}, \overline{\lambda})$ as in Lemma 3.23, let $k \geq 1$ be so large that $64 \cdot 10^5 C^1 \frac{\overline{\lambda}^4}{\underline{\lambda}^4} 2^{-k} \leq \frac{1}{8}$, and set $\eta := 2^{-k}$.

We argue by contradiction and assume that the Lemma is false for this number η, i.e., we assume that there exists a sequence of (c, α)-almost-minimisers m_i for $|A_i \nabla \cdot|^2$, and a sequence of balls $B_{r_i}(a_i) \subset \Omega$, such that $\lim_{i \to \infty} r_i = 0$ and

$$\epsilon_i^2 \;:=\; \frac{1}{r_i} \int_{B_{r_i}(a_i)} |\nabla m_i|^2 \;\to\; 0, \tag{3.50}$$

$$\frac{1}{\eta r_i} \int_{B_{\eta r_i}(a_i)} |\nabla m_i|^2 \;>\; \frac{\eta^\alpha}{2} \epsilon_i^2, \tag{3.51}$$

$$r_i^{-\alpha} \frac{1}{r_i} \int_{B_{\eta r_i}(a_i)} |\nabla m_i|^2 \;\to\; \infty. \tag{3.52}$$

Because of (3.52) we have $\lim_{i \to \infty} r_i^\alpha \epsilon_i^{-2} = 0$, so we can use Lemma 3.23. Thus there exists a subsequence, denoted with $(m_i)_{i \in \mathbb{N}}$ as well, such that for all $\beta \in [\eta, 1]$ we have

$$\frac{1}{(\beta r_i)^3} \int_{B_{\kappa r_i}(a_i)} |m_i - \overline{m}_i| \;\leq\; C_1 \beta^2 \epsilon_i^2.$$

Inserting this estimate into inequality (3.39), we get for all $\beta \in [\eta, 1]$

$$\frac{1}{\frac{1}{2}\beta r_i} \int_{B_{\frac{1}{2}\beta r_i}} |\nabla m_i|^2 \;\leq\; \frac{\tau}{\beta r_i} \frac{\overline{\lambda}^2}{\underline{\lambda}^2} \int_{B_{\beta r_i}} |\nabla m_i|^2 + \frac{\overline{\lambda}^2}{\underline{\lambda}^2} \frac{10^5 C_1 \epsilon_i^2}{\tau} \beta^2 + c r_i^\alpha \beta^\alpha.$$

We iterate this formula for $\beta := 1, \frac{1}{2}, \ldots 2^{-k+1}$ and obtain

$$\begin{aligned}
&\frac{1}{\eta r_i} \int_{B_{2^{-k} r_i}} |\nabla m_i|^2 \\
&\leq\; \tau^k \epsilon_i^2 + 10^5 C_1 \epsilon_i^2 \left(\sum_{j=1}^{k} \left(\frac{\overline{\lambda}^2}{\underline{\lambda}^2} \right)^j \tau^{j-2} 2^{-2(k-j)} \right) \\
&\quad + c r_i^\alpha \left(\sum_{j=1}^{k} \left(\frac{\overline{\lambda}^2}{\underline{\lambda}^2} \tau \right)^{j-1} 2^{-\alpha(k-j)} \right) \\
&=\; \tau^k \epsilon_i^2 + 10^5 C_1 \epsilon_i^2 \frac{\eta^2}{\tau^2} \left(\sum_{j=1}^{k} \left(\frac{4\tau \overline{\lambda}^2}{\underline{\lambda}^2} \right)^j \right) + c r_i^\alpha \frac{\eta^\alpha \underline{\lambda}^2}{\overline{\lambda}^2 \tau} \left(\sum_{j=1}^{k} \left(\frac{\overline{\lambda}^2}{\underline{\lambda}^2} 2^\alpha \tau \right)^j \right).
\end{aligned}$$

Setting $\tau := \frac{1\underline{\lambda}^2}{8\overline{\lambda}^2}$ we have

$$\frac{1}{\eta r_i}\int_{B_{2^{-k}r_i}} |\nabla m_i|^2 \;\leq\; 4^{-k}\eta\epsilon_i^2 + 64\; 10^5\frac{\overline{\lambda}^4}{\underline{\lambda}^4}\eta^2 C_1\epsilon_i^2 + 4c(\eta r_i)^\alpha$$
$$\leq\; \frac{1}{4}\eta\epsilon_i^2 + \frac{1}{8}\eta\epsilon_i^2 + 4c(\eta r_i)^\alpha.$$

Since $\lim_{i\to\infty} r_i^\alpha \epsilon_i^{-2} = 0$, for large i the right hand side of the inequality is strictly smaller than $\frac{\eta}{2}\epsilon_i^2 \leq \frac{\eta^\alpha}{2}\epsilon_i^2$, and we have a contradiction to (3.51). This shows that there exists $\tilde{\epsilon}_0$, $\tilde{r}_0$, C_2 such that (3.48) holds.

Now set $\epsilon_0 := \eta\tilde{\epsilon}_0$, and let $r_0 \leq \tilde{r}_0$ be so small that $C_2 r_0^\alpha \leq \epsilon_0$. Moreover, assume that $0 < \rho \leq r_0$ and $\frac{1}{\rho}\int_{B_\rho(a)} |\nabla m|^2 \leq \epsilon_0$. By induction and because of (3.48), we have for all $n \in \mathbb{N}$ the relation $\frac{1}{\eta^n \rho}\int_{B_{\eta^n\rho}} |\nabla m|^2 \leq \epsilon_0$ which implies $\frac{1}{r}\int_{B_r} |\nabla m|^2 \leq \frac{\epsilon_0}{\eta} = \tilde{\epsilon}_0$ for all $r \leq \rho$. Thus we can use Lemma 3.18 to get the estimate

$$\frac{1}{r^{1+\alpha}}\int_{B_r(a)} |\nabla m|^2 \;\leq\; \frac{1}{\eta^{1+\alpha}\rho^{1+\alpha}}\left(\int_{B_\rho} |\nabla m|^2\right) \;+\; \frac{2}{\eta^{1+2\alpha}}C_2$$

for all $r \leq \rho$. □

Combining this decay estimate with the characterization theorem for Hölder continuous functions (Theorem 3.17), we prove that the minimizers m^R are Hölder continuous. For $x_0 \in \mathbb{R}$ we set

$$Z_R(x_0) \;:=\; [x_0 - R, x_0 + R] \times D_R \tag{3.53}$$

Theorem 3.25. *There exists $C_s > 0$ such that for all R small enough and all $p, q \in \Sigma(\frac{3}{2}R)$ with $|p - q| \leq R$ we have $m^R(p) - m^R(q) \leq C_s|p - q|^{\frac{1}{4}}$.*

Proof. The functions m^R are (c, α)-almost-minimisers for $|\frac{1}{\sqrt{\beta}}A^R\nabla\cdot|^2$ in $\Sigma(\frac{3}{2}R)$ with $c = 4 + \sqrt{E(m)}$ and $\alpha = \frac{1}{2}$ (Theorem 3.16). Thus for $R \leq 1$ we have uniform constants ϵ_0, r_0, C_3 in Theorem 3.24. Assume that R is so small that

$$R \leq 1, \quad R \leq r_0, \quad \frac{1}{R}E(m^R) \leq \frac{\epsilon_0}{4}, \quad \frac{1}{R^{1+\frac{1}{2}}}E(m^R) \leq 1.$$

Then we have for all $p \in \Sigma(R)$ the estimate

$$\frac{1}{\frac{1}{2}R}\int_{B_{\frac{1}{2}R}1(p)} |\nabla m^R|^2 \;\leq\; \frac{2}{R}\int_{\Sigma(\frac{3}{2}R)} |\nabla m^R|^2 \;\leq\; \frac{4}{R}\|\nabla m^R\|^2_{L^2(\Sigma)} \;\leq\; \epsilon_0,$$

so we can apply Theorem 3.24. Thus

$$\begin{aligned}\int_{B_r(p)} |\nabla m^R|^2 &\le C_3 r^{1+\frac{1}{2}} \left(1 + \frac{1}{R^{1+\frac{1}{2}}} \int_{B_R} |\nabla m^R|^2\right)\\ &\le C_3 r^{1+\frac{1}{2}} \left(1 + \frac{1}{R^{1+\frac{1}{2}}} E(m^R)\right) \le 2C_3 r^{1+\frac{1}{2}},\end{aligned}$$

and the Pointcaré inequality (3.26) implies

$$\int_{B_r(p)} |m^R - \langle m^R \rangle_{B_r(p)}|^2 \le 128 C_3 r^{3+\frac{1}{2}}.$$

If $p, q \in \Sigma(\frac{3}{2}R)$ and $|p-q| < R$, then there exists $x_0 \in \mathbb{R}$ such that $p^*, q^* \in Z_R(x_0)$. We apply Theorem 3.17 for $\Omega := Z_1$ and $\eta := R$ and get

$$\begin{aligned}|m(p) - m(q)| &= |m(p^*) - m(q^*)| \le 128 C_3 C_{\text{Camp}} \left(Z_1, \tfrac{1}{2}\right) |p^* - q^*|^{\frac{1}{4}}\\ &\le 128 C_3 C_{\text{Camp}} \left(Z_1, \tfrac{1}{2}\right) |p - q|^{\frac{1}{4}}.\end{aligned}$$

□

3.6 Uniform $C^{1,\beta}$ regularity of the functions m^R and convergence results

The aim of this section is to prove good regularity of the functions m^R to get strong convergence results. In the first subsection we show that we do not have strong oscillations of ∇m^R on the balls of diameter R. In the second subsection we use methods as in the Morrey-Campanato approach to regularity (cf. [17]) to show that on smaller scales we have even less oscillations.

3.6.1 Convergence on balls of radius R

In this section we will use the fact that the functions m^R are energy minimizing to bound ∇m^R on the scale of R. We know that m^{red} satisfies the differential equation $|\partial_x m^{\text{red}}| - \frac{1}{\sqrt{2}}|m_y^{\text{red}}| = 0$. The first lemma shows that the functions m^R satisfy this differential equation approximately, i.e., that the difference $\left||\partial_x m^R| - \frac{1}{\sqrt{2}}|m_y^R|\right|$ is small. As described in Subsection 2.1.4, for $m\colon \Sigma \to \mathbb{R}^3$ we set

$$\overline{m}(x) := \frac{1}{|D_R|} \int_{D_R} m(x,y)\,dy, \qquad \tilde{m}(x,y) := m(x,y) - \overline{m}(x).$$

Lemma 3.26. *For all $\beta < 1$ we have*

$$\lim_{R\to 0} \frac{1}{R^{3+\beta}} \int_{\Sigma(R)} \left(|\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}} |\overline{m}_y^R| \right)^2 = 0.$$

Proof. Because of Lemma 2.28, the functions $\overline{m}^R$ converges in $H^1(\mathbb{R})$ to m^{red}. Thus, using the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$, we can assume $|\overline{m}^R| \geq \frac{1}{2}$ for sufficiently small R. Moreover, after reducing R, we can assume $\|\tilde{m}^R\|_{L^2(\Sigma)} \leq CR^3$ (Lemma 2.29) and $\|\nabla m^R\|_{L^2(\Sigma)} \leq 3R$ (Theorem 2.27). Since

$$\begin{aligned} \left|\partial_x \overline{m}^R\right| &= \left| \partial_x \left(\frac{\overline{m}^R}{|\overline{m}^R|} |\overline{m}^R| \right) \right| = \sqrt{ |\overline{m}^R|^2 \left| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right|^2 + (\partial_x |\overline{m}^R|)^2 } \\ &\geq |\overline{m}^R| \left| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right|^2 \end{aligned}$$

with (2.23) we have

$$\begin{aligned} \int_\Sigma |\partial_x \overline{m}^R| \cdot |\overline{m}_y^R| &\geq \int_\Sigma |\overline{m}^R|^2 \left| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right| \frac{|\overline{m}_y^R|}{|\overline{m}^R|} \\ &= \underbrace{\int_\Sigma \left| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right| \frac{|\overline{m}_y^R|}{|\overline{m}^R|}}_{T_1} \underbrace{- |\tilde{m}^R|^2 \left| \partial_x \frac{\overline{m}^R}{|\overline{m}^R|} \right| \frac{|\overline{m}_y^R|}{|\overline{m}^R|}}_{T_2}. \end{aligned}$$

To bound the first term T_1 from below, we set $\theta := \arccos\left(\frac{\overline{m}_x^R}{|\overline{m}^R|}\right)$ and calculate

$$T_1 = \int_\Sigma |\partial_x \theta| \cdot |\sin(\theta)| \geq 2\pi R^2.$$

For the second term T_2 we use the assumptions on R and the pointwise estimates $|\tilde{m}^R| \leq 1$, $|\overline{m}^R| \leq 1$. We have

$$\begin{aligned} T_2 &\geq -\int_\Sigma |\tilde{m}^R|^2 \left| \frac{\partial_x \overline{m}^R}{|\overline{m}^R|} \right| \cdot \frac{|\overline{m}_y^R|}{|\overline{m}^R|} \geq -4 \int_\Sigma |\tilde{m}^R| \left|\partial_x \overline{m}^R\right| \\ &\geq -4 \|\tilde{m}^R\|_{L^2(\Sigma)} \|\nabla m^R\|_{L^2(\Sigma)} \geq -12CR^4. \end{aligned}$$

Using this calculation and the equality $E^0(m^{\text{red}}, R) = \sqrt{8}\pi R^2$, we obtain

$$\begin{aligned}
&\left\| |\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}} |\overline{m}_y^R| \right\|_{L^2(\Sigma)}^2 \\
&= \int_{\Sigma(R)} \left|\partial_x \overline{m}^R\right|^2 + \frac{1}{2}\left|\overline{m}_y^R\right|^2 - \sqrt{2}\,|\partial_x \overline{m}^R| \cdot |\overline{m}_y^R| \\
&\le E^0(\overline{m}^R, R) - E^0(m^{\text{red}}, R) + 12CR^4 \\
&= E^0(\overline{m}^R, R) - E(\overline{m}^R, R) + E(\overline{m}^R, R) - E(m^R, R) \\
&\quad + E(m^R, R) - E^0(m^{\text{red}}) + 12CR^4 \\
&\le \underbrace{R^2 \frac{\pi}{2} \|\overline{m}_y^R\|_{L^2(\mathbb{R})} - E_H(\overline{m}^R, R)}_{S_1} + \underbrace{E_H(\overline{m}^R, R) - E_H(m^R, R)}_{S_2} \\
&\quad + \underbrace{E(m^R, R) - E^0(m^{\text{red}})}_{S_3} + 12CR^4.
\end{aligned}$$

We consider each summand separately. From Lemma 2.24 we know

$$\lim_{R\to 0} \frac{1}{R^{3+\beta}} S_1 \;\le\; \lim_{R\to 0} 2\pi R^{1-\beta}\, |\ln(R)|\, \|m_y\|_{H^1(\mathbb{R})}^2 \;=\; 0.$$

Since $\lim_{R\to 0} \frac{1}{R^2} E(m^R, R) = \sqrt{8}\pi$ (Theorem 2.27), for the second summand, (2.14) and Lemma 2.29 imply

$$\begin{aligned}
\lim_{R\to 0} \frac{1}{R^{3+\beta}} S_2 \;&\le\; \lim_{R\to 0} \frac{1}{R^{3+\beta}} \left(2\sqrt{E(m^R, R)}\, \|\tilde{m}^R\|_{L^2(\Sigma(R))} + \|\tilde{m}^R\|_{L^2(\Sigma(R))}^2 \right) \\
&=\; 0.
\end{aligned}$$

Finally, using Theorem 2.30 we get

$$\lim_{R\to 0} \frac{1}{R^{3+\beta}} S_3 \;=\; 0.$$

□

Combining the bound on $\nabla \tilde{m}^R$ of Lemma 2.29 with the estimate for $\partial_x \overline{m}$ of Lemma 3.26 we get the following Lemma.

Lemma 3.27. *Let Z_R as in* (3.53). *For all $\beta < 1$ we have uniformly in x_0*

$$\lim_{R\to 0} \frac{1}{|R|^{3+\beta}} \int_{Z_R(x_0)} \left| \nabla m^R - \left\langle \nabla m^R \right\rangle_{Z_R(x_0)} \right|^2 \;=\; 0.$$

Proof. With (3.27) we have

$$\begin{aligned}
&\int_{Z_R(x_0)} |\nabla m^R - \langle \nabla m^R \rangle_{Z_R(x_0)}|^2 \\
&\quad \le \int_{Z_R(x_0)} \left| |\nabla \tilde{m}^R| + |\partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \langle \overline{m}^R_y \rangle_{Z_R(x_0)}| \right|^2 \\
&\quad \le 3 \left(\int_\Sigma |\nabla \tilde{m}^R|^2 + |\partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \overline{m}^R_y|^2 \right) + \frac{3}{2} \int_{Z_R(x_0)} |\overline{m}^R_y - \langle \overline{m}^R_y \rangle_{Z_R(x_0)}|^2.
\end{aligned}$$

Lemma 2.29 and Lemma 3.26 imply

$$\lim_{R\to 0} \frac{3}{R^{3+\beta}} \left(\int_\Sigma |\nabla \tilde{m}^R|^2 + |\partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \overline{m}^R_y|^2 \right) \; = \; 0,$$

and the Poincaré inequality [18, p.164] yields

$$\begin{aligned}
&\lim_{R\to 0} \frac{3}{R^{3+\beta}} \int_{Z_R(x_0)} |\overline{m}^R_y - \langle \overline{m}^R_y \rangle_{Z_R(x_0)}|^2 \\
&= \lim_{R\to 0} \frac{3}{R^{3+\beta}} |D_R| \int_{x_0-R}^{x_0+R} \left| \overline{m}^R_y - \langle \overline{m}^R_y \rangle_{[x_0-R,x_0+R]} \right|^2 \\
&\le \lim_{R\to 0} \frac{12}{R^{1+\beta}} \|\partial_x \overline{m}^R_y\|^2_{L^2(Z_R(x_0))} \;\le\; \lim_{R\to 0} \frac{12}{R^{1+\beta}} E(m^R, R) \; = \; 0.
\end{aligned}$$

□

From Lemma 3.27 we can deduce the main theorem of this subsection.

Theorem 3.28. *For all $\beta < 1$ we have*

$$\lim_{R\to 0} \sup_{p\in\Sigma(R)} \frac{1}{R^{3+\beta}} \int_{B_{\frac{R}{2}}(p)} |\nabla m^R - \langle \nabla m^R \rangle_{B_{\frac{R}{2}}(p)}|^2 \; = \; 0, \tag{3.54}$$

$$\lim_{R\to 0} \sup_{p\in\Sigma(R)} \frac{1}{R^{2+\beta}} \int_{B_{\frac{R}{2}}(p)} |\partial_x m^R|^2 \; = \; 0. \tag{3.55}$$

Proof. With (3.29) and (3.4.1) we have for all $x_0 \in \mathbb{R}$

$$\begin{aligned}
\int_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}\setminus Z_R(x_0)} |\nabla m^R|^2 \;&\le\; \int_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}\setminus Z_R(x_0)} \frac{3}{\kappa} \left|A^R \nabla m^R\right|^2 \\
&=\; 3 \int_{Z_R(x_0)\setminus [x_0-R,x_0+R]\times D_{\frac{R}{2}}} |\nabla m^R|^2 \\
&\le\; 3 \int_{Z_R(x_0)} |\nabla m^R|^2.
\end{aligned}$$

Because of the special form of A^R we have the same estimate for $\nabla_y m^R$ instead of ∇m^R. This implies for all $(x_0, y_0) \in \Sigma$

$$\int_{B_{\frac{R}{2}}(x_0,y_0)} |\nabla_y m^R|^2 \;\le\; \int_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}} |\nabla m^R|^2 \;\le\; 4\int_{\Sigma(R)} |\nabla_y m^R|^2.$$

Therefore, using (3.27) and (2.34), we obtain

$$\begin{aligned}
&\lim_{R\to 0}\sup_{p\in\Sigma(R)} \frac{1}{R^{3+\beta}}\int_{B_{\frac{R}{2}}(p)} |\nabla_y m^R - \langle \nabla_y m\rangle_{B_{\frac{R}{2}}(p)}|^2 \\
&\le \lim_{R\to 0}\sup_{p\in\Sigma(R)} \frac{1}{R^{3+\beta}}\int_{B_{\frac{R}{2}}(p)} |\nabla_y m^R|^2 \;=\; 0. \qquad (3.56)
\end{aligned}$$

To show the analogous statement for $\partial_x m^R$, we note that we have $\partial_x m^R(p) = \partial_x m^R(p^*)$, where p^* as in (3.28). As a consequence of (3.27), (3.29), and Lemma 3.27 we obtain

$$\begin{aligned}
&\lim_{R\to 0}\sup_{(x_0,y_0)\in\Sigma(R)} \frac{1}{R^{3+\beta}}\int_{B_{\frac{R}{2}}(x_0,y_0)} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{B_{\frac{R}{2}}(x_0,y_0)}|^2 \\
&\le \lim_{R\to 0}\sup_{(x_0,y_0)\in\Sigma(R)} \frac{1}{R^{3+\beta}}\int_{B_{\frac{R}{2}}(x_0,y_0)} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{Z_R(x_0)}|^2 \\
&\le \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{1}{R^{3+\beta}}\int_{[x_0-R,x_0+R]\times D_{\frac{3}{2}R}} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{Z_R(x_0)}|^2 \\
&\le \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{4}{R^{3+\beta}}\int_{Z_R(x_0)} |\partial_x m^R - \left\langle \partial_x m^R\right\rangle_{Z_R(x_0)}|^2 \\
&= 0.
\end{aligned}$$

We now prove (3.55). For all $x_0 \in \mathbb{R}$ we have

$$\begin{aligned}
\left|\int_{Z_R(x_0)} \partial_x m^R\right| &\le \int_{Z_R(x_0)} \left(\left||\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}}|\overline{m}_y^R|\right| + \frac{1}{\sqrt{2}}|\overline{m}_y^R|\right) \\
&\le \sqrt{|Z_R|}\left\||\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}}|\overline{m}_y^R|\right\|_{L^2(Z_R(x_0))} + \frac{1}{\sqrt{2}}|Z_R| \\
&\le \sqrt{|Z_R|}\left\||\partial_x \overline{m}^R| - \frac{1}{\sqrt{2}}|\overline{m}_y^R|\right\|_{L^2(\Sigma(R))} + \frac{1}{\sqrt{2}}|Z_R|.
\end{aligned}$$

Thus Lemma 3.26 implies

$$\lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{1}{R^{2+\beta}}\left|\int_{Z_R(x_0)} \partial_x m^R\right| \;=\; 0,$$

and with Lemma 3.27 we obtain

$$\begin{aligned}
\lim_{R\to 0} \sup_{(x_0,y_0)\in\Sigma(R)} \frac{1}{R^{2+\beta}} \int\limits_{B_{\frac{R}{2}}(x_0,y_0)} |\partial_x m^R|^2 \;&\leq\; \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{4}{R^{2+\beta}} \int_{Z_R(x_0)} \|\partial_x m^R|^2 \\
= \lim_{R\to 0}\sup_{x_0\in\mathbb{R}} \frac{4}{R^{2+\beta}} &\left(\int_{Z_R(x_0)} \left|\partial_x m^R - \langle \partial_x m^R\rangle_{Z_R(x_0)}\right|^2 + \left| \int_{Z_R(x_0)} \partial_x m^R \right|^2 \right) \\
= 0.
\end{aligned}$$

□

3.6.2 Bounds on balls with radius smaller than R

In this subsection, let ζ be as in Theorem 3.16. Moreover, choose $a \in \Sigma(R)$, $\rho \leq \frac{1}{2}R$ and define $v, w \colon B_\rho(a) \to \mathbb{R}^3$ as the weak solutions of

$$\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\,\nabla v\right) = 0 \qquad \text{in } B_\rho(a), \tag{3.57}$$

$$v = m^R \qquad \text{on } \partial B_\rho(a), \tag{3.58}$$

$$\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\,\nabla w\right) = \frac{1}{\kappa}\,|A^R\nabla m^R|^2 m^R + \zeta \qquad \text{in } B_\rho(a), \tag{3.59}$$

$$w = 0 \qquad \text{on } \partial B_\rho(a). \tag{3.60}$$

These definitions will be valid for the rest of this Section. For v we have the estimates of Lemma 3.20 uniformly in R.

Lemma 3.29. *There exists a constant C_{InEst} such that for all $R \leq 1$, $a \in \Sigma(R)$, $\rho \leq \frac{1}{2}R$, $0 < \eta < 1$,*

$$\frac{1}{(\eta\rho)^3}\int_{B_{\eta\rho}} |\nabla v|^2 \;\leq\; \frac{C_{\text{inEst}}}{\rho^3}\int_{B_\rho} |\nabla v|^2 \tag{3.61}$$

$$\frac{1}{(\eta\rho)^5}\int_{B_{\eta\rho}} \left|\nabla v - \langle\nabla v\rangle_{B_{\eta\rho}}\right|^2 \;\leq\; \frac{C_{\text{inEst}}}{\rho^5}\int_{B_\rho} \left|\nabla v - \langle\nabla v\rangle_{B_\rho}\right|^2 \tag{3.62}$$

Proof. A^R is symmetric and

$$\frac{1}{3}|\nabla v| \;\leq\; \frac{1}{\kappa}|\nabla v|^2 \;\leq\; \left|\frac{1}{\sqrt{\kappa}}A^R\nabla v\right|^2 \;\leq\; \kappa|\nabla v|^2 \;\leq\; 3|\nabla v|^2.$$

For each radius R we have a different function κ. Removing this ambiguity and writing κ_R we have

$$\left\|D^n\left(\frac{1}{\sqrt{\kappa}_R}A^R\right)\right\|_{C_0(\Sigma(\frac{3}{2}R))} = \frac{1}{R^n}\left\|D^n\left(\frac{1}{\sqrt{\kappa}_1}A^1\right)\right\|_{C_0(\Sigma(\frac{3}{2}))}$$

Therefore we get, with the notation of Lemma 3.20, a uniform lower bound for $\underline{\lambda}$ and uniform upper bounds for $\overline{\lambda}$ and K. Thus Lemma 3.20 implies the result. $\square$

Using the techniques described in Section 3.4, we will prove $C^{1,\beta}$-regularity by showing scaled L^2-estimates. First, in Lemma 3.30, we will consider $\frac{1}{r^\gamma}\int_{B_r}\left|\nabla m^R\right|^2$ for $\gamma < 3$. Then, in Lemma 3.31, we will use Lemma 3.30 to treat $\frac{1}{r^\gamma}\int_{B_r}|\nabla m^R - \langle\nabla m^R\rangle_{B_r}|^2$ for $3 \le \gamma \le 3 + \frac{1}{8}$.

Lemma 3.30. *For each $\gamma < 3$ there exists $R_0 = R_0(\gamma)$ such that for all $R \le R_0$, all $a \in \Sigma(R)$ and all $r \le \frac{1}{2}R$ we have*

$$\frac{1}{r^\gamma}\int_{B_r(a)}\left|\nabla m^R\right|^2 \;\le\; 1.$$

Proof. We assume that $R \le 1$ is so small that with Theorem 3.25

$$|m^R(p) - m^R(p')| \le C_s|p-p'|^{\frac{1}{4}} \qquad \text{for all } p,p' \in \Sigma(\tfrac{3}{2}R) \text{ with } |p-p'| < R.$$

First, we show that $|w|$ is small. Since $\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\nabla v_i\right) = 0$ in $B_\rho(a)$ for $i \in \{1,2,3\}$ the maximum principle yields

$$\sup_{p\in B_\rho(a)} v_i(p) \;=\; \sup_{p\in\partial B_\rho(a)} m_i^R(p), \qquad \inf_{p\in B_\rho(a)} v_i(p) \;=\; \inf_{p\in\partial B_\rho(a)} m_i^R(p).$$

Thus

$$|w| \;=\; |m^R - v| \;\le\; \sum_{i=1}^{3} \sup_{p\in\partial B_\rho(a)} m_i^R(p) - \inf_{p\in\partial B_\rho(a)} m_i^R(p) \;\le\; 6C_s\rho^{\frac{1}{4}}. \quad (3.63)$$

Since $|\zeta(p)| \le |\nabla u(p)|$ for all $p \in \mathbb{R}^3$, Lemma 3.21 implies

$$\int_{B_\rho(a)} |\zeta| \;\le\; C_\gamma\rho^\gamma. \quad (3.64)$$

Testing (3.59) with w yields

$$\begin{aligned}
\int_{B_\rho(a)} |\nabla w|^2 \;&\le\; 3\int_{B_\rho(a)} \frac{1}{\kappa}|A^R\nabla w|^2 \;=\; 3\int_{B_\rho(a)} \operatorname{div}\left(\frac{1}{\kappa}(A^R)^2\nabla w\right)w \\
&=\; 3\int_{B_\rho(a)} \frac{1}{\kappa}|A^R\nabla m^R|^2 m w + \zeta w \\
&\le\; 3\,\|w\|_{C_0(B_\rho(a))}\int_{B_\rho(a)} 3|\nabla m^R|^2 + |\zeta| \\
&\le\; 54C_s\rho^{\frac{1}{4}}\left(\int_{B_\rho(a)}\left|\nabla m^R\right|^2\right) + 18C_sC_\gamma\rho^{\frac{1}{4}+\gamma}.
\end{aligned}$$

Moreover, using Lemma 3.29, for all $\eta \le 1$ we have

$$\int_{B_{\eta\rho}(a)} |\nabla v|^2 \;\le\; C_{\mathrm{inEst}}\eta^3 \int_{B_\rho(a)} |\nabla v|^2 \;\le\; 3C_{\mathrm{inEst}}\eta^3 \int_{B_\rho(a)} \frac{1}{\kappa}|A^R\nabla v|^2$$
$$\overset{*}{\le}\; 3C_{\mathrm{inEst}}\eta^3 \int_{B_\rho(a)} \frac{1}{\kappa}|A^R\nabla m^R|^2 \;\le\; 9C_{\mathrm{inEst}}\eta^3 \int_{B_\rho(a)} |\nabla m^R|^2.$$

For inequality $(*)$ we have used that v is the minimiser of

$$g \mapsto \int_{B_\rho(a)} \frac{1}{\kappa}|A^R\nabla g|^2 \quad \text{in } \left\{g \in H^1(B_\rho(a)) \,:\, g|_{\partial B_\rho(a)} = m^R|_{\partial B_\rho(a)}\right\}.$$

Combining the estimates, we get for all $0 < \eta \le 1$

$$\frac{1}{(\eta\rho)^\gamma}\int_{B_{\eta\rho}(a)} |\nabla m^R|^2 \;\le\; \frac{2}{(\eta\rho)^\gamma}\left(\int_{B_{\eta\rho}(a)} |\nabla v|^2\right) + \frac{2}{(\eta\rho)^\gamma}\left(\int_{B_\rho(a)} |\nabla w|^2\right)$$
$$\le\; \left(18C_{\mathrm{inEst}}\eta^{3-\gamma} + \frac{108C_s\rho^{\frac14}}{\eta^\gamma}\right)\left(\frac{1}{\rho^\gamma}\int_{B_\rho(a)} |\nabla m^R|^2\right) + \frac{36C_sC_\gamma}{\eta^\gamma}\rho^{\frac14}.$$

Now let η be the largest number such that

$$18C_{\mathrm{inEst}}\eta^{3-\gamma} \;\le\; \frac14,$$

and let r_0 be the largest number such that

$$\frac{108C_s}{\eta^\gamma}r_0^{\frac14} \;\le\; \frac14, \qquad \frac{36C_sC_\gamma}{\eta^\gamma}r_0^{\frac14} \;\le\; \frac14\eta^\gamma.$$

Then, if $\rho \le r_0$, we have

$$\frac{1}{(\eta\rho)^\gamma}\int_{B_{\eta\rho}(a)} |\nabla m^R|^2 \;\le\; \frac{1}{2\rho^\gamma}\int_{B_{\eta\rho}(a)} |\nabla m^R|^2 + \frac14\eta^\gamma,$$

so Lemma 3.18 yields for all $r \le \rho \le r_0$

$$\frac{1}{r^\gamma}\int_{B_r(a)} |\nabla m^R|^2 \;\le\; \frac{1}{(\eta\rho)^\gamma}\left(\int_{B_\rho(a)} |\nabla m^R|^2\right) + \frac12.$$

Using Theorem 3.28, we can find $R_0 \le r_0$ such that for all $R \le R_0$ and all $a \in \Sigma(R)$ we have $\frac{1}{R^\gamma}\int_{B_{\frac{R}{2}}(a)} |\nabla m^R|^2 \le \frac12\eta^\gamma$. Then we have for all $r \le \frac12 R$ the estimate

$$\frac{1}{r^\gamma}\int_{B_r(a)} |\nabla m^R|^2 \;\le\; 1.$$

□

Lemma 3.31. *For all $\beta < \frac{1}{8}$ there exist positive constants $R_1 = R_1(\beta)$, $C = C(\beta)$ such that for all $R \leq R_1$, all $r \leq \frac{1}{2}R$ and all $a \in \Sigma(R)$ we have*

$$\int_{B_r(a)} |\nabla m^R - \langle \nabla m^R \rangle_{B_r(a)}|^2 \;\leq\; C r^{3+2\beta}.$$

Proof. Set $\gamma := 3 + 2\beta - \frac{1}{4}$, let $R_0(\gamma)$ as in Lemma 3.30 and assume that $R \leq R_0(\gamma)$. Then, using (3.63), (3.64), and Lemma 3.30, we have for all $a \in \Sigma(R)$, $\rho \leq \frac{1}{2}R$

$$\begin{aligned}\int_{B_\rho(a)} |\nabla w|^2 \;&\leq\; 3\int_{B_\rho(a)} \frac{1}{\kappa}|A^R\nabla w|^2 \;=\; 3\int_{B_\rho(a)} \left(\frac{1}{\kappa}|A^R\nabla m^R|^2 m^R\cdot w + \zeta\cdot w\right)\\ &\leq\; 6C_s\rho^{\frac{1}{4}}\int_{B_\rho} 3|\nabla m^R|^2 + |\zeta| \;\leq\; 6C_s(3+C_\gamma)\rho^{\frac{1}{4}+\gamma}\\ &=\; 6C_s(3+C_\gamma)\rho^{3+2\beta}.\end{aligned}$$

The function $s : p \mapsto v(p) - \langle \nabla m \rangle_{B_\rho(a)} \cdot (p-a)$ is in

$$S := \left\{ g : B_\rho(a) \to \mathbb{R}^3 \mid g_i(p) = m_i(p) - \langle \nabla m_i \rangle_{B_\rho(a)} \cdot (p-a) \text{ on } \partial B_\rho(a) \right\}$$

and satisfies

$$\operatorname{div}\left(\frac{1}{\kappa}(A^R)^2 s\right) = 0 \quad \text{in } B_\rho(a).$$

Thus s is a minimizer of $m \mapsto \int_{B_\rho(a)} \left|\frac{1}{\sqrt{\kappa}}A^R\nabla m\right|^2$ in S. We have in particular

$$\int\limits_{B_\rho(a)} \left|\frac{1}{\sqrt{\kappa}}A^R\left(\nabla v - \langle \nabla m \rangle_{B_\rho(a)}\right)\right|^2 \;\leq\; \int\limits_{B_\rho(a)} \left|\frac{1}{\sqrt{\kappa}}A^R\left(\nabla m - \langle \nabla m \rangle_{B_\rho(a)}\right)\right|^2.$$

Therefore, with Lemma 3.29 and (3.27), we obtain

$$\begin{aligned}\int_{B_{\eta\rho}(a)} \left|\nabla v - \langle \nabla v \rangle_{B_{\eta\rho}(a)}\right|^2 \;&\leq\; C_{\text{inEst}}\eta^5 \int_{B_\rho(a)} \left|\nabla v - \langle \nabla v \rangle_{B_\rho(a)}\right|^2\\ &\leq\; C_{\text{inEst}}\eta^5 \int_{B_\rho(a)} \left|\nabla v - \langle \nabla m \rangle_{B_\rho(a)}\right|^2\\ &\leq\; 3C_{\text{inEst}}\eta^5 \int_{B_\rho(a)} \left|\frac{1}{\sqrt{\kappa}}A^R\left(\nabla v - \langle \nabla m \rangle_{B_\rho(a)}\right)\right|^2\\ &\leq\; 3C_{\text{inEst}}\eta^5 \int_{B_\rho(a)} \left|\frac{1}{\sqrt{\kappa}}A^R\left(\nabla m - \langle \nabla m \rangle_{B_\rho(a)}\right)\right|^2\\ &\leq\; 9C_{\text{inEst}}\eta^5 \int_{B_\rho(a)} \left|\nabla m^R - \langle \nabla m^R \rangle_{B_\rho(a)}\right|^2.\end{aligned}$$

Thus, using again (3.27),

$$\begin{aligned}
&\frac{1}{(\eta\rho)^{3+2\beta}} \int_{B_{\eta\rho}(a)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{B_{\eta\rho}(a)}\right|^2 \\
&\le \frac{1}{(\eta\rho)^{3+2\beta}} \int_{B_{\eta\rho}(a)} \left|\nabla m^R - \langle \nabla v \rangle_{B_{\eta\rho}(a)}\right|^2 \\
&\le \frac{2}{(\eta\rho)^{3+2\beta}} \int_{B_{\eta\rho}(a)} \left|\nabla v - \langle \nabla v \rangle_{B_{\eta\rho}(a)}\right|^2 + \frac{2}{(\eta\rho)^{3+2\beta}} \int_{B_\rho(a)} |\nabla w|^2 \\
&\le \frac{18 C_{\text{inEst}} \eta^{2-2\beta}}{\rho^{3+2\beta}} \left(\int_{B_\rho(a)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{B_\rho(a)}\right|^2 \right) + \frac{12 C_s (3 + C_\gamma)}{\eta^{3+2\beta}}.
\end{aligned}$$

Now let η be the largest numbers such that $18 C_{\text{inEst}} \eta^{2-2\beta} \le \frac{1}{2}$. Then, with Lemma 3.18, for all $0 < r < \rho$, we get

$$\begin{aligned}
&\frac{1}{r^{3+2\beta}} \int_{B_r(a)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{B_r(a)}\right|^2 \\
&\le \frac{1}{\eta^{3+2\beta}} \frac{1}{\rho^{3+2\beta}} \int_{B_\rho(a)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{B_\rho(a)}\right|^2 + \frac{24 C_s (3 + C_\gamma)}{\eta^{6+4\beta}}.
\end{aligned}$$

In particular, setting $\rho := R$ and using Theorem 3.28, we see that there exists a constant C such that for all $r \le R \le R_0(\gamma)$

$$\int_{B_r(a)} \left|\nabla m^R - \left\langle \nabla m^R \right\rangle_{B_r(a)}\right|^2 \le C r^{3+2\beta}.$$

□

Now we come to the main regularity theorem. We prove uniformly good regularity for the functions m^R. These bounds of strong norms then imply convergence in slightly weaker norms.

Theorem 3.32. *For each $\beta < \frac{1}{8}$ there exist positive constants $R_{C^{1,\beta}}$, $C_{C^{1,\beta}}$, such that for all $R \le R_{C^{1,\beta}}$*

Proof. Let $R_1 = R_1(\beta)$ as in Lemma 3.31, and let $R \le \min\{R_1, 1\}$ be so small that Lemma 2.29 and Lemma 3.26 yield

$$\|\partial_x \overline{m}^R - \frac{1}{\sqrt{2}} \overline{m}^R_y\|^2_{L^2(\Sigma)} \le R^{3+\frac{1}{2}}, \qquad \|\nabla \tilde{m}^R\|^2_{L^2(\Sigma)} \le R^{3+\frac{1}{2}}. \tag{3.65}$$

Using the integral characterisation of Hölder continuous functions (Theorem 3.17) we get locally, on the scale of R, a uniform bound of ∇m^R in $C^{0,\beta}$, i.e., there exists C depending only on β, such that

$$|\nabla m^R(p) - \nabla m^R(p')| \le C |p - p'|^\beta \qquad \text{if } |p - p'| \le R. \tag{3.66}$$

In particular, we have $\Big(\nabla m^R(p) - \langle \nabla m^R \rangle_{Z_R(x)}\Big) \le 2CR^\beta$ for all $p \in Z_R(x)$ where $Z_R(x)$ as in (3.53). Therefore

$$\begin{aligned}
&\left|\nabla m^R(p) - \frac{1}{\sqrt{2}}\langle \overline{m}_y^R\rangle_{Z_R(x)}\right| \\
&\le \ \Big| \Big(\nabla m^R(p) - \langle \nabla m^R\rangle_{Z_R(x)}\Big) + \langle \nabla \tilde{m}^R\rangle_{Z_R(x)} \\
&\quad + \Big(\langle \partial_x \overline{m}^R\rangle_{Z_R(x)} - \frac{1}{\sqrt{2}}\langle \overline{m}_y^R\rangle_{Z_R(x)}\Big) \Big| \\
&\overset{*}{\le} \ 2CR^\beta + \frac{\sqrt{|Z_R|}}{|Z_R|}\left\|\nabla \tilde{m}^R\right\|_{L^2(Z_R)} + \frac{\sqrt{|Z_R|}}{|Z_R|}\left\|\partial_x \overline{m}^R - \frac{1}{\sqrt{2}}\overline{m}_y^R\right\|_{L^2(\Sigma)} \\
&\le \ 2CR^\beta + \sqrt{\frac{R^{3+\frac{1}{2}}}{|Z_R|}} + \sqrt{\frac{R^{3+\frac{1}{2}}}{|Z_R|}} \ \le \ (2C+1)R^\beta.
\end{aligned}$$

For the estimate $(*)$ we have used (3.65).
Since $\langle \overline{m}_y^R\rangle_{Z_R(x)} \le 1$, this calculation shows that $|\nabla m^R|$ is bounded by some constant $\tilde{C}$, which then yields

$$\begin{aligned}
&|\nabla m^R(p) - \nabla m^R(p')| \\
&\quad \le \ \left|\nabla m^R(p) - \langle \overline{m}_y^R\rangle_{Z_R(x)}\right| + \left|\langle \overline{m}_y^R\rangle_{Z_R(x)} - \langle \overline{m}_y^R\rangle_{Z_R(x')}\right| \\
&\qquad + \left|\nabla m^R(p') - \langle \overline{m}_y^R\rangle_{Z_R(x')}\right| \\
&\quad \le \ 2(2C+1)R^\beta + \tilde{C}|x-x'|.
\end{aligned}$$

Thus we have for all $p, p' \in \Sigma(R)$ the estimate

$$\left|\nabla m^R(p) - \nabla m^R(p')\right| \ \le \ \begin{cases} C|p-p'|^\beta & \text{if } |p-p'| \le R \\ (4C+2+\tilde{C})|p-p'|^\beta & \text{if } R < |p-p'| \le 1 \\ 2\tilde{C}|p-p'|^\beta, & \text{if } 1 < |p-p'|. \end{cases}$$

□

Theorem 3.33. *(i) For R small enough, $m^R \in H^2(\Sigma(R)) \cap C^1(\Sigma(R))$.*
(ii) We have

$$\lim_{R\to 0} \frac{1}{R}\|m^R - m^{\text{red}}\|_{H^1(\Sigma(R))} \ = \ 0, \tag{3.67}$$

$$\lim_{R\to 0} \frac{1}{R}\|\Phi(m^R) - \Phi(m^{\text{red}})\|_{L^2(\Sigma(R))} \ = \ 0, \tag{3.68}$$

$$\lim_{R\to 0} \|m^R - m^{\text{red}}\|_{C^1(\Sigma(R))} \ = \ 0, \tag{3.69}$$

$$\lim_{R\to 0} \|\Phi(m^R) - \Phi(m^{\text{red}})\|_{C^0(\Sigma(R))} \ = \ 0. \tag{3.70}$$

Proof. (i) Because of Theorem 3.32 we have $m^R \in C^1(\Sigma(R))$, and because of Lemma 3.19 we have $m^R \in H^2(\Sigma(R))$.

(ii) (3.67) and (3.68) are a consequence of Lemma 2.28. (3.67) and (3.68) are a consequence of Theorem 3.32. □

Chapter 4

The vortex mode: Travelling waves with a singularity

4.1 Introduction

In this chapter we study a simplified model for the vortex mode which captures the highest order terms with respect to the derivatives. This model is related to the harmonic map heat flow equation (1.4). Using variational methods, we show that there exist travelling wave solutions with corotational symmetry and thus a moving singularity.

The harmonic map heat flow equation has been extensively studied, but most of the time on bounded domains (cf. [37] and references therein). Bertsch, Muratov and Primi [6] consider it in an infinite cylinder and investigate travelling wave solutions. Since the equation itself does not contain any driving force, the travelling waves have to be "pulled" by the boundary conditions. Our methods to prove existence of travelling waves rely on the same variational principles as the methods used in that article.

4.1.1 The model

We consider (1.4) with an additional external magnetic field $h\vec{e}_x$ in direction of $\vec{e}_x$ to model the reversal of magnetic nanowires via the vortex mode. By this, we get travelling wave solutions without imposing any special boundary conditions that would be physically difficult to realize.

$$E(m) := E_{\mathrm{ex}}(m) - hJ_s\vec{e}_x \cdot m, \tag{4.1}$$

$$\begin{aligned} \partial_t m &= \alpha\left(-\delta E_{\mathrm{ex}}(m) + (\delta E_{\mathrm{ex}}(m)\cdot m)m\right) \\ &= \alpha A_{\mathrm{ex}}(\Delta m - (\Delta m \cdot m)m) + \alpha h J_s(\vec{e}_x - (\vec{e}_x \cdot m)m). \end{aligned} \tag{4.2}$$

Moreover, we assume that the magnetisation in each point is tangential to the closest boundary. This ensures that we have a magnetisation without surface charges.

At first glance this model seems inappropriate because the vortex mode only appears in thick wires, where the stray field energy is important (cf. Chapter 2). However, we have the following picture in mind: The *existence* of the singularity in the vortex mode is due to the strong influence of the stray field energy that prevents surface charges, but the properties of the *evolution* of a magnetisation with a singularity is mainly determined by the highest order terms with respect to the derivatives.

We now change to a coordinate system that is better adapted our problem. As before, let $\Sigma := \mathbb{R} \times D_R := \mathbb{R} \times \{y \in \mathbb{R}^2 \ : \ |y| \leq R\}$ be the infinite cylinder with radius R. We describe the magnetisation $m\colon \Sigma \to \mathbb{S}^2$ by spherical coordinates $(\gamma, \theta)\colon \Sigma \to [0, 2\pi[\times[0, \pi]$ (cf. Figure 4.1). Then

$$m = \begin{pmatrix} -\cos\theta \\ \sin\theta\,\cos\gamma \\ \sin\theta\,\sin\gamma \end{pmatrix} \quad \text{and} \quad \nabla m = \begin{pmatrix} \sin\theta\nabla\theta \\ \cos\theta\,\cos\gamma\,\nabla\theta - \sin\theta\,\sin\gamma\,\nabla\gamma \\ \cos\theta\,\sin\gamma\,\nabla\theta + \sin\theta\cos\gamma\,\nabla\gamma \end{pmatrix}.$$

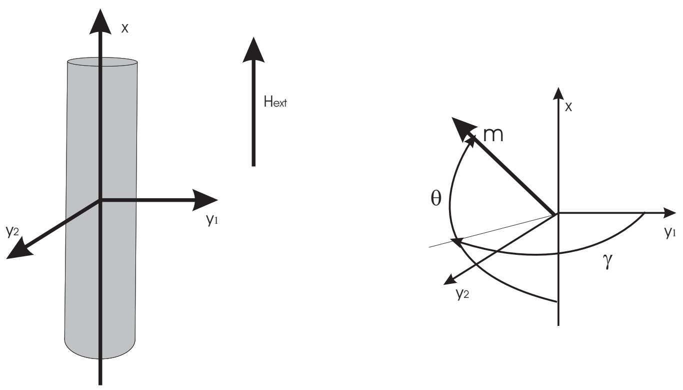

Figure 4.1: The coordinate system in the domain and in the range

In spherical coordinates, the assumption that the magnetisation is tangential to the closest boundary in each point (x, y) with $y \neq 0$ is equivalent to the equality $\gamma = \arctan\left(\frac{-y_1}{y_3}\right)$. In this case we have $|\nabla\gamma|^2 = \frac{1}{|y|^2}$ and the energy (4.1) is

$$E(\theta) \ = \ \int_\Sigma A_{\text{ex}}|\nabla\theta|^2 + \frac{A}{|y^2|}\sin^2(\theta) + HJ_s(\cos\theta). \tag{4.3}$$

To get the harmonic map heat flow equation with additional external magnetic field in sperical coordinates, we can transform (4.2) or use the variation of the energy (4.3). Choosing the second possibility we get

$$\begin{aligned}\partial_t\theta &= -\alpha\left(\delta_m E - (\delta_m E \cdot m)m\right)\partial_\theta m \\ &= -\alpha\delta_\theta E \;=\; \alpha\left(2A_{\text{ex}}\Delta\theta - \frac{A_{\text{ex}}}{|y^2|}\sin(2\theta) + hJ_s\sin\theta\right).\end{aligned}$$

Since we are interested in travelling wave solutions of this equation, we replace $\partial_t\theta$ by $-c\partial_x\theta$, where c is the speed of the travelling wave. We rescale to get rid of the constants, and to have θ in the interval $[0,1]$. We measure length in multiples of the radius R, the magnetic field in multiples of $\frac{A_{\text{ex}}}{R^2 J_s}$, time in multiples of $\frac{R^2}{2\alpha A_{\text{ex}}}$ and θ in multiples of π. Since there is no influence from the rest of the space on the magnetisation of the wire, the appropriate boundary conditions are Neumann boundary conditions, which are also the natural boundary conditions for the energy E. Thus we get the equation

$$\Delta\theta + c\partial_x\theta + f^0(y,\theta) = 0 \text{ in } \Sigma, \qquad \partial_\nu\theta = 0 \text{ on } \partial\Sigma, \tag{4.4}$$

where

$$f^0(\theta, y) := -\frac{1}{2\pi y^2}\sin(2\pi\theta) + \frac{h}{\pi}\sin(\pi\theta).$$

Note that we have normalised such that the radius of the wire is 1.

4.1.2 The main results

In this chapter we show results about the existence and the speed of travelling waves and discuss possible end states.

In Section 4.2 and Section 4.3 we will treat the problem of existence and speed of travelling waves. The following theorem summarises our results.

Theorem 4.1. *For all $h > 0$ there exists a monotone solution (c,θ) of* (4.4) *such that $u(x,0) \in \{0,1\}$ almost everywhere and $\underline{c} \le c \le \overline{c}$ with*

$$\overline{c} := 2\sqrt{h} \quad \textit{and} \quad \underline{c} := \begin{cases} \frac{2h}{5\pi} & \textit{for } 0 < h < h_c \\ 2\sqrt{h - k_0^2} & \textit{for } h_c \le h \end{cases}$$

where $k_0 \approx 1.8$ is the first root of the Bessel function J_1 and $h_c \approx 0.05 + k_0^2$ is the smaller one of the two solutions of the equation $\frac{2h}{5\pi} = 2\sqrt{h - k_0^2}$.

There are two possibilities:

- *In the variational case there exists a solution $(c^\dagger, \theta^\dagger)$ such that $\theta^\dagger$ is a minimiser of the functional $\Phi^0_{c^\dagger}$ (See* (4.10)*) and $(c^\dagger)^2 > 4(h - k_0^2)$.*
- *In the non-variational case there is a solution (c^*, θ^*) such that $c^* = 2\sqrt{h - k_0^2}$.*

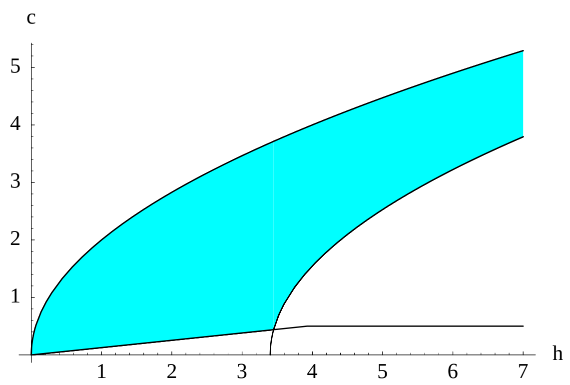

Figure 4.2: Upper bound $\overline{c}$ and lower bound $\underline{c}$ for the speed

There exists some $h_0 \in [h_c, \infty]$ such that for all $h < h_0$ we have the variational case and for all $h \geq h_0$ we have the non-variational case.

In Section 4.4 and Section 4.5 we consider possible end states and find the following theorem.

Theorem 4.2. *We have*

$$\lim_{x \to -\infty} \theta(x, \cdot) = \theta_- = 0 \qquad \text{in } C^\infty_{\text{loc}}(D_1 \setminus \{0\}),$$
$$\lim_{x \to \infty} \theta(x, \cdot) = \theta_+ \qquad \text{in } C^\infty_{\text{loc}}(D_1 \setminus \{0\})$$

for some semistable stationary θ_+ state (see Definition 4.31). For $h \leq 2$ and $h \geq k_0^2 + 1$ we know $\theta_+ \equiv 1$. Here k_0 is as in Theorem 4.1. In the variational case, $\theta(x, \cdot)$ converges also in $L^\infty(D_1)$ to the endstates $\theta_\pm$.

Remark 4.3. In the variational case with $\theta_+ \equiv 1$, because of convergence in $L^\infty(D_1)$, there has to exist some point x_0 such that $\theta^\dagger$ jumps in x_0 form zero to one . Thus in this case we have a discontinuous travelling wave. In the other cases we may have this jump as well, but it is also possible that on the whole x-axis we have $\theta(\cdot, 0) \equiv 0$ or $\theta(\cdot, 0) \equiv 1$.

Remark 4.4. Theorem 4.1 is a combination of Theorem 4.25 and Theorem 4.28 in Section 4.3.2, Theorem 4.2 is a short version of Theorem 4.52 in Section 4.5.2.

4.1.3 Outline of the chapter

In Section 4.2 we recall results of [31] concerning variational methods for travelling wave problems.

In Section 4.3 we apply these results to our problem and show that for each $h > 0$ there exist solutions of (4.4). Moreover we derive some properties of the travelling waves, including bounds on the speed.

Clearly, the possible end states are stationary states, i.e., solutions of (4.4) that do not depend on x. In Section 4.4, we analyse properties of stationary states.

In Section 4.5 we first show that the stationary state at $-\infty$ is semistable. Then we combine the results about stationary states with the results of Section 4.3. We conclude that for $h < 2$ and $h \geq 4.38$ the solutions of (4.4) found in Section 4.3 converge to zero at $+\infty$ and converge to one at $-\infty$.

4.2 Variational methods for travelling wave problems

The difficulty of (4.4) is the singularity of the function f at $y = 0$. Therefore the usual methods to get travelling wave solutions are not applicable. When we replace f by a continuous function, the problem becomes much simpler. We set

$$\begin{aligned} \Delta\theta + c\partial_x\theta + f^\epsilon(y,\theta) &= 0 \quad \text{in } \Sigma, \\ \partial_\nu\theta &= 0 \quad \text{on } \partial\Sigma, \end{aligned} \tag{4.5}$$

where

$$f^\epsilon(\theta, y) := -\frac{1}{2\pi y^2 + \epsilon^2}\sin(2\pi\theta) + \frac{h}{\pi}\sin(\pi\theta).$$

Now we could use standard methods to find solutions of (4.5), applying techniques relying on the maximum principle, as described in [5]. However, these methods do not give uniform bounds on the speed c^ϵ, so it is impossible to prove the convergence of a subsequence of solutions of (4.5) for $\epsilon \to 0$. Therefore we use a variational principle for travelling wave equations. Such a principle was developed by Heinze [21] and by Lucia, Muratov and Novaga [30, 29, 28, 31].

In this section we present some of the results of [31]. In that article the authors give a comprehensive treatment of the properties of the travelling waves that correspond to minimisers of a certain functional. In addition they use variational methods to find travelling wave solutions in the case where minimisers of the functional do not exist.

To be consistent with the other chapters, we use a different orientation than [31]. In [31], waves have a positive speed and connect an energetically favourable stable state v_- with $0 < v_- \leq 1$ at $-\infty$ with the state $v_+ = 0$ at $+\infty$. In this thesis, waves have a negative speed and connect the state $w_- = 0$ at $-\infty$ with an energetically favourable stable state w_+ with $0 < w_+ \leq 1$ at $+\infty$.

Muratov and Novaga [31] study the solutions (c, u) of the equations

$$\left.\begin{aligned} \Delta u + c\,\partial_x u + \nabla_y \phi \cdot \nabla_y u + f(u,y) &= 0 \quad \text{in } \Sigma_\Omega \\ u &= 0 \quad \text{on } \partial\Sigma_\Omega^{\pm}, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Sigma_\Omega^{0}. \end{aligned}\right\} \tag{4.6}$$

Here, $\Sigma_\Omega = \mathbb{R} \times \Omega$, the boundary $\partial\Sigma_\Omega$ is of class C^2 and the disjoint union of $\partial\Sigma_\Omega^0$ and $\partial\Sigma_\Omega^{\pm}$. The term $\nabla\phi$ is a convection term that is important for the application in combustion theory that is studied by the authors. We need the results of [31] only in the case $\phi = 0$ and $\Sigma_\Omega^{\pm} = \emptyset$, thus we consider the equation

$$\begin{aligned} \Delta u + c\,\partial_x u + f(u,y) &= 0 \quad \text{in } \Sigma_\Omega, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Sigma_\Omega. \end{aligned} \tag{4.7}$$

In order to guarantee that there is a solution of (4.7) such that u has values in $]0,1[$, we have to make the following assumptions:

(H1) The function $f\colon [0,1] \times \overline{\Omega} \to \mathbb{R}$ satisfies for all $y \in \Omega$ the equations

$$f(0,y) = 0, \qquad f(1,y) \leq 0.$$

(H2) For some $\gamma \in]0,1[$,

$$f \in C^{0,\gamma}([0,1] \times \Omega), \qquad \partial_u f \in C^{0,\gamma}([0,1] \times \Omega).$$

To formulate a third hypothesis, we need some definitions. Let $L_c^2(\Sigma_\Omega)$ be the Hilbert space with the weighted norm

$$\|u\|_{L_c^2(\Sigma_\Omega)} := \sqrt{\int_{\Sigma_\Omega} e^{cx} u^2 \, dp}$$

and let $H_c^1(\Sigma_\Omega)$ be the Hilbert space of all functions for which

$$\|u\|_{H_c^1(\Sigma_\Omega)} := \sqrt{\|u\|^2_{L_c^2(\Sigma_\Omega)} + \|\nabla u\|^2_{L_c^2(\Sigma_\Omega)}}$$

is finite. We define the functional

$$\Phi_c : H_c^1(\Sigma_\Omega) \to \mathbb{R}, \qquad u \mapsto \int_{\Sigma_\Omega} e^{cx} \left(\frac{1}{2} |\nabla u|^2 + V(u,y) \right) dp,$$

where

$$V(u,y) = \begin{cases} 0 & \text{for } u < 0, \\ -\int_0^u f(s,y) ds & \text{for } 0 \leq u \leq 1, \\ -\int_0^1 f(s,y) ds & \text{for } 1 < u. \end{cases}$$

so that $\partial_u V = -f(u)\mathbb{1}_{[0,1]}$. Note that (4.7) is the Euler-Lagrange equation of the functional Φ_c, thus every critical point u of Φ_c that satisfies $0 \leq u \leq 1$ is a solution of (4.7). Finally, define the auxiliary functional

$$I : H^1(\Omega) \to \mathbb{R}, \qquad v \mapsto \int_\Omega \frac{1}{2}|\nabla_y v|^2 + V(v,y)\, dy.$$

In order to investigate (4.7), it is important to consider the linearisation of (4.7) at the end states at $\pm\infty$.

Let μ be the smallest eigenvalue of $-\Delta u - \partial_u f(0,y)$ and let ψ be the corresponding eigenfunction. Analogously, let $\tilde{\mu}$ be the smallest eigenvalue of $-\Delta\tilde{\psi} - \partial_u f(u_+,y)\tilde{\psi}$, where u_+ is the end state at $+\infty$, and let $\tilde{\psi}$ be the corresponding eigenfunction.

Using these definitions, we can formulate the hypothesis that is crucial for the existence of a number $c^\dagger$ such that $\Phi_{c^\dagger}$ has a nontrivial minimiser.

(H3) There exists $c < 0$ satisfying $c^2 + 4\mu > 0$ and $u \in H^1_c(\Sigma_\Omega)$, such that $\Phi_c(u) \leq 0$ and $u \not\equiv 0$.

Definition 4.5. The number c in (H3) is called *admissible trial velocity.*

We now present the two main results about the existence of travelling waves and their properties. Theorem 4.6 corresponds to [31, Theorem 3.3] and considers the case when (H3) is satisfied, Theorem 4.7 corresponds to [31, Theorem 4.2] and considers the case when (H3) is not satisfied.

Theorem 4.6. *Under hypotheses (H1)-(H3) there exists a unique $c^\dagger < 0$ such that there exists a minimiser $u \not\equiv 0$ of $\Phi_{c^\dagger}$ in $H^1_{c^\dagger}(\Sigma_\Omega)$. This minimiser is unique up to translation. Moreover we have:*

(1.) $c^\dagger \leq c < 0$, where c is the admissible trial velocity given by assumption (H3). We have $u \in C^2(\Sigma_\Omega) \cap W^{1,\infty}(\overline{\Sigma}_\Omega)$, and u solves (4.7) with $c = c^\dagger$.

(2.) $u(x,y)$ is strictly increasing in x for each $y \in \Omega$. We have

$$\lim_{x\to-\infty} u(x,\cdot) = 0 \quad \text{in } C^1(\overline{\Omega}), \qquad \lim_{x\to+\infty} u(x,\cdot) = u_+ \quad \text{in } C^1(\overline{\Omega}),$$

where $u_+\colon \Omega \to]0,1]$ is a critical point of I with $I(u_+) < 0$.

(3.) Set $\lambda_- := \frac{1}{2}\left(c^\dagger + \sqrt{(c^\dagger)^2 + 4\mu}\right)$. There exists $a > 0$ and $\lambda > \lambda_-$ such that

$$\lim_{t\to-\infty} \left\| \left(u(x,y) - a\psi(y)e^{\lambda_- x}\right) e^{-\lambda x} \right\|_{C^1(]-\infty,t]\times\overline{\Omega})} = 0.$$

(*4.*) *We have* $\tilde\mu \geq 0$. *Set* $\lambda_+ := \frac{1}{2}\left(c^\dagger - \sqrt{(c^\dagger)^2 + 4\tilde\mu}\right)$. *If* $\tilde\mu > 0$, *then there exists* $a > 0$ *and* $\lambda > -\lambda_+$ *such that*

$$\lim_{t\to+\infty} \left\| \left(u_+ - u(x,y) - a\psi(y)e^{\lambda_+ x}\right) e^{\lambda x} \right\|_{C^1([t,\infty[\times\overline{\Omega})} = 0.$$

Theorem 4.7. *Assume that hypotheses (H1) and (H2) hold, whereas hypothesis (H3) is not satisfied. Assume in addition that there exists a function* $v \in H^1(\Omega)$, *such that* $I(v) < 0$. *Then there exists* $u \in C^2(\Sigma_\Omega) \cap W^{1,\infty}(\Sigma_\Omega)$ *which solves* (4.7) *with* $c = c_0 := -2\sqrt{|\mu|}$. *Furthermore,* u *has the asymptotic behaviour*

$$u(x,y) = (a - bx)\psi e^{-\frac{1}{2}c_0 x} + O(e^{\lambda x}) \quad \text{as } x \to -\infty$$

for some $\lambda > \frac{c_0}{2}$, *and either* $b > 0$ *or* $b = 0$ *and* $a > 0$. *Assertions (2) and (4) of Theorem 4.6 still hold.*

Since we will work only with solutions provided by the two theorems above, we make the following definition.

Definition 4.8. A solution (c,u) of (4.7) is called an *MN-solution* if it is a solution provided by Theorem 4.6 or in Theorem 4.7. In the first case it is called a *variational MN-solution*, in the latter a *non-variational MN-solution.*

There is no easy criterion to decide whether or not (H3) is satisfied. However, there are necessary and sufficient conditions for (H3). On the one hand, [31, Remark 3.8] and [31, Theorem 3.9] yield the following result.

Proposition 4.9. *If* $\mu \geq 0$, *and if there exists a function* $v \in H^1(\Omega)$ *such that* $I(v) < 0$, *then hypothesis* $(H3)$ *is satisfied.*

On the other hand, in a certain case we know that (H3) is not satisfied ([31, Proposition 4.1]).

Proposition 4.10. *Under hypotheses (H1) and (H2) assume that* $\mu < 0$ *and*

$$\frac{2}{u_0^2}\int_0^{u_0} f(s,y)\,ds \;\leq\; \partial_u f(0,y) \quad \text{for all } y \in \Omega,\ u_0 \in [0,1]. \tag{4.8}$$

Then the hypothesis $(H3)$ *is not satisfied.*

Remark 4.11. If $f(s,y) \leq s\partial_u f(0,y)$ for all $s \in [0,1]$, then (4.8) is satisfied.

Both in the variational and in the non-variational case we have the existence of a solution of (4.7), and a statement about the speed c of the travelling wave. For variational MN-solutions we know $c^2 \geq -4\mu$, for non-variational MN-solutions we have $c = -2\sqrt{-\mu}$. It is easy to see that the speed depends monotonously on the potential V.

Corollary 4.12. *For $f = f_1$, $f = f_2$, let (u_1, c_1) and (u_2, c_2) be MN-solutions of* (4.7). *If $V_1 \leq V_2$ then $c_1 \leq c_2$, i.e., $|c_1| \geq |c_2|$.*

Proof. First, assume that (u_2, c_2) is a non-variational MN-solution. Then

$$c_2 \;=\; -2\sqrt{-\mu_2} \;\geq\; -2\sqrt{-\mu_1} \;\geq\; c_1.$$

Now assume that (u_2, c_2) is a variational MN-solution. If $c_2^2 + 4\mu_1 \leq 0$ then surely $c_1 \leq -2\sqrt{-\mu_1} \leq c_2$. Otherwise the combination

$$c_1 \leq -2\sqrt{-\mu_1} < c_2, \qquad \Phi^1_{c_2}(u_2) \leq \Phi^2_{c_2}(u_2) \leq 0$$

implies that c_2 is an admissible trial velocity for f_1 and we again have $c_1 \leq c_2$. □

4.2.1 A sketch of the proof of Theorem 4.6

We now give a short sketch of the proof of Theorem 4.6. This gives us the opportunity to present some lemmas, used in the proof and necessary for us later.

To show Theorem 4.6, Muratov and Novaga use an auxiliary constrained variational problem. They minimise the functional Φ_c over the set

$$B_c := \left\{ u \in H^1_c(\Sigma_\Omega) \;\middle|\; \frac{1}{2}\|\partial_x u\|^2_{L^2_c(\Sigma_\Omega)} = 1 \right\}. \tag{4.9}$$

First they show that there is a minimiser of Φ_c over the set B_c, and then how the minimiser of the constrained problem is related to the global minimiser of $\Phi_{c^\dagger}$. To show that the mininmiser of the constrained problem exists, they use the direct method. They prove that for an arbitrary function $w \in H^1_c(\Sigma_\Omega)$, the $H^1_c(\Sigma_\Omega)$ norm of w is bounded by $\Phi_c(w)$ and $\|\partial_x w\|^2_{L^1_c(\Sigma_\Omega)}$ (Lemma 4.13). and that the functional Φ_c is weakly lower semi-continuous (Lemma 4.14).

Lemma 4.13. *Assume that f satisfies (H1) and (H2), and define*

$$C_1 := \min_{(u,p)\in[0,1]\times\Sigma_\Omega} \left(\frac{V(u(p), y)}{x^2} \right).$$

Then for each $u \in H^1_c(\Sigma_\Omega)$ we have

$$\begin{aligned} \|u\|^2_{L^2_c(\Sigma_\Omega)} &\leq \frac{4}{c^2}\|\partial_x u\|^2_{L^2_c(\Sigma_\Omega)}, \\ \|\nabla_y u\|^2_{L^2_c(\Sigma_\Omega)} &\leq 2\Phi_c(u) - \frac{8C_1}{c^2}\|\partial_x u\|^2_{L^2_c(\Sigma_\Omega)}. \end{aligned}$$

Proof. This lemma corresponds to [28, Lemma 5.1 and Lemma 5.2]. □

Lemma 4.14. *Let f satisfy hypotheses (H1) and (H2), and let $c^2 + 4\mu > 0$. Then the functional Φ_c is sequentially weakly lower semi-continuous on $H^1_c(\Sigma_\Omega)$.*

Proof. See [28, Prop. 5.5] and the proof of [31, Thm 3.3]. □

To transfer the results for the constrained problem to the full problem, they use the following relation between the functionals Φ_{c_1} and Φ_{c_2}.

Lemma 4.15. *Define $v(x) := u\left(\frac{c_1}{c_2}x\right)$. Then*

$$\Phi_{c_1}(v) \;=\; \frac{c_2}{c_1}\left(\Phi_{c_2}(u) + \frac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\int_{\Sigma_\Omega} e^{c_2 x}|\partial_x u(x)|^2\,dx\right).$$

Proof. This Lemma can be verified by simple calculation,

$$\begin{aligned}
\Phi_{c_1}(v) &= \frac{c_2}{c_1}\Phi_{c_2}(u) - \frac{c_2}{c_1}\frac{1}{2}\int_{\Sigma_\Omega} e^{c_2 x}|\partial_x u(x)|^2\,dx \;+\frac{1}{2}\int_{\Sigma_\Omega} e^{c_1 x}|\partial_x v(x)|^2\,dx \\
&= \frac{c_2}{c_1}\Phi_{c_2}(u) - \frac{c_2}{c_1}\frac{1}{2}\int_{\Sigma_\Omega} e^{c_2 x}|\partial_x u(x)|^2\,dx + \frac{1}{2}\int_{\Sigma_\Omega} \frac{c_1}{c_2}e^{c_2 x}|\partial_x u(x)|^2\,dx \\
&= \frac{c_2}{c_1}\Phi_{c_2}(u) + \frac{1}{2}\left(\frac{c_1}{c_2} - \frac{c_2}{c_1}\right)\int_{\Sigma_\Omega} e^{c_2 x}|\partial_x u(x)|^2\,dx \\
&= \frac{c_2}{c_1}\left(\Phi_{c_2}(u) + \frac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\int_{\Sigma_\Omega} e^{c_2 x}|\partial_x u(x)|^2\,dx\right).
\end{aligned}$$

□

Remark 4.16. Note that Lemma 4.15 implies that (H3) is equivalent to:

(H3') There exists $c < 0$ satisfying $c^2 + 4\mu > 0$ and $u \in H^1_c(\Sigma_\Omega)$ such that $\Phi_c(u) < 0$ and $u \not\equiv 0$.

Remark 4.17. Let $u \not\equiv 0$ be a minimiser of $\Phi_{c^\dagger}$. Then $\Phi_{c^\dagger}(u) = 0$, so Lemma 4.15 implies $c^\dagger = \sup\{c > 0 : c \text{ is admissible trial velocity}\}$.

Summarising their result about the constrained minimiser we have the following lemma.

Lemma 4.18. *Assume that (H1)-(H3) are satisfied, let $c^\dagger$ be as in Theorem 4.6 and let c be an admissible trial velocity for (H3). Then there exists a minimiser u of Φ_c over the set B_c, we have $c^\dagger = c\sqrt{1 - \Phi_c(u_c)}$, and $v(x,y) := u(\frac{c^\dagger}{c}x, y)$ is a minimiser of $\Phi_{c^\dagger}$.*

Proof. The proof is part of the proof of [31, Theorem 3.3]. □

4.3 Existence and properties of travelling wave solutions modelling the vortex mode

In this section we show the existence of solutions of (4.4). In the first subsection we use the theorems of the preceding section to find and investigate solutions of (4.5) and in particular to find good bounds on the speed. In the second subsection we pass to the limit $\epsilon \to 0$.

4.3.1 Preliminary lemmas and properties of travelling wave solutions for a regularised equation

In view of the variational problem, we define for all $\epsilon \geq 0$

$$
\begin{aligned}
V_h^\epsilon \colon [0,1] \times D_1 \to \mathbb{R}, \quad & (u,r) \mapsto \frac{1}{2\pi^2(r^2+\epsilon^2)} \sin^2(\pi u) + \frac{h}{\pi^2}(\cos(\pi u) - 1), \\
I_h^\epsilon \colon H^1(D_1) \to \overline{\mathbb{R}}, \quad & u \mapsto \int_{D_1} \frac{1}{2}|\nabla_y u|^2 + V_h^\epsilon(u,y)\, dy, \quad I_h := I_h^0, \\
\Phi_{h,c}^\epsilon \colon H_c^1(\Sigma) \to \overline{\mathbb{R}}, \quad & u \mapsto \int_\Sigma \left(\frac{1}{2}|\nabla u|^2 + V_h^\epsilon(u,y)\right) e^{cx}\, dp,
\end{aligned}
$$

$$
V_h := V_h^0, \quad I_h := I_h^0, \quad \Phi_{h,c} := \Phi_{h,c}^0. \tag{4.10}
$$

V_h^ϵ can be split in two parts: a positive part V_+^ϵ which is monotonously decreasing in ϵ and independent of h, and a negative part V_{h-} that is independent of ϵ:

$$
\begin{aligned}
V_+^\epsilon \colon [0,1] \times [0,1] \to \mathbb{R}, \quad & (u,r) \mapsto \frac{1}{2\pi^2(r^2+\epsilon^2)} \sin^2(\pi u), \\
V_{h-} \colon [0,1] \times [0,1] \to \mathbb{R}, \quad & (u,r) \mapsto \frac{h}{\pi^2}(\cos(\pi u) - 1).
\end{aligned}
$$

Definition 4.19. For $\epsilon \geq 0$, let μ_ϵ be the smallest eigenvalue of

$$
\Delta u - \frac{1}{|y|^2+\epsilon^2} u + \mu u = 0 \ \text{ in } D_1, \quad \partial_\nu u = 0 \ \text{ on } \partial D_1,
$$

and let ψ_ϵ be the corresponding positive eigenfunction with $\|\psi_\epsilon\|_{L^2(D_1)} = 1$.

Remark 4.20. The smallest eigenvalue of

$$
\Delta u - \frac{1}{|y|^2+\epsilon^2} u + hu + \mu u = 0 \ \text{ in } D_1, \quad \partial_\nu u = 0 \ \text{ on } \partial D_1, \tag{4.11}
$$

is $\mu_\epsilon - h$. A simple inspection of the defining equation yields $\mu_0 = k_0^2$ and $\psi(r) = J_1(k_0 r)$. Here and in the following, J_1 is the first Bessel function of first kind (cf.[26]), and $k_0 \approx 1.84$ is the first root of the derivative of J_1.

Lemma 4.21. *For $\epsilon \geq 0$, the smallest eigenvalue μ_ϵ depends continuously on ϵ.*

Proof. First assume that $\epsilon_0 > 0$ and let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence converging to ϵ_0. Then for all functions $v \in H^1(\Sigma)$ we obviously have

$$\lim_{n\to\infty} \int_{D_1} \frac{1}{2}|\nabla_y v|^2 + \frac{v^2}{|y|^2+\epsilon_n^2} \;=\; \int_{D_1} \frac{1}{2}|\nabla_y v|^2 + \frac{v^2}{|y|^2+\epsilon_0^2},$$

so in particular

$$\begin{aligned} \mu_{\epsilon_0} &= \inf_{\{v:\|v\|_{H^1(\Omega)}=1\}} \int_{D_1} \frac{1}{2}|\nabla_y v|^2 + \frac{v^2}{|y|^2+\epsilon_0^2} \\ &= \lim_{n\to\infty} \inf_{\{v:\|v\|_{H^1(\Omega)}=1\}} \int_{D_1} \frac{1}{2}|\nabla_y v|^2 + \frac{v^2}{|y|^2+\epsilon_n^2} \;=\; \lim_{n\to\infty} \mu_{\epsilon_n}. \end{aligned}$$

Now assume $\epsilon_0 = 0$. Since $\epsilon_1 \geq \epsilon_2$ implies $\mu_{\epsilon_1} \leq \mu_{\epsilon_2}$, it suffices to show that there is a positive sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, such that $\lim_{n\to\infty}\mu_{\epsilon_n} \geq \mu_0$. Since $\|\nabla\psi^\epsilon\|^2_{L^2(D_1)} \leq \mu_\epsilon$ is bounded, there is a positive sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, such that $(\psi^{\epsilon_n})_{n\in\mathbb{N}}$ converges weakly in $H^1(D_1)$ and strongly in $L^2(D_1)$ to some function v_{lim}. We have

$$\begin{aligned} \mu_0 &\leq \lim_{\delta\to 0} \int_{D_1\setminus D_\delta} \frac{1}{2}|\nabla_y v_{\text{lim}}|^2 + \frac{v_{\text{lim}}^2}{|y|^2} \\ &= \lim_{\delta\to 0}\lim_{n\to\infty} \int_{D_1\setminus D_\delta} \frac{1}{2}|\nabla_y \psi_{\epsilon_n}|^2 + \frac{\psi_{\epsilon_n}^2}{|y|^2+\epsilon_n^2}\, dp \;\leq\; \lim_{n\to\infty}\mu_{\epsilon_n}. \end{aligned}$$

□

Using the monotonicity properties of the speed (Corollary 4.12) and special trial functions, we find upper and lower bounds on the speed of the solutions of (4.5).

Lemma 4.22. *For all $\epsilon > 0$ there exists an MN-solution (c^ϵ, u^ϵ) of (4.5). Moreover there exist constants $\overline{c}$, $\underline{c} > 0$, such that $\underline{c} < |c^\epsilon| \leq \overline{c}$ if ϵ small enough. We have*

$$\overline{c} := 2\sqrt{h} \quad \textit{and} \quad \underline{c} := \begin{cases} \frac{2h}{5\pi} & \text{for } 0 < h < h_c \\ 2\sqrt{h-k_0^2} & \text{for } h_c \leq h, \end{cases}$$

where $h_c \approx 0.05 + k_0^2$ is the smaller one of the two solutions of the equation $\frac{2h}{5\pi} = 2\sqrt{h-k_0^2}$.

Proof. The existence of MN-solutions follows from Theorem 4.6 and Theorem 4.7. Corollary 4.12 implies that the MN-solution (c, u) of (4.5) is slower than the MN-solution $(\overline{c}, \overline{u})$ of

$$\begin{aligned} \Delta\overline{u} + \overline{c}\partial_x\overline{u} + \frac{h}{\pi}\sin(\pi\overline{u}) &= 0 \quad \text{in } \Sigma, \\ \partial_\nu\overline{u}(x,y) &= 0 \quad \text{on } \partial\Sigma. \end{aligned} \tag{4.12}$$

The term $\frac{h}{\pi}\sin(\pi\overline{u})$ satisfies equation (4.8), so Proposition 4.10, Theorem 4.7, and Definition 4.8 yield that $(\overline{c}, \overline{u})$ is a non-variational MN-solution. The smallest eigenvalue of $-\Delta v - hv$ in D_1 with Neumann boundary conditions is $-h$, thus we have $\overline{c} = 2\sqrt{h}$.

If $h \geq k_0^2$, using Remark 4.20 and the statements about the speed in Theorem 4.6 and Theorem 4.7, we find the lower bound

$$|c| \;\geq\; 2\sqrt{h - \mu_\epsilon} \;\geq\; 2\sqrt{h - \mu_0} \;=\; 2\sqrt{h - k_0^2} \;=:\; c_b.$$

To find a lower bound for $h < k_0^2$, we use a trial function v and find some c_s such that (H3) is satisfied for f^ϵ when ϵ is small enough. We define

$$v(x,r) = \begin{cases} e^{\frac{\lambda x}{\sqrt{r}}} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

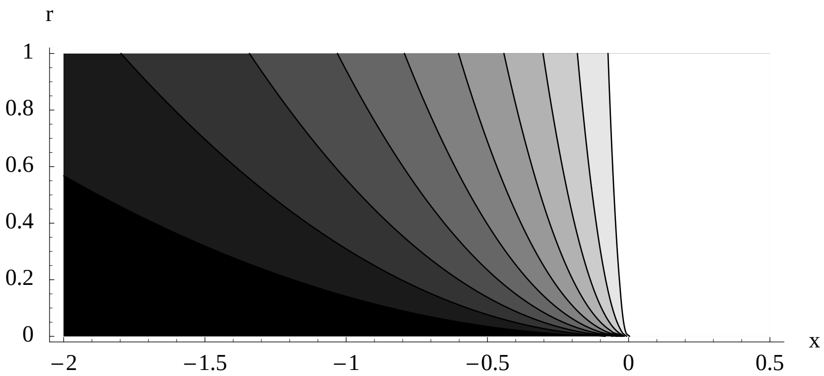

Figure 4.3: contour plot of v

Then we can estimate the terms of $\Phi_c(v)$ by

$$\begin{aligned} \int_\Sigma \frac{1}{2}(\partial_x v(p))^2 e^{cx}\, dp &= \int_0^1 \int_{\mathbb{R}^-} \pi r \frac{\lambda^2}{r} e^{\left(\frac{2\lambda}{\sqrt{r}}+c\right)x}\, dx\, dr \\ &\leq \int_0^1 \pi \frac{\lambda^2\sqrt{r}}{2\lambda + c}\, dr \;=\; \frac{2\pi\lambda^2}{3(2\lambda + c)}, \end{aligned}$$

$$
\begin{aligned}
\int_\Sigma \frac{1}{2}|\nabla_y v|^2 e^{cx}\,dp &= \int_0^1 \int_{\mathbb{R}^-} 2\pi r\ \lambda^2 x^2 r^{-3}\ e^{\left(\frac{2\lambda}{\sqrt{r}}+c\right)x}\,dx\,dr \\
&\le \int_0^1 2\pi\lambda^2 r^{-2}\, 2\left(\frac{\sqrt{r}}{2\lambda+c}\right)^3 dr \\
&\le \int_0^1 4\pi\lambda^2 r^{-\frac{1}{2}}\ \frac{\lambda^2}{(2\lambda+c)^3}\,dr \ \le\ \frac{8\pi\lambda^2}{(2\lambda+c)^3},
\end{aligned}
$$

$$
\begin{aligned}
\int_\Sigma V_h^0(v,y)\ e^{cx}\,dp &\le \int_0^1\int_{\mathbb{R}^-} \frac{r}{\pi}\frac{e^{\left(\frac{2\lambda}{\sqrt{r}}+c\right)x}}{r^2}\,dx\,dr + \int_{\mathbb{R}^+}\frac{-2h}{\pi}e^{cx}\,dx \\
&\le \frac{2h}{\pi c} + \int_0^1 \frac{1}{\pi r}\frac{\sqrt{r}}{2\lambda+c}\,dr \ \le\ \frac{2h}{\pi c}+\frac{2}{\pi(2\lambda+c)}.
\end{aligned}
$$

Now choose $\lambda := 2$, and assume $c \ge -0.5$. Then for all $\epsilon \ge 0$ we have

$$
\Phi^\epsilon_{h,c}\ \le\ \Phi^0_{h,c}(v)\ <\ 5+\frac{2h}{\pi c}.
$$

Thus, for $c < 0$ and $|c| \le \underline{c}_s := \min\left(\frac{2}{5\pi}h, \frac{1}{2}\right)$, the functional $\Phi^\epsilon_{h,c}(v)$ is negative, and for $h < k_0^2+1$ we have $c_s^2 > 4(h-k_0^2)$. Now continuity of μ_ϵ yields that, for $h < k_0^2+1$ and ϵ small enough, hypothesis (H3) is satisfied and we have $|c| \ge c_s$. □

Lemma 4.23. *For $\epsilon > 0$, let (c^ϵ, u^ϵ) be a monotone solution of* (4.5) *and set $u^\epsilon_+ = \lim_{x\to+\infty} u(x,\cdot)$. Then u^ϵ is rotationally symmetric and*

$$
\begin{aligned}
c^\epsilon\,\|\partial_x u^\epsilon\|^2_{L^2([a,b]\times D_1)} &= \int_{D_1} V_h^\epsilon(u^\epsilon,y)+\frac{1}{2}(\nabla_y u^\epsilon)^2-\frac{1}{2}(\partial_x u^\epsilon)^2\,dy\,\Big|_{x=a}^{x=b}, \\
c^\epsilon\,\|\partial_x u^\epsilon\|^2_{L^2(\Sigma)} &= I_h^\epsilon(u_+^\epsilon)
\end{aligned}
$$

Proof. To show rotational symmetry, we parametrise D_1 by polar coordinates (r,ϕ), and differentiate (4.5) by x and by ϕ:

$$
\begin{aligned}
&\Delta\partial_x u^\epsilon + c\partial_x\partial_x u^\epsilon\ \partial_x u^\epsilon + \partial_u f^\epsilon = 0 \quad \text{in } \Sigma, \qquad \partial_\nu\partial_x u^\epsilon = 0 \quad \text{on } \partial\Sigma \\
&\Delta\partial_\phi u^\epsilon + c\partial_x\partial_\phi u^\epsilon + \partial_x u^\epsilon \partial_u f^\epsilon\ = 0 \quad \text{in } \Sigma, \qquad \partial_\nu\partial_\phi u^\epsilon = 0 \quad \text{on } \partial\Sigma.
\end{aligned}
$$

Thus, both $\partial_x u^\epsilon$ and $\partial_\phi u^\epsilon$ are eigenfunctions of the operator

$$
u \mapsto -\Delta u + c\partial_x u + u\partial_u f
$$

with Neumann boundary conditions. Since u^ϵ is monotone in x, the function $\partial_x u$ is nonnegative, so 0 is the smallest eigenvalue and all eigenfunctions for the eigenvalue zero are multiples of $\partial_x u$. This has to hold in particular for $\partial_\phi u$. Since $u^\epsilon(x,r,\phi) = u^\epsilon(x,r,\phi+2\pi)$, the function $\partial_\phi u^\epsilon$ can not be entirely positive or entirely negative, thus $\partial_\phi u^\epsilon \equiv 0$.

The equations in the statement of Lemma 4.23 can be verified by a simple calculation. We test equation (4.5) with $\partial_x u^\epsilon$ and then use partial integration:

$$
\begin{aligned}
c^\epsilon \int_{[a,b]\times D_1} (\partial_x u^\epsilon)^2 \, dx\, dy \;&=\; \int_{[a,b]\times D_1} -\Delta u^\epsilon\, \partial_x u^\epsilon - f^\epsilon(u^\epsilon, y)\, \partial_x u^\epsilon \;\, dx\, dy \\
= \int_{[a,b]\times D_1} & -\mathrm{div}(\nabla u^\epsilon\, \partial_x u^\epsilon) + \nabla u^\epsilon\, \nabla \partial_x u^\epsilon + \partial_x V_h^\epsilon(u^\epsilon, y) \;\, dx\, dy \\
= \int_{D_1} & -(\partial_x u^\epsilon)^2 + \frac{1}{2}|\nabla u^\epsilon|^2 + V_h^\epsilon(u^\epsilon, y) \;\, dy \;\Bigg|_{x=a}^{x=b} \\
= \int_{D_1} & -\frac{1}{2}(\partial_x u^\epsilon(x,y))^2 + \frac{1}{2}|\nabla_y u^\epsilon(x,y)|^2 + V_h^\epsilon(u^\epsilon(x,a), y) \;\, dy \;\Bigg|_{x=a}^{x=b}
\end{aligned}
$$

The second equation follows from the first as the limit for $a \to -\infty$ and $b \to +\infty$. $\square$

4.3.2 Travelling wave solutions for the original equation

In this subsection we will construct solutions of (4.4) as a limit of solutions of (4.5). We could do this purely on the level of differential equations, but if the travelling waves in the sequence are variational MN-solutions, we have more information, and we would like transfer this information to the limit. In particular, we want to have a limit that is the minimiser of the limit functional. So we will first show a lower semi-continuity result.

Lemma 4.24. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence converging to zero and let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions converging weakly in $H^1_c(\Sigma)$ to some function $v \in H^1_c(\Sigma)$. If $\delta := c^2 + 4(k_0^2 - h) > 0$, then*

$$\Phi_{h,c}(v) \;\leq\; \liminf_{n\to\infty} \Phi^{\epsilon_n}_{h,c}(v_n) \;\leq\; \liminf_{n\to\infty} \Phi_{h,c}(v_n).$$

Proof. Since μ_ϵ depends continuously on ϵ (Lemma 4.21), and since $\mu_0 = k_0^2$ (Remark 4.20), there is some ϵ_0 such that $c^2+4(\mu_\epsilon - h) \geq \frac{\delta}{2} > 0$ for all $\epsilon < \epsilon_0$. So for all $\epsilon < \epsilon_0$ the functionals $\Phi^{\epsilon_n}_{h,c}$ are weakly lower semi-continuous on

$H^1_c(\Sigma)$ (Lemma 4.14), and thus

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\Phi_{h,c}(v)-\Phi^{\epsilon_n}_{h,c}(v_n)\right)\\
&\quad\le\ \limsup_{n\to\infty}(\Phi_{h,c}(v)-\Phi^{\epsilon_n}_{h,c}(v))+\underbrace{\limsup_{n\to\infty}(\Phi^{\epsilon_n}_{h,c}(v)-\Phi^{\epsilon_n}_{h,c}(v_n))}_{\le 0}\\
&\quad\le\ \limsup_{n\to\infty}\frac{1}{2\pi^2}\int_\Sigma e^{cx}\left(\frac{1}{|y|^2}-\frac{1}{|y|^2+\epsilon_n^2}\right)\sin^2(\pi v)\,dp \qquad (4.13)\\
&\quad\overset{(*)}{\le}\ \lim_{\rho\to 0}\limsup_{n\to\infty}\frac{1}{2\pi^2}\int_{\Sigma\setminus(\mathbb{R}\times D_\rho)} e^{cx}\left(\frac{1}{|y|^2}-\frac{1}{|y|^2+\epsilon_n^2}\right)\sin^2(\pi v)\,dp \qquad (4.14)\\
&\quad\le\ \lim_{\rho\to 0}\limsup_{n\to\infty}\frac{1}{2\pi^2}\left(\frac{1}{\rho^2}-\frac{1}{(\rho^2+\epsilon_n^2)}\right)\|v\|^2_{L^2_c(\Sigma)}\ =\ 0.
\end{aligned}$$

The estimate $(*)$ needs some explanation. The positive parts of (4.13) and (4.14) are independent of n and thus equal. For the negative part, we have for all $\delta > 0$ the estimate

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{2\pi^2}\int_\Sigma -e^{cx}\frac{1}{|y|^2+\epsilon_n^2}\sin^2(\pi v)\,dp\\
&\quad\le\ \limsup_{n\to\infty}\frac{1}{2\pi^2}\int_{\Sigma\setminus(\mathbb{R}\times D_\rho)} -e^{cx}\frac{1}{|y|^2+\epsilon_n^2}\sin^2(\pi v).
\end{aligned}$$

Thus we can pass to the limit $\delta\to 0$ on the right hand side. Since $\Phi^{\epsilon_n}_{h,c}(v_n)\le\Phi_{h,c}(v_n)$ for all $\epsilon\ge 0$, we have proved the statement. □

Theorem 4.25. *Assume that there exists $c<0$ satisfying $c^2+4(k_0^2-h)\ge\delta>0$, and $\tilde{u}\in H^1_c(\Sigma)$ such that $\Phi_{h,c}(\tilde{u})\le 0$. Then there exists $(c^\dagger,u)$ such that $u\in H^1_{c^\dagger}(\Sigma)$ is a minimiser of $\Phi_{h,c^\dagger}$. Moreover, $(c^\dagger,u^\dagger)$ satisfies (4.4), and we have:*

(i) $\Phi_{h,c^\dagger}(u)=0$ and $e^{\frac{c^\dagger x}{2}}u(x,y)$ is bounded.

(ii) u is monotonously increasing in x and rotationally symmetric and $\underline{c}\le|c^\dagger|\le\overline{c}$ with $\underline{c},\overline{c}$ as in Lemma 4.22.

(iii) There is at most one point $x\in\mathbb{R}$ where $u(x,0)$ is neither zero nor one.

(iv) The minimiser u is unique up to translation.

(v) $|c^\dagger|=|c^\dagger(h)|$ depends monotonously increasing and continuously on h. Up to translation, the minimiser u depends continuously in $H^1_{\mathrm{loc}}(\overline{\Sigma})$ on h.

Proof. Let $c<c^\dagger$ be an admissible trial velocity in (H3) for $\epsilon=0$. Since μ_ϵ depends continuously on ϵ (Lemma 4.21) and since $\Phi^\epsilon_{h,c}(\tilde{u})\le\Phi_{h,c}(\tilde{u})$ (Lemma 4.24), there exists $\epsilon_0>0$ such that c is an admissible trial velocity for all $\epsilon\le\epsilon_0$. Thus for all $\epsilon\le\epsilon_0$ there exists $(c^\dagger_\epsilon,u_\epsilon)$ such that $u_\epsilon\not\equiv 0$ is a minimiser of $\Phi^\epsilon_{h,c^\dagger_\epsilon}$ (Theorem 4.6).

Since the functions u_ϵ are in different spaces $H^1_{c^\dagger_\epsilon}$, we rescale. We set $v_\epsilon(x,y) := u_\epsilon(\frac{c}{c^\dagger_\epsilon}(x-a_\epsilon),y)$, where a_ϵ is chosen such that $v_\epsilon(x,y) \in B_c$ with B_c as in (4.9). Since the minimisers u_ϵ of $\Phi^\epsilon_{h,c^\dagger_\epsilon}$ are unique up to translation (Theorem 4.6), Lemma 4.18 implies that for $\epsilon \le \epsilon_0$ the functions v_ϵ are minimisers of $\Phi^\epsilon_{h,c}$ in B_c. We have $\Phi^\epsilon_{h,c}(v_\epsilon) = \frac{c}{c^\dagger_\epsilon} - \frac{c^\dagger_\epsilon}{c} \le \frac{c}{c^\dagger} - \frac{c^\dagger}{c} < 0$ (Lemma 4.15), so $\|v_\epsilon\|_{H^1_c(\Sigma)}$ is uniformly bounded and there is a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to zero such that $(v_{\epsilon_n})_{n\in\mathbb{N}}$ converges weakly in $H^1_c(\Sigma)$ to some $v \in H^1_c(\Sigma)$. Lemma 4.24 implies $0 > \lim_{n\to\infty} \Phi_{h,c}(v_{\epsilon_n}) \ge \Phi_{h,c}(v)$.

We show that v is a minimiser of $\Phi_{h,c}$ in B_c. Because of weak convergence in $H^1_c(\Sigma)$, we have $\int_\Sigma \frac{1}{2}|\partial_x v|^2 \le 1$. If $\int_\Sigma \frac{1}{2}|\partial_x v|^2 < 1$ there exists $a > 0$ such that $\tilde{v}\colon (x,y) \mapsto v(x+a,y)$ is in B_c and we have

$$\Phi_{h,c}(\tilde{v}) \;=\; \frac{\int_\Sigma \frac{1}{2}|\partial_x \tilde{v}|^2}{\int_\Sigma \frac{1}{2}|\partial_x v|^2}\Phi_{h,c}(v) \;<\; \Phi_{h,c}(v) \le \lim_{n\to\infty} \Phi_{h,c}(v_{\epsilon_n}).$$

Thus there exists $n_0 \in \mathbb{N}$ such that $\Phi_{h,c}(\tilde{v}) < \Phi_{h,c}(v_{\epsilon_n})$ for all $n \ge n_0$. This is a contradiction to the fact that the functions v_{ϵ_n} were chosen as minimisers of $\Phi^{\epsilon_n}_{h,c}$ in B_c.

If $\Phi_{h,c}(v) < \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c}(v_{\epsilon_n})$ or if v is not a minimiser of $\Phi_{h,c}$ in B_c, then, again, there is a function $\tilde{v} \in B_c$ such that $\Phi^{\epsilon_n}_{h,c}(\tilde{v}) \le \Phi_{h,c}(\tilde{v}) < \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c}(v_{\epsilon_n})$ for all $n \in \mathbb{N}$ and, again, this is impossible. Thus v is a minimiser of $\Phi_{h,c}$ in B_c.

Now define $c^\dagger := c\sqrt{1-\Phi_{h,c}(v)}$ and set $u(x,y) := v(\frac{c^\dagger}{c}x,y)$. Arguing as in the proof of [31, Theorem 3.3], we show that u is a minimiser of $\Phi_{h,c^\dagger}$: For any $w \in H^1_c(\Sigma)$, $w \not\equiv 0$ define $\tilde{w}(x,y) := w\left(\frac{c(x-a)}{c^\dagger}, y\right)$, where a is chosen such that $\tilde{w} \in B_c$. Using Lemma 4.15 we have

$$\begin{aligned} e^{c^\dagger a}\Phi_{h,c^\dagger}(w) \;&=\; \frac{c}{c^\dagger}\left(\Phi_{h,c}(\tilde{w}) + \left(\frac{(c^\dagger)^2}{c^2} - 1\right)\frac{1}{2}\int_\Sigma e^{cx}|\partial_x \tilde{w}(x)|^2\,dx\right) \\ &=\; \frac{c}{c^\dagger}\left(\Phi_{h,c}(\tilde{w}) - \Phi_{h,c}(v)\right). \end{aligned}$$

Since v was a minimiser of $\Phi_{h,c}(v)$ in B_c, the right hand side of the equation is non-negative and the minimum is attained for $\tilde{w} = v$, i.e. $w = u$. Thus u is a minimiser of $\Phi_{h,c^\dagger}$, which immediately implies that u satisfies (4.4). Moreover, $v \in H^1_c(\Sigma)$ implies $u \in H^1_{c^\dagger}(\Sigma)$

(i) If $\Phi_{h,c^\dagger}(u) \ne 0$ we could decrease $\Phi_{h,c^\dagger}$ by translating u. This contradicts the assumption that u is a minimiser of $\Phi_{h,c^\dagger}$, thus $\Phi_{h,c^\dagger}(u) = 0$. Lemma 4.32 below implies $I_0(w) \ge \frac{1}{\pi}\left(1 - \cos(\pi\|w\|_{L^\infty(D_1)})\right)$ for all $w\colon D_1 \to \mathbb{R}$. Since, in addition, $2t^2 \leq 1 - \cos(\pi t) \leq \frac{\pi^2}{2} t^2$ for all $t \in [0,1]$, we have

$$\begin{aligned}
0 &= \Phi_{h,c^\dagger}(u) \\
&\geq \int_{\mathbb{R}} e^{c^\dagger x} \left(I_0(u(x,\cdot)) + \frac{1}{2} \|\partial_x - \int_{D_1} \frac{hr}{\pi^2} (1 - \cos(\pi u(x,y)) \, dy \right) \, dx \\
&\geq \int_{\mathbb{R}} e^{c^\dagger x} \frac{1}{\pi} \left(1 - \cos(\|\pi u(x,\cdot)\|_{L^\infty(D_1)}) \right) \, dx \; - \; \frac{\pi}{2} h \|u\|^2_{L^2_{c^\dagger}(\Sigma)} \\
&\geq \int_{\mathbb{R}} \frac{2}{\pi} e^{c^\dagger x} \|u(x,\cdot)\|^2_{L^\infty(D_1)} \, dx - \; h\pi \|u\|^2_{L^2_{c^\dagger}(\Sigma)}.
\end{aligned}$$

Since the second summand is finite, the first summand has to be finite as well, and since u is monotone we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{2}{\pi} e^{c^\dagger x} \|u(x,\cdot)\|^2_{L^\infty(D_1)} \, dx &\geq \sup_{x_0 \in \mathbb{R}} \int_{-\infty}^{x_0} e^{c^\dagger x} \|u(x_0,\cdot)\|^2_{L^\infty(D_1)} \, dx \\
&\geq \sup_{x_0 \in \mathbb{R}} \frac{2}{\pi c^\dagger} e^{c^\dagger x_0} \|u(x_0,\cdot)\|^2_{L^\infty(D_1)}.
\end{aligned}$$

This can only be finite if $\|u(x,\cdot)\|_{L^\infty(D_1)} e^{\frac{c^\dagger}{2} x}$ is bounded.

(ii) The statements follow immediately from the fact that for $\epsilon > 0$ the functions v_ϵ are monotonously increasing in x and rotationally symmetric (Lemma 4.23) and from the bounds on $c^\dagger_\epsilon$ of Lemma 4.22.

(iii) Since $\Phi_{h,c^\dagger}(u)$ is finite and u is monotone, there is at most one point where u is neither zero or one.

(iv) The proof is due to S. Heinze [21]. Let u_1, u_2 be nontrivial minimisers of $\Phi_{h,c^\dagger}$. After a translation we can assume that there is some point (x^*, y^*), $y^* \neq 0$ such that $u_1(x^*, y^*) = u_2(x^*, y^*)$. Set $\overline{u} = \max(u_1, u_2)$ and $\underline{u} = \min(u_1, u_2)$. Then $\Phi_{h,c^\dagger}(\overline{u}) + \Phi_{h,c^\dagger}(\underline{u}) = 0$, thus $\overline{u}$ and $\underline{u}$ are minimisers as well. Set $w := \overline{u} - \underline{u}$ and set

$$g(x,y) := \begin{cases} \frac{f^0(\overline{u}(x,y),y) - f^0(\underline{u}(x,y),y)}{\overline{u}(x,y) - \underline{u}(x,y)} & \text{if } \overline{u}(x,y) \neq \underline{u}(x,y), \\ 0 & \text{otherwise.} \end{cases}$$

Then $|g(x,y)| \leq \frac{1}{|y|^2} + h$, and the function w satisfies

$$w \geq 0, \quad w(x^*, y^*) = 0, \quad \Delta w + c\partial_x w + \min(0, g(x,y)) w \leq 0.$$

So the strong maximum principle [18, Theorem 3.5] implies $w \equiv 0$.

(v) Corollary 4.12 implies that $|c^\dagger|$ is monotonously increasing in h. Let h_0 be such that (H3) is satisfied. Since, for any $v \in L^2_c$ such that $\Phi_{c,h}(v)$ is finite, $\Phi_{c,h}(v)$ depends continuously on h, for any admissible trial velocity $c < c^\dagger(h_0)$ there exists a neighborhood $U_c(h_0)$ such that c is an admissible trial velocity for all $h \in U_c(h_0)$. Thus Remark 4.17 implies

$c^\dagger(h_0) \leq \lim_{h \to h_0} c^\dagger(h)$. Now fix some admissible trial velocity $c_0 < c^\dagger(h_0)$ and let $(h_n)_{n\in\mathbb{N}}$, $h_n \in U(h_0)$ be a sequence converging to h_0. As we have seen at the beginning of this proof, there exist minimisers v_n of $\Phi_{h_n,c}$ in B_c. These minimisers of the constrained problem are uniformly bounded in $H^1_c(\Sigma)$ (Lemma 4.13), thus there exists a subsequence converging weakly in $H^1_c(\Sigma)$ to some $v \in H^1_c(\Sigma)$. Since the functional $\Phi_{h_0,c}$ is lower semi-continuous (Lemma 4.24), we have

$$\begin{aligned}\lim_{n\to\infty} \Phi_{h_n,c}[v_n] &\geq \lim_{n\to\infty} \left(\Phi_{h_0,c}[v_n] - (h_n - h_0)\|v_n\|^2_{L^2_c(\Sigma)} \right) \\ &\geq \Phi_{c,h_0}[v] \geq \Phi_{h_0,c}(\tilde{v}),\end{aligned}$$

where $\tilde{v}$ is the function v translated in a way that $\frac{1}{2}\|v\|^2_{L^2_c(\Sigma)} = 1$. Thus

$$\begin{aligned}c_0^\dagger &= c\sqrt{1 - \Phi_{h_0,c}[v_0]} \geq c\sqrt{1 - \Phi_{h_0,c}[\tilde{v}]} \\ &\geq \lim_{n\to\infty} c\sqrt{1 - \Phi_{h_n,c}[v_n]} = \lim_{n\to\infty} c_n^\dagger \geq c_0^\dagger,\end{aligned}$$

and we conclude that all inequalities have to be equalities. In particular, $v = \tilde{v}$ has to be a constrained minimiser of $\Phi_{h_0,c}$. Since minimisers are unique up to translation, constrained minimisers are unique, thus $v = v_0$. So all convergent subsequences of $(v_n)_{n\in\mathbb{N}}$ have to converge weakly in $H^1_c(\Sigma)$ to v_0, which implies that $(v_n)_{n\in\mathbb{N}}$ itself converges to v_0. The minimisers u_n of $\Phi_{h_n,c_n^\dagger}$ are the functions v_n, rescaled by a factor that depends continuously on $c_n^\dagger$. We can conclude that, up to translation, the functions u_n depend continuously in $H^1_{\text{loc}}(\overline{\Sigma})$ on h. □

Remark 4.26. For $h \leq h_c$, where h_c as in Lemma 4.22, we can use the trial function described in the proof of Lemma 4.22, so the conditions of Theorem 4.25 are certainly satisfied. Note that $h_c > k_0^2$.

If the conditions of Theorem 4.25 are not satisfied, we have to work with the differential equation and use the following lemma.

Lemma 4.27. *Let u^ϵ satisfy* (4.5). *Then there is a constant $C = C(l, \rho, |c|)$ independent of ϵ and monotonously increasing in $|c|$ such that*

$$\|u^\epsilon\|_{H^2([-l,l]\times D_1 \setminus D_\rho)} \leq C(l, \rho, |c|) \left(\|f^\epsilon\|_{L^\infty([-2l,2l]\times D_1 \setminus D_{\frac{\rho}{2}})} + 1 \right).$$

Proof. Like in Subsection 3.4.1 we can extend the functions u^ϵ, f^ϵ by reflection to $(u^\epsilon)^*, (f^\epsilon)^* : \mathbb{R} \times D_{\frac{3}{2}} \to \mathbb{R}$. Since u^ϵ satisfies Neumann boundary conditions, the function $(u^\epsilon)^*$ fulfills an elliptic equation $Au^\epsilon = f^\epsilon$ in $D_{\frac{3}{2}}$ where the elliptic operator A is independent of ϵ (see Subsection 3.4.1, in particular (3.30)). Thus with standard elliptic estimates, as they can be found for example in [18, Theorem 9.11], there exists a constant $C_1 = C_1(\rho, l\|c|)$ monotonously increasing in $|c|$ such that

$$\begin{aligned}
&\|u^\epsilon\|_{H^2([-l,l]\times D_1\setminus D_\rho)} \\
&\quad\leq\ C_1(\rho, l, |c|)\left(\|(u^\epsilon)^*\|_{L^2([-2l,2l]\times D_1\setminus D_{\frac{\rho}{2}})} + \|(f^\epsilon)^*\|_{L^2([-2l,2l]\times D_{1.5}\setminus D_{\frac{\rho}{2}})}\right) \\
&\quad\leq\ 4l\,|D_{1.5}|\,C_1(\rho, l, |c|)\left(\|u^\epsilon\|_{L^\infty([-2l,2l]\times D_1\setminus D_{\frac{\rho}{2}})} + \|f^\epsilon\|_{L^\infty([-2l,2l]\times D_1\setminus D_{\frac{\rho}{2}})}\right) \\
&\quad\leq 9\pi l C_1(\rho, l, |c|)\left(\|f^\epsilon\|_{L^\infty([-2l,2l]\times D_1\setminus D_{\frac{\rho}{2}})} + 1\right).
\end{aligned}$$

□

Theorem 4.28. *Assume that there exists no pair $(c, \tilde{u}) \in \mathbb{R} \times H^1_c(\Sigma)$ such that $c^2 + 4(k_0^2 - h) > 0$ and $\Phi_{h,c}(\tilde{u}) \leq 0$. Then there exists a solution (c^*, u) of* (4.4) *such that:*

(i) u is monotonously increasing in x and rotationally symmetric. Moreover,

$$\inf\left\{x \in \mathbb{R} \mid \exists\, y \in D_1 \setminus D_{\frac{1}{2}} \text{ s.th. } u(x, y) \geq 0.33\right\} = 0.$$

(ii) $c^* = -2\sqrt{h - k_0^2}$.

(iii) $\|\partial_x u\|_{L^2(\mathbb{R}\times D_1)} \ \leq\ \sqrt{\frac{2h}{\pi|c^*|}}$ and

$$\int_{[a,b]\times D_1} \frac{1}{2}(\nabla u)^2 + V_+^0(u, y) \ \leq\ \frac{h}{\pi}\left(4(2-a) + \frac{1}{|c^*|}\right) \quad \text{for all } a, b \in \mathbb{R}.$$

(iv) There is at most one point $x \in \mathbb{R}$ where $u(x, 0)$ is neither zero nor one.

Proof. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence converging to zero and let $(u_n, c_n)_{n\in\mathbb{N}}$ be a sequence of MN-solutions of (4.5) with $\epsilon = \epsilon_n$ such that

$$0 \ =\ \inf\left\{x \in \mathbb{R} \mid \exists\, y \in D_1 \setminus D_{\frac{1}{2}} \text{ s.th. } u_n(x, y) \geq 0.33\right\}. \tag{4.15}$$

With Lemma 4.27, the bound on c^n (Lemma 4.22) implies for each $l, \rho > 0$ a bound on $\|u_n\|_{H^2([-l,l]\times D_1\setminus D_\rho)}$.

Thus we have weak convergence of a subsequence of $(u_n)_{n\in\mathbb{N}}$ in $H^2_{\text{loc}}(\overline{\Sigma} \setminus \mathbb{R} \times \{0\})$. A bootstrap argument gives convergence in $H^k_{\text{loc}}(\overline{\Sigma} \setminus \mathbb{R} \times \{0\})$ for all $k \in \mathbb{N}$ and thus convergence in $C^\infty_{\text{loc}}(\overline{\Sigma} \setminus \mathbb{R} \times \{0\})$.

After passing to a subsequence, we can assume that $(u_n, c_n)_{n\in\mathbb{N}}$ converges to (u, c).

(i) Since all MN-solutions are monotonously increasing, the functions u_n are monotonously increasing. They are rotationally symmetric (Lemma 4.23), and, by assumption, (4.15) is satisfied. Thus the statement follows from convergence in $C^\infty_{\text{loc}}(\Sigma \setminus \mathbb{R} \times \{0\})$.

(ii) Either

$$c^* \;=\; \lim_{n\to\infty} c_n \;=\; \lim_{n\to\infty} -2\sqrt{h-\mu_{\epsilon_n}} \;=\; -2\sqrt{h-k_0^2}$$

or

$$c^* \;=\; \lim_{n\to\infty} c_n \;<\; \lim_{n\to\infty} -2\sqrt{h-\mu_{\epsilon_n}}.$$

In the latter case we can assume that for all $n \in \mathbb{N}$ the functions u_n are minimisers of $\Phi^{\epsilon_n}_{h,c_n}$. Set $v_n(x) := u_n(\frac{c^*}{c_n}x - a_n)$ where a_n is chosen such that $v_n \in B^c$. Then $\|v_n\|_{H^1_c(\Sigma)}$ is uniformly bounded, so the functions v_n converge, up to a subsequence, weakly in H^1_c to some function v. Since the velocity c^ϵ is montonously decreasing in ϵ (Corollary 4.12), we have $c^* \geq c_n$ for all $n \in \mathbb{N}$. Thus, using Lemma 4.15 and Lemma 4.24, we have

$$0 \;=\; \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c_k}(u_n) \;\geq\; \lim_{n\to\infty} \Phi^{\epsilon_n}_{h,c^*}(v_n) \;\geq\; \Phi_{h,c^*}(v).$$

This contradicts the assumptions.

(iii) For all $n \in \mathbb{N}$, Lemma 4.23 implies

$$c_n \, \|\partial_x u_n\|^2_{L^2(\mathbb{R}\times D_1)} \;\geq\; \int_{D_1} V_h^{\epsilon_n}(1,y)\,dy \;=\; -\frac{2}{\pi}h.$$

Thus

$$\|\partial_x u\|^2_{L^2(\Sigma)} \;=\; \lim_{\rho\to 0}\lim_{n\to\infty} \|\partial_x u_n\|^2_{L^2(\Sigma\setminus(\mathbb{R}\times D_\rho))} \;\leq\; \lim_{n\to\infty} \|\partial_x u_n\|^2_{L^2(\Sigma)} \;\leq\; \frac{2h}{\pi|c^*|}. \tag{4.16}$$

For the second equation we use Lemma 4.23 again. Letting b tend to ∞ in the first equation of Lemma 4.23, for all $a \in \mathbb{R}$ we have

$$\begin{aligned}
&\int_{D_1} V_+^{\epsilon_n}(u_n(a,y),y) + \frac{1}{2}|\nabla_y u_n(a,y)|^2\,dy \\
&= \; -\|\partial_x u_n\|_{L^2([a,\infty[\times D_1} + \frac{1}{2}\|\partial_x u_n(a,\cdot)\|^2_{L^2(D_1)} - \int_{D_1} V_{h-}(u_n(a,y),y)\,dy \\
&\leq \; \frac{1}{2}\|\partial_x u_n(a,\cdot)\|^2_{L^2(D_1)} + \frac{2h}{\pi}.
\end{aligned}$$

Integration from a_1 to a_2 and passing to the limit $n \to \infty$ yields

$$\begin{aligned}
&\int_{[a_1,a_2]\times D_1} V_+^0(u,y) + \frac{1}{2}|\nabla_y u|^2 \\
&= \lim_{\rho\to 0}\lim_{n\to\infty}\int_{[a_1,a_2]\times(D_1\setminus D_\rho)} V_+^{\epsilon_n}(u_n,y) + \frac{1}{2}|\nabla_y u_n|^2 \\
&\le \lim_{n\to\infty}\int_{[a_1,a_2]\times D_1} V_+^{\epsilon_n}(u_n,y) + \frac{1}{2}|\nabla_y u_n|^2 \\
&\le \lim_{n\to\infty}\frac{2h(a_2-a_1)}{\pi} + \frac{1}{2}\|\partial_x u_n\|^2_{L^2([a_1,a_2]\times D_1)} \\
&\le \frac{2h(a_2-a_1)}{\pi} + \frac{h}{\pi|c^*|}.
\end{aligned}$$

(iv) Set $r = |y|$. Since u is rotationally symmetric, we can write $u(x,r)$ instead of $u(x,y)$ in the calculation below. We know from (iii) that for almost every $a \in \mathbb{R}$

$$\begin{aligned}
\infty &> \int_{D_1} 2V_+^0(u(a,y),y) + |\nabla_y u(a,y)|^2 \, dy \\
&= \int_0^1 \frac{1}{\pi r}\sin^2(\pi u(a,r)) + \pi r(\partial_r u(a,r))^2 \, dr \\
&\ge \int_0^1 2|\sin(\pi u(x,r))\partial_r u(a,r)| \, dr \quad \ge \int_0^1 \frac{2}{\pi}|\partial_r \cos(\pi u(a,r))| \, dr
\end{aligned}$$

and see that $u(a,\cdot)$ is continuous for almost all $a \in \mathbb{R}$. If $u(a,\cdot)$ is continuous and $\int_{D_1} V_+(u,y)$ is finite, either $u(a,0) = 0$ or $u(a,0) = 1$. Since $u(x,0)$ is monotonously increasing in x, there is at most one point a^* such that $u(a,0)$ is neither zero nor one. □

4.4 Stationary states

In this section we investigate stationary states, i.e. solutions of (4.4) that do not depend on x and therefore solve

$$\Delta u + f^0(y,u) = 0 \ \text{ in } D_1, \qquad \partial_\nu u = 0 \ \text{ on } \partial D_1. \tag{4.17}$$

We restrict our attention to radially symmetric solutions u with values in $[0,1]$ for which $I_h(u)$ as defined in (4.10) is finite. Such functions u depend only on the scalar variable $r = |y|$. For $a,b \in \mathbb{R}$, $v : [a,b] \to \mathbb{R}$ we define

$$\tilde{I}_h(v,[a,b]) := \int_a^b \frac{\pi r}{2}(v')^2 + \frac{1}{2\pi r}\sin^2(\pi v) + \frac{h\,r}{\pi}(\cos(\pi u)+1)\,dr,$$

and for $v : [0,1] \to \mathbb{R}$ we set

$$\tilde{I}_h(v) := \tilde{I}_h(v,[0,1]).$$

Then, as functions of r, the maps u are the critical points of $\tilde{I}_h$. Moreover, they are exactly the solutions of the ordinary differential equation

$$-u'' - \frac{1}{r}u' + \frac{1}{2\pi r^2}\sin(2\pi u) - \frac{h}{\pi}\sin(\pi u) = 0, \quad u'(1) = 0 \tag{4.18}$$

that are contained in

$$\mathcal{A} := \left\{u \in H^1_{\text{loc}}(]0,1],[0,1]) \ : \ \tilde{I}_h(u) < \infty\right\}.$$

Lemma 4.29. *If* $u \in \mathcal{A}$ *then* $\lim_{r\to 0} u(r) = 0$ *or* $\lim_{r\to 0} u(r) = 1$.

Proof. By assumption, $\int_0^1 \frac{1}{r}\sin^2(\pi u(r))\, dr < \infty$. Thus, if $\lim_{r\to 0} u(r)$ exists, then $\lim_{r\to 0} u(r) = 0$ or $\lim_{r\to 0} u(r) = 1$. We prove by contradiction that this limit exists and assume that there is a number $\epsilon > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ converging to 0 such that $|u(t_{n+1}) - u(t_n)| > \epsilon$. Then we have $|\cos(\pi u(t_{n+1}) - \cos(\pi u(t_n))| \geq 2\epsilon^2$ and therefore

$$\begin{aligned}
\tilde{I}_h(u) &\geq \int_0^1 \frac{\pi r}{2}|u(r)'|^2 + \frac{1}{2\pi r}\sin^2(\pi u(r))\, dr \\
&\geq \int_0^1 |\sin(\pi u(r))\, u(r)'|\, dr \\
&= \frac{1}{\pi}\int_0^1 |(\cos(\pi u(r)))'|\, dr \\
&\geq \frac{1}{\pi}\sum_{n=1}^{\infty} |\cos(\pi u(t_{n+1})) - \cos(\pi u(t_n))| = \infty.
\end{aligned}$$

This is in contradiction to the assumption $u \in \mathcal{A}$. □

We define

$$\mathcal{A}_0 := \{u \in \mathcal{A} : \lim_{r\to 0} u(r) = 0\}, \qquad \mathcal{A}_1 := \{u \in \mathcal{A} : \lim_{r\to 0} u(r) = 1\}.$$

With Lemma 4.29 we have $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$. Moreover, if u is a local minimiser of $\tilde{I}_h$ in $\mathcal{A}_i$ ($i \in \{0,1\}$) then u is a local minimiser of $\tilde{I}_h$ in $H^1_{\text{loc}}(]0,1])$.
For $\rho, k \in]0,1]$ we set

$$\begin{aligned}
\mathcal{W}(k,\rho) &:= \{v \in \mathcal{A}_0 : v[0,\rho] \subset [0,k],\ v(\rho) = k\}, \\
E_h(k,\rho) &:= \inf\{\tilde{I}_h(u,[0,\rho]) : u \in \mathcal{W}(k,\rho)\}.
\end{aligned}$$

Remark 4.30. For $r \in \left]0, \sqrt{\frac{1}{2h}}\right]$, $t \in \left[0, \frac{1}{8}\right]$ we have

$$\begin{aligned}
&\partial_{tt}\left(\frac{1}{2\pi r}\sin^2(\pi t) + \frac{hr}{\pi}(\cos(\pi t) + 1)\right) \\
&\quad = \frac{\pi}{r}\cos(2\pi t) - h\pi r\cos(\pi t) \ \geq \ \left(\frac{\pi}{r} - \frac{h\pi r}{2}\right)\cos(2\pi t) \ > \ 0.
\end{aligned}$$

Thus for $\rho \in \left]0, \sqrt{\frac{1}{2h}}\right]$, $k \in \left[0, \frac{1}{8}\right]$ the functional $\tilde{I}(\cdot, [0,\rho])$ is convex on $\mathcal{W}(k,\rho)$.

Definition 4.31. A function u is called a *semistable stationary state* or a *semistable solution* of (4.18) if u is a solution of (4.18) whose second variation is nonnegative, i.e., where for all $v \in \mathcal{A}_0$

$$\pi \int_0^1 r(v')^2 + \left(\frac{1}{r}\cos(2\pi u) - hr\cos(\pi u)\right) v^2 \, dr \;\geq\; 0. \tag{4.19}$$

This is equivalent to all eigenvalues of L_u being nonnegative, where

$$L_u(\phi) \;:=\; -\phi'' - \frac{1}{r}\phi' + \left(\frac{1}{r^2}\cos(2\pi u) - h\cos(\pi u)\right)\phi. \tag{4.20}$$

4.4.1 Stationary states without outer magnetic field

We consider the functional $\tilde{I}_0$ for fixed boundary value $u(1) = k$. Since $\tilde{I}_0(u) = \tilde{I}_0(1-u)$ for all $u \in \mathcal{A}$, we can assume $u \in \mathcal{A}_0$ without loss of generality. Using the Modica-Mortola trick, we can determine the value of $E_0(k,\rho)$ as well as the minimisers. In a second lemma we show that these minimisers are the only solutions of (4.18) for $h = 0$.

Lemma 4.32. *For all $k \in [0,1]$ we have $E_0(k,\rho) = \frac{1}{\pi}(1-\cos(\pi k))$. The minimum is attained, and the minimiser is*

$$\xi_a \colon [0,\rho] \to [0,1], \quad r \mapsto \frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{a^2r^2+1}}\right) \qquad \text{where } a = \frac{1}{\rho}\tan\left(\frac{\pi}{2}k\right).$$

It satisfies the differential equation

$$r\xi_a' = \frac{1}{\pi}\sin(\pi\xi_a),$$

and we have

$$\xi_a(r) \sim \frac{2}{\pi}ar \quad \text{for } r \to 0, \qquad \xi_a(r) \sim 1 - \frac{2}{\pi}\cdot\frac{1}{ar} \quad \text{for } r \to \infty.$$

For a sketch of ξ_a for different values of a see Figure 4.4.

Proof. For all functions $\xi \in \mathcal{A}_0$ we have

$$\begin{aligned} \tilde{I}_0(\xi,[0,\rho]) \;&=\; \int_0^\rho \frac{\pi r}{2}|\xi'|^2 + \frac{1}{2\pi r}\sin^2(\pi\xi)\,dr \;\geq\; \int_0^\rho |\sin(\pi\xi)\,\xi'|\,dr \\ &=\; \frac{1}{\pi}\int_0^\rho |(\cos(\pi\xi))'|\,dr \;\geq\; \frac{1}{\pi}(1-\cos(\pi\xi(\rho))). \end{aligned} \tag{4.21}$$

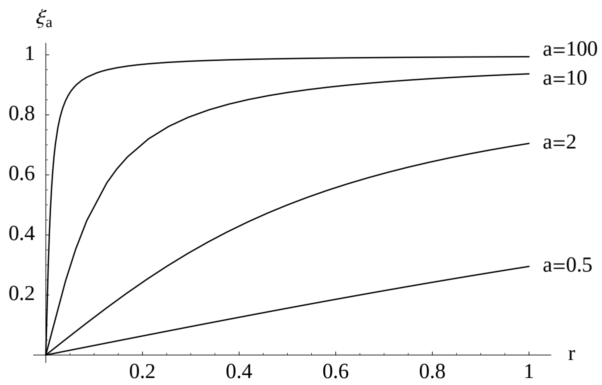

Figure 4.4: The functions ξ_a for different values of a

Assume that ξ is a minimiser with $\xi(\rho) = k$. Then ξ is a monotonously increasing function that satisfies (4.21) with equality. The latter is the case if and only if

$$r|\xi'(r)|^2 \;=\; \frac{1}{\pi^2 r}\sin^2(\pi\xi(r)) \qquad \text{for all } r \in \mathbb{R},$$

that is, if ξ satisfies the differential equation

$$r\partial_r\xi \;=\; \frac{1}{\pi}\sin(\pi\xi), \qquad \xi(0) = 0, \qquad \xi(\rho) \;=\; k. \tag{4.22}$$

A solution of this equation is

$$\xi_a\colon \mathbb{R}_0^+ \longrightarrow [0,1], \qquad r \mapsto \frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{a^2r^2+1}}\right),$$

where a can be calculated from k via

$$\frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{a^2\rho^2+1}}\right) \;=\; k, \quad \text{i.e.,} \quad a \;=\; \frac{1}{\rho}\tan\left(\frac{\pi}{2}k\right).$$

Applying the uniqueness theorem for differential equations in $r = k$, we see that the function ξ_a is the only solution. The statements about the asymptotic behaviour of ξ_a can be found by direct inspection. □

In the following, ξ_a will always refer to the function described above.

Lemma 4.33. *The only functions $u \in \mathcal{A}$ that satisfy*

$$-u'' - \frac{1}{r}u' + \frac{1}{2\pi r^2}\sin(2\pi u) = 0 \tag{4.23}$$

are the functions ξ_a and $1 - \xi_a$ where $a \in \mathbb{R}_0^+$. In particular, besides $u \equiv 0$ and $u \equiv 1$, there is no solution $u \in \mathcal{A}$ of (4.23) with $u'(1) = 0$.

Proof. Let u be a solution of (4.23), and without loss of generality assume $u \in \mathcal{A}_0$. Moreover, let $\rho > 0$ be such that $u([0,\rho]) \subset [0, \frac{1}{4}[$. Then the functional $\tilde{I}_0$ is convex on $\mathcal{W}(\rho, u(\rho))$. Lemma 4.32 implies

$$u|_{[0,\rho]} = \xi_a|_{[0,\rho]} \qquad \text{where } a = \frac{1}{\rho}\tan\left(\frac{\pi}{2}u(\rho)\right),$$

and since u and ξ_a solve the same differential equation, we have $u = \xi_a$ on the whole interval $[0,1]$. Since $\xi_a'(r) \neq 0$ for all $a, r \in \mathbb{R}^+$, the second statement follows immediately. □

4.4.2 Monotonicity properties of stationary states

In this subsection we use the functions ξ_a as comparison functions to find properties of solutions of (4.18). As a result of this subsection we will obtain the following theorem.

Theorem 4.34. *Let u be a solution of* (4.18)*. Then either $u \equiv 1$ or $u \in \mathcal{A}_0$ and for all $r_0 \in]0,1]$ we have $u(r) \geq \frac{u(r_0)}{r_0} r$. If u is semistable, then u is monotonously increasing.*

Lemma 4.35. *The function $u \equiv 1$ is the only solution of* (4.18) *in $\mathcal{A}_1$.*

Proof. Assume $u \in \mathcal{A}_1$, $u \not\equiv 1$ is a solution of (4.18). Define

$$\tilde{\mathcal{W}}(\rho) \;:=\; \left\{ v \in \mathcal{A}_1 : \, v(r) \geq \frac{3}{4} \text{ for all } r < \rho, \; v(\rho) = u(\rho) \right\}.$$

Since u is continuous there exists a $\rho_0 > 0$ such that $u(r) > \frac{3}{4}$ for all $r < \rho_0$. For all $\rho < \rho_0$ the functional $\tilde{I}_h(\cdot, [0,\rho])$ is convex on $\tilde{\mathcal{W}}(\rho)$ and u is a minimiser of $\tilde{I}_h(\cdot, [0,\rho])$ in $\tilde{\mathcal{W}}(\rho)$. For all $a \in \mathbb{R}^+$ for which u and $1 - \xi_a$ coincide in some point $\rho \in [0, \rho_0]$, we have $u|_{[0,\rho]} \geq 1 - \xi_a|_{[0,\rho]}$, since otherwise the map $\tilde{u}(r) := \max\{u(r), 1 - \xi_a(r)\}$ satisfies the inequality $\tilde{I}_h(\tilde{u}, [0,\rho]) < \tilde{I}_h(u, [0,\rho])$, which contradicts the minimality of u. We define

$$a_0 \;:=\; \sup\left\{a : \text{there exists } r > 0 \text{ such that } 1 - \xi_a(r) > u(r)\right\}.$$

The number a_0 is finite: The map u attains some minimum $w > 0$ and setting $a_1 := \frac{1}{\rho_0}\tan(\frac{\pi}{2}(1-w))$ we have $1 - \xi_{a_1}|_{[\rho_0,1]} \leq w \leq u$. With the above considerations we have $1 - \xi_{a_1}|_{[0,\rho_0]} \leq u|_{[0,\rho_0]}$ as well, thus $a_0 \leq a_1$. Figure 4.5 shows a sketch of the situation.

For all $r \in [0,1]$ we have $1 - \xi_{a_0}(r) \leq u(r)$. There are three possibilities:

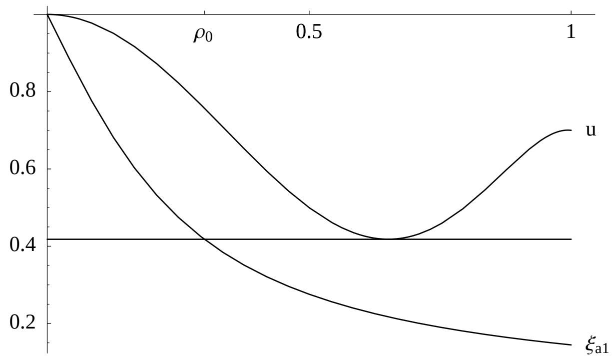

Figure 4.5: The function ξ_{a_1}

(1.) $1 - \xi_{a_0}(r) < u(r)$ for all $0 < r \le 1$. This is the case if and only if the intersection point s_a of $1 - \xi_a$ and u goes to zero for a to a_0 from below. But since we have shown that s_a cannot be smaller than ρ_0, this is impossible.

(2.) $1 - \xi_{a_0}(r) < u(r)$ for all $0 < r < 1$, but $1 - \xi_{a_0}(1) = u(1)$. Since $1 - \xi'_{a_0}(1) < 0 = u'(1)$, there is $\rho_1 < 1$ with $1 - \xi_{a_0}(r) > u(r)$ for all $\rho_1 < r < 1$. This is a contradiction to $1 - \xi_{a_0} \le u$.

(3.) There is some $r_0 \in]0, 1[$ such that $1 - \xi_{a_0}(r_0) = u(r_0)$. Then $u'(r_0) = \xi'_{a_0}(r_0)$ and the equations (4.18) and (4.23) imply

$$u''(r_0) + \frac{h}{\pi}\sin(\pi r_0) = -\xi''_{a_0}(r_0), \quad \text{i.e., } u''(r_0) < -\xi''_{a_0}.$$

Thus there is some neighbourhood U of r_0 such that the inequality $(1 - \xi_{a_0})(r) > u(r)$ holds for all $r \in U \setminus \{r_0\}$. This is a contradiction to $1 - \xi_a \le u$.

So all three possibilities lead to a contradiction, and we have shown that there is no nontrivial solution of (4.18) in $\mathcal{A}_1$. □

Lemma 4.36. *Let u be a solution of* (4.18) *in $\mathcal{A}_0$, $r_0 \in]0, 1]$ and $a_0 \in \mathbb{R}^+$ such that $u(r_0) = \xi_{a_0}(r_0)$. Then we have $u(r) \ge \xi_a(r)$ for all $r < r_0$.*

Proof. The proof of this lemma is similar to the proof of Lemma 4.35. There is some $\rho_0 > 0$ such that for all $\rho < \rho_0$ the functional $\tilde{I}_h(\cdot, [0, \rho])$ is convex on the set $\mathcal{W}(\rho, u(\rho))$. Therefore the statement of the lemma is obvious for all $r_0 \le \rho_0$. We define

$$a_1 := \inf\{a : \text{there exists } r \in [0, r_0] \text{ such that } \xi_a(r) > u(r)\}.$$

Then $\xi_{a_1}(r) \le u(r)$ for all $r \in [0, r_0]$.

If $a_1 = a_0$ the statement of the lemma holds. Otherwise there are two possibilities:

(1.) ξ_{a_1} touches u in some point $r_1 \in]0, r_0[$.

(2.) $\xi_{a_1}(r) < u(r)$ for all $0 < r \leq r_0$.

We can exclude both possibilities like in Lemma 4.35. $\square$

Since for all $a \in \mathbb{R}^+$ the second derivative of ξ_a is negative, Lemma 4.36 implies:

Corollary 4.37. *Let u be a solution of* (4.18) *in $\mathcal{A}_0$. Then we have for all $r_0 > 0$ and all $r \in [0, r_0]$ the inequality $u(r) \geq \frac{u(r_0)}{r_0} r$.*

If we assume additionally that u is semistable, we obtain the following monotonicity result.

Lemma 4.38. *Let $u \not\equiv 1$ be a semistable solution of* (4.18)*. Then u is in $\mathcal{A}_0$ and monotonously increasing.*

Proof. Since u is not in $\mathcal{A}_1$ (Lemma 4.35), u is in $\mathcal{A}_0$. We assume $u \not\equiv 0$ and define

$$e^t := r, \qquad \tilde{u}(t) := u(e^t), \qquad \phi := \partial_t \tilde{u} = r\partial_r u.$$

Then $\partial_t \phi = r^2 \partial_{rr} u + r\partial_r u$. On the one hand, equation (4.18) can be transformed to

$$-\partial_{tt}\tilde{u} + \frac{1}{2\pi}\sin(2\pi\tilde{u}) - \frac{he^{2t}}{\pi}\sin(\pi\tilde{u}) \;=\; 0,$$

and differentiation with respect to t yields

$$-\partial_{tt}\phi + \cos(2\pi\tilde{u})\phi - he^{2t}\cos(\pi\tilde{u})\phi \;=\; \frac{2he^{2t}}{\pi}\sin(\pi\tilde{u}). \tag{4.24}$$

On the other hand, (4.19) implies for all $v \in \mathcal{A}_0$, $\tilde{v}(t) := v(e^t)$

$$\begin{aligned} 0 \;&\leq\; \int_0^1 r(\partial_r v(r))^2 + \left(\frac{1}{r}\cos(2\pi u(r)) - hr\cos(\pi u(r))\right) v(r)^2 \, dr \\ &= \int_{-\infty}^0 \left(e^t(e^{-t}\partial_t v(e^t))^2 + \left(e^{-t}\cos(2\pi u(e^t)) - he^t\cos(\pi u(e^t))\right) v(e^t)^2\right) e^t \, dt \\ &= \int_{-\infty}^0 (\partial_t\tilde{v}(t))^2 + \cos(2\pi\tilde{u}(t))\tilde{v}(t)^2 - he^{2t}\cos(\pi\tilde{u}(t))\tilde{v}(t)^2 \, dt. \end{aligned} \tag{4.25}$$

Assume that u is not monotonously increasing. Then there exists $t_0 < t_1 \leq 0$ such that $\phi(t_0) = \phi(t_1) = 0$ and $\phi(t) < 0$ for $t_0 < t < t_1$. Set

$$\tilde{\phi}(t) := \begin{cases} \phi(t) & \text{if } t_0 \leq t \leq t_1 \\ 0 & \text{otherwise.} \end{cases}$$

We test equation (4.24) with $\tilde{\phi}$.

$$\int_{-\infty}^{0} (\partial_t\tilde{\phi})^2 + \cos(2\pi\tilde{u})\tilde{\phi}^2 - he^{2t}\cos(\pi\tilde{u})\tilde{\phi}^2\,dt = \int_{-\infty}^{0} \underbrace{\frac{2he^{2t}}{\pi}\sin(\pi\tilde{u})}_{>0}\,\tilde{\phi}\,dt < 0.$$

This is a contradiction to (4.25). ☐

4.4.3 Nonexistence of minimisers in $\mathcal{A}_0$

Theorem 4.39. *Set*

$$\begin{aligned} b_h &:= \inf\{\tilde{I}_h(u)\,:\,u\in\mathcal{A}_0\},\\ b_\infty &:= \inf\{\tilde{I}_0(u,\mathbb{R}_0^+)\,:\,u(0)=0,\ \lim_{r\to\infty}u(r)=1\}. \end{aligned}$$

If $h \le 2$ then the constant zero function is the only minimiser of $\tilde{I}_h$ in $\mathcal{A}_0$. If $h > 2$ there exists no minimiser of $\tilde{I}_h$ in $\mathcal{A}_0$ and $b_h = b_\infty = \frac{2}{\pi}$.

Proof. In view of Lemma 4.32 we see that $b_\infty = \frac{2}{\pi}$. We show the inequality $b_h \le b_\infty$ by calculating $\tilde{I}_h(\xi_a)$ for large a. We have

$$\tilde{I}_h(\xi_a) \le b_\infty + \int_0^1 \frac{h\,r}{\pi}(\cos(\pi\xi_a)+1).$$

Since $\lim_{a\to\infty}\xi_a(r) = \lim_{a\to\infty}\xi_1(ar) = 1$, for $a\to\infty$ the functions ξ_a converge locally uniformly to 1 and $\int_0^1 hr(\cos(\pi\xi_a)+1)$ becomes arbitrarily small. Thus

$$b_h \;\le\; \lim_{a\to\infty}\tilde{I}_h(\xi_a) \;=\; b_\infty.$$

According to Lemma 4.38, we can limit our search for minimisers to monotonously increasing functions. So let $u\in\mathcal{A}_0$, $u\not\equiv 0$ be a monotonously increasing function. We have

$$\tilde{I}_h(u) \;=\; \tilde{I}_0(u) + \int_0^1 \frac{h\,r}{\pi}(\cos(\pi u(r))+1)\,dr.$$

To bound the first summand from below, we calculate as in the proof of Lemma 4.32

$$\tilde{I}_0(u) \;\ge\; \frac{1}{\pi}\int_0^1 |\cos(\pi u)'| \;=\; \frac{1-\cos(\pi u(1))}{\pi}.$$

To bound the second summand from below, we use the monotonicity of u

$$\begin{aligned} \int_0^1 \frac{h\,r}{\pi}(\cos(\pi u(r))+1)\,dr \;&>\; \int_0^1 \frac{h\,r}{\pi}(\cos(\pi u(1))+1)\,dr\\ &=\; \frac{h}{2\pi}(\cos(\pi u(1))+1). \end{aligned}$$

Combining theses estimates, we obtain

$$\begin{aligned}\tilde{I}_h(u) &> \frac{-\cos(\pi u(1))+1}{\pi} + \frac{h}{2\pi}(\cos(\pi u(1))+1)\\ &= \frac{2}{\pi} + \left(\frac{h}{2\pi} - \frac{1}{\pi}\right)(\cos(\pi u(1))+1).\end{aligned}$$

We can conclude that if $h > 2$ then $\tilde{I}_h(u) > \frac{2}{\pi} = b_\infty$, i.e., $b_h = b_\infty$ but the infimum is not attained. Otherwise, if $h \le 2$, $\tilde{I}_h(u) > \frac{2h}{\pi} = \tilde{I}_h(0)$. Thus a function $u \in \mathcal{A}_0$, $u \not\equiv 0$ is never a minimiser of $\tilde{I}_h$ in $\mathcal{A}_0$: Either the constant zero function has a smaller energy, or the infimum of the energy b_h is not attained at all. □

Corollary 4.40. *If $h \le 2$ then $u \equiv 1$ is the only solution u of* (4.18) *such that $\tilde{I}_h(u) \le \tilde{I}_h(0)$.*

4.4.4 Nonexistence of semistable stationary states for a large external magnetic field

Recall the definiton of k_0 from Remark 4.20. The main result in this subsection is:

Theorem 4.41. *(i) If $h > k_0^2$, the only solution $u \in \mathcal{A}$ with $u|_{[\frac{1}{k_0},1]} \le 0.33$ is $u \equiv 0$.*

(ii) If $h \ge k_0^2 + 1$, the only semistable solution $u \in \mathcal{A}$ of (4.18) *is $u \equiv 1$.*

Here, as introduced in Remark 4.20, $k_0 \approx 1.84$ is the first root of J_1, the first Bessel function of first kind. We prove the result in two steps and distinguish the regimes "$u(1)$ is small" and "$u(1)$ is large".

Lemma 4.42. *(i) If $h > k_0^2$ the only solution $u \in \mathcal{A}$ of* (4.18) *such that $u|_{[\frac{1}{k_0},1]} \le 0.33$ is $u \equiv 0$.*

(ii) If $h \ge k_0^2 + 1$ there exists no semistable solution $u \in \mathcal{A}$ of (4.18) *with $u(1) \le 0.62$.*

Proof. Let $u \in \mathcal{A}_0$, $u \not\equiv 0$ be a solution of (4.18) and set $v(r) := J_1(\frac{r}{k_0})$. Then

$$-v'' - \frac{1}{r}v' + \left(-k_0^2 + \frac{1}{r^2}\right) v \;=\; 0, \qquad v'(1) \;=\; 0. \tag{4.26}$$

Since $\int_0^1 r u'(r)^2 \, dr$ is finite, there is a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to zero such that $\epsilon_n u'(\epsilon_n)$ converges to zero. Thus

$$\begin{aligned}\int_0^1 ru''v + u'v &= \lim_{n\to\infty}\left(ru'v|_{\epsilon_n}^1 - \int_{\epsilon_n}^1 ru'v'\right)\\ &= \lim_{n\to\infty}\left(-ruv'|_{\epsilon_n}^1 + \int_{\epsilon_n}^1 ruv'' + uv'\right) = \int_0^1 ruv'' + uv'.\end{aligned}$$

We test equation (4.18) with $rv(r)$

$$\begin{aligned}0 &= \int_0^1 -(u'' + \frac{1}{r}u')rv + \left(-\frac{h}{\pi}\sin(\pi u) + \frac{1}{2\pi r^2}\sin(2\pi u)\right) rv\\ &= \underbrace{\int_0^1 -ru(v'' + \frac{1}{r}v') + \left(-k_0^2 + \frac{1}{r^2}\right) ruv}_{T_1} + \underbrace{\int_0^1 (k_0^2 - h)\frac{\sin(\pi u)}{\pi} rv}_{T_2}\\ &\quad + \underbrace{\int_0^1 \underbrace{\left(k_0^2\left(u - \frac{\sin(\pi u)}{\pi}\right) + \frac{1}{r^2}\left(-u + \frac{\sin(2\pi u)}{2\pi}\right)\right)}_{t_3(u,r)} rv}_{T_3} \qquad (4.27)\end{aligned}$$

Because of (4.26) the term T_1 is zero and since $h > k_0^2$ the term T_2 is negative.

(i) For $r < \frac{1}{k_0}$ and arbitrary $0 < s < 1$ the term $t_3(s, r)$ is negative. A numerical calculation shows that $t_3(s, 1) < 0$ for all $0 < s < 0.33$. Since the term $t_3(s, r)$ is monotonously increasing in r, this implies $t_3(s, r) < 0$ for all $0 < s \leq 0.33$, $0 < r \leq 1$. Thus if $u|_{[\frac{1}{k_0},1]} \leq 0.33$ then $T_3 < 0$ and $T_1 + T_2 + T_3 < 0$. This is a contradiction to (4.27), so there is no solution of (4.18) with $u|_{[\frac{1}{k_0},1]} \leq 0.33$.

(ii) Assume that u is a semistable solution of (4.18) with $u(1) \leq 0.62$. Since u is monotonously increasing (Theorem 4.34), with $u(r) \geq r\,u(1)$ (Corollary 4.37) we have

$$\sin(\pi u(r)) \geq 2u(r) \geq 2r\,u(1).$$

Moreover, a numerical calculation gives $t_3(u(1), 1) \leq 0.4u(1)$, so

$$t_3(u(r), r) \leq t_3(u(r), 1) \leq t_3(u(1), 1) \leq 0.4\,u(1).$$

Since $t_3(r, s) \leq 0$ for $r \leq \frac{1}{k_0}$, and since $v(1)r \leq v(r) \leq v(1)$, we can calculate

$$\begin{aligned}T_2 + T_3 &\leq \int_0^1 (k_0^2 - h)\frac{\sin(\pi u(r))}{\pi} rv(r)\, dr + \int_{\frac{1}{k_0}}^1 t_3(u(r), r)\, rv(r)\, dr\\ &\leq -\int_0^1 \frac{2}{\pi}u(1)v(1)\, r^3\, dr + \int_{\frac{1}{k_0}}^1 0.4u(1)v(1)\, r\, dr\\ &\leq \left(-\frac{1}{2\pi} + 0.2\left(1 - \frac{1}{k_0^2}\right)\right) u(1)v(1) < 0.\end{aligned}$$

Again we have a contradiction to (4.27). □

To show the nonexistence of semistable stationary solutions of (4.18) for large values of $u(1)$ we need an additional lemma.

Lemma 4.43. *Let u be a solution of* (4.18)*. Then*

$$\int_0^1 hr\ (\cos(\pi u(r)) - \cos(\pi u(1)))\ dr \ = \ \frac{1}{4}\sin^2(\pi u(1)).$$

Proof. For $\Omega \subset [0,1]$ and $v\colon [0,1] \to \mathbb{R}$ we define the functional J_h by

$$J_h(u,\Omega) \ := \ \int_\Omega \frac{hr}{\pi}(\cos(\pi u) + 1)\ dr.$$

For $\epsilon > 0$ we moreover set

$$u_\epsilon\colon [0,1] \to \mathbb{R}, \qquad u_\epsilon(r) := \begin{cases} u(\frac{r}{1-\epsilon}) & \text{if } 0 \le r < 1-\epsilon \\ u(1) & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \tilde{I}_h(u_\epsilon) \ &= \ \tilde{I}_0(u_\epsilon,[0,1-\epsilon]) + J_h(u_\epsilon,[0,1-\epsilon]) \\ &\quad + \tilde{I}_0(u_\epsilon,[1-\epsilon,1]) + J_h(u_\epsilon,[1-\epsilon,1]). \end{aligned}$$

We calculate the summands

$$\tilde{I}_0(u_\epsilon,[0,1-\epsilon]) \ = \ \tilde{I}_0(u)$$

$$\begin{aligned} J_h(u_\epsilon,[0,1-\epsilon]) \ &= \ \int_0^{1-\epsilon} \frac{hr}{\pi}\cos\left(\pi u\left(\tfrac{r}{1-\epsilon}\right)\right)\ dr \\ &= \ \int_0^1 \frac{ht(1-\epsilon)}{\pi}\cos(\pi u(t))\ (1-\epsilon)\ dt \\ &= \ J_h(u,[0,1]) \cdot (1 - 2\epsilon + \epsilon^2) \end{aligned}$$

$$\begin{aligned} \tilde{I}_0(u_\epsilon,[1-\epsilon,1]) \ &= \ \int_{1-\epsilon}^1 \frac{1}{2\pi r}\sin^2(\pi u(1))\ dr \\ &= \ \frac{1}{2\pi}\sin^2(\pi u(1)) \cdot (-\log(1-\epsilon)) \\ &= \ \frac{1}{2\pi}\sin^2(\pi u(1)) \cdot \left(\epsilon - \frac{1}{2}\epsilon^2 + O(\epsilon^3)\right) \end{aligned}$$

$$\begin{aligned} J_h(u_\epsilon,[1-\epsilon,1]) \ &= \ \int_{1-\epsilon}^1 \frac{h\,r}{\pi}(\cos(\pi u(1)) + 1)\ dr \\ &= \ \frac{h}{\pi}(\cos(\pi u(1)) + 1) \cdot \left(\epsilon - \frac{1}{2}\epsilon^2\right). \end{aligned}$$

The derivative with respect to ϵ is

$$\partial_\epsilon \tilde{I}_h(u_\epsilon)|_{\epsilon=0} \;=\; -2J_h(u,[0,1]) + \frac{1}{2\pi}\sin^2(\pi u(1)) + \frac{h}{\pi}(\cos(\pi u(1)) + 1)).$$

Since u is a stationary state we have $\partial_\epsilon I_h(u_\epsilon)|_{\epsilon=0} = 0$. This implies

$$J_h(u,[0,1]) \;=\; \frac{1}{4\pi}\sin^2(\pi u(1)) + \frac{h}{2\pi}(\cos(\pi u(1)) + 1),$$

which is equivalent to

$$\int_0^1 hr\,(\cos(\pi u(r)) - \cos(\pi u(1)))\,dr \;=\; \frac{1}{4}\sin^2(\pi u(1)).$$

□

Using this lemma, we now show that there are no nontrivial semistable solutions for $h \geq k_0^2 + 1$ and large $u(1)$.

Lemma 4.44. *For $h \geq k_0^2 + 1$ there exist no semistable solutions $u \in \mathcal{A}_0$ of (4.18) with $u(1) \geq 0.62$.*

Proof. Assume that $u(1) \geq 0.62$ and set $r_0 := 0.69$. Then for $a := \tan(\frac{\pi}{2}u(1))$ we have $\xi_a(1) = u(1)$ and for all $r \in [r_0, 1]$ we have $u(r) \geq \xi_a(r) \geq 0.5$. In particular we have the estimates

$$\sin(\pi u(r)) \geq \sin(\pi u(1)), \quad \sin(2\pi u(r)) \leq 0 \quad \text{for all } r \in [r_0, 1]\,. \tag{4.28}$$

We prove the lemma in two steps: First, using Lemma 4.43, we show

$$\frac{u(1) - u(r_0)}{\sin(\pi u(1))} \;\leq\; \frac{1}{2\pi\, h r_0^2} \;\leq\; \frac{1.0502}{\pi\, h}, \tag{4.29}$$

then, using equation (4.18), we show

$$\frac{u(1) - u(r_0)}{\sin(\pi u(1))} \;\geq\; 0.05455\frac{h}{\pi}. \tag{4.30}$$

Combining (4.29) and (4.30) we get

$$h \;\leq\; \sqrt{\frac{1.0502}{0.05455}} < 4.388 < k_0^2 + 1.$$

Step 1: Lemma 4.43 states that

$$\underbrace{\int_0^1 hr\,(\cos(\pi u) - \cos(\pi u(1)))\,dr}_{LS} \;=\; \frac{1}{4}\sin^2(\pi u(1)). \tag{4.31}$$

Using (4.28) and the fact that u is monotone (Theorem 4.34), we can bound the left hand side LS from below:

$$\begin{aligned} LS &\geq \int_0^{r_0} hr(\cos(\pi u(r)) - \cos(\pi u(1))\, dr \\ &\geq \int_0^{r_0} hr\left(\cos\left(\pi u\left(r_0\right)\right) - \cos(\pi u(1))\right)\, dr \\ &\geq \frac{\pi}{2} hr_0^2 \left(u(1) - u\left(r_0\right)\right) \sin(\pi u(1)). \end{aligned}$$

Combining this estimate with equation (4.31) yields (4.29).

Step 2: We test equation (4.18) with $r\ln(\frac{r}{r_0}) \cdot 1_{[r_0,1]}$

$$\begin{aligned} 0 &= \int_{r_0}^1 \left(-u''(r) - \frac{1}{r}u'(r)\right) r\ln\left(\tfrac{r}{r_0}\right)\, dr \\ &+ \int_{r_0}^1 \left(\frac{1}{2\pi r^2}\sin(2\pi u(r)) - \frac{h}{\pi}\sin(\pi u(r))\right) r\ln\left(\tfrac{r}{r_0}\right)\, dr. \end{aligned} \tag{4.32}$$

Since $(ru')' = u' + u''r$, partial integration of the first summand yields

$$\begin{aligned} &\int_{r_0}^1 \left(-u''(r) - \frac{1}{r}u'(r)\right) r\ln\left(\frac{r}{r_0}\right)\, dr \\ &= \underbrace{-ru'\ln\left(\frac{r}{r_0}\right)\Big|_{r=r_0}^{r=1}}_{=0} + \int_{r_0}^1 u'(r)\, dr \;=\; u(1) - u(r_0). \end{aligned}$$

Using (4.28), we can bound the second summand

$$\begin{aligned} &\int_{r_0}^1 \left(\frac{1}{2\pi r^2}\sin(2\pi u(r)) - \frac{h}{\pi}\sin(\pi u(r))\right) r\ln\left(\frac{r}{r_0}\right)\, dr \\ &\leq -\frac{h}{\pi}\sin(\pi u(1)) \int_{r_0}^1 r\ln\left(\frac{r}{r_0}\right) \;\leq\; -0.05455\,\frac{h}{\pi}\sin(\pi u(1)). \end{aligned}$$

So equation (4.32) implies

$$u(1) - u(r_0) \;\geq\; 0.05455\,\frac{h}{\pi}\sin(\pi u(1)),$$

which is equivalent to (4.30). □

4.4.5 Stationary states have a bounded derivative

In this subsection we will often use the following version of the maximum principle.

Lemma 4.45. *Given $r_0 < r_1$, let*

$$a, c :]r_0, r_1[\to \mathbb{R}^+, \quad b :]r_0, r_1[\to \mathbb{R}, \quad u, v : [r_0, r_1] \to \mathbb{R}$$

be functions such that

$$\begin{aligned} u(r_0) = v(r_0), \quad u(r_1) = v(r_1), \\ au'' + bu' - cu \ > \ av'' + bv' - cv \qquad \text{on }]r_0, r_1[. \end{aligned}$$

Then $u \leq v$ on $[r_0, r_1]$.

Proof. Assume the conditions of the lemma hold but the conclusion is false. Then $u - v$ has a positive maximum in some point x. Since $u'(x) - v'(x) = 0$, the conditions imply $u''(x) - v''(x) = \frac{c(x)}{a(x)}(u(x) - v(x)) > 0$. This is in contradiction to the assumption that $u - v$ attains a maximum in x. □

Let u be a solution of (4.18). Using the maximum principle, we can prove bounds of the form $u(r) \leq Cr^\alpha$ for any $\alpha < 1$. Unfortunately, we cannot prove $u(r) \leq Cr$ directly. So we first prove the bound $u(r) \leq C\sqrt{r}$ using the maximum principle and then use the fact that for small r_0 the function u is a minimiser of $I_h(\cdot, [0, r_0])$ in $\mathcal{W}(k, \rho)$.

Lemma 4.46. *For each solution $u \in \mathcal{A}_0$ of (4.18) there exists a number K_0 such that $u(r) \leq K_0\sqrt{r}$.*

Proof. We choose $\rho_0 < \frac{1}{2\sqrt{h}}$ such that $u(r) \leq \frac{1}{4}$ for all $r \leq \rho_0$. Then, for $0 < r \leq \rho_0$, we have the estimates

$$\begin{aligned} \frac{h}{\pi}\sin(\pi u(r)) \ &\leq \ \frac{1}{4\pi r^2}\sin(\pi u(r)) \ \leq \ \frac{1}{4\pi r^2}\sin(2\pi u(r)), \\ \sin(2\pi u(r)) \ &> \ \pi u(r), \end{aligned}$$

and therefore

$$\begin{aligned} 0 \ &= \ u''(r) + \frac{1}{r}u'(r) - \frac{1}{2\pi r^2}\sin(2\pi u(r)) + \frac{h}{\pi}\sin(\pi u(r)) \\ &\leq \ u''(r) + \frac{1}{r}u'(r) - \frac{1}{4\pi r^2}\sin(2\pi u(r)) \\ &\leq \ u''(r) + \frac{1}{r}u'(r) - \frac{1}{4r^2}u(r). \end{aligned}$$

Set $v : [0, \rho_0] \to \mathbb{R}, r \mapsto \frac{u(\rho_0)}{\sqrt{\rho_0}}\sqrt{r}$. Then v satisfies the differential equation $v''(r) + \frac{1}{r}v'(r) - \frac{1}{4r^2}v(r) = 0$, so the maximum principle (Lemma 4.45) implies

$$u(r) \ \leq \ v(r) \ = \ \frac{u(\rho_0)}{\sqrt{\rho_0}}\sqrt{r} \ \leq \ \frac{\sqrt{r}}{\sqrt{\rho_0}} \qquad \text{for } 0 < r \leq \rho_0.$$

Since for all $r \geq \rho_0$ the estimate $u(r) \leq 1 \leq \frac{\sqrt{r}}{\sqrt{\rho_0}}$ is trivially true, we can set $K_0 := \frac{1}{\sqrt{\rho_0}}$. □

Lemma 4.47. *Let $u \in \mathcal{A}_0$ be a solution of (4.18) and set $\alpha(r) := u'(r) - \frac{1}{\pi r}\sin(\pi u)$. Then the function $r \mapsto \frac{1}{r}|\alpha(r)|$ is integrable.*

Proof. First we calculate

$$\begin{aligned}
&\tilde{I}_h(u,[0,\rho]) \\
&= \int_0^\rho \frac{r}{2}(u'(r))^2 + \frac{1}{2\pi^2 r}\sin^2(\pi u(r)) + \frac{h\,r}{\pi^2}(\cos(\pi u(r)) - 1)\,dr \\
&= \int_0^\rho \frac{1}{\pi}\sin(\pi u(r))\,u'(r) + \frac{r}{2}\left(u'(r) - \frac{1}{\pi r}\sin(\pi u(r))\right)^2 \\
&\quad + \frac{h\,r}{\pi^2}(\cos(\pi u(r)) - 1)\,dr \\
&= 1 - \cos(\pi u(\rho)) + \int_0^\rho \frac{1}{2} r\,\alpha(r)^2 + \frac{h\,r}{\pi^2}(\cos(\pi u(r)) - 1)\,dr.
\end{aligned}$$

Choose ρ_0 such that $\rho_0 \le \sqrt{\frac{1}{2h}}$ and $u|_{[0,\rho_0]} \le \frac{1}{8}$. Then, in particular, for all $\rho \le \rho_0$ the function u is a minimiser of $\tilde{I}_h(\cdot,[0,\rho])$ in $\mathcal{W}(u(\rho),\rho)$ (Remark 4.30). It suffices to show that $\int_0^{\rho_0}\frac{1}{r}|\alpha(r)|$ is finite.

Using Lemma 4.32 we have for all $\rho \le \rho_0$

$$\tilde{I}_h(u,[0,\rho]) < \inf_{v\in\mathcal{W}(u(\rho),\rho)} \tilde{I}_0(v,[0,\rho]) = 1 - \cos(\pi u(\rho)).$$

Since $u \le K_0\sqrt{r}$, we have in particular

$$\begin{aligned}
\int_0^\rho r\,\alpha(r)^2 &\le 2\int_0^\rho \frac{hr}{\pi^2}\left(1 - \cos(\pi K_0\sqrt{r})\right)\,dr \\
&\le \int_0^\rho h\,K_0^2\,r^2\,dr \;=\; \frac{h}{3}K_0^2\,\rho^3.
\end{aligned}$$

We calculate $\int \frac{1}{r}\alpha\,dr$ on the intervals $I_k := [2^{-k-1}\rho_0, 2^{-k}\rho_0]$.

$$\begin{aligned}
\int_{I_k}\frac{1}{r}|\alpha(r)|\,dr &= \int_{I_k} r^{\frac{1}{2}}\,|\alpha(r)|\,r^{-\frac{3}{2}}\,dr \\
&\le \sqrt{\int_0^{2^{-k}\rho_0} r\,\alpha(r)^2\,dr}\;\sqrt{\int_{I_k} r^{-3}\,dr} \\
&\le \sqrt{\frac{1}{3}h\,K_0^2\,2^{-3k}\,\rho_0^3}\;\sqrt{2^{(2k+1)}\rho_0^{-2}} \\
&\le \sqrt{h\rho_0}\,K_0\,2^{-\frac{k}{2}}.
\end{aligned}$$

Thus we have the estimate

$$\int_0^{\rho_0}\frac{1}{r}\,|\alpha(r)|\,dr \;\le\; \sqrt{h\rho_0}\,K_0\sum_{k=0}^{\infty}2^{-\frac{k}{2}} \;\le\; 4\sqrt{h\rho_0}\,K_0.$$

□

Theorem 4.48. *For each solution u of* (4.18) *there exists a number K_1 such that we have $u(r) \le K_1 r$ and $u'(r) \le K_1$ for all $r \in [0,1]$.*

Proof. We define $v(r) := \frac{1}{r}u(r)$ and set $\alpha(r) := u'(r) - \frac{1}{\pi r}\sin(\pi u)$. Then

$$\begin{aligned} |v'(r)| &= \left|\frac{1}{r}u'(r) - \frac{1}{r^2}u(r)\right| = \left|\frac{1}{\pi r^2}\sin(\pi u) + \frac{1}{r}\alpha(r) - \frac{1}{r^2}u(r)\right| \\ &\le \frac{1}{r}|\alpha(r)| + \frac{\pi}{2r^2}\, u(r)^3 \le \frac{1}{r}|\alpha(r)| + \frac{\pi K_0^3}{2\sqrt{r}}. \end{aligned}$$

Since $r \mapsto \frac{1}{r}|\alpha(r)|$ is integrable (Lemma 4.47), v' is also integrable, and $v(r) = \frac{1}{r}u(r)$ is bounded by some number K_1. Now Corollary 4.37 implies $u'(r) \le \frac{u}{r} \le K_1$. □

We can use this information on solutions u of (4.18) to show a similar bound for the eigenfunctions of A_u.

Theorem 4.49. *Let $u \in \mathcal{A}_0$ be a solution of* (4.18)*, and let ϕ be an eigenfunction for some eigenvalue λ of A_u as defined in* (4.20)*. Then there is a number K_2 such that $\phi(r) \le K_2 r$ and $\phi'(r) \le K_2$ for all $r \in [0,1]$.*

Proof. To prove this lemma, we use the fact that $u(r) \le K_1 r$ and the maximum principle. For all $r \le r_0 := \frac{1}{2K_1}$ we have

$$\begin{aligned} 0 &= \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\cos(2\pi u_-) + h\cos(\pi u_-)\phi + \lambda\phi \\ &\le \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\cos(2\pi K_1 r)\phi + (h+\lambda)\phi \\ &\le \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}(1 - \frac{1}{2}(2\pi K_1 r)^2)\phi + (h+\lambda)\phi \\ &= \phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\phi + c_1^2\phi, \qquad \text{where } c_1 := \sqrt{2\pi K_1^2 + h + \lambda} \end{aligned}$$

As before let J_1 denote the first Bessel function of first kind. Then $j\colon x \mapsto J_1(c_1 x)$ is a solution of $j'' + \frac{1}{r}j' - \frac{1}{r^2}j + c_1^2 j = 0$ with $j(0) = 0$ and bounded derivative on $[0,1]$. Using the maximum principle (Lemma 4.45) and setting $c_2 := \frac{\phi(r_0)}{J_1(r_0)}$ we have the inequality $c_2(c_1 r) \ge \phi(r)$ for all $r \le r_0$. In particular, since the derivative of J_1 is bounded, there is a number K_2 such that $\phi(r) \le K_2 r$.

On the other hand, $\phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\phi < 0$. Since linear functions g are solutions of $g'' + \frac{1}{r}g' - \frac{1}{r^2}g = 0$, the maximum principle yields the estimate $\phi(r) \ge \frac{\phi(\rho)}{\rho}r$ for all $\rho \in [0,1]$ and all $r \in [0,\rho]$. Thus we get $\phi'(\rho) \le \frac{\phi(\rho)}{\rho} \le K_2$ for all $\rho \in [0,1]$. □

4.5 Possible end states of travelling waves modelling the vortex mode

In this section we consider possible end states of solutions of (4.4). In the first subsection we discuss the properties of end state $u_\pm$ at $\pm\infty$ and show that they are rotationally symmetric, semistable, stationary states with finite energy. In the second subsection we use the results about stationary states of Section 4.4 to show that $u_+ \equiv 1$ for small and large external magnetic field.

4.5.1 Properties of the end states

It is easy to see that end states are rotationally symmetric stationary states with finite energy.

Lemma 4.50. *Let* (u, c) *be a solution of* (4.4) *provided by Theorem 4.25 or Theorem 4.28. Then* $u(x, \cdot)$ *converges for* $x \to \pm\infty$ *in* $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$ *to maps* $u_{\pm\infty}$. *These are rotationally symmetric and satisfy*

$$\begin{aligned} \Delta_y u_\pm + f^0(u_\pm, y) &= 0 \quad \text{in } D_1, \\ \partial_\nu u_\pm &= 0 \quad \text{on } \partial D_1. \end{aligned} \tag{4.33}$$

For the energy of the end states we have $I_h(u_\pm) < \infty$.

Proof. Monotonicity of u in x, Lemma 4.27 and a bootstrap argument imply that $u(x, \cdot)$ converges for $x \to \pm\infty$ in $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$ to some functions $u_\pm : D_1 \to [0, 1]$. Thus in particular $\partial_x u(x, y)$ and $\partial_{xx} u(x, y)$ converge to zero for all $y \in D_1 \setminus \{0\}$, and passing to the limit in (4.4) yields (4.33). Since u is rotationally symmetric, the functions $u_\pm$ are rotationally symmetric as well.

If u is a minimiser of $\Phi^\dagger_c$ then $I_h(u_\pm)$ is obviously finite. If u is a non-variational solution, we use Theorem 4.28 (iii). Because of convergence in $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$ we have

$$\begin{aligned} I_0(u_\pm) &= \lim_{\delta\to 0} \lim_{x\to\pm\infty} \int_{D_1\setminus D_\delta} \int_x^{x+1} \frac{1}{2}|\nabla_y u(t, y)|^2 + V(u(t, y))\, dt\, dy \\ &\le \lim_{x\to\pm\infty} \int_x^{x+1} I_0(u(t, \cdot))\, dt \le \frac{h}{\pi}\left(4 + \frac{1}{|c^*|}\right) < \infty. \end{aligned} \tag{4.34}$$

□

The difficulty in the proof of the following theorem lies in the singularity of $f^0(u, y)$ for $y = 0$. If the function f^0 was smooth we could use the proof of Heinze [21, Thm. 2.4]. To overcome the problem we will use that, close to $y = 0$, Theorem 4.48 and Theorem 4.49 provide good bounds for the functions we are considering.

Theorem 4.51. *Let u be a solution of* (4.4) *provided by Theorem 4.25 or Theorem 4.28. Then u_+ is a semistable stationary state.*

Proof. First, we introduce some notation for this proof. For functions $w, \tilde{w} \colon D_1 \to \mathbb{R}$ we set

$$\begin{aligned} L_{u_+}(w) &:= -\Delta_y w - \partial_u f^0(u_+, y) w \\ &= -\Delta w + \frac{1}{|y|^2}\cos(2\pi u_+)w - h\cos(\pi u_+)w, \\ N(w) &:= -\Delta_y w - (f^0(u_+, y) - f^0(u_+ - w, y)). \end{aligned}$$

Let μ be the smallest eigenvalue of L_{u_+}, and let ϕ be the corresponding eigenfunction. Without loss of generality we can assume $u_+ \in A_0$. Then we have the bounds $u_+(y) \leq K_1|y|$ and $\phi(y) \leq K_2|y|$ for some constants K_1, K_2 (Theorem 4.48, Theorem 4.49).

For $v(x, y) := u_+(y) - u(x, y)$, Equation (4.4) yields

$$\partial_{xx}v + c\partial_x v = -\partial_{xx}u - c\partial_x u = \Delta_y u \underbrace{-\Delta_y u_+ - f^0(u_+, y)}_{=0} + f^0(u, y) = N(v).$$

Our strategy is to prove that, if μ is negative, then there is $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$ we have $\langle N(v(x,\cdot), \phi\rangle_{D_1} < 0$. Since $c\partial_x v$ is positive, this will imply

$$\langle \partial_{xx} v(x,\cdot), \phi\rangle_{D_1} < -\langle \partial_x v(x,\cdot), \phi\rangle_{D_1} < 0.$$

Thus $\partial_x\langle v(x,\cdot), \phi\rangle_{D_1}$ goes exponentially to $-\infty$ as x tends to $+\infty$ and we get a contradiction to the fact that v is bounded.

The operator N can be written as

$$\begin{aligned} N(w) &= -\Delta_y w - \int_0^w \partial_u f^0(u_+ - t, y)\, dt \\ &= -\Delta_y w - \left(\int_0^w \partial_u f^0(u_+, y) - \int_0^t \partial_{uu} f^0(u_+ - s, y)\, ds\, dt\right) \\ &= L(w) + \int_0^w \int_0^t \frac{2\pi}{|y|^2}\sin(2\pi u_+ - s) - h\pi\sin(\pi u_+ - s)\, ds\, dt, \end{aligned}$$

and we have the estimate

$$\begin{aligned} N(w) &\leq L(w) + \int_0^w \int_0^t \frac{2\pi}{|y|^2}(2\pi u_+ - s)\, ds\, dt \\ &\leq L(w) + \frac{4\pi^2}{|y|^2}u_+ \int_0^w \int_0^t 1\, ds\, dt \\ &= Lw + \frac{2\pi^2}{|y|^2}u_+ w^2 \leq L(w) + \frac{2\pi^2 K_1}{|y|}w^2. \end{aligned} \tag{4.35}$$

Claim 1: For each $\lambda > 0$ there exists $\delta > 0$ such that for all functions $w\colon D_1 \to \mathbb{R}$ with Neumann boundary values and $0 \le w \le u_+$ the inequality $\langle w, \phi\rangle_{D_1} \le \delta \|w\|_{L^1}$ implies $\int_{D_1} N(w) \ge \lambda \|w\|_{L^1}$.

We have

$$\begin{aligned} N(w) &= -\Delta w + \frac{1}{|y|^2}\cos(2\pi u_+)w + \int_0^w \int_0^t \frac{2\pi}{|y|^2}\sin(2\pi u_+ - s)\, ds\, dt \\ &\quad + \frac{h}{\pi}\underbrace{\left(-\cos(\pi u_+) + \cos(\pi u_+ - w)\right)}_{\ge -\pi w}. \end{aligned}$$

If $u_+ < \frac{1}{6}$ then $\sin(2\pi u_+ - s)$ is positive for all $s < u_+$ and $\cos(2\pi u_+) \ge \frac{1}{2}$. Choose r_0 such that $r_0 \le \frac{1}{\sqrt{4\lambda + 4h}}$ and $u_+|_{D_{r_0}} \le \frac{1}{6}$. Then

$$\begin{aligned} N(w)|_{D_{r_0}} &\ge -\Delta w(y)\, dy + \frac{1}{2r_0^2} w - hw \\ \int_{D_1} N(w) &\ge \int_{D_1} -\Delta w(y)\, dy + \frac{1}{2r_0^2}\|w\|_{L^1(D_{r_0})} - h\|w\|_{L^1(D_1)} \\ &\ge (2\lambda + 2h)\|w\|_{L^1(D_{r_0})} - h\|w\|_{L^1(D_1)}. \end{aligned} \tag{4.36}$$

Set $\delta := \frac{1}{2}\min_{y\in D_1\setminus D_{r_0}} \phi(y)$. Then we have $\langle \phi, w\rangle_{D_1} \ge 2\delta\|w\|_{L^1(D_1\setminus D_{r_0})}$, so the inequality $\langle \phi, w\rangle_{D_1} \le \delta\|w\|_{L^1(D_1)}$ implies $\|w\|_{L^1(D_{r_0})} \ge \frac{1}{2}\|w\|_{L^1(D_1)}$, and in particular (4.36) yields $\int_{D_1} N(w) \ge \lambda\|w\|_{L^1(D_1)}$.

Claim 2: There are numbers $x_0 \in \mathbb{R}$, $\delta > 0$ such that for all $x > x_0$ we have $\langle v(x,\cdot), \phi\rangle_{D_1} \ge \delta\|v(x,\cdot)\|_{L^1(D_1)}$.

Choose $\tilde{\delta}$ such that Claim 1 holds for $\lambda := 2\pi^2 K_1^2$ and set

$$\delta := \min\left(\tilde{\delta}, \frac{\langle v(0,\cdot), \phi\rangle}{\|v(0,\cdot)\|_{L^1(D_1)}}\right).$$

If $\langle v(x,\cdot), \phi\rangle_{D_1} \ge \delta\|v(x.\cdot)\|_{L^1(D_1)}$ for all $x \ge 0$ there is nothing to show. Otherwise set

$$\begin{aligned} x_0 &:= \inf\left\{x \ge 0 \mid \langle v(x,\cdot), \phi\rangle_{D_1} < \delta\|v(x,\cdot)\|_{L^1(D_1)}\right\}, \\ x_1 &:= \sup\left\{x > x_0 \mid \langle v(\tilde{x},\cdot), \phi\rangle_{D_1} < \delta\|v(x,\cdot)\|_{L^1(D_1)} \text{ for all } \tilde{x} \in]x_0, x_1[\right\}. \end{aligned}$$

Then $x_0 \in [0,\infty[$, $x_1 \in]0,\infty]$. By the choice of $\tilde{\delta}$, we have the relation $\int_{D_1} N(v(x,\cdot)) \ge 2\pi^2 K_1^2\|v(x,\cdot)\|_{L^1(D_1)}$ for all $x \in]x_0, x_1[$, thus

$$\begin{aligned} 0 &= \int_{D_1} \partial_{xx} v(x,y) + c\partial_x v(x,y) - N(v(x,y),y)\, dy \\ &\ge \partial_{xx}\|v(x,\cdot)\|_{L^1(D_1)} + c\,\partial_x\|v(x,\cdot)\|_{L^1(D_1)} - 2\pi^2 K_1^2\|v(x,\cdot)\|_{L^1(D_1)}. \end{aligned}$$

On the other hand, (4.35) implies

$$\begin{aligned} 0 &= \langle \partial_{xx} v(x,\cdot), \phi \rangle_{D_1} + \langle c\partial_x v(x,\cdot), \phi \rangle_{D_1} - \langle N(v(x,\cdot),\cdot), \phi \rangle_{D_1} \\ &\le \partial_{xx} \langle v(x,\cdot), \phi \rangle_{D_1} + c\partial_x \langle v(x,\cdot), \phi \rangle_{D_1} - 2\pi^2 K_1^2 \langle v(x,\cdot), \phi \rangle_{D_1}. \end{aligned}$$

If x_1 is finite, by definition $\langle v(x,\cdot), \phi\rangle_{D_1}$ and $\delta \|v(x,\cdot)\|_{L^1(D_1)}$ agree at x_0 and x_1. Otherwise $\langle v(x,\cdot), \phi\rangle_{D_1}$ and $\delta \|v(x,\cdot)\|_{L^1(D_1)}$ agree at x_0, and we have $0 = \lim_{x\to\infty} \langle v(x,\cdot), \phi\rangle_{D_1} = \lim_{x\to\infty} \delta \|v(x,\cdot)\|_{L^1(D_1)}$. In both cases the maximum principle yields $\delta \|v(x,\cdot)\|_{L^1(D_1)} \le \langle v(x,\cdot), \phi\rangle_{D_1}$ for all $x \in]x_1, x_0]$. This is a contradiction to the definition of x_0.

Claim 3: The eigenvalues of L_{u_+} are nonnegative.

Assume that the smallest eigenvalue μ of L_{u_+} is negative. Using (4.35), we have

$$\begin{aligned} \langle N(v(x,\cdot)), \phi\rangle_{D_1} &\le \langle L(v(x.\cdot)), \phi\rangle_{D_1} + \int_{D_1} \frac{2\pi^2 K_1}{|y|} v(x,y)^2 \phi(y)\, dy \\ &\le \mu \langle \phi, v(x.\cdot)\rangle_{D_1} + 2\pi^2 K_1 K_2 \|v(x,\cdot)\|_{L^\infty(D_1)} \|v(x,\cdot)\|_{L^1(D_1)}. \end{aligned}$$

So with Claim 2 we have for all $x \le x_0$ the relation

$$\langle N(v(x,\cdot)), \phi\rangle_{D_1} \le \left(\delta\mu + 2\pi^2 K_1 K_2 \|v(x,\cdot)\|_{L^\infty(D_1)}\right) \|v(x,\cdot)\|_{L^1(D_1)}.$$

Since $v(x,y) \le K_1|y|$ and $v(x,\cdot)$ converges to zero in $C^\infty_{\mathrm{loc}}(D_1 \setminus \{0\})$, we have that $\|v(x,\cdot)\|_{L^\infty}$ converges to zero as well, that is, the first summand inside the brackets is negative, and the second converges to 0. Thus there is some $x_2 \in \mathbb{R}$ such that $\langle N(v(x.\cdot))\phi\rangle < 0$ for all $x < x_2$. As discussed at the beginning of the proof, this implies that $\partial_x \langle v(x,\cdot), \phi\rangle$ goes exponentially to $-\infty$ as x tends to $+\infty$, which is a contradiction to the fact that $v \le u_+$ is bounded. □

4.5.2 Combining the facts

In this subsection we combine the results about stationary states with the results about travelling wave solutions.

Theorem 4.52. *Let u be a solution provided by Theorem 4.25 or Theorem 4.28, and set $u_\pm(y) := \lim_{x\to\pm\infty} u(x,y)$. Then*

(1.) $u_- \equiv 0$.

(2.) If $h \le 2$, we have $u_+ \equiv 1$.

(3.) If $h \ge k_0^2 + 1$, we have $u_+ \equiv 1$.

(4.) If $2 < h < k_0^2 + 1$, the function u_+ could be 1 or some other semistable stationary solution of (4.18)*. For such a solution we have $u_+(0) = 0$, u_+ is monotonously increasing, u_+' is bounded and $u_+'(0) > 0$.*

(5.) If u is a variational solution, $u(x, \cdot)$ converges in $L^\infty(D_1)$ to u_+.

Proof. For variational solutions (1.) is clear and for non-variational solutions it is a consequence of Theorem 4.28 and Theorem 4.41.

(2.) follows from Corollary 4.40 and Lemma 4.50.

(3.) follows from Theorem 4.41 and Theorem 4.51.

(4.) follows from Theorem 4.34 and Theorem 4.51.

(5.) For $u_+ \in \mathcal{A}_0$ the statement is obvious. Otherwise $u \equiv 1$ (Theorem 4.34), and we show the statement by contradiction. We assume that there is a minimiser u and a number $\delta > 0$ such that $\|1 - u(x, \cdot)\|_{L^\infty(D_1)} > \delta$ for all $x \in \mathbb{R}$. Because of convergence in $C^\infty_{\text{loc}}(D_1 \setminus \{0\})$, for each $\epsilon > 0$ there is an x_ϵ, such that $u|_{D_1 \setminus D_\epsilon} > 1 - \epsilon$ for all $x \geq x_\epsilon$. For θ_ϵ as in (2.37) we set

$$
\begin{aligned}
v_\epsilon(x, y) &:= \begin{cases} \theta_\epsilon\left(x - x_\epsilon - \frac{\sqrt{\epsilon}}{2}, |y|\right) & \text{for } |y| < \epsilon, \\ \theta_\epsilon\left(x - x_\epsilon - \frac{\sqrt{\epsilon}}{2}, \epsilon\right) & \text{for } |y| \geq \epsilon, \end{cases} \\
\tilde{u}_\epsilon(x, y) &:= \max(v_\epsilon(x, y), u(x, y)), \\
S_1 &:= \{(p \in \Sigma : v(p) > u(p)\}.
\end{aligned}
$$

By definition $\tilde{u}_\epsilon \geq u$, so

$$
\begin{aligned}
\Phi^0_{h,c^\dagger}(u) - \Phi^0_{h,c^\dagger}(\tilde{u}_\epsilon) \;&\geq\; \Phi^0_{0,c^\dagger}(u) - \Phi^0_{0,c^\dagger}(\tilde{u}_\epsilon) \\
= & \underbrace{\int_{S_1} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2\pi^2 r^2}\sin^2(\pi u)\right) e^{c^\dagger x}}_{a} \\
& - \underbrace{\int_{S_1} \left(\frac{1}{2}|\nabla v_\epsilon|^2 + \frac{1}{2\pi^2 r^2}\sin^2(\pi v_\epsilon)\right) e^{c^\dagger x}}_{b}.
\end{aligned}
$$

We have $S_1 \supseteq [x_\epsilon + 1, \infty[\times D_R$, and a calculation like in the proof of Lemma 4.29 yields for all $x > x_\epsilon$

$$
\begin{aligned}
\int_{D_1} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2\pi^2 r^2}\sin^2(\pi u)\right) \;&\geq\; \frac{1}{\pi}\left(\cos(\pi(1-\delta)) - \cos(\pi(1-\epsilon))\right) \\
&=\; \frac{1}{\pi}\left(\cos(\pi\epsilon) - \cos(\pi\delta)\right).
\end{aligned}
$$

Thus

$$
a \;\geq\; \frac{1}{c^\dagger} e^{c^\dagger(x_\epsilon + 1)} \frac{1}{\pi}\left(\cos(\pi\epsilon) - \cos(\pi\delta)\right).
$$

To estimate the second term b we define

$$\begin{aligned} S_2 &:= \{(x,y) \in \Sigma \setminus (\mathbb{R} \times D_\epsilon) : 1-\epsilon \le v_\epsilon(x,y) < 1\} \\ &= \{(x,y) \in \Sigma : |y| \ge \epsilon,\ x_\epsilon + \sqrt{\epsilon} - \epsilon\sqrt{\epsilon} \le x \le x_\epsilon + \sqrt{\epsilon}\}. \end{aligned}$$

Then

$$\begin{aligned} b \ \le\ & e^{c^\dagger x_\epsilon} \int_{[x_\epsilon, x_\epsilon+\sqrt{\epsilon}]\times D_\epsilon} \left(\frac{1}{2}|\nabla v_\epsilon|^2 + \frac{1}{2\pi^2 r^2}\sin^2(\pi v_\epsilon)\right) \\ & + e^{c^\dagger x_\epsilon} \int_{S_2} \left(\frac{1}{2}|\nabla v_\epsilon|^2 + \frac{1}{2\pi^2 r^2}\sin^2(\pi v_\epsilon)\right). \end{aligned}$$

The calculations in the proof of Lemma 2.34 imply

$$\int_{\mathbb{R}\times D_\epsilon} \left(\frac{1}{2}|\nabla v_\epsilon|^2 + \frac{1}{2\pi^2 r^2}\sin^2(\pi v_\epsilon)\right) \ \le\ \left(\frac{\pi^3}{3} + \frac{\pi^3}{12} + 2\pi\right)\epsilon \ =:\ C\epsilon,$$

and, in addition, we have

$$\int_{S_2} \left(\frac{1}{2}|\nabla v_\epsilon|^2 + \frac{1}{2\pi^2 r^2}\sin^2(\pi v_\epsilon)\right) \ \le\ \frac{\pi\epsilon\sqrt{\epsilon}}{2}\frac{\epsilon^2}{\epsilon^3} + \frac{\sqrt{\epsilon}\epsilon^2}{4\pi\epsilon^2} \ \le\ 2\sqrt{\epsilon}.$$

Combining the inequalities yields

$$\Phi^0_{h,c^\dagger}(u) - \Phi^0_{h,c^\dagger}(\tilde{u}_\epsilon) \ \ge\ e^{c^\dagger x_\epsilon}\left(\frac{1}{e\pi c^\dagger}(\cos(\pi\epsilon) - \cos(\pi\delta)) - (C+2)\sqrt{\epsilon}\right).$$

This is positive for small ϵ, contradicting the assumption that u is a minimiser of $\Phi^0_{h,c^\dagger}$. □

Chapter 5

Conclusion

In this chapter we interpret our results. We compare them to simulations and suggest new simulations to verify or disprove our conjectures. Moreover, we point out possible next steps in the mathematical analysis.

5.1 Static domain walls

In Chapter 2 we have analysed the properties of static domain walls in magnetic nanowires in dependence of the radius. In this section we compare our results to a recent review article [38]. Our results match numerical simulations very well.

In [38], Thiaville and Nakatani discuss the reversal modes from a physicist's point of view. This is the only article that describes the simulation of static domain walls in nanowires. The cross section of the wires are quadratic. The results match the dynamic simulations of reversal modes in wires with circular cross sections [16, 22, 40] and the results of this thesis. This suggests that the phenomena are robust and that assuming a circular cross section is not an illegal idealisation.

We have proved that for $R \to 0$ the micromagnetic energy functional Γ-converges to a reduced local energy functional

$$E_{\text{red}}(m) = \pi \|\partial_x m\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2} \|m_y\|^2_{L^2(\mathbb{R})}.$$

This functional is known in the physics community. Thiaville and Nakatani [38] use it to approximate the micromagnetic energy functional in nanowires. Our convergence result clarifies why and in which sense this approximation is valid.

We have established the convergence of the energy minimising domain walls to the minimiser m^{red} of the reduced problem,

$$m^{\text{red}}(x) = \left(\tanh\left(\frac{x}{\sqrt{2}}\right), \frac{1}{\cosh\left(\frac{x}{\sqrt{2}}\right)}, 0 \right).$$

Thiaville and Nakatani [38] compare their simulations with this profile and observe that for thin wires the agreement is very good.

We have shown that there exists a minimiser in the class of vortex walls (Theorem 2.21). Because of its symmetry, it is a critical point of the energy in the class of all walls. For thick wires, its energy is at most a constant factor larger than the energy of the optimal domain wall (Theorem 2.3). Thus we expect that vortex walls should be seen in simulations.

In [38] this is the case. In that article this type of wall is called Bloch point wall. It is remarkable that the vortex wall is seen even in wires without rotational symmetry.

The considerations in Section 2.7 and Section 2.8 suggest that the vortex wall should have a square root type singularity and that the length of the transition region scales with $R^2 \ln(R)$. Indeed, in [38, Fig. 2a] the simulated vortex walls seem to have such a singularity. It would be interesting to make a sequence of simulations to verify or disprove this conjecture.

For further mathematical research, we suggest to determine the Γ-limit for wires with other types of cross sections (elliptic, rectangular, etc) and to investigate whether the vortex wall can also be understood as a Γ-limit. It should be possible to solve the first problem with the methods used in this thesis.

5.2 The transverse mode

The results of Chapter 3 are a step towards understanding the transverse mode. We have proved that, for thin wires and weak external magnetic fields, there exists a travelling wave solution of the overdamped limit of the Landau-Lifshitz-Gilbert (LLG) equation. Moreover, we have shown that for thin wires the static domain walls have uniformly good regularity properties.

The final goal is to prove that for thin magnetic nanowires there exist, possibly rotating, travelling wave solutions of the full LLG equation and to describe them with an effective theory. For thin wires and weak external magnetic field the methods developed in this thesis should be applicable. In particular, we expect the regularity results developed in Section 3.4–3.5 to be helpful.

A candidate for an effective theory would be to replace the effective field corresponding to the micromagnetic energy by an effective field corresponding to the reduced energy of Chapter 2. Thiaville and Nakatani [38] have investigated travelling wave solutions to this simplified equation. Our results support this approach: Since E_{red} is the Γ-limit of the micromagnetic energy, it is reasonable to belive that this model is indeed an effective model for the reversal of thin wires. It would be interesting to make this relation more rigorous and investigate, whether and in which sense travelling wave solutions of the full Landau-Lifshitz-Gilbert equation converge to travelling wave solutions of the reduced problem.

In the theory of thin magnetic films, such effective models have been derived [14, 25, 7]. When the damping constant α is held constant, the effective equation of the in-plane magnetisation is the overdamped limit of the Landau-Lifshitz-Gilbert equation. Maybe there is a similar close relation between the LLG equation and its overdamped limit for magnetic nanowires.

5.3 The vortex mode

In Chapter 4 we have used the harmonic map heat flow equation with an additional external magnetic field as a qualitative model for the vortex mode.

We have proved the existence of travelling wave solutions with a moving singularity. Since the other terms in the energy are lower order, such solutions should exist also for the full gradient flow equation and even for the Landau-Lifshitz-Gilbert equation. However, the variational methods of Chapter 4 use exponentially weighted spaces. Therefore we cannot apply them to equations that contain a nonlocal term.

Quantitatively, the bounds on the speed established in Chapter 4 cannot be compared to the domain wall speeds of vortex walls found in simulations.

This is the first analytic model that takes into account the three dimensional structure of moving domain walls in nanowires. For the vortex mode, the analytic considerations are especially important because numerical calculations have problems resolving the singularity. Thiaville and Nakatani [38] observe for strong external magnetic field that the domain wall becomes tilted and curved, leading to a reduction in velocity. They wonder whether this an instability of purely numerical origin or real and suggest a systematic variation of the mesh. An instability at very high fields has also been seen in [22].

Similarly there exist two types of vortex waves in our model, depending on the strength of the external magnetic field. In the physical units, that is without rescaling, the critical field is proportional to $\frac{1}{R^2}$. We suggest a systematic variation also of the diameter and the magnetic field to see whether this transition to an instable regime can be explained by our model.

List of symbols

This list contains symbols that are valid in some parts of the thesis. They may have different meanings in different parts, and they also may denote local objects.

Bibliography

[1] G. Alberti. Some remarks about a notion of rearrangement. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(2):457–472, 2000.

[2] D. Atkinson, D. Allwood, G. Xiong, M. Cooke, C. Faulkner, and R. Cowburn. Magnetic domain wall dynamics in a submicrometer structure. *Nat. Mater.*, 2:85–87, 2003.

[3] D. Atkinson, G. Xiong, C. Faulkner, D. Allwood, D. Petit, and R. Cowburn. Magnetic domain wall logic. *Science*, 309:1688–1692, 2005.

[4] G. Beach, C. Nistor, C. Knutson, M. Tsoi, and J. Erskine. Dynamics of field-driven domain-wall propagation in ferromagnetic nanowires. *Nat. Mater.*, 4:741–744, 2005.

[5] H. Berestycki and L. Nirenberg. Travelling fronts in cylinders. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9(5):497–572, 1992.

[6] M. Bertsch, C. B. Muratov, and I. Primi. Traveling wave solutions of harmonic heat flow. *Calc. Var. Partial Differential Equations*, 26(4):489–509, 2006.

[7] A. Capella, C. Melcher, and F. Otto. Wave type dynamics in ferromagnetic thin film and the motion of neel walls. Preprint.

[8] G. Carbou. Regularity for critical points of a nonlocal energy. *Calc. Var. Partial Differential Equations*, 5(5):409–433, 1997.

[9] G. Carbou and P. Fabrie. Regular solutions for Landau-Lifschitz equation in $\mathbb{R}^3$. *Commun. Appl. Anal.*, 5(1):17–30, 2001.

[10] G. Carbou and S. Labbé. Stability for static walls in ferromagnetic nanowires. *Discrete Contin. Dyn. Syst. Ser. B*, 6(2):273–290 (electronic), 2006.

[11] J.-M. Coron. The continuity of the rearrangement in $W^{1,p}(\mathbf{R})$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 11(1):57–85, 1984.

[12] G. Dal Maso. *An introduction to Γ-convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[13] A. DeSimone, R. V. Kohn, S. Müller, and F. Otto. Recent analytical developements in micromagnetics. In G. Bertotti and I. Mayergoyz, editors, *The Science of Hysteresis*. Academic Press, Inc., 2005.

[14] W. E and C. J. García-Cervera. Effective dynamics for ferromagnetic thin films. *J. Appl. Phys.*, 90:370–374, 2001.

[15] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[16] H. Forster, T. Schrefl, D. Suess, W. Scholz, V. Tsiantos, R. Dittrich, and J. Fidler. Domain wall motion in nanowires using moving grids. *J. Appl. Phys.*, 91:6914–6919, 2002.

[17] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, volume 105 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1983.

[18] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[19] R. Hardt and D. Kinderlehrer. Some regularity results in ferromagnetism. *Comm. Partial Differential Equations*, 25(7-8):1235–1258, 2000.

[20] R. Hardt, D. Kinderlehrer, and F.-H. Lin. Existence and partial regularity of static liquid crystal configurations. *Comm. Math. Phys.*, 105(4):547–570, 1986.

[21] S. Heinze. A variational approach to travelling waves. *MPI Preprints*, 85/2001.

[22] R. Hertel and J. Kirschner. Magnetization reversal dynamics in nickel nanowires. *Physica B*, 343:206–210, 2004.

[23] K. Hong and N. Giodano. New effects in ferromagnetic nanostructures. *J. Magn. Magn. Mat*, 151:396–402, 1995.

[24] A. Hubert and R. Schäfer. *Magnetic Domains*. Springer-Verlag, New York, 2000.

[25] R. V. Kohn and V. V. Slastikov. Effective dynamics for ferromagnetic thin films: a rigorous justification. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2053):143–154, 2005.

[26] B. G. Korenev. *Bessel functions and their applications*, volume 8 of *Analytical Methods and Special Functions*. Taylor & Francis Ltd., London, 2002. Translated from the Russian by E. V. Pankratiev.

[27] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[28] M. Lucia, C. B. Muratov, and M. Novaga. Existence of traveling wave solutions for Ginzburg-Landau-type problems in infinite cylinders. *Arch. Rat. Mech. Anal.* To appear.

[29] M. Lucia, C. B. Muratov, and M. Novaga. Linear vs. nonlinear selection for the propagation speed of the solutions of scalar reaction-diffusion equations invading an unstable equilibrium. *Comm. Pure Appl. Math.*, 57(5):616–636, 2004.

[30] C. B. Muratov. A global variational structure and propagation of disturbances in reaction-diffusion systems of gradient type. *Discrete Contin. Dyn. Syst. Ser. B*, 4(4):867–892, 2004.

[31] C. B. Muratov and M. Novaga. Front propagation in infinite cylinders. I. A variational approach. submitted to *Commun. Math. Sci.*

[32] Y. Nakatani, A. Thiavile, and J. Miltat. Faster magnetic walls in rough wires. *Nat. Mater. Lett.*, 2:521–523, 2003.

[33] K. Nielsch, R. B. Wehrspohn, J. Barthel, J. Kirschner, U. Gösele, S. F. Fischer, and H. Kronmüller. Hexogonally ordered 100nm period nickel nanowire arrays. *Appl. Phys. Lett.*, 79(9):1360–1362, 2001.

[34] L. Piraux, J. M. George, J. F. Despres, C. Leroy, E. Ferain, R. Legras, K. Ounadjela, and A. Fert. Giant magnetoresistance in magnetic multilayered nanowires. *Appl. Phys. Lett.*, 65(19):2484–2486, 1994.

[35] D. Sanchez. Behaviour of the Landau-Lifschitz equation in a ferromagnetic wire. *Math. Methods Appl. Sci.* to appear.

[36] E. Snoeck, R. E. Dunin-Borkowski, F. Dumestre, P. Renaud, C. Amiens, B. Chaudret, and P. Zurcher. Quantitative magnetization measurements on nanometer ferromagnetic cobalt wires using electron holography. *Appl. Phys. Lett.*, 82(1):88–90, 2003.

[37] M. Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, second edition, 1996.

[38] A. Thiaville and Y. Nakatani. Domain-wall dynamics in nanowires and nanostripes. In *Spin dynamics in confined magnetic structures III*,

volume 101 of *Topics in applied physics*, pages 161–205. Springer Verlag, Berlin, 2006.

[39] D. Werner. *Funktionalanalysis.* Springer-Verlag, Berlin, 2000.

[40] R. Wieser, U. Nowak, and K. D. Usadel. Domain wall mobility in nanowires: Transverse versus vortex walls. *Phys. Rev. B*, 69:0604401, 2004.

[41] E. Zeidler. *Nonlinear functional analysis and its applications. IV.* Springer-Verlag, New York, 1988.

Acknowledgement

Firstly, I would like to express my gratitude to my supervisor Prof. Stefan Müller for suggesting the topic, for his support and for many stimulating discussions. I also owe him and the MPI for Mathematics in the Science the chance to work in an inspiring environment. I benefited a lot from the interaction with scientists in Leipzig and while attending conferences and summer schools.

Many thanks are due to Helmut Abels, Wendy Kress, Anja Schlömerkemper, and, especially, to Carl-Friedrich Kreiner for proofreading parts of this thesis. I also would like to thank Anja Schlömerkemper for her helpful advice while writing this thesis.

I am grateful to Alexander Krauss for his constant encouragement and interest, and, in particular, for his practical and moral support during the last weeks before the date of submission.

I acknowledge the financial support by the DFG Priority Program 1095 "Analysis, Modeling and Simulation of Multiscale Problems".

Bibiliographische Daten

Kühn, Katharina
Reversal Modes in Magnetic Nanowires
(Umklappvorgänge in magnetischen Nanodrähten)
Universität Leipzig, Dissertation
162 Seiten, 9 Abbildungen, 41 Referenzen

Wissenschaftlicher Werdegang

1998	Abitur am Markgrafengymnasium Karlsruhe
1998–2003	Studium der Mathematik und Physik an der TU Darmstadt
09/2000	Vordiplom Mathematik
09/2000	Vordiplom Physik
10/2003	Diplom Mathematik Diplomarbeit ,,Über direkte Limites diagonaler Ketten vom Typ U, O und SP und ihre Homotopiegruppen" Betreuer: Prof. Dr. Karl-Hermann Neeb
04/2004–heute	Doktorandin am Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig

Selbstständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichen oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

Leipzig, 09. Juli 2007

. . . .
(Katharina Kühn)