The Completion of the Manifold of Riemannian Metrics with Respect to its $L^2$ Metric

by

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CHAPTER 1

Introduction

1.1. Summary of results

Let $M$ be a smooth, closed, finite-dimensional, oriented manifold, and denote by $\mathcal{M}$ the Fréchet manifold of smooth Riemannian metrics on $M$. There is a natural Riemannian metric on $\mathcal{M}$ called the $L^2$ metric and denoted by $(\cdot, \cdot)$. It is the primary goal of this thesis to give a description of the completion of $(\mathcal{M}, (\cdot, \cdot))$. There are three main results we will summarize here. The first is the following:

**Theorem.** With the Riemannian distance function $d$ induced from $(\cdot, \cdot)$, $(\mathcal{M}, d)$ is a metric space.

This is indeed a theorem that needs proving, as the $L^2$ metric on $\mathcal{M}$ is an example of a so-called weak Riemannian metric. This means that the tangent spaces of $\mathcal{M}$ are not complete with respect to $(\cdot, \cdot)$, and so many general theorems from the usual theory of Riemannian Hilbert manifolds do not hold. In this theory, typically only so-called strong metrics are considered, with respect to which the tangent spaces are complete. Of course, for a strong Riemannian Hilbert manifold, the Riemannian metric induces a metric space structure on the manifold. However, for weak Riemannian manifolds, there are examples (cf. [36], [37]) where this does not hold, as the distance between some points may be zero. Therefore, one must explicitly prove that a given weak Riemannian manifold is a metric space.

Given a metric space structure on $\mathcal{M}$, we know that it has a completion, and the second result—which can be seen as the main result of the thesis—gives a concrete description of this. Let $\mathcal{M}_f$ denote the set of semimetrics on $M$ (i.e., sections of the bundle $S^2T^*M$ that induce a positive semidefinite scalar product on each tangent space of $M$) that have measurable coefficients and finite volume. Define an equivalence relation on $\mathcal{M}_f$ by saying $g_0 \sim g_1$ if the following statement holds for almost every $x \in M$: if $g_0(x)$ and $g_1(x)$ differ, then both $g_0(x)$ and $g_1(x)$ fail to be positive definite. If we let $\overline{\mathcal{M}}$ denote the completion of $\mathcal{M}$ with respect to $(\cdot, \cdot)$, then:

**Theorem.** There is a natural bijection $\Omega : \overline{\mathcal{M}} \to \mathcal{M}_f/\sim$ that is the identity when restricted to $\mathcal{M} \subset \overline{\mathcal{M}}$.

This completion fits in with the general philosophy that in order to complete a space of objects, one must allow objects of a somewhat more general type. Note that we start with smooth metrics, yet in order to complete $\mathcal{M}$, we must add in points corresponding to metrics with far worse properties. This essentially arises from the fact that the $L^2$ metric—as its name implies—induces the $L^2$ topology on the tangent spaces of $\mathcal{M}$, which themselves only consist of smooth objects. Thus, the extreme incompleteness of the tangent spaces is reflected in the incompleteness of the space $\mathcal{M}$ itself.

The final result we describe here is an application of the completion of $\mathcal{M}$ to Teichmüller theory. If the base manifold $M$ is additionally assumed to be a Riemann surface of genus larger than one, then the Teichmüller space $T$ of $M$ can be identified with the space of conformal classes of metrics on $M$ modulo $D_0$, by which we denote the diffeomorphisms of $M$ that are homotopic to the identity. Let $\mathcal{N}$ be a smooth submanifold of $\mathcal{M}$ which is invariant under the action (by pull-back) of the diffeomorphism group, and
which contains exactly one representative from each conformal class. Then we have a diffeomorphism \( T \cong \mathcal{N}/D_0 \), and the \( L^2 \) metric restricted to \( \mathcal{N} \) induces a Riemannian metric on \( T \). As a corollary of the last theorem, we have:

**Theorem.** Each point in the completion of \( T \) with respect to the Riemannian metric described above can be identified with an element of \( \mathcal{M}_f/\sim \). This identification is not unique.

The metrics on \( T \) we have just constructed generalize the Weil-Petersson metric on Teichmüller space, and this theorem generalizes what is already known about the completion of Teichmüller space with respect to the Weil-Petersson metric.

### 1.2. Motivation

The original motivation for studying this problem comes from Teichmüller theory, and that is why this application in particular is given. Let us describe how our considerations arose from similar ones in Teichmüller theory.

As above, let the base manifold \( M \) be a Riemann surface of genus greater than one. Consider the group \( \mathcal{P} \) of positive functions on \( M \); it acts on \( \mathcal{M} \) by pointwise multiplication. The quotient space \( \mathcal{M}/\mathcal{P} \) is a smooth manifold, called the manifold of conformal classes on \( M \). Furthermore, the pull-back action of a diffeomorphism on \( M \) descends to an action on \( \mathcal{M}/\mathcal{P} \).

Fischer and Tromba \([55]\) have given a description of Teichmüller space \( T \) in this context. They show that there exists a diffeomorphism 

\[
T \cong (\mathcal{M}/\mathcal{P})/D_0.
\]

Thus, Teichmüller theory can be considered, in their words, in a “purely Riemannian” way.

A crucial step in Fischer and Tromba’s approach is using the Poincaré uniformization theorem to show a diffeomorphism between \( \mathcal{M}/\mathcal{P} \) and the space \( \mathcal{M}_{-1} \) of hyperbolic metrics (those with constant scalar curvature \(-1\)) on \( M \). By the Poincaré uniformization theorem, there exists exactly one hyperbolic metric in each conformal class on \( M \). Thus, Teichmüller space can just as well be described as 

\[
T \cong \mathcal{M}_{-1}/D_0.
\]

The advantage of using \( \mathcal{M}_{-1} \) is that the submanifold \( \mathcal{M}_{-1} \subset \mathcal{M} \) is easier to work with than the quotient space \( \mathcal{M}/\mathcal{P} \). Furthermore, the above construction allows us to define a metric on Teichmüller space by first restricting the \( L^2 \) metric to \( \mathcal{M}_{-1} \) and then looking at the metric it induces on the quotient \( \mathcal{M}_{-1}/D_0 \), and of course also on \( T \). The metric thus defined coincides, up to a constant scalar factor, with the well-known Weil-Petersson metric on Teichmüller space.

The Weil-Petersson metric has been the object of much study, and its completion has very interesting properties. Wolpert \([56]\) and Chu \([7]\) independently proved that it is incomplete, as there are geodesics that cannot be indefinitely extended—in finite time, they hit a singular limit surface.

Masur \([33]\) computed the asymptotics of the Weil-Petersson metric as one approaches the boundary of Teichmüller space. He did so in order to describe an extension of the metric to the completion of Teichmüller space. The completion of Teichmüller space with respect to the Weil-Petersson metric also induces a compactification of the moduli space of \( M \), and this compactification coincides with the Deligne-Mumford compactification, which arises in the context of algebraic geometry \([9]\). Thus, the Weil-Petersson metric links the differential geometric and algebraic geometric approaches to moduli space, which on the surface seem quite disparate.
Habermann and Jost [22], [23] later generalized the Weil-Petersson metric in the following way. The correspondence between \( \mathcal{M}/\mathcal{P} \) and \( \mathcal{M}_{-1} \) is basically given by the fact that \( \mathcal{M}_{-1} \) is a smooth global section of the principal \( \mathcal{P} \)-bundle
\[
\mathcal{M} \to \mathcal{M}/\mathcal{P}.
\]
The question that naturally arises is, what if one were to take a different section of this bundle? To retain the correspondence with Teichmüller and moduli space, the section should be smooth and invariant under the diffeomorphism group, but as long as these requirements are satisfied, any section will give a metric on Teichmüller space. Note that though we take direct inspiration from Habermann and Jost, the authors did not treat this exactly the same way as we described in Section 1.1, but rather retained some of the structures from the complex analytic definition of Teichmüller space (described, e.g., in [26]). The construction we described in Section 1.1 is, in the spirit of Fischer and Tromba, a purely Riemannian one, and it was chosen primarily because it allows us to directly apply our main result. The differences between our construction and that of Habermann and Jost are described in Chapter 6.

In [22], Habermann and Jost first considered the section given by the so-called Bergman metric in each conformal class. They gave a description of the completion of Teichmüller space with respect to their generalization of the Weil-Petersson metric for this special case. In [23], they considered all possible choices of sections, giving a sufficient analytic criterion for incompleteness of their generalized Weil-Petersson metric.

Thus, our application as described in Section 1.1 is in the same spirit as the papers of Habermann and Jost, and we expect that our theorems will have other, similar applications, especially to Teichmüller theory. That we have nevertheless chosen to prove the other main results listed in Section 1.1 for base manifolds of all dimensions and topologies has various reasons. First, the generalization to arbitrary dimension was mostly straightforward. Second, the manifold of Riemannian metrics on an \( n \)-dimensional manifold arises in various other contexts, and it is possible that our theorems might find applications there. The manifold of metrics has been considered in general relativity by, e.g., DeWitt [10]. Furthermore, critical points of functionals on the manifold of metrics have been used to determine “best metrics” on a given base manifold—for a nice survey of this topic with compendious references, see [2] Ch. 11. Finally, the manifold of metrics is itself of great intrinsic interest, as it exhibits interesting geometry. We will review the work that has been done on this last aspect in the next section.

1.3. Overview of previous work

Geometric structures on the manifold of metrics were perhaps first considered by DeWitt [10], who, as mentioned above, was interested in applications to general relativity. The metric on \( \mathcal{M} \) considered by DeWitt is quantitatively similar to the \( L^2 \) metric, but has different signature.

Ebin [11] shortly thereafter used the \( L^2 \) metric on \( \mathcal{M} \) to obtain local slices for the action of the diffeomorphism group on \( \mathcal{M} \). He used this, for one, to obtain results about the topology of so-called superspace, which is the quotient \( \mathcal{M}/\mathcal{D} \) of \( \mathcal{M} \) by the group \( \mathcal{D} \) of smooth, orientation-preserving diffeomorphisms of \( M \). Superspace can be viewed as the space of Riemannian geometries on \( M \). Ebin also used his slice theorem to show that the set of metrics \( \mathcal{M}' \) with trivial isometry group is an open, dense subset of \( \mathcal{M} \).

Later, Freed and Groisser [19] studied the basic geometry of \( \mathcal{M} \) with the \( L^2 \) metric. They computed the curvature and geodesics of \( \mathcal{M} \) and two related manifolds, the submanifold \( \mathcal{M}_\mu \subset \mathcal{M} \) of metrics inducing a fixed volume form \( \mu \), and the manifold \( V \) of smooth volume forms on \( M \). Additionally, Freed and Groisser used their results to study the curvature and geodesics of the quotient manifold \( \mathcal{M}'/\mathcal{D} \), where \( \mathcal{M}' \) is again the set of metrics with trivial isometry group.
Gil-Medrano and Michor \cite{20} generalized the results of Freed and Groisser on \( \mathcal{M} \) to the case of base manifolds \( M \) that are not necessarily compact. Though the Ricci and scalar curvature of \( \mathcal{M} \) cannot be defined in the usual way, they define and compute curvatures on \( M \) which they call “Ricci-like” and “scalar-like” curvature. Moreover, Gil-Medrano and Michor give a detailed analysis of the exponential mapping, which serves as an excellent illustration of the problems that can arise when considering weak instead of strong Riemannian manifolds. They also prove the existence and uniqueness of Jacobi fields on \( \mathcal{M} \), and give an explicit expression for these fields.

A generalized version of \cite{20} is the paper \cite{21} by Gil-Medrano, Michor, and Neuwirther. The results of this paper are also given in \cite{29}, §45.

We will rely heavily on the work of the above-mentioned authors and are indebted to all of them for laying the foundations upon which this thesis is built.

1.4. Outline of the thesis

The thesis is arranged as follows. In Chapter 2, we summarize the preliminary knowledge necessary to carry out and understand the work that we will do in the remainder of the thesis. We begin with a discussion of the completion of a metric space, which is meant to recall fundamental results on this topic and collect all the facts we will need into a coherent form. Following that, we give the definition of Fréchet manifolds. This is the category in which we will work, and we describe how spaces of smooth mappings, like \( \mathcal{M} \), can be viewed as Fréchet manifolds. We then go over a few somewhat nonstandard facts from Riemannian geometry for which we could find no complete reference.

In Chapter 2, we also discuss weak Riemannian manifolds, a class of manifolds that includes \((\mathcal{M},(\cdot,\cdot))\), as we already mentioned. In particular, we sketch an example by Michor and Mumford \cite{37} of the potentially pathological properties of such manifolds, as well as giving our own proofs of some standard results from the theory of Riemannian Hilbert manifolds that we have weakened so that they hold for weak Riemannian manifolds as well. With knowledge of these structures at hand, we then go into details on the manifold of metrics itself, more explicitly describing many of the previously known facts mentioned in Section 1.3. Chapter 2 closes with a list of conventions and notation that we use throughout the thesis.

In Chapter 3 we begin by proving the first of the main results given in Section 1.1, namely that \( \mathcal{M} \) with its \( L^2 \) metric has the structure of a metric space. One of the steps in this proof, which is also of use in later chapters, is the fact that the function on \( \mathcal{M} \) assigning to a metric the square root of its total volume is Lipschitz. The second half of the chapter initiates the study of the completion of \((\mathcal{M},(\cdot,\cdot))\), where we first try to complete “nice” subspaces of \( \mathcal{M} \). We show that if we take a subset of metrics satisfying certain uniformity conditions, then the completion of such a subset with respect to \((\cdot,\cdot)\) coincides with its completion with respect to the \( L^2 \) norm (not to be confused with the \( L^2 \) metric). This fact is used as a springboard for our further investigations of the completion.

The completion of a metric space is a quotient space of the set of Cauchy sequences in the space. Since we wish to identify \( \overline{\mathcal{M}} \) with (a quotient of) the space \( \mathcal{M}_f \) of measurable, finite-volume, positive semidefinite sections of \( S^2 T^* M \), we need a rigorous notion for how a Cauchy sequence in \( \mathcal{M} \) converges to an element of \( \mathcal{M}_f \). This notion, which we call \( \omega \)-convergence, is described in Chapter 4, where we prove that every Cauchy sequence in \( \mathcal{M} \) subconverges to a unique element of \( \mathcal{M}_f \) (cf. Section 1.1). This allows us to define the map \( \Omega : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_f \) mentioned in Section 1.1, as well as to show that \( \Omega \) is an injection. This chapter is the most technically challenging of the thesis.

In the definition of \( \omega \)-convergence, we basically have pointwise convergence of the metrics in a Cauchy sequence \( \{g_k\} \) almost everywhere, with the exception that on any set \( E \) with \( \text{Vol}(E, g_k) \to 0 \), there is no convergence required and none can be asked for. The
reason for this is the following proposition, which is in our eyes one of the most striking and unexpected results of the thesis:

**Proposition.** Suppose that $g_0, g_1 \in \mathcal{M}$, and let $E := \text{carr}(g_1 - g_0) = \{ x \in M \mid g_0(x) \neq g_1(x) \}$. Let $d$ be the Riemannian distance function of the $L^2$ metric $(\cdot, \cdot)$. Then there exists a constant $C(n)$ depending only on $n := \dim M$ such that

$$d(g_0, g_1) \leq C(n) \left( \sqrt{\text{Vol}(E, g_0)} + \sqrt{\text{Vol}(E, g_1)} \right).$$

In particular, we have

$$\text{diam} \left( \{ \tilde{g} \in \mathcal{M} \mid \text{Vol}(M, \tilde{g}) \leq \delta \} \right) \leq 2C(n)\sqrt{\delta}.$$

The surprising thing about this proposition is that it says that two metrics can vary wildly, but as long as they do so on a set that has small volume with respect to each, they are close together in the $L^2$ metric. For example, if $M = T^2$, the two-dimensional torus, with its standard chart $([0, 1] \times [0, 1]$ with edges identified), we consider the metrics

$$g_0 = \begin{pmatrix} 10 & 0 \\ 0 & 10^{-5} \end{pmatrix} \quad g_1 = \begin{pmatrix} 10^{10} & 0 \\ 0 & 10^{-14} \end{pmatrix}.$$ 

By the above proposition, these two very different metrics satisfy $d(g_0, g_1) \leq C(n)/100$, simply because they define tori with small volume. The difference between the geometries defined by $g_0$ and $g_1$ is depicted very qualitatively in Figure 1.

The above proposition is the reason why, in the second theorem of Section 1.1, we identify $\overline{\mathcal{M}}$ with a quotient space of the space $\mathcal{M}_f$ of semimetrics with measurable coefficients and finite volume, instead of $\mathcal{M}_f$ itself. The reasons for this are discussed in more detail in Chapter 4.

In Chapter 5 we complete the proof of the second main result of Section 1.1 by showing that the map $\Omega : \overline{\mathcal{M}} \to \overline{\mathcal{M}_f}$ is a surjection. Combined with the already mentioned results of Chapter 4 we thus see that $\Omega$ is a bijection, proving the main result on the completion of $\mathcal{M}$.

Finally, in Chapter 6 we give a more detailed overview of the aspects of Teichmüller theory mentioned in Section 1.2. One novelty of our presentation of this well-tread area of mathematics is a compact and relatively elementary proof of the existence of horizontal lifts for the principal bundle $\mathcal{M}_{-1} \to \mathcal{M}_{-1}/D_0$. We note, though, that this existence has been long-known to experts in the field.

After presenting the known facts about Teichmüller theory and the Weil-Petersson metric that we need, we give the generalizations of the Weil-Petersson metric mentioned in Section 1.1 rigorously stating and proving the result on the completion of Teichmüller space with respect to these metrics.

**Figure 1.** Two tori that are close together in $\mathcal{M}$ (with base manifold $M = T^2$) purely by virtue of having small volume.
At this point, we would like to draw the reader’s attention to two reading aids that should help to avoid confusion. First, on page 125 we lay out the relations between the various Riemannian metrics, distance functions and convergence notions used in the thesis. The second aid is the list of symbols on page 127 where we have attempted to include all symbols used with any frequency throughout the text. We hope that that these two guides provides the reader with at least a trail of bread crumbs to avoid getting lost while navigating the thesis.

1.5. Outlook

We have given just one application of the main result of our thesis, the application to Teichmüller theory. However, we envision more applications to arise in the future, in particular applications to determining the completion of superspace $\mathcal{M}/\mathcal{D}$, the space of Riemannian geometries on $M$ mentioned in Section 1.3.

In particular, the $L^2$ metric is invariant under the pull-back action of the group $\mathcal{D}$ of orientation-preserving diffeomorphisms of $M$ (see Section 6.1.2), and so it induces a well-defined distance function on the quotient. Note that we do not get a Riemannian metric, since the quotient is a singular space due to non-freeness of the $\mathcal{D}$-action at any metric with nontrivial isometry group. Of course, we nevertheless hope that information about the completion of $\mathcal{M}$ can lead us to information about the completion of $\mathcal{M}/\mathcal{D}$.

These results are not immediate, however, for a number of reasons. The simple fact that $\mathcal{M}$ is a metric space does not necessarily imply that the orbit space $\mathcal{M}/\mathcal{D}$ carries a metric space structure as well—the induced distance function may only be a pseudometric, as a priori two $\mathcal{D}$-orbits may be infinitesimally close to one another. The singular nature of $\mathcal{M}/\mathcal{D}$ makes it difficult to relate distances on $\mathcal{M}/\mathcal{D}$ to those on $\mathcal{M}$. Here, the existence of Ebin’s slice [11] might be helpful. Alternatively, one could adopt the philosophy analogous to using Teichmüller space for studying moduli space and first study an intermediate, smooth space like Fischer’s resolution of the singularities of $\mathcal{M}/\mathcal{D}$ [14].

These considerations are, however, extremely preliminary, and are merely given to illustrate one potential future direction this work might lead us in.
CHAPTER 2

Preliminaries

In this chapter, we define and explore the concepts necessary to carry out the main body of the work. The chapter is structured as follows:

We go over the most basic material in Section 2.1, where we briefly recall the definitions and fundamental facts regarding completions of metric spaces. We also give an alternative definition of the completion of a metric space that is more suited to studying Riemannian manifolds.

We give a definition of Fréchet manifolds in Section 2.2, since this will be the category in which we work. We go into depth on the class of Fréchet manifolds that plays the greatest role in global analysis, that of manifolds of mappings (actually, manifolds of sections of finite-dimensional fiber bundles).

In Section 2.3, we briefly review some of the geometry that will be needed for the subsequent portions of the thesis.

We then generalize the notion of a Riemannian metric to Fréchet manifolds in Section 2.4. In particular, we are interested in so-called weak Riemannian metrics on Hilbert and Fréchet manifolds, as the $L^2$ metric on the manifold of metrics is such an object. Weak Riemannian metrics are, as we will argue, fundamental objects in global analysis, though the lack of a good general theory for them makes their study more difficult than the tamer strong Riemannian manifolds. With some notable exceptions, the research on weak Riemannian manifolds focuses on studying specific cases, and the difficulties arising from the weak nature of the metrics are often only implicit. General results on weak Riemannian manifolds are often given without proof, as the statements are typically just a weakening of the corresponding statements for strong Riemannian manifolds. Nevertheless, we felt a precise treatment was appropriate for this work. Therefore, at the end of Section 2.4 we present some results which are relatively straightforward generalizations of analogous results for strong Riemannian manifolds—though necessarily weaker—and which the author has not found explicitly proved anywhere else in the literature.

The study of weak Riemannian metrics will allow us to define the Riemannian manifold $M$ of Riemannian metrics in Section 2.5, as well as to discuss what is already known about this manifold, in particular what is already known about its metric geometry. For example, we will give a description of its exponential mapping and discuss its curvature. We will also discuss two important classes of submanifolds, the orbits of the conformal group (i.e., the group of positive functions) and the manifolds of metrics that induce the same volume form.

Finally, we end the chapter with Section 2.6, which describes the nonstandard conventions that will be in place throughout the text.

2.1. Completions of metric spaces

In this short section, we look at completions of metric spaces. We will simply state the definition and explore a couple of consequences of it, then give an alternative, equivalent viewpoint for path metric spaces.

For the rest of the section, let $(X, \delta)$ be a metric space.

Recall that $X$ is called complete if every Cauchy sequence converges. Even if $X$ is incomplete, there is a very natural way to construct a complete space from $X$. The basic
idea is that if we want a space in which every Cauchy sequence converges, then we should replace $X$ with a space in which each point represents a Cauchy sequence in $X$. Then each Cauchy sequence in this new space “converges to itself” in a certain sense. This idea can be made more precise as follows.

The precompletion of $(X, \delta)$ is the set $\overline{X}^\text{pre}$, usually just denoted by $\overline{X}^\text{pre}$, consisting of all Cauchy sequences of $X$, together with the distance function

$$\delta(\{x_k\}, \{y_k\}) := \lim_{k \to \infty} \delta(x_k, y_k).$$

(We denote the distance function of the precompletion of a space using the same symbol as for the space itself; which distance function is meant will always be clear from the context.) We claim that $\delta$ is well-defined by the above definition, as $\delta(x_k, y_k)$ is a Cauchy sequence in $\mathbb{R}$, so the limit exists. To see this, choose $K$ large enough that $k, l \geq K$ implies $\delta(x_k, x_l) < \epsilon/2$ and $\delta(y_k, y_l) < \epsilon/2$. Then

$$\delta(x_l, y_l) \leq \delta(x_l, x_k) + \delta(x_k, y_k) + \delta(y_k, y_l) < \delta(x_k, y_k) + \epsilon,$$

and similarly with $k$ and $l$ swapped, showing $|\delta(x_k, y_k) - \delta(x_l, y_l)| < \epsilon$.

It is immediate from the definition that $\delta$ defines a pseudometric on $\overline{X}^\text{pre}$ (i.e., $\delta$ satisfies all properties of a metric except that two distinct points may have $\delta$-distance zero from one another). Therefore, as with any pseudometric space, we can define a metric space by declaring all points with distance zero from one another to be equal. In this case, the resulting space $\overline{X}$ is called the completion of $X$. In symbols, its definition is

$$\overline{X} := \overline{X}^\text{pre} / \sim,$$

where $\sim$ is the equivalence relation defined by

$$\{x_k\} \sim \{y_k\} \iff \delta(\{x_k\}, \{y_k\}) = 0. \tag{2.1}$$

Of course, we wouldn’t call it the completion if we didn’t have good reason to. The next theorem proves this and shows two other important properties of the completion $\overline{X}$ of $X$.

Before we state the theorem, we simply remark that if $\{x_k\}$ is a Cauchy sequence in $X$ and $\{x_{m_k}\}$ is a subsequence, then clearly $\{x_{m_k}\} \sim \{x_k\}$. Thus, given an element of the precompletion of $X$, we can always pass to a subsequence and still be talking about the same element of the completion.

**Theorem 2.1.** The completion $\overline{X}$ of $X$ has the following properties:

1. $\overline{X}$ is a complete metric space.
2. The canonical embedding of $X$ into $\overline{X}$ mapping a point $x$ to the constant sequence $\{x\}$ is an isometry, and the image of $X$ is a dense subspace of $\overline{X}$.
3. Any uniformly continuous function $f : X \to Y$, where $Y$ is a complete metric space, has a unique extension to a uniformly continuous function on $\overline{X}$.

**Proof.** To prove (1), let any Cauchy sequence $\{x_k^l\}$ in $\overline{X}$ be given. The index $l$ is meant to be the index in $\overline{X}$, while $k$ is meant to be the index in $X$. Thus, for each fixed $l$, $\{x_k^l\}$ is a Cauchy sequence in $X$ with index $k$. The square brackets in the above represent that each element of $\overline{X}$ is an equivalence class of Cauchy sequences.

We claim that $\{x_k^l\}$ converges to the equivalence class of the diagonal sequence $\{x_k^k\}$; that is, for any $\epsilon > 0$, we can find representatives $\{x_l^l\} \in \{x_k^l\}$ and $M \in \mathbb{N}$ such that $l \geq M$ implies

$$\delta(\{x_l^l\}, \{x_k^k\}) = \lim_{k \to \infty} \delta(x_l^l, x_k^k) < \epsilon.$$

To put it one last way, given $\epsilon > 0$, we must find an $M$ such that for each $l \geq M$, there exists $N \in \mathbb{N}$ such that for $k \geq N$,

$$\delta(x_k^k, x_l^l) < \epsilon. \tag{2.2}$$
Choose any representatives \( \{ x^l_k \} \in \{ [x^l_k] \} \); by passing to subsequences if necessary, we can assume that for each \( l \in \mathbb{N}, k, n \geq K \) implies that
\[
\delta(x^l_k, x^l_n) < 2^{-K} \quad (2.3)
\]
Let’s fix a particular \( K \) that is large enough that \( 2^{-K} \leq \epsilon/3 \).

Now, since \( \{ x^l_k \} \) is a Cauchy sequence, we can find \( M \geq K \) such that if \( l, m \geq M \), then
\[
\delta(x^l_k, x^m_m) = \lim_{r \to \infty} \delta(x^l_r, x^m_r) < \epsilon/6. \quad (2.4)
\]
Now simply set \( N := M \), and let \( k, l \geq M = N \) be given. By (2.4), we can find \( R \in \mathbb{N} \) such that \( r \geq R \) implies
\[
\delta(x^l_r, x^k_k) < \epsilon/3. \quad (2.5)
\]
Thus, by the triangle inequality, if \( k, l \geq M = N \) and \( r \geq R \),
\[
\delta(x^l_k, x^k_k) \leq \delta(x^l_k, x^l_r) + \delta(x^l_r, x^k_r) + \delta(x^k_r, x^k_k) < \epsilon,
\]
where we have used (2.5) to estimate the middle term and (2.3) to estimate the two other terms. As this proves (2.2), statement (1) is shown.

Statement (2) is not difficult, since
\[
\delta(\{ x \}, \{ y \}) = \lim_{k \to \infty} \delta(x, y) = \delta(x, y),
\]
and to find a constant sequence arbitrarily close to any Cauchy sequence, we can simply take an appropriate element of said sequence.

As for statement (3), this follows directly from (2) and the fact that a uniformly continuous function \( f \) on a dense subset \( A \) of a metric space \( X \) always has a unique uniformly continuous extension to the entire space, provided the target space \( Y \) is complete. This fact is readily verified by noting that a uniformly continuous function maps Cauchy sequences to Cauchy sequences. The extension of the function to a point \( x \in X \setminus A \) is defined as follows. Take any sequence \( x_k \to x \). This is then a Cauchy sequence in \( X \), so \( \{ f(x_k) \} \) is a Cauchy sequence in \( Y \). But \( Y \) is complete, so we can define \( f(x) := \lim f(x_k) \). It is straightforward to check that the extension thus defined is uniformly continuous. \( \square \)

Recall that a path metric space is a metric space for which the distance between any two points coincides with the infimum of the lengths of curves joining the two points. Given this definition, we expect that there be a description of the completion of a path metric space that uses curves instead of Cauchy sequences, and indeed this is so. Before we give it, though, let’s give the definition of a path metric space in more detail.

Let \( \alpha : [0, 1] \to X \) be a continuous path, and let \( 0 = t_1 < t_2 < \cdots < t_n = 1 \) be any finite partition of the interval \( [0, 1] \). Then the length of the polygonal path given by \( \{ \alpha(t_1), \ldots, \alpha(t_n) \} \) is defined to be
\[
L_{t_1, \ldots, t_n}(\alpha) := \sum_{k=1}^{n-1} \delta(\alpha(t_k), \alpha(t_{k+1})).
\]
Finally, we define the length of \( \alpha \) to be
\[
L(\alpha) := \sup \{ L_{t_1, \ldots, t_n}(\alpha) \mid (t_1, \ldots, t_n) \text{ is a partition of } [0, 1] \}.
\]
We take the supremum since as we add vertices to a polygonal path, i.e., improve the approximation of \( \alpha \), the triangle inequality implies the lengths of the polygonal paths are nondecreasing. Thus, this definition will match up with, say the length of a differentiable path in a Riemannian manifold.

We call a path \( \alpha \) with \( L(\alpha) < \infty \) rectifiable and say that \( (X, \delta) \) is a path metric space if for any \( x, y \in X \),
\[
\delta(x, y) = \inf \{ L(\alpha) \mid \alpha \text{ is a rectifiable curve joining } x \text{ and } y \}.
\]
If the domain of $\alpha$ is an open interval, e.g., $(0, 1)$, then we define the length of $\alpha$ to be

$$L(\alpha) := \lim_{\epsilon \to 0} L(\alpha|_{[\epsilon, 1-\epsilon]}),$$

and similarly if the domain is a half-open interval. We again call such a curve rectifiable if its length is finite.

We will also call a rectifiable curve a finite-length path or simply a finite path.

Given these definitions, we can formulate an alternate definition of the completion of a path metric space. Just as we can imagine a Cauchy sequence to be "open-ended" but convergent in some larger space containing $X$, we can imagine a path to be open on one end and view the path as representing its endpoint, which may or may not exist within $X$.

**Theorem 2.2.** Let $(X, \delta)$ be a path metric space. Then the following description of the completion of $(X, \delta)$ is equivalent to the definition given above.

Define the precompletion $\overline{X}^{\text{pre}}$ of $X$ to be the set of rectifiable curves $\alpha : (0, 1] \to X$.

It carries the pseudometric

$$\delta(\alpha_0, \alpha_1) := \lim_{t \to 0} \delta(\alpha_0(t), \alpha_1(t)).$$

Then the completion of $(X, \delta)$ is the metric space associated to $\overline{X}^{\text{pre}}$. That is,

$$\overline{X} := \overline{X}^{\text{pre}} / \sim,$$

where $\alpha_0 \sim \alpha_1 \iff \delta(\alpha_0, \alpha_1) = 0$.

**Proof.** First, let’s show that the limit in (2.6) exists—this will follow if, for every sequence $t_k \to 0$, $\delta(\alpha_0(t_k), \alpha_1(t_k))$ is a Cauchy sequence. But given $\epsilon > 0$, by rectifiability of the two curves, we can find $K \in \mathbb{N}$ such that $k \geq K$ implies that $L(\alpha_i|_{(0, t_k)}) < \epsilon/2$

for $i = 0, 1$. Thus, if $k, l \geq K$, we have

$$\delta(\alpha_0(t_k), \alpha_1(t_k)) \leq \delta(\alpha_0(t_k), \alpha_0(t_l)) + \delta(\alpha_0(t_l), \alpha_1(t_l)) + \delta(\alpha_1(t_l), \alpha_1(t_k))$$

$$< \delta(\alpha_0(t_l), \alpha_1(t_l)) + \epsilon.$$

Doing the same computation with $k$ and $l$ swapped proves that $\delta(\alpha_0(t_k), \alpha_1(t_k))$ is a Cauchy sequence.

Now, to show equivalence of the two definitions, we demonstrate an isometry from the completion as defined using sequences to the completion as defined using paths. For completeness (excusing the pun), we also write down the inverse mapping of this isometry.

So let a Cauchy sequence $\{x_k\}$ be given. Choose a subsequence $\{x_{k_l}\}$ (which, as previously noted, is equivalent to $\{x_k\}$) such that

$$\sum_{l=1}^{\infty} \delta(x_{k_l}, x_{k_{l+1}}) < \infty.$$

Since $X$ is a path metric space, we can choose paths $\alpha_l$ joining $x_{k_l}$ and $x_{k_{l+1}}$ such that $L(\alpha_l) \leq 2\delta(x_{k_l}, x_{k_{l+1}})$. Then the concatenated path $\alpha_{\{x_{k_l}\}} := \alpha_1 \ast \alpha_2 \ast \alpha_3 \ast \cdots$

is rectifiable.

To get a Cauchy sequence from a curve $\alpha : (0, 1] \to X$, simply take any monotonically decreasing sequence $t_k \searrow 0$ in $(0, 1]$ and define $x_k^\alpha := \alpha(t_k)$.
Then it is easy to see that finite length of $\alpha$ implies that $x_k^\alpha$ is a Cauchy sequence, for
given $\epsilon > 0$, we can find $K \in \mathbb{N}$ such that $k \geq K$ implies $L(\alpha|_{[0,t_k]}) < \epsilon$. Thus $l \geq k \geq K$
implies
\[
\delta(x_k^\alpha, x_l^\alpha) \leq L(\alpha|_{[0,t_k]}) < \epsilon.
\]
To see that these two mappings are well-defined on the completion, as defined via
sequences on the one side and paths on the other, and to show that they are isometries,
we need to show the following:

(1) If $\{x_k\}$ and $\{y_k\}$ both satisfy (2.7) (with $x_k$ and $y_k$, respectively, in place of $x_k$),
then $\delta(\alpha(x_k), \alpha(y_k)) = \delta(\{x_k\}, \{y_k\})$.

(2) If $\alpha$ and $\beta$ are equivalent finite paths and $t_k \searrow 0$, then $\delta(\{x_k^\alpha\}, \{x_k^\beta\}) = 0$.
Furthermore, different choices of sequences $t_k \searrow 0$ give rise to equivalent Cauchy
sequences.

(3) If $\alpha$ is a finite path and $t_k \searrow 0$, then $\alpha(x_k) \sim \alpha$.
From (1), we see that $\{x_k\} \mapsto \alpha(x_k)$ is well-defined and an isometry from one completion
to the other. From (2), it follows that $\alpha \mapsto \{x_k\}$ is well-defined on the completions, and
(3) implies that these two mappings are inverses of one another.

To prove (1), reparametrize $\alpha(x_k)$ and $\alpha(y_k)$ so that
\[
\alpha(x_k)(1/k) = x_k \quad \text{and} \quad \alpha(y_k)(1/k) = y_k.
\]
Let $\epsilon > 0$ be given, and choose $L \in \mathbb{N}$ such that
\[
\sum_{l=L}^{\infty} \delta(x_{k_l}, x_{k_{l+1}}) < \epsilon/4 \quad \text{and} \quad \sum_{l=L}^{\infty} \delta(y_{k_l}, y_{k_{l+1}}) < \epsilon/4,
\]
so that by the construction of $\alpha(x_k)$ and $\alpha(y_k)$,
\[
L(\alpha(x_k)|_{[0,1/L]}) < \epsilon/2 \quad \text{and} \quad L(\alpha(y_k)|_{[0,1/L]}) < \epsilon/2.
\]
Then for $t \leq 1/L$,
\[
\delta(\alpha(x_k)(t), \alpha(y_k)(t)) \leq \delta(\alpha(x_k)(t), \alpha(x_k)(1/L)) + \delta(\alpha(x_k)(1/L), \alpha(y_k)(1/L))
\quad + \delta(\alpha(y_k)(1/L), \alpha(y_k)(t))
\leq \delta(\{x_L\}, \{y_L\}) + L(\alpha(x_k)|_{[t,1/L]}) + L(\alpha(y_k)|_{[t,1/L]})
\leq \delta(\{x_L\}, \{y_L\}) + \epsilon,
\]
where we have used (2.8) in the second inequality and (2.9) in the third. Similarly, one
can prove that for $t \leq 1/L$,
\[
\delta(\{x_L\}, \{y_L\}) \leq \delta(\alpha(x_k)(t), \alpha(y_k)(t)) + \epsilon.
\]
From the two above inequalities, it is easy to see that
\[
\lim_{t \to 0} \delta(\alpha(x_k)(t), \alpha(y_k)(t)) = \lim_{k \to \infty} \delta(x_k, y_k),
\]
as was to be proved.

The proofs of (2) and (3) are very similar, yet simpler, and so we omit them. Besides,
we have already proved the statement of the theorem, so these are just “bonus” statements
about the inverse to the isometry $\{x_k\} \mapsto \alpha(x_k)$.

We are now equipped with all of the metric space tools we need to study the completion
of the manifold of metrics.
2. Fréchet manifolds

The manifold of smooth metrics is itself a Fréchet manifold, and so these will play an extremely important role in this work. However, we will not need to go into depth on Fréchet manifolds. This is because the manifold of metrics is an extremely simple type of Fréchet manifold, namely an open subset of a Fréchet space.

An excellent source on Fréchet manifolds and the implicit function theorem in the category of Fréchet spaces is [24], and this is our main reference for the first two subsections. For more in-depth and recent results on this and related categories, see [43], which focuses mainly on Fréchet Lie groups.

After introducing Fréchet spaces and Fréchet manifolds, we will discuss a particular class of Fréchet manifolds, namely manifolds of smooth mappings. The main result, which will allow us to define the manifold of metrics, is that if \( N \) is a finite-dimensional manifold and \( F \) is a finite-dimensional fiber bundle over \( N \), then the set of \( C^\infty \) sections of \( F \) carries the structure of a smooth Fréchet manifold. If \( F \) is a vector bundle, then the set of \( C^\infty \) sections has a linear structure, so it even forms a Fréchet space.

There is another category incorporating manifolds of smooth mappings, the so-called convenient setting [29]. This setting is highly developed and allows one to deal with more general spaces than Fréchet spaces. We chose to use the Fréchet category because we need only basic facts, and Fréchet manifolds are the most familiar and simplest to introduce.

So, without further delay, we get into the definitions.

2.2.1. Fréchet spaces.

Definition 2.3. Let \( E \) be a vector space over a field \( K \). A seminorm on \( E \) is a function \( \| \cdot \| : E \to K \) with the following properties for all \( v, w \in E \) and \( \lambda \in K \):

1. \( \|v\| \geq 0 \),
2. \( \|v + w\| \leq \|v\| + \|w\| \) and
3. \( \|\lambda v\| = |\lambda|\|v\| \).

Given a collection of seminorms \( \{\| \cdot \|_i \mid i \in I\} \) on \( E \), we can define a topology on \( E \) by declaring that a sequence or net \( \{v_k\} \) converges to \( v \) if and only if \( \|v - v_k\|_i \to 0 \) for all \( i \in I \). A locally convex topological vector space (or LCTVS) is a vector space together with a topology defined in this way. It happens that the topology of an LCTVS is metrizable if and only if it is defined by a countable collection of seminorms, and it is Hausdorff if and only if \( v = 0 \) whenever \( \|v\|_i = 0 \) for all \( i \in I \). In a metrizable LCTVS, it suffices to use sequences instead of nets when describing the topology via convergence.

In a metrizable LCTVS, we call a sequence \( \{v_k\} \) a Cauchy sequence if given any \( i \in \mathbb{N} \) and \( \epsilon > 0 \), we can find \( N(i, \epsilon) \in \mathbb{N} \) such that \( \|v_k - v_l\|_i < \epsilon \) for all \( k, l \geq N(i, \epsilon) \). We call the space complete if every Cauchy sequence converges.

With these preparations, we can make the following definition.

Definition 2.4. A Fréchet space is an LCTVS that is Hausdorff, metrizable and complete.

For example, every Banach or Hilbert space is a Fréchet space, with topology given by a single norm. For a more interesting example, consider the interval \( [0, 1] \subset \mathbb{R} \) and the space \( C^\infty[0,1] \) of smooth functions on this interval. If we give this space the topology defined by the \( C^k \) norms,

\[
\|f\|_k = \sum_{l=1}^{k} \sup_{x \in [0, 1]} \left| \frac{d^l}{dx^l} f(x) \right|,
\]

then \( C^\infty[0,1] \) becomes a Fréchet space. The Hausdorff property and metrizability are clear, and completeness follows from the fact that \( C^k[0,1] \) is a Banach space with the \( \| \cdot \|_k \).
norm. Therefore, if a sequence is Cauchy in each $\| \cdot \|_k$ norm, it converges to a function that is $C^k$ for each $k \in \mathbb{N}$, i.e., a smooth function.

Note that we could have also used the $H^s$ norms to define $C^\infty[0,1]$. The proof that this defines a Fréchet space topology is the same, but we have to make the extra step of using the Sobolev embedding theorem to show that a Cauchy sequence converges to a smooth limit function. The advantage of using the $H^s$ norms is that they come from scalar products, which yields some extra structure to work with. However, the topology on $C^\infty[0,1]$ is the same as when we use the $C^k$ norms, which we can again see using the Sobolev embedding theorem.

As suggested by the term locally convex topological vector space above, a Fréchet space is a topological vector space, meaning that vector addition and scalar multiplication are continuous maps.

Fréchet spaces have some fundamental differences from Banach spaces. For example, the dual of a Fréchet space $E$ is not always a Fréchet space. In fact, the dual is a Fréchet space if and only if $E$ is a Banach space! This implies that the space $L(E,F)$ of linear maps between two Fréchet spaces $E$ and $F$ is a Fréchet space if and only if $F$ is a Banach space. Additionally, naive generalizations of the Banach space implicit function theorem to Fréchet spaces fail—instead, one must work in the category of so-called tame Fréchet spaces, which require additional estimates on maps between them that are not present in the Banach case, to get a satisfactory implicit function theorem. However, these matters are not important to our concerns.

Despite the difficulties in working with Fréchet spaces, many results from the theory of Banach spaces and Banach manifolds carry over. For example, the Hahn-Banach theorem holds, as does the open mapping theorem.

Calculus in Fréchet spaces works in almost exactly the same manner as it does in Banach spaces, if we define the derivative in the following way. Let $E$ and $F$ be Fréchet spaces, let $U \subseteq E$ be open, and let $f : U \subseteq E \to F$ be a continuous map. We define the differential of $f$ at the point $x \in U$ in the direction $v \in E$ to be

$$Df(x)v := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$ 

We define $f$ to be differentiable at $x$ in the direction $v$ if the limit exists. We define $f$ to be $C^1$ (or continuously differentiable) if the limit exists for all $x \in U$ and $v \in E$, and the map

$$Df : U \times E \to F$$

is continuous in both its arguments. Note that $Df$ is a map from the product $U \times E$ to $F$. We do not consider it as a map $U \to L(E,F)$, because as we mentioned above, $L(E,F)$ is not necessarily a Fréchet space—even though $Df(x)$ is indeed a linear map from $E$ to $F$ for each $x \in U$.

To define the second derivative, we take the partial derivative of the map $Df$ in the first component, i.e., we take the derivative as $Df$ varies only over $U$. This is because $Df$ is linear in the second component, and hence this partial derivative just gives $Df$ again. It is also done to match up with the usual definition of the derivative in Banach spaces. Thus, the second derivative is a map

$$D^2f : U \times E \times E \to F.$$ 

We can iterate the definitions above to define $C^k$ and $C^\infty$ maps between Fréchet spaces. The chain rule holds for the differential as thus defined. We can also define integrals over curves in the usual way, and if we do so then the fundamental theorem of calculus holds.

With all of these results at hand, it is clear that calculus in Fréchet spaces is formally very similar to that in Banach spaces. Thus, we will not go into any more detail at
2.2.2. Fréchet manifolds. Again, our reference for this subsection is [24].

Just as the usual rules for calculus generalize to Fréchet spaces, so does the definition of a manifold. Thus, a Fréchet manifold modeled on a Fréchet space $E$ is a Hausdorff topological space $M$ with an atlas of coordinates $\{(U_i, \phi_i) \mid i \in I\}$, where each $U_i \subseteq M$ is open and each $\phi_i : U_i \subseteq M \to E$ is a homeomorphism onto its image. Furthermore, if $U_i \cap U_j \neq 0$, we require that the transition map

$$\phi_j|_{U_i \cap U_j} \circ \phi_i^{-1}|_{\phi_i(U_i \cap U_j)} : \phi_j(U_i \cap U_j) \subseteq E \to E$$

is a smooth mapping of Fréchet spaces.

Tangent spaces/bundles, smooth/differentiable mappings, vector bundles, fiber bundles, and so on are defined in the category of Fréchet manifolds exactly analogously to the case of Banach manifolds.

Fréchet Lie groups are Fréchet manifolds that are also groups and on which the operations of multiplication and taking the inverse are smooth. One example of a Fréchet Lie group is the diffeomorphism group of a compact manifold. For more facts on these fascinating and difficult objects, which are so important in global analysis, see [42] and [43].

2.2.3. Manifolds of mappings. The fundamental object in the field of global analysis is the set of sections of a smooth fiber bundle $F$ with $m$-dimensional fibers over a smooth, $n$-dimensional manifold $M$. Typically, one is interested in restricting to sections with a certain regularity, say $C^k$ for $0 \leq k \leq \infty$ or $H^s$ for $s \geq 0$. $C^k$ regularity is, of course, well understood, and it is the goal of this section to outline the notion of $H^s$ regularity. Additionally, as analysts and geometers, we prefer to work with smooth manifolds, and so we will sketch the useful fact that the by restricting to certain types of sections, we get a Hilbert/Banach/Fréchet manifold.

Let’s get down to defining manifolds of mappings. The facts presented here are taken from the texts [51]. [45] and [44] Chap. 4. For a very concise but readable outline, see [11] §3. It will simplify the presentation somewhat, and is in fact sufficient for our purposes, to assume that the base manifold $M$ is closed and oriented.

Manifolds of sections are constructed using the notion of a jet bundle, which is essentially a bundle that contains information about the Taylor expansions of sections of $F$. The precise definition is as follows.

Suppose we are given two local, $k$-times differentiable sections $\varphi$ and $\psi$ of $F$. Suppose that $\varphi$ and $\psi$ are both defined on an open neighborhood of $p \in M$. We say that $\varphi$ and $\psi$ are $k$-equivalent at $p$ if $\varphi(p) = \psi(p)$ and the following holds. Let $(x^i, u^\alpha)$ be coordinates on $F$ around $p$ such that $(x^i)$ are coordinates on the base manifold $M$ and $(u^\alpha)$ are coordinates in the fiber directions—i.e., $(x^i, u^\alpha)$ are the coordinates of a local trivialization. We require that for all multi-indices $I$, taking values in $\{1, \ldots, n\}$, with $1 \leq |I| \leq k$ and all $1 \leq \alpha \leq m$ (recall $m$ is the dimension of the fibers):

$$\frac{\partial^{I_1}|\varphi^\alpha_{\alpha}}{\partial x^I} \bigg|_p = \frac{\partial^{I_1}|\psi^\alpha_{\alpha}}{\partial x^I} \bigg|_p.$$  \hspace{1cm} (2.10)

Thus, two local sections are $k$-equivalent at $p$ if and only if their values at $p$ are equal, as are their first $k$ derivatives at $p$ in some local coordinate system around $p$. Note that while the value of the derivatives depends on the local coordinates, equality of the derivatives as in (2.10) does not (see [51] Lemma 6.2.1).

The equivalence class containing the local section $\varphi$ is denoted by $J^k_p\varphi$ and is called the $k$-jet of $\varphi$ at $p$. The equivalence class of a local section $\varphi$ thus consists of all local
sections having Taylor expansion up to order $k$—in local coordinates at $p$—equal to that of $\varphi$.

The set of all $k$-jets of local sections of $F$, denoted

$$ J^k F := \{ j_p^k \varphi \mid p \in M, \varphi \text{ is a local section of } F \text{ around } p \}, $$

is called the $k$-th jet bundle, as it has a natural structure of a smooth, finite-dimensional fiber bundle over both $M$ and $F$. To see this, we first write down the coordinate atlas that makes it into a manifold. As above, let $(x^i, u^\alpha)$ be coordinates on an open set $U \subseteq F$, with $(x^i)$ coordinates on the base and $(u^\alpha)$ coordinates on the fibers. Let $U^k$ be the subset of $J^k F$ given by

$$ U^k := \{ j_p^k \varphi \mid \varphi(p) \in U \}. $$

Then we get coordinates $(x^i, u^\alpha, u^\alpha_j)$ on $U^k$, where $I$ runs through all unordered multi-indices taking values in $\{1, \ldots, n\}$ with $1 \leq |I| \leq k$, and

\begin{align*}
x^i(j_p^k \varphi) &= x^i(p), \\
|u^\alpha(j_p^k \varphi) &= u^\alpha(\varphi(p)), \\
u^\alpha_j(j_p^k \varphi) &= \left. \frac{\partial^{|I|} \varphi^\alpha}{\partial x^I} \right|_p.
\end{align*}

(2.11)

(The reason we require $I$ to be unordered is the symmetry of the derivatives in local coordinates, i.e., because differentiations in different coordinate directions commute with one another.) We will not show that this does indeed define a smooth atlas on $J^k F$, but refer the interested reader to [51]. We do note, however, that since there are only finitely many multi-indices of order not greater than $k$ taking values in $\{1, \ldots, n\}$, there are only finitely many coordinates $u^\alpha_j$, and hence $J^k F$ is finite-dimensional.

The bundle structures $J^k F \rightarrow M$ and $J^k F \rightarrow F$ are given by the so-called source and target projections:

$$ \pi_k : J^k F \rightarrow M, \quad j_p^k \varphi \mapsto p, $$

and

$$ \pi_{k,0} : J^k F \rightarrow F, \quad j_p^k \varphi \mapsto \varphi(p), $$

respectively. We can also view $J^l F$ as a bundle over $J^k F$ for any $1 \leq k \leq l$; the bundle structure is given by the $k$-jet projection:

$$ \pi_{l,k} : J^l F \rightarrow J^k F $$

$$ j_p^l \varphi \mapsto j_p^k \varphi. $$

There is a natural mapping, denoted $j^k$, sending local $C^l$ sections of $F \rightarrow M$ (for $l \geq k$) to local $C^{l-k}$ sections of $J^k F \rightarrow M$. If $\varphi$ is a local section of $F$, then this map is defined by

$$ j^k \varphi(p) := j_p^k \varphi. $$

The section $j^k \varphi$ is sometimes called the $k$-th prolongation of $\varphi$, and $j^k$ is sometimes called the $k$-jet extension map. If we only consider global sections of $F$, then $j^k$ defines a map from $C^\infty(F)$ to $C^\infty(J^k F)$, where for a fiber bundle $E \rightarrow M$, $C^\infty(E)$ denotes the space of smooth sections of $E$.

\textbf{Remark 2.5.} The notation $C^\infty(E)$ for the set of smooth sections of the fiber bundle $E \rightarrow M$ should not be confused with the oft-used identical notation for the set of smooth functions on the manifold $E$. In this thesis, whenever we consider a bundle structure on a space $E$, by $C^k(E)$, $C^\infty(E)$, $H^k(E)$ (the last one we have yet to define), and so on, we will always mean the appropriate space of sections of the bundle.

This point will hardly arise outside this chapter, though, so we hope this admittedly suboptimal notation will cause no large problems.
At this point, let us restrict to the case where $F$ is a vector bundle over $M$, as it will simplify the exposition somewhat and will still be sufficient for our purposes. With this assumption, $J^kF$ has the structure of a vector bundle over $M$, not just a fiber bundle. This can be seen, heuristically, from the fact that the values of any section $\varphi$ at a point $p$ belong to the vector space $F_p$, and the $j$-th total differential (with respect to $(x^i)$, as in the $y^j$-coordinates of (2.11)) of the section $\varphi$ at $p$ can be seen in local coordinates as a $j$-linear map from $\mathbb{R}^n$ to $F_p$ (recall $n = \dim M$). This is an extremely sketchy “proof” and not at all rigorous, so we refer the reader to [45, pp. 5–6] for details.

Since $F$ and $J^kF$ are both vector bundles, $C^\infty(F)$ and $C^\infty(J^kF)$ are both vector spaces. It is then easy to see that the $k$-jet extension map $j^k : C^\infty(F) \to C^\infty(J^kF)$ is a linear map.

A Riemannian metric $\gamma$ on $J^kF$ is given by a smooth choice of positive-definite scalar product $\gamma(p)$ on $J^k_F$, the fiber of $\pi_k : J^kF \to M$ at $p$, for each $p \in M$. Given a Riemannian metric $\gamma$ on $J^kF$ and a smooth volume form $\mu$ on $M$, we get a scalar product $(\cdot, \cdot)_\gamma$ on $C^\infty(J^kF)$ via

$$(\varphi, \psi)_\gamma = \int_M \gamma(p)(\varphi(p), \psi(p)) \mu(p).$$

We can pull this scalar product back along the $k$-jet extension map $j^k$ to get a scalar product on $C^\infty(F)$. We denote by $H^k(F)$ the completion of $C^\infty(F)$ with respect to this scalar product. The space $H^k(F)$ is a Hilbert space over the reals, and its norm depends on our choices of $\gamma$ and $\mu$. However, the topology of $H^k(F)$ does not, as [44, §IX.2] shows. Therefore, we are justified in omitting $\gamma$ and $\mu$ from our notation and calling $H^k(F)$ the space of $H^k$ sections of $F$.

**Remark 2.6.** The scalar product $(\cdot, \cdot)_\gamma$ is essentially an $L^2$ scalar product on sections $\varphi$ of the $k$-th jet bundle. Since these sections contain the all derivatives of $\varphi$ of order $k$ and lower, it can be seen that the definitions above match up with the definitions of Sobolev spaces of functions on open sets of $\mathbb{R}^n$. If we allow ourselves to speak imprecisely by mixing global and local notions, we can say that the completion of $C^\infty(F)$ with respect to the above-described scalar product contains all sections with $L^2$-integrable partial derivatives up to order $k$.

In a similar but simpler way, we can define a Banach space structure on the space $C^k(F)$ of $C^k$ sections of $F$. To do this, we again choose a Riemannian structure $\gamma$ on $J^kF$, but this time define a norm on $C^0(J^kF)$ by

$$\|\varphi\|_\gamma = \sup_{p \in M} \sqrt{\gamma(p)(\varphi(p), \varphi(p))}.$$

Since the $k$-jet extension map $j^k$ is a linear map defined on $C^k(F)$, we pull the above norm back to $C^k(F)$ along $j^k$. Then $C^k(F)$ is a Banach space with respect to the pulled-back norm.

With these definitions, the Sobolev embedding theorem holds for spaces of sections of vector bundles, just as it does for spaces of functions over $\mathbb{R}^n$. Thus if $s > n/2 + k$, there is a continuous linear inclusion $H^s(F) \to C^k(F)$. (See [44, §X.4, Thm. 4].) A consequence is the following statement. Define $C^\infty(F)$ to be the Fréchet space of smooth sections of $F$ with the topology given by the family of $C^k(F)$-norms for $k \in \mathcal{M}$. Then this topology on $C^\infty(F)$ coincides with the one given by the family of $H^s(F)$-norms for $s \in \mathbb{N}$. This latter view is the one we will take in this thesis, since it allows us to work with the chain of Hilbert manifolds $H^0(F), H^1(F), \ldots$, which have nicer properties than the Banach manifolds $C^0(F), C^1(F), \ldots$.

To recap, for any vector bundle $F \to M$, we can consider the set of all sections of $F$. By taking sections with certain properties, we can build Hilbert spaces $H^s(F)$ of Sobolev sections of $F$, Banach spaces $C^k(F)$ of $k$-times differentiable sections, and the Fréchet
space $C^\infty(F)$ of smooth sections. The latter has the topology coming either from the family of $C^k$ norms or the family of $H^s$ norms.

We have restricted the discussion to vector bundles for simplicity, but we end this section by briefly remarking on the situation when $F$ is a fiber bundle. In this case, we can build a Banach manifold (which will in general not be a linear space) $C^k(F)$ for $k = 0, 1, 2, \ldots$. We can also define the sets $H^s(F)$ of $H^s$ sections of $F$ for $s = 0, 1, 2, \ldots$, but if we want $H^s(F)$ to be a (Hilbert) manifold, then for technical reasons we have to restrict to $s > n/2$, i.e., we have to require that $H^s(F) \subseteq C^{0}(F)$. (See [45], §11ff.) Using either the $H^s$ or $C^k$ norms, we can give $C^\infty(F)$ a Fréchet manifold structure. The way that all of these results are proved is by locally reducing the analysis of $H^s$ sections of a related vector bundle. We do not need this directly, however, so instead of proving it we refer to [45], §13 for the general case and [24] Ex. 4.1.2 for a nice, concise description of the $C^k$ and $C^\infty$ cases.

**Remark 2.7.** It is also worth noting that if $N$ is another finite-dimensional manifold, then the set of mappings from $M$ to $N$ can be treated as in this subsection by viewing a map $M \rightarrow N$ as a section of the trivial bundle $M \times N$ over $M$. If $N = M$, then we can construct the manifold $C^\infty(M, M)$ of smooth self-mappings of $M$. It is not hard to see that the set $\mathcal{D}$ of smooth diffeomorphisms of $M$ is open in $C^\infty(M, M)$, and we therefore get a Fréchet manifold structure on $\mathcal{D}$. As we mentioned above, $\mathcal{D}$ is even a Fréchet Lie group [42], [43].

### 2.3. GEOMETRIC PRELIMINARIES

At this point, we will go over some geometric notions and notation that we will be using later in the thesis. We’ll first look at the endomorphism bundle of a finite-dimensional manifold and the eigenvalues of its sections. Then we’ll discuss a few concepts from measure theory, and finish with the description of two special manifolds of mappings that will play a role in what is to come.

**Convention 2.8.** For the remainder of this thesis, we work over a fixed, finite-dimensional, oriented, closed base manifold $M$, and set $n := \dim M$.

#### 2.3.1. The endomorphism bundle of $M$.

The endomorphism bundle $\text{End}(M)$ is the bundle of $(1, 1)$-tensors on $M$. A $(1, 1)$-tensor at $p \in M$ is an element of $T_p M \otimes T_p M$, and so it can be identified with an endomorphism of $T_p M$. A smooth section of $\text{End}(M)$ is therefore a smooth vector bundle map of $TM$ into itself. Furthermore, $(1, 1)$-tensors and sections of $\text{End}(M)$ have a well-defined multiplication, which is simply the multiplication of matrices (in local coordinates) or the composition of linear transformations (invariantly described). As a $(1, 1)$-tensor $H$ is a linear transformation, any property of matrices that is invariant under a change of basis will be well-defined (i.e., coordinate-independent) for an endomorphism of $T_p M$. Especially important for us is that this includes the determinant, the trace, and the eigenvalues of $H$.

This also implies that if we are given a section $H$ of $\text{End}(M)$, then the determinant, trace, and eigenvalues of $H$ are well-defined functions over $M$. Furthermore, if $H$ is measurable/continuous/smooth, then the determinant and trace will be so as well, since they are smooth functions from the space of $n \times n$ matrices into $\mathbb{R}$.

The regularity properties of the eigenvalues of a section of the endomorphism bundle are not so immediate, but there are a couple of things that we need to understand better. To do this, we first prove a statement about the eigenvalues of symmetric matrices, then “globalize” the statement. We do this in two lemmas, after reviewing a fact from linear algebra in the following proposition.
PROPOSITION 2.9 \([25 \text{ Thm. 7.2.1}]\). A symmetric \(n \times n\) matrix \(T\) is positive definite (resp. positive semidefinite) if and only if all eigenvalues of \(T\) are positive (resp. nonnegative).

In particular, if \(T\) is positive definite (resp. positive semidefinite), then \(\det T > 0\) (resp. \(\det T \geq 0\)). If \(T\) is positive semidefinite but not positive definite, then \(\det T = 0\).

LEMMA 2.10. Let \(\langle \cdot, \cdot \rangle\) be any scalar product on \(\mathbb{R}^n\), and let \(\lambda^A_{\min}\) and \(\lambda^A_{\max}\) denote the smallest and largest eigenvalues, respectively, of an \(n \times n\) matrix \(A\). Then the map \(A \mapsto \lambda^A_{\min}\) is a concave function from the space of self-adjoint \(n \times n\) matrices to \(\mathbb{R}\). (Of course, we define “self-adjoint” with respect to \(\langle \cdot, \cdot \rangle\).) Furthermore, \(A \mapsto \lambda^A_{\max}\) is convex.

In particular, each map is continuous.

PROOF. Consider the following formula for the minimal eigenvalue of a self-adjoint matrix, which follows from the min-max theorem \([49 \text{ Thm. XIII.1}]\):

\[
\lambda^A_{\min} = \min_{v \in \mathbb{R}^n} \langle v, Av \rangle. \tag{2.12}
\]

Therefore, if \(A \) and \(B\) are self-adjoint matrices, we have

\[
\lambda^{(1-t)A+tB}_{\min} = \min_{v \in \mathbb{R}^n} \langle v, ((1-t)A + tB)v \rangle
\]

\[
\geq \min_{v \in \mathbb{R}^n} \langle v, (1-t)Av \rangle + \min_{v \in \mathbb{R}^n} \langle v, tBv \rangle
\]

\[
= (1-t)\lambda^A_{\min} + t\lambda^B_{\min}.
\]

That the map sending a self-adjoint matrix to its maximal eigenvalue is convex follows in exactly the same way from the formula

\[
\lambda^A_{\max} = \max_{v \in \mathbb{R}^n} \langle v, Av \rangle, \tag{2.13}
\]

which again follows from the min-max theorem.

Continuity of the maps follows from the well-known result that a convex or concave function on a real, finite-dimensional vector space is continuous \([50 \text{ Thm. 10.1}]\). □

LEMMA 2.11. Let \(h\) be any continuous, symmetric \((0,2)\)-tensor field. Suppose \(g\) is a Riemannian metric on \(M\), and let \(H\) be the \((1,1)\)-tensor field obtained from \(h\) by raising an index using \(g\). (That is, locally \(H^i_j = g^{ik}h_{kj}\).) Then \(H\) is a continuous section of the endomorphism bundle \(\text{End}(M)\). Denote by \(\lambda^H_{\min}(x)\) the smallest eigenvalue of \(H(x)\). We have that

1. \(\lambda^H_{\min}\) is a continuous function and
2. if \(h\) is positive definite, then \(\min_{x \in M} \lambda^H_{\min}(x) > 0\).

Furthermore, if \(\lambda^H_{\max}(x)\) denotes the largest eigenvalue of \(H(x)\), then \(\lambda^H_{\max}\) is a continuous and hence bounded function.

PROOF. For any fixed \(p \in M\), let a neighborhood \(U\) of \(p\) be given with the property that we can find a frame field for \(TM|_U\), i.e., there exist \(n\) smooth vector fields \(X_1, \ldots, X_n\) over \(U\) that together form a basis of \(T_x M\) for each \(x \in U\). For every nonzero \(n\)-tuple \(\alpha = (\alpha^1, \ldots, \alpha^n) \in \mathbb{R}^n\), we define a vector field \(X_\alpha\) over \(U\) via

\[
X_\alpha := \alpha^1 X_1 + \cdots + \alpha^n X_n.
\]

For each such \(\alpha \in \mathbb{R}^n\), consider the function

\[
Q^p_\alpha : U \to \mathbb{R}
\]

\[
x \mapsto \frac{g(x)(X_\alpha(x), H(x)X_\alpha(x))}{g(x)(X_\alpha(x), X_\alpha(x))}.
\]
Thus, $Q^g_\alpha(x)$ is the Rayleigh quotient, with respect to $g(x)$, of $H(x)$ on the vector $X_\alpha(x)$. It does not depend on $\alpha$, but only on the line on which $\alpha$ lies. Thus, the family $\{Q^g_\alpha \mid \alpha \in \mathbb{R}^n\}$ can be seen as a family of functions for $\alpha \in S^{n-1} \subset \mathbb{R}^n$.

By the continuity of $g$, $H$ and $X_\alpha$, as well as the relative compactness of $U$ and the compactness of $S^1$, it is not hard to show that

$$Q^g_{\alpha \max}(x) := \max_{\alpha \in S^{n-1}} Q^g_\alpha(x) \quad \text{and} \quad Q^g_{\alpha \min}(x) := \min_{\alpha \in S^{n-1}} Q^g_\alpha(x)$$

are continuous functions defined on $U$. On the other hand, since $H(x)$ is self-adjoint with respect to $g(x)$, we can use the formulas in the proof of Lemma 2.10 to see that

$$Q^g_{\alpha \max}(x) = \lambda^H_{\alpha \max}(x) \quad \text{and} \quad Q^g_{\alpha \min}(x) = \lambda^H_{\alpha \min}(x).$$

From this, and since $p$ was chosen arbitrarily, the continuity of $\lambda^H_{\alpha \max}$ and $\lambda^H_{\alpha \min}$ is immediate.

The upper bound on $\lambda^H_{\alpha \max}$ follows from its continuity. That $\lambda^H_{\alpha \min}$ is bounded away from zero if $h$ is positive definite follows from the fact that if $p$ is the (arbitrary) point chosen above, then

$$\lambda^H_{\alpha \min}(p) = \min_{\alpha \in S^{n-1}} h(p)(X_\alpha(p), H(p)X_\alpha(p)) = \min_{\alpha \in S^{n-1}} h(p)(X_\alpha(p), X_\alpha(p)) > 0,$$

so $\lambda^H_{\alpha \min}$ is a continuous positive function on $M$. 

\section{2.3. Geometric Preliminaries}

### 2.3.2. Lebesgue measure on manifolds. The concept of Lebesgue measurability carries over from $\mathbb{R}^n$ to smooth (or even topological) manifolds very simply. Let a maximal atlas of coordinate charts $\{(U_\alpha, \phi_\alpha) \mid U_\alpha \rightarrow V_\alpha \subseteq \mathbb{R}^n\}$ for $M$ be given. We say a subset $E \subset M$ is Lebesgue measurable if we can find a covering of $E$ by charts $\{(U_\beta, \phi_\beta)\}$ such that $\phi_\beta(E \cap U_\beta)$ is Lebesgue measurable for each $\beta$. This concept is independent of the particular choice of covering: if $\{U_\gamma, \phi_\gamma\}$ is a second covering of $E$ by charts, then each transition function

$$\phi_\beta \gamma = \phi_\gamma \mid_{U_\gamma \cap U_\beta} \circ \phi_\beta^{-1} \mid_{\phi_\beta(U_\beta \cap U_\gamma)}$$

is a smooth diffeomorphism, and hence it maps Lebesgue measurable sets to Lebesgue measurable sets. Of course, the transition function will not necessarily preserve the quantitative measure of a Lebesgue measurable set, but it \textit{will} map nullsets to nullsets. Therefore, we can speak about nullsets on a smooth, finite-dimensional manifold.

\textbf{Convention 2.12.} Whenever we refer to a measure-theoretic concept on $M$, we implicitly mean that we work with Lebesgue measure or Lebesgue sets, unless we explicitly state otherwise. (At some points Borel measures will also come up.)

With Lebesgue measurable sets well-defined, the concept of a measurable function or a measurable map between manifolds is also well-defined—these are simply those maps for which the preimage of any measurable set is measurable. If we have a local $(r, s)$-tensor field $t$ on $M$, defined over a set $E \subseteq M$, we say that it is \textit{measurable} or \textit{has measurable coefficients} if there is a covering of $\{(U_\beta, \phi_\beta)\}$ of $E$ by coordinate charts such that over each $U_\beta$, the coefficients of $t$ are measurable. This is again independent of the covering chosen, since in a different coordinate chart $(U_\gamma, \phi_\gamma)$, the coefficients of $t$ are determined from the coefficients in the original chart and the transition function $\phi$ via

$$t^{i_1 \cdots i_r}_{j_1 \cdots j_s} = (D \phi^{-1})^{i_1}_{k_1} \cdots (D \phi^{-1})^{i_r}_{k_r} (t^{k_1 \cdots k_r}_{l_1 \cdots l_s} \circ \phi_\beta^{-1})(D \phi)^{l_1}_{j_1} \cdots (D \phi)^{l_s}_{j_s}.$$  

That is, the new coefficients are obtained from the old via composition, addition, and multiplication with smooth functions, so they are again measurable.

We can also speak about Lebesgue measures. By the above definition, it is immediate that any volume form $\mu$ on $M$ with measurable coefficients induces a Lebesgue measure on $M$. (By volume form, we simply mean any $n$-form with positive coefficient. Saying the coefficient is positive is coordinate-independent because of the orientation of $M$.) We can also allow $\mu$ to be a nonnegative $n$-form—i.e., one for which the coefficient is everywhere
Furthermore, and \( \mu \) again induces a Lebesgue measure on \( M \). (Nonnegativity is again a coordinate-independent notion thanks to orientability of \( M \).)

We next mention the relation of the Lebesgue measurable sets \( \mathcal{L} \) to the Borel measurable sets \( \mathcal{B} \). It is not hard to see that the same general relationship between these sets that holds on \( \mathbb{R}^n \) holds on \( M \) as well. Let’s recall this relationship——namely, that Lebesgue measure on \( \mathbb{R}^n \) coincides with the outer measure induced by Borel measure \([47] \S.1.7\). For the reader’s convenience, we briefly review this notion, as well as that of the completion of a measure space. All facts are taken from \([4] \S.1.5\) unless otherwise mentioned.

Let \( (X, \Sigma, \mu) \) be a measure space. The measure \( \mu \) is called complete if for every \( B \in \Sigma \) with \( \mu(B) = 0 \) and every subset \( A \subset B \), we have \( A \in \Sigma \) (and therefore of course \( \mu(A) = 0 \)). That is, \( \mu \) is complete if every subset of a nullset is \( \mu \)-measurable.

If \( \mu \) is not complete, we can extend it to a complete measure as follows. We define the outer measure \( \mu^* \) induced by \( \mu \) to be, for any \( E \subseteq X \),

\[
\mu^*(E) := \inf \left\{ \sum_{k=1}^{\infty} E_k \in \Sigma, \ E \subseteq \bigcup_{k=1}^{\infty} E_k \right\}
\]

for any set \( E \subseteq X \). We note that if \( E \in \Sigma \), then \( \mu^*(E) = \mu(E) \). Define a set \( E \subseteq X \) to be \( \mu^* \)-measurable if for every \( Y \subseteq X \),

\[
\mu^*(Y) = \mu^*(Y \cap E) + \mu^*(Y \cap E^c).
\]

We denote the class of \( \mu^* \)-measurable sets by \( \Sigma^* \), and note that this is a \( \sigma \)-algebra. Furthermore, \( \Sigma \subseteq \Sigma^* \), and \( (X, \Sigma^*, \mu^*) \) is complete. Thus \( \mu^* \) is an extension of \( \mu \) to a complete measure.

Now, let’s see what this means in the special case of \( M \) with a measure \( \mu \) on the Borel sets \( \mathcal{B} \) of \( M \). Firstly, since the above statement that Lebesgue measure coincides with the outer measure induced by Borel measure can be localized, the outer measure corresponding to \( \mu \) is a measure \( \mu^* \) on the Lebesgue sets \( \mathcal{L} \) of \( M \). We also see that a Borel measurable set is Lebesgue measurable. Furthermore, by \([47] \S.3.11\), the following holds.

**Lemma 2.13.** For every \( E \in \mathcal{L} \), there exist \( F \in \mathcal{B} \) and \( G \in \mathcal{L} \) such that

1. \( E = F \cup G \),
2. there exists a \( \mu \)-nullset \( A \in \mathcal{B} \) such that \( G \subset A \) and
3. \( \mu^*(E) = \mu(F) \).

In other words, Lebesgue measurable sets can always be built from the union of a Borel measurable set and a subset of a Borel nullset.

To close this subsection, for convenience we recall two standard results from measure theory: the Lebesgue dominated convergence theorem and Fatou’s lemma.

**Theorem 2.14 (The Lebesgue dominated convergence theorem \([47] \text{Thm. 5.4.9}\)).** Let \( (X, \Sigma, \nu) \) be a measure space, and let \( \{f_k\} \) be a sequence of measurable functions on \( X \) converging a.e. to a function \( f \). Suppose further that there exists an \( L^1 \) function \( g \) with \( |f_k| \leq g \) a.e. Then \( f \) is also \( L^1 \) and

\[
\int_X f \, d\mu = \lim_{k \to \infty} \int_X f_k \, d\mu.
\]

Furthermore,

\[
\lim_{k \to \infty} \int_X |f - f_k| \, d\mu = 0.
\]

**Theorem 2.15 (Fatou’s lemma \([4] \text{Thm. 2.8.3}\)).** Let \( (X, \Sigma, \nu) \) be a measure space, and let \( \{f_k\} \) be a sequence of nonnegative measurable functions on \( X \) converging a.e. to a function \( f \). Suppose that there exists a constant \( K < \infty \) such that

\[
\sup_k \int_X f_k \, d\mu \leq K.
\]
2.3. GEOOMETRIC PRELIMINARIES

Then the function \( f \) is integrable and

\[
\int_X f \, d\mu \leq K.
\]

In addition,

\[
\int_X f \, d\mu \leq \liminf_{k \to \infty} \int_X f_k \, d\mu.
\]

2.3.3. The manifolds of positive functions and volume forms. We will denote the set of positive \( C^\infty \) functions on the base manifold \( M \) by \( \mathcal{P} \). By the considerations of Subsection 2.2.3, \( \mathcal{P} \) is a Fréchet manifold, since it can be identified with the space of smooth sections of the trivial fiber bundle \( M \times (0, \infty) \). (Alternatively, one can view it as an open set in the Fréchet space of smooth sections of the vector bundle \( M \times \mathbb{R} \).) It is not hard to see that \( \mathcal{P} \) is even a Fréchet Lie group with respect to the group operation of pointwise multiplication—that is, it is a Fréchet manifold such that the multiplication of two elements is a smooth map, as is the map sending an element to its multiplicative inverse.

Similarly, if we denote by \( \mathcal{V} \) the set of smooth volume forms on \( M \), then this is a Fréchet manifold. One can see this either by viewing it as an open set of \( \Omega^n(M) \), the Fréchet space of highest-order differential forms on \( M \) (it is the sections of the line bundle \( \Lambda^n T^* M \)), or by viewing \( \mathcal{V} \) as the smooth sections of the fiber bundle of positive \( n \)-forms on \( M \).

Given any volume form \( \mu \in \mathcal{V} \) and any \( n \)-form \( \alpha \in \Omega^n(M) \), there exists a unique \( C^\infty \) function, denoted by \( (\alpha/\mu) \), such that

\[
(2.14) \quad \alpha = \left( \frac{\alpha}{\mu} \right) \mu.
\]

This fact is easy to deduce from the coordinate representations of \( \alpha \) and \( \mu \), along with the fact that the coefficient of \( \mu \) is positive in any coordinate chart because \( \mu \) is a volume form.

If \( \alpha \) is also a smooth volume form, then \( (\alpha/\mu) \) is additionally a positive function. If we consider the measures induced by \( \alpha \) and \( \mu \), then \( (\alpha/\mu) \) coincides with the Radon-Nikodym derivative [47, Dfn. 9.1.16] of \( \alpha \) with respect to \( \mu \). That is, for any measurable set \( E \subseteq M \), we have

\[
\int_E \alpha = \int_E \left( \frac{\alpha}{\mu} \right) \mu.
\]

We just note that the Radon-Nikodym derivative is defined in general as follows. Say we are given a space \( X \) with a \( \sigma \)-algebra \( \Sigma \), as well as two \( \sigma \)-finite measures \( \nu \) and \( \nu_0 \) on \( \Sigma \). Furthermore, suppose that \( \nu_0 \) is absolutely continuous with respect to \( \nu \), that is, \( \nu_0(E) = 0 \) for all \( E \subseteq X \) with \( \nu(E) = 0 \). Then there exists a nonnegative measurable function \( f \) on \( X \), called the Radon-Nikodym derivative, such that

\[
\int_E d\nu_0 = \int_E f \, d\nu.
\]

The considerations above suggest a natural diffeomorphism between \( \mathcal{V} \) and \( \mathcal{P} \). Namely, if we choose any volume form \( \mu \in \mathcal{V} \), then we can define a map

\[
(2.15) \quad \nu \mapsto \left( \frac{\nu}{\mu} \right),
\]

which as we have seen maps \( \mathcal{V} \) into \( \mathcal{P} \). It is not hard to see that this map is bijective. To see that it is smooth, we simply note that it is the restriction to \( \mathcal{V} \) of the linear map \( \alpha \mapsto (\alpha/\mu) \), which maps \( \Omega^n(M) \) into \( C^\infty(M) \).
Remark 2.16. Note that the function \((\alpha/\mu)\) can be more generally defined for any \(n\)-form \(\alpha\) and any volume form \(\mu\), including those that are not smooth, or continuous, or even measurable. The function will be smooth/continuous/measurable if both \(\alpha\) and \(\mu\) are, as is easily seen in a coordinate chart. It is easy to show (or one may consult [47, Prop. 5.2.6]) that if \(\alpha\) is measurable and nonnegative, then it induces a measure on \(M\), defined by fixing any volume form \(\mu \in \mathcal{V}\) and setting
\[
\int_E \alpha := \int_E \left(\frac{\alpha}{\mu}\right) \mu
\]
for any measurable \(E \subseteq M\). Furthermore, this measure is absolutely continuous with respect to \(\mu\).

If \(g\) is a Riemannian metric on \(M\), then it induces a volume form on \(M\) given in local coordinates \(x^1, \ldots, x^n\) by
\[
\mu_g = \sqrt{\det g} \, dx^1 \cdots dx^n.
\]
If \(g_0\) and \(g_1\) are two Riemannian metrics on \(M\), then locally, the Radon-Nikodym derivative of \(\mu_{g_1}\) with respect to \(\mu_{g_0}\) is given by
\[
\left(\frac{\mu_{g_1}}{\mu_{g_0}}\right) = \sqrt{\frac{\det g_1}{\det g_0}} = \sqrt{\det(g_0^{-1} g_1)}.
\]

Note that this is a well-defined function on \(M\) by the discussion of Subsection 2.3.1.

This completes our general geometric considerations. We now move on to the study of Riemannian metrics on \(\mathcal{F}\)r\(\acute{e}\)chet manifolds.

2.4. Weak Riemannian manifolds

The manifold of metrics with its \(L^2\) metric, the object of study of this thesis, is an example of what is called a weak Riemannian manifold. In this section, we will describe and explore these objects a little bit.

It is well known that on a finite-dimensional vector space, all positive-definite scalar products are equivalent—i.e., every positive-definite scalar product induces the same topology on the space. In infinite-dimensional vector spaces, this is no longer the case—there are many inequivalent positive-definite scalar products, with differing topologies—a simple example might be the \(C^k\) and \(H^s\) topologies on the space of smooth functions \([0, 1] \to \mathbb{R}\). As one might naturally expect, this linear phenomenon has an analog in nonlinear spaces, i.e., manifolds.

In manifold theory, the nonlinear analog of a positive-definite scalar product on a vector space is a Riemannian metric on a manifold. Of course, this is just a positive-definite scalar product on the linearization of the manifold at each point (i.e., its tangent spaces) that varies in a smooth way as we move from point to point. (We’ll make a formal definition of infinite-dimensional Riemannian metrics soon; for the moment, let’s just take this as our heuristic definition for purposes of the introductory discussion.) There is also a nonlinear analog of the difference between finite- and infinite-dimensional spaces as described above.

On a finite-dimensional Riemannian manifold \((N, \gamma)\) modeled on \(\mathbb{R}^n\), the tangent space \(T_x N\) is, via a choice of coordinates, isomorphic to \(\mathbb{R}^n\) for each \(x \in N\). The equivalence of all scalar products on \(\mathbb{R}^n\) implies that the scalar product induced by the Riemannian metric, when viewed as a scalar product on \(\mathbb{R}^n\), is equivalent to the Euclidean scalar product. In particular, it induces the standard topology on \(\mathbb{R}^n\).

In the case of an infinite-dimensional Riemannian manifold \((N, \gamma)\) modeled on a Hilbert space \(E\), we cannot necessarily say that the scalar product induced by \(\gamma\) on a tangent space \(T_x N\) is equivalent to the Hilbert space scalar product of \(E\). Therefore, the topology that
We can distinguish two types of Riemannian metrics on a Fréchet (or, as a special case, Hilbert) manifold. We call $\gamma$ a strong Riemannian metric if it induces the model space topology on each tangent space, and a weak Riemannian metric if it induces a weaker topology.

This subtle but important distinction between the two types of metrics leads to a vast gulf in the two theories one can develop around each structure. For a strong Riemannian metric, one can reproduce most of the important results in finite-dimensional Riemannian geometry. For example, the Levi-Civita connection, geodesics, and the exponential mapping exist. A strong Riemannian metric induces a distance function that gives a metric space structure on the manifold. In addition, the topology induced from this metric space structure agrees with the manifold’s intrinsic topology.

None of the above-mentioned results hold in general for weak Riemannian manifolds. In this section, we will go into detail on these and other differences between weak and strong Riemannian manifolds, as well as explore what statements one can make about weak Riemannian metrics in the cases where the corresponding statements for strong metrics break down.

Before we continue with formal definitions and results, though, we make a couple of philosophical remarks. We have found relatively few references that systematically cover what results of standard Riemannian geometry do and do not hold in this context, as most authors naturally treat only those aspects that arise in the examples they are considering. Furthermore, to the author’s knowledge, all standard textbooks about Riemannian geometry on Hilbert manifolds, such as [28] and [30], work only with strong Riemannian metrics, without explicitly mentioning the distinction between the two types of metrics. Therefore, even the most basic results on weak Riemannian metrics seem not to have been formally written down. Later in the section, we will prove a few general results that will come in useful to us.

Despite there being, to our knowledge, no comprehensive formal treatment of them, weak Riemannian metrics are fundamental objects in global analysis, which deals primarily with manifolds of sections of fiber bundles over a finite-dimensional manifold. Of course, one is typically most interested in $C^\infty$ sections, and the space of $C^\infty$ sections of a vector bundle is a proper Fréchet space—proper meaning that the topology does not come from a (single) norm. Such a space carries only weak Riemannian metrics, since the existence of a strong Riemannian metric would give, via a coordinate chart, a norm inducing the topology of the model space—but this is impossible.

If one considers $H^s$ sections of a vector/fiber bundle, then strong Riemannian metrics can be found. However, since the choice of $s \in \mathbb{N}$ is essentially arbitrary, one would have to choose a different Riemannian metric on the manifold of sections for each $s$. This somewhat unsatisfactory situation leads one to generally pick a single metric (i.e., use the same formula for each $s$), often one inducing the $L^2$ topology. Thus, one is again led back to working with weak Riemannian metrics.

Hopefully we have convinced the reader of the importance of weak Riemannian metrics. We now move on to defining them, exploring some of their deficiencies as compared with strong Riemannian metrics, and then elaborating what weaker results one can prove about them in general.

### 2.4.1. (Weak) Riemannian Fréchet manifolds

A Riemannian metric on a Fréchet manifold is defined exactly analogously to one on a finite-dimensional manifold, modulo the distinction between weak and strong metrics mentioned above.

Recall that on a Banach manifold $N$ modeled on a Banach space $E$, each tangent space $T_xN$ is naturally isomorphic to the model space $E$, the isomorphism being given by any choice of coordinates around $x$ (this choice is, of course, usually very non-canonical). The
same holds true for Fréchet manifolds, and we keep this in mind as we make the following definition.

**Definition 2.17.** Let $N$ be a Fréchet manifold modeled on a Fréchet space $E$. A *Riemannian metric* $\gamma$ on $N$ is a choice of scalar product $\gamma(x)$ on $T_x N$ for each $x \in N$, such that for each $x \in N$, the following holds:

1. $\gamma$ is smooth in the sense that if $U$ is any open neighborhood of $x$ and $V, W$ are vector fields defined on $U$, then $\gamma(\cdot)(V, W) : U \to \mathbb{R}$ is a smooth local function;
2. $\gamma(x)$ is a continuous (i.e., bounded) bilinear mapping; and
3. $\gamma(x)$ is positive definite on $T_x N$.

Furthermore, $\gamma$ is called

1. strong if the topology induced by $\gamma$ coincides with the topology of the model space $E$; and
2. weak otherwise, i.e., if the topology induced by $\gamma$ is weaker than the model space topology.

The pair $(N, \gamma)$ is called a *Riemannian Fréchet manifold*.

To put it another way, $(N, \gamma)$ is a strong Riemannian Fréchet manifold if its tangent spaces are complete with respect to $\gamma$, and it is weak if the tangent spaces are incomplete with respect to $\gamma$.

**Remark 2.18.** There is no such thing as a Riemannian metric inducing a topology on the tangent space that is stronger than the manifold topology. This is because in that case some vectors would have infinite norm—just think of the $H^{s+1}$ norm on $H^s$ functions, for example.

The first definition of weak Riemannian Hilbert manifolds that we know of (though our knowledge is surely incomplete) is in \[11\], the paper that founded the study of the geometry of the manifold of metrics. The generalization to weak Riemannian Fréchet manifolds is natural and has been used in several works. In no particular order, here is a list of papers that consider weak Riemannian manifolds (specifically, those that explicitly deal with the questions posed by “weakness” and are not mentioned elsewhere in this thesis): \[3, 8, 12, 32, 35, 38, 39, 40\] and \[41\]. We have made no attempt to make this list complete—it is simply a smattering of examples.

Let $(N, \gamma)$ be a Riemannian Fréchet manifold. Just as in the case of finite-dimensional Riemannian manifolds, we can use $\gamma$ to define a distance between points of $N$ by taking the infima of lengths of paths.

Let $a \leq b$ be real numbers, and let $\alpha : [a, b] \to N$ be a piecewise $C^1$ path. Define

$$L(\alpha) := \int_a^b \sqrt{\gamma(\alpha(t))(\dot{\alpha}(t), \dot{\alpha}(t))} \, dt.$$  

Then, for any $x, y \in N$, we define

$$d_\gamma(x, y) := \inf_{\alpha} L(\alpha),$$

where the infimum is taken over all piecewise $C^1$ paths that start at $x$ and end at $y$.

It is easy to see that $d_\gamma$ is a *pseudometric*. That is, it has all the properties of a metric (in the sense of metric spaces) other than positive-definiteness. That $d_\gamma(x, y) = d_\gamma(y, x)$, $d_\gamma(x, y) \geq 0$ and $d_\gamma(x, x) = 0$ for all $x, y \in N$ is clear. The triangle inequality for $d_\gamma$ then follows from the fact that if we have a path from $x$ to $y$ and a path from $y$ to $z$, the concatenation of the two is a path from $x$ to $z$ with length the sum of the two original paths.

Positive definiteness of the distance function is a trickier issue, and in fact it only holds in general for strong Riemannian metrics! For weak metrics, it may fail. In fact,
the example of the next subsection shows that it may fail in the most spectacular way possible—for some weak Riemannian manifolds, $d_{\gamma}(x, y) = 0$ for all points $x, y \in N$.

After we have described the example and seen how bad things can get, we will see what parts of the theory break down and allow such things to happen. After that we will try to partially rebuild.

### 2.4.2. Pathological behavior of a weak Riemannian metric on the manifold of embeddings of $S^1$ into $\mathbb{R}^2$

The following example is from [37], to which we refer for more details. We will give only a very sketchy and conceptual presentation of one of their results. There is no harm in skipping this subsection and continuing on to the discussion of the Levi-Civita connection in the next subsection. On the other hand, the reader interested in a complete description of this example should consult [37].

Let $C^\infty(S^1, \mathbb{R}^2)$ denote the vector space of all smooth mappings of $S^1$ into $\mathbb{R}^2$. This is a Fréchet space, as we saw in Subsection 2.2.3. We consider the open set $E \subset C^\infty(S^1, \mathbb{R}^2)$ of smooth embeddings of $S^1$ into $\mathbb{R}^2$—in other words, this is the space of smooth, parametrized, closed curves in $\mathbb{R}^2$. As an open set of a Fréchet space, it is trivially a Fréchet manifold.

Let $D$ denote the group of smooth diffeomorphisms of the circle. It is a Fréchet Lie group, and it acts on $C^\infty(S^1, \mathbb{R}^2)$ from the right by composition, i.e., pull-back: for $\varphi \in D$ and $f \in C^\infty(S^1, \mathbb{R}^2)$, the action is $\varphi \cdot f = f \circ \varphi$. If we restrict this action to $E$, then it is free, and it turns out that the quotient $E/D$ is a smooth Fréchet manifold.

There exists a natural $D$-invariant Riemannian metric on $E$. It is a weak metric, as it induces the $L^2$ topology on the tangent spaces. To define it, let $f \in E$ be any embedding. Since $E$ is an open set of $C^\infty(S^1, \mathbb{R}^2)$, the tangent space $T_f E$ is canonically isomorphic to $C^\infty(S^1, \mathbb{R}^2)$ itself, and we can think of $T_f E$ as the space of vector fields on $f(S^1) \subset \mathbb{R}^2$. That is, if $\pi : S^1 \times \mathbb{R}^2 \to S^1$ is the projection, then $T_f E$ consists of maps $h : S^1 \to S^1 \times \mathbb{R}^2$ with $\pi \circ h = f$. With this in mind, we define for any $h, k \in C^\infty(S^1, \mathbb{R}^2) \cong T_f E$:

$$\langle (h, k) \rangle_f := \int_{S^1} \langle h(\theta), k(\theta) \rangle |\partial_\theta f(\theta)| d\theta,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product on $\mathbb{R}^2$. Describing this metric in words, we integrate the scalar product of $h$ and $k$ with respect to the Euclidean volume form pulled back along $f$.

Since $\langle \cdot, \cdot \rangle$ is $D$-invariant (as is relatively easily computed), it descends to a weak Riemannian metric on $E/D$. Though it is outside the scope of this thesis to prove this here, the Riemannian metric thus obtained induces a distance function as described above, but the distance between any two points vanishes! Thus the Riemannian metric $\langle \cdot, \cdot \rangle$ is, in some sense, a very bad metric on $E/D$.

Rather than prove this fact, we will simply give the idea of the proof. We can bound the distance between two points in $E/D$ from above by the distance between any two points of their preimages in $E$. So take any path $f_t$ of curves interpolating between $f_0$ and $f_1$—it happens that one can modify this path to get a path whose image has arbitrarily small length when projected to $E/D$, showing that the distance between the endpoints in $E/D$ is zero. To do this, we simply construct a path $f_{n,t}$ from $f_t$ in which the curves oscillate $n$ times as they interpolate between $f_0$ and $f_1$. These oscillating curves are illustrated in Figure 1. It then happens that the length of $f_{n,t}$, when projected onto $E/D$, goes to zero as $n \to \infty$.

So now we have an extremely pathological example of how bad the distance function of a weak Riemannian metric can be. Our next task is to understand how such a phenomenon, which is impossible in the finite-dimensional case, can occur. To do so, we need to reexamine some of the standard theorems of Riemannian geometry and see what can be said about them in the infinite-dimensional case.
2. PRELIMINARIES

2.4.3. The Levi-Civita connection. On a finite-dimensional Riemannian manifold \((N, \gamma)\), and even on a strong Riemannian Hilbert manifold, there is a unique connection \(\nabla\) that is both

1. metric, i.e., \(X \gamma(Y, Z) = \gamma(\nabla_X Y, Z) + \gamma(Y, \nabla_X Z)\) for all vector fields \(X, Y\) and \(Z\); and

2. torsion-free, i.e., \(\nabla_X Y - \nabla_Y X = [X, Y]\) for all vector fields \(X\) and \(Y\).

The existence and uniqueness of this connection relies on the Koszul formula, which states that a connection is both metric and torsion-free if and only if the following equation holds for all vector fields \(X, Y, Z\):

\[
\gamma(\nabla_X Y, Z) = X \gamma(Y, Z) + Y \gamma(X, Z) - Z \gamma(X, Y) - \gamma(X, [Y, Z]) + \gamma(Y, [X, Z]) + \gamma(Z, [X, Y]).
\]

Existence and uniqueness of the element \(\nabla_X Y\) at the point \(x \in N\) now follows from the Riesz representation theorem applied to the Hilbert space \((T_x N, \gamma)\).

The Levi-Civita connection is then used to define geodesics as those paths \(\alpha\) for which \(\nabla_{\dot{\alpha}} \dot{\alpha} = 0\). Geodesics, in turn, are used to define the exponential mapping, as is well known.

On a weak Riemannian manifold, this picture breaks down, as (2.18) fails to guarantee existence of the Levi-Civita connection. (If it exists, though, (2.18) does guarantee its uniqueness.) Since the tangent spaces of \((N, \gamma)\) are incomplete with respect to \(\gamma\), only guarantees the existence of \(\nabla_X Y\) at \(x \in N\) as an element of the completion of \(T_x N\) with respect to \(\gamma\). This is of course because the Riesz representation theorem does not hold on incomplete spaces.

The result of this is: On a weak Riemannian manifold, the Levi-Civita connection does not exist in general. As a consequence, geodesics and the exponential mapping do not exist in general, either.

The usual strategy when dealing with weak Riemannian manifolds is the following. Without general theorems at one’s disposal, various properties that are automatic for strong Riemannian manifolds have to be directly verified. For example, in the next section, we will sketch how, in [11], the existence of the Levi-Civita connection for the manifold...
of metrics was shown. In essence, an explicit formula for $\nabla_X Y$ was computed using the Koszul formula, and it was shown that the result is in fact a section of the tangent bundle.

2.4.4. The exponential mapping and distance function on a strong Riemannian manifold. Subsection 2.4.2 gave an example of a weak Riemannian manifold with an induced distance function that is not a metric—i.e., that fails to be positive definite. In contrast, for a strong Riemannian manifold, the following theorem holds, as it does in the finite-dimensional case:

**Theorem 2.19** ([28 Thm. 1.9.5]). Let $(N, \gamma)$ be a strong Riemannian (Hilbert) manifold. Then the induced distance function $d_{\gamma}$ is a metric on $N$, and the topology of $d_{\gamma}$ coincides with the topology of $N$.

The natural question that arises is, what goes wrong in the case of a weak Riemannian manifold? To answer this, we recall the main steps in the proof of Theorem 2.19. The first is:

**Theorem 2.20** ([28 Thm. 1.8.15]). Let $(N, \gamma)$ be a strong Riemannian (Hilbert) manifold. Then there exists an open neighborhood $U \subseteq TN$ of $N$ such that the exponential mapping is defined and differentiable on $U$.

Furthermore, for every $x \in N$, there exist positive numbers $\epsilon = \epsilon(x)$ and $\eta = \eta(x)$, with $\epsilon < \eta$, and a neighborhood $V$ of $x$ such that the following holds:

1. The mapping
   
   $$\exp_x |_{B_\epsilon(0)} : B_\epsilon(0) \to V,$$
   
   where $B_\epsilon(0)$ is the open ball of radius $\epsilon$ (w.r.t. $\gamma$) around $0 \in T_x N$, is a diffeomorphism.

2. For any $y, z \in V$, there exists a unique geodesic from $y$ to $z$ with length less than $\eta$.

3. For each $y \in V$, $\exp_y |_{B_\eta(0)}$ is a diffeomorphism onto an open neighborhood $V_y$ of $y$, with $V \subseteq V_y$.

Using this theorem, we have some control over the domain of definition and the range of the exponential mapping. The next step is to control the lengths of paths contained within the image of the exponential mapping:

**Theorem 2.21** ([28 Thm. 1.9.2]). Suppose $(N, \gamma)$ is a strong Riemannian manifold. Let $x \in N$, and suppose that $\exp_x$ is defined on an open neighborhood $U_x$ of $0 \in T_x N$. Let $v : [0, 1] \to U_x$ be any path with $v(0) = 0$, and let $\tilde{v} : [0, 1] \to U_x$ be the straight-line path in $T_x N$ between $0$ and $v(1)$. Finally, define paths in $N$ by $\alpha(t) := \exp_x(v(t))$ and $\tilde{\alpha}(t) := \exp_x(\tilde{v}(t))$.

Then $L(\tilde{\alpha}) \leq L(\alpha)$, and equality holds if $v(t) = \tilde{v}(t(s))$, where $t(s)$ is a reparametrization with $t'(s) \geq 0$.

Conversely, if $L(\tilde{\alpha}) = L(\alpha)$ and $D_{sv(t)} \exp_x$ has maximal rank for all $0 \leq s, t \leq 1$, then $v(t) = \tilde{v}(t(s))$, where $t(s)$ is a reparametrization with $t'(s) \geq 0$.

What this theorem essentially says is the following. Let $\exp_x$ be defined on $U_x \subseteq T_x N$, with range $V_x \subseteq N$, and let $y \in V_x$. Then among the class of paths in $V_x$ from $x$ to $y$, the unique shortest path (up to reparametrization) is the radial geodesic emanating from $x$ and ending at $y$.

What Theorem 2.21 does not tell us is that the radial geodesic from $x$ to $y$ is the shortest path among the class of all paths in $N$ from $x$ to $y$. However, combining Theorems 2.20 and 2.21 gives us what we want:

**Theorem 2.22** ([28 Thm. 1.9.3]). Suppose $(N, \gamma)$ is a strong Riemannian manifold. Let $x \in N$, and let $\epsilon$, $\eta$ and $V$ be as in Theorem 2.20. Suppose that $y \in V$. Then
any geodesic starting from \( y \) of length less than \( \eta \) is a path of minimal length between its endpoints.

Thus, geodesics are, locally, length-minimizing paths. Using Theorem 2.22 and our previous statement that the distance function of a Riemannian manifold is always a pseudometric, it is then trivial to prove Theorem 2.19.

2.4.5. The exponential mapping and distance function on a weak Riemannian manifold. We now return to weak Riemannian manifolds. The question remains: What goes wrong when we try to extend the results of Subsection 2.4.4?

The theorem that breaks down, it turns out, is Theorem 2.20. This is true even if we assume that the Levi-Civita connection exists. It even breaks down if we assume that the exponential mapping exists and is a diffeomorphism when restricted to some open neighborhood of the zero section in \( TM \)—none of which are guaranteed on a weak Riemannian manifold!

The problem is the following: on a strong Riemannian manifold \((N, \gamma)\), a neighborhood of \( 0 \in T_x N \) contains an open \( \gamma \)-ball of some sufficiently small radius. However, if \((N, \gamma)\) is a weak Riemannian manifold, since the topology induced by \( \gamma \) is weaker than the manifold topology of \( T_x N \), an open neighborhood of \( 0 \) (in the manifold topology) need not necessarily contain any open \( \gamma \)-balls.

This phenomenon does indeed occur—it is not too hard to see that it occurs for the example of Subsection 2.4.2, and we will see below, in Section 2.5, that the manifold of metrics also exhibits this phenomenon.

In the case of the manifold of metrics, we will eventually be able to show, in Section 3.1, that the \( L^2 \) metric does in fact induce a metric space structure. However, the metric space topology does not agree with the manifold topology, and so strange phenomena that are absent for strong Riemannian metrics occur. For example, we will later show in Lemma 5.19 that there is no metric ball of any positive radius around any point of the manifold of metrics! This is, of course, tied very closely to the analogous fact about the tangent space.

For now, though, we put aside the nastier behavior of weak Riemannian manifolds and show what results actually do hold for them in general. They will necessarily be weaker than the results for strong Riemannian manifolds, but they will still come in hand later on and are of interest in their own right.

Our goal is to prove statements analogous to, but weaker than, the theorems of Subsection 2.4.4. We will follow a very similar course, making only minor modifications to the statements and proofs in [28] as necessary.

Our first theorem is familiar from finite-dimensional Riemannian geometry and is quite simple to prove.

**Proposition 2.23.** Let \((N, \gamma)\) be a weak Riemannian manifold on which the Levi-Civita connection exists. Let \( p \in N \) and \( v \in T_p N \), and suppose that \( v \) is in the domain of \( \exp_p \). Then the geodesic \( \alpha(t) := \exp_p(tv), t \in [0, 1] \), has length \( ||v||_\gamma \).

**Proof.** The proof for Riemannian Hilbert manifolds is algebraic in nature and so carries over to weak Riemannian manifolds—here we just give a sketch. Since the Levi-Civita connection is metric, its parallel transport along any curve is an isometry of the tangent spaces. That \( \alpha \) is a geodesic implies that \( \alpha'(t) \) is parallel along \( \alpha \), and hence \( \alpha'(t) \) has constant length. Since \( \alpha'(0) = v \), this length is \( ||v||_\gamma \). \( \square \)

Unfortunately, we cannot prove much more that is useful about weak Riemannian manifolds without first making a couple of assumptions on the exponential mapping. Basically, we want it to exist and to be a diffeomorphism between some open sets—so we’ll have to assume that as well. The next bit of terminology incorporates this, and also adds one technical detail that we’ll soon need.
2.4. WEAK RIEMANNIAN MANIFOLDS

Definition 2.24. We call a weak Riemannian manifold \((N, \gamma)\) normalizable at \(x\) if there are open neighborhoods \(U_x \subseteq T_x N\) and \(V_x \subseteq N\) containing \(0\) and \(x\), respectively, such that

1. the exponential mapping \(\exp_x\) exists and is a \(C^1\)-diffeomorphism between \(U_x\) and \(V_x\); and

2. the following function is continuous:

\[
R : T_x N \to \mathbb{R}_+
\]

\[
v \mapsto \sup \{ r \in \mathbb{R}_+ \mid r \cdot v \in U_x \}.
\]

Note that the neighborhoods \(U_x\) and \(V_x\) are required to be open in the manifold topology of \(N\). We do not require that \(U_x\) be open in the topology induced by \(\gamma\).

We call \((N, \gamma)\) normalizable if it is normalizable at each \(x \in N\).

Definition 2.25. Let \((N, \gamma)\) be a weak Riemannian manifold and let \(x \in N\). We denote by \(S_x N \subset T_x N\) the unit sphere, i.e.,

\[
S_x N = \{ v \in T_x N \mid \|v\|_{\gamma} = 1 \}.
\]

For the rest of this section, let \((N, \gamma)\) be a weak Riemannian manifold that is normalizable at a point \(x \in N\), and retain the notation of Definition 2.24.

The following lemma shows that the exponential mapping of a weak Riemannian manifold that is normalizable at \(x\) is defined on some nonzero vector pointing in each direction in \(T_x N\).

Lemma 2.26. For each \(v \in T_x N\), \(R(v) > 0\).

Proof. Let \(v \in T_x N\) be given. Since \(T_x N\) with its manifold topology is a topological vector space and \(U_x\) is a neighborhood of the origin, there is some \(\epsilon > 0\) such that \(\epsilon \cdot v \in U_x\).

Remark 2.27. Lemma 2.26 does not imply that \(R(v)\) is uniformly bounded away from zero, even if we restrict the domain of \(R\) to \(S_x N\) at each \(x \in N\).

This next proposition is the analog of Theorem 2.21, and is proved similarly.

Proposition 2.28. Let \(r(s) \cdot v(s) \in U_x\), \(s \in [0, 1]\), be a path in \(U_x\) such that \(v(s) \in S_x N\), \(r(s) \in \mathbb{R}_{\geq 0}\). (That is, we express the path in polar coordinates.) We define a path \(\alpha\) by \(\alpha(s) := \exp_x(r(s)v(s))\), \(s \in [0, 1]\). Then

\[
L(\alpha) \geq |r(1) - r(0)|,
\]

with equality if and only if \(v(s)\) is constant and \(r'(s) \geq 0\).

Proof. By Definition 2.24 and Lemma 2.26, as well as the compactness of \([0, 1]\), there exist \(\epsilon, \delta > 0\) such that if

\[
(s, t) \in U_{\epsilon, \delta} := \{(s, t) \in \mathbb{R}^2 \mid s \in [0, 1], \ t \in [-\epsilon, r(s) + \delta]\},
\]

then \(t \cdot v(s) \in U_x\).

We define a one-parameter family of paths in \(N\) by

\[
c_s(t) := \exp(t \cdot v(s)), \quad (s, t) \in U_{\epsilon, \delta}
\]

Note that for each fixed \(s\), the path \(t \mapsto c_s(t)\) is a geodesic with

\[
\|\partial_t c_s(t)\|_{\gamma} \equiv \|\partial_t c_s(0)\|_{\gamma} = \|v(s)\|_{\gamma} = 1.
\]

Note also that the image of the family of paths \(c_\cdot(\cdot)\) is a singular surface in \(N\) parametrized by the coordinates \((s, t)\).
Keeping this in mind, we compute

\[
\partial_t \gamma (\partial_s c_s(t), \partial_t c_s(t)) = \gamma \left( \frac{\nabla}{\partial t} \partial_s c_s(t), \frac{\nabla}{\partial t} \partial_t c_s(t) \right) + \gamma \left( \partial_s c_s(t), \frac{\nabla}{\partial t} \partial_t c_s(t) \right)
\]

\[
= \gamma \left( \frac{\nabla}{\partial s} \partial_t c_s(t), \partial_t c_s(t) \right)
\]

\[
= \frac{1}{2} \partial_s \gamma (\partial_t c_s(t), \partial_t c_s(t))
\]

\[
= 0.
\]

(2.20)

Here, the second line holds because

\* $s$ and $t$ are coordinate functions, and hence (covariant) derivatives in the two directions commute, and

\* $t \mapsto c_s(t)$ is a geodesic, hence $\frac{\partial}{\partial t} \partial_t c_s(t) = 0$.

The last line follows directly from (2.19).

From (2.20), we immediately see that

\[
\gamma (\partial_s c_s(t), \partial_t c_s(t))
\]

is independent of $t$. However, we also have that $c_s(0) = x$ for all $s$, implying that $\partial_s c_s(0) = 0$, thus

\[
0 = \gamma (\partial_s c_s(0), \partial_t c_s(0)) = \gamma (\partial_s c_s(t), \partial_t c_s(t))
\]

for all $t$. That is, $\partial_s c_s(t)$ and $\partial_t c_s(t)$ are orthogonal for all $s$ and $t$.

We now estimate:

\[
\|\alpha'(s)\|_\gamma^2 = \left\| \frac{d}{ds} c_s(r(s)) \right\|_\gamma^2
\]

\[
= \|\partial_s c_s(r(s)) + r'(s) \partial_t c_s(r(s))\|_\gamma^2
\]

\[
= \|\partial_s c_s(r(s))\|_\gamma^2 + |r'(s)|^2 \|\partial_t c_s(r(s))\|_\gamma^2
\]

\[
\geq |r'(s)|^2.
\]

Here, in the third line, we have used orthogonality of $\partial_s c_s(t)$ and $\partial_t c_s(t)$. In the last line, we have used (2.19). Note that equality holds if and only if $\|\partial_s c_s(r(s))\|_\gamma \equiv 0$.

Finally, we see that

\[
L(\alpha) = \int_0^1 \|\alpha'(s)\|_\gamma \, ds \geq \int_0^1 |r'(s)| \, ds \geq \left| \int_0^1 r'(s) \, ds \right| = |r(1) - r(0)|,
\]

which proves the desired inequality. We note that the first inequality is an equality if and only if $\|\partial_s c_s(r(s))\|_\gamma \equiv 0$ (see the previous paragraph) and the second inequality is an equality if and only if $r'(s) \geq 0$ for all $s$. \qed

Finally, we get the analog of Theorem 2.22. The remark afterwards points out in what way this is weaker than that theorem, however.

**Proposition 2.29.** Suppose $y \in V_x$ with $\exp^{-1}_x(y) = v$. Then the path

\[ \alpha : [0, 1] \to V_x, \quad \alpha(t) = \exp_x(t \cdot v) \]

satisfies $L(\alpha) = \|v\|_\gamma$, and $\alpha$ is of minimal length among all paths in $V_x$ from $x$ to $y$. Furthermore, $\alpha$ is the unique minimal path (up to reparametrization) in $V_x$ from $x$ to $y$.

**Remark 2.30.** Note that we will only show that $\alpha$ is minimal only among paths (or geodesics) in $V_x$, not all paths (or geodesics) in $N$. In particular, we cannot conclude from Proposition 2.29 that $d(\gamma(x, y) = L(\alpha)$. 
2.4. WEAK RIEMANNIAN MANIFOLDS

Figure 2. A path between two points in \( V \subset N \) that travels out of and back into \( V \) in possibly very short distance. The dashed circle on the left represents the sphere of radius \( \| \exp_x^{-1}(y) \|_{\gamma(x)} \) in \( T_x N \).

**Proof.** The equality \( L(\alpha) = \| v \|_\gamma \) holds by Proposition 2.23.

A path \( \eta(s), s \in [0,1] \), in \( V \) from \( x \) to \( y \) corresponds via \( \exp_x^{-1} \) to a path \( r(s) \cdot v(s) \) in \( U_x \) with \( v(s) \in S_x N \), \( r(0) = 0 \) and \( r(1) \cdot v(1) = v \), implying \( |r(1)| = \| v \|_\gamma \). By Proposition 2.28 we therefore have that

\[
L(\eta) \geq \| v \|_\gamma = L(\alpha),
\]

immediately implying minimality of \( \alpha \).

Let equality hold in (2.21). Again by Proposition 2.28 this implies \( v(s) \) is constant and \( r'(s) \geq 0 \) for all \( s \). However, this means that \( \eta \) is just a reparametrization of \( \alpha \), proving the second statement. \( \square \)

As an obvious result of Proposition 2.29, we get the following criterion for a weak Riemannian manifold to be a metric space. It requires rather strong assumptions which could probably be weakened significantly, but it will be sufficient for some purposes that we have in mind—specifically, we will use it to show that certain submanifolds of the manifold of metrics are metric spaces.

**Theorem 2.31.** Let \( (N, \gamma) \) be a weak Riemannian manifold. Suppose that for some \( x \in N \), the exponential mapping \( \exp_x \) is a diffeomorphism between open (in the manifold topology) neighborhood \( U_x \) of 0 \( \in T_x N \) and \( N \).

Then \( (N, d_\gamma) \), where \( d_\gamma \) is the Riemannian distance function of \( \gamma \), is a metric space.

**Proof.** Let \( y \in N \). It remains to show that if \( y \neq x \), then \( d(x,y) > 0 \). But if \( \exp_x^{-1}(y) = v \), then Proposition 2.29 shows that the shortest path from \( x \) to \( y \) in \( N \) is \( \exp_x(t \cdot v) \), which has length \( \| v \| \). Therefore \( d(x,y) = \| v \| > 0 \). \( \square \)

Proposition 2.29 of course cannot tell us anything about whether a general weak Riemannian manifold \( (N, \gamma) \) is a metric space, and given the example of Subsection 2.4.2, neither can any other theorem, since at a point \( x \in N \), the exponential mapping \( \exp_x \) need not be defined on any \( \gamma(x) \)-open neighborhood of 0 \( \in T_x N \). This means that we cannot use the exponential mapping directly to control the lengths of curves between two chosen points.

Let’s be more precise about this. Assume that for some point \( x \in N \), \( \exp_x \) is a diffeomorphism between open sets \( U \subset T_x N \) and \( V \subset N \), but that \( U \) contains no \( \gamma(x) \)-open ball. Say we are given a point \( y \in N \), and let’s even assume that \( y \in V \) to illustrate our point most dramatically. We know that \( U \) does not contain any \( \gamma(x) \)-open ball around zero, and from Proposition 2.29 a radial path in \( U \) is mapped by \( \exp_x \) to a minimal geodesic between its endpoints (minimal among the class of paths remaining within \( V \)). Thus we can imagine a radial path that starts at \( x \), leaves \( V \) after an arbitrarily short distance, then reenters \( V \) such that its image under \( \exp_x^{-1} \) lies on the sphere of radius
\[ \| \exp_x^{-1}(y) \|_{\gamma(x)} \]. This is illustrated in Figure 2. In this case, the results stated so far do not allow us to control the length of our path outside of \( V \) or on the second piece inside \( V \), since Proposition 2.28 does not tell us anything about paths with \( r(t) \) constant (in the notation of that proposition). Our results therefore do not rule out paths of arbitrarily small length.

2.5. The manifold of metrics \( \mathcal{M} \)

In this section, we define the manifold of smooth Riemannian metrics \( \mathcal{M} \) over a closed, finite-dimensional base manifold \( M \). We are especially interested in the geometry of the so-called \( L^2 \) metric \((\cdot,\cdot)\) on \( \mathcal{M} \), which is a weak Riemannian metric on a Fréchet manifold. We will also discuss \( \mathcal{M}^s \), the manifold of Riemannian metrics with \( H^s \) coefficients, which is a weak Riemannian Hilbert manifold. These objects will be defined in the first two subsections. In the third subsection, we will give a useful decomposition of \( \mathcal{M} \) into a product manifold, a decomposition that we will refer back to later in the thesis. Finally, we will mention some facts about the geometry of \((\mathcal{M},(\cdot,\cdot))\) that are already known, such as formulas for its curvature and geodesics.

All of the facts in this section are culled from the three papers [11], [19] and [20]. We refer the reader to these for more details, and we will also reference specific theorems at appropriate points. We point out a few differences between the papers. The study of the geometry and topology of \((\mathcal{M},(\cdot,\cdot))\), as well as that of superspace (the quotient of \( \mathcal{M} \) by the action of the diffeomorphism group) was initiated in [11] in the \( H^s \) setting—i.e., this paper studied the manifold of metrics with \( H^s \) coefficients (see Subsection 2.5.1). Much later, [19] computed the curvature and geodesics of \( \mathcal{M} \) using some general theorems from the context of strong Riemannian Hilbert manifolds. Most of these general theorems carry over to weak Riemannian manifolds, however, and the explicit formulas of [19] all match up with those of [20], which computed the same things using tools strictly from the theory of weak Riemannian manifolds. Furthermore, [20] computed the analogs of Ricci curvature, scalar curvature and Jacobi fields in this setting, and additionally did not require the base manifold \( M \) to be compact—simply without boundary.

2.5.1. Definition of the manifold of metrics. Let \( S^2 T^* M \) denote the second symmetric tensor power of the cotangent bundle, and let \( \mathcal{S} := \Gamma(S^2 T^* M) \) denote the vector space of smooth, symmetric \((0,2)\)-tensor fields on \( M \). By the discussion in Subsection 2.2.3, \( \mathcal{S} \) is a Fréchet space with topology coming from the \( H^s \) norms induced by any smooth Riemannian metric \( g \) on \( M \). Furthermore, for \( s \in \mathbb{N} \cup \{0\} \), we define \( \mathcal{S}^s := H^s(S^2 T^* M) \), i.e., \( \mathcal{S}^s \) is the vector space of \( H^s \) sections of \( S^2 T^* M \). We equip \( \mathcal{S}^s \) with the \( H^s \) norm induced by any smooth Riemannian metric \( g \).

The first thing we note is that while the norm on \( \mathcal{S}^s \) (and the collection of norms on \( \mathcal{S} \)) depend on our choice of \( g \), the topologies of \( \mathcal{S} \) and \( \mathcal{S}^s \) do not. This was pointed out in Subsection 2.2.3.

Now, let \( \mathcal{M} \subset \mathcal{S} \) and \( \mathcal{M}^s \subset \mathcal{S}^s \) denote the subsets of smooth Riemannian metrics and Riemannian metrics with \( H^s \) coefficients, respectively. That is, \( \mathcal{M} \) and \( \mathcal{M}^s \) consist of those elements that induce positive definite scalar products at each point. We claim that for \( s > n/2 \), \( \mathcal{M}^s \) is an open subset of \( \mathcal{S}^s \), implying also that \( \mathcal{M} \) is an open subset of \( \mathcal{S} \). This follows easily from the Sobolev embedding theorem, if for \( s > n/2 \), then a bound on the \( H^s \) norm of a tensor field implies a bound on the \( C^0 \) norm. Thus it is easy to see that if \( g \in \mathcal{M}^s \) is any \( H^s \) (and hence continuous) metric and \( h \in \mathcal{S}^s \) is any tensor field with sufficiently small \( H^s \) (and hence \( C^0 \)) norm, then \( g + h \) will also be positive definite. Note also that \( \mathcal{M} \) and \( \mathcal{M}^s \) are positive cones, i.e., if \( g_0 \) and \( g_1 \) are metrics and \( \lambda, \mu > 0 \), then \( \lambda g_0 + \mu g_1 \) is also a metric.

As open subsets of vector spaces, we trivially have that \( \mathcal{M} \) is a Fréchet manifold and \( \mathcal{M}^s \) is a Hilbert manifold. For the remainder of the section, we will only discuss the
2.5. THE MANIFOLD OF METRICS $\mathcal{M}$

manifold of smooth metrics $\mathcal{M}$, as this is our main object of interest. This is in the interest of brevity and clarity of presentation only. All results hold for $\mathcal{M}^s$ as well if one uses $H^s$ objects instead of smooth objects and puts a superscript “$s$” on all manifolds of mappings, i.e., considers spaces of $H^s$ instead of smooth mappings. We will point out a couple of examples along the way to show what we mean by this.

Since $\mathcal{M}$ is an open subset of $S$, its tangent space at any point $g \in \mathcal{M}$ is canonically identified with $S$, i.e., $T_g \mathcal{M} \cong S$. We will use this identification over and over throughout the thesis.

2.5.2. The $L^2$ metric. Since $S^2 T^* M$ is a vector bundle associated to the tangent bundle, a Riemannian metric $g \in \mathcal{M}$ induces a Riemannian metric on $S^2 T^* M$. Let’s take a look at some fundamental linear algebra before we write down this metric.

If $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ are vector spaces over the same ground field with scalar products, we can form a scalar product on their tensor product $V \otimes W$ in the following way. For tensors of the form $v \otimes w$, we define

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{V \otimes W} := \langle v_1, v_2 \rangle_V \langle w_1, w_2 \rangle_W,$$

and this definition is then extended via bilinearity to all of $V \otimes W$.

The scalar product induced by the Riemannian metric $g$ on the cotangent space $T^*_x M$ is given in local coordinates by

$$g(x)(\alpha, \beta) = g^{ij} \alpha_i \beta_j.$$

Hence, by (2.22), on the tensor product $T^*_x M \otimes T^*_x M$, the scalar product on elements of the form $\alpha \otimes \beta$ is given by

$$g(x)(\alpha \otimes \beta, \gamma \otimes \delta) = g(x)(\alpha, \gamma) \cdot g(x)(\beta, \delta) = g^{ij} \alpha_i \gamma_j g^{kl} \beta_k \delta_l.$$

It is easy to see that the general formula, obtained by extending via bilinearity, is the following. For $h, k \in T^*_x M \otimes T^*_x M$,

$$g(x)(h, k) = g^{ij} h_{il} g^{lm} k_{mj}.$$  

Remark 2.32. By the considerations above, a Riemannian metric $g \in \mathcal{M}$ gives rise to a Riemannian metric on any bundle associated to the tangent bundle (i.e., any bundle that can be built from $TM$ using tensor products, taking the dual, symmetrization, antisymmetrization etc.), since we know that $g^{-1}$ is a metric on $T^*M$ and (2.22) shows us how to form a scalar product on tensor products of vector spaces.

Let’s now restrict to symmetric tensors. Denote by $S_x := S^2 T^*_x M$ the symmetrization of $T^*_x M \otimes T^*_x M$. Let $h, k \in S_x$, and let $H$ and $K$ be the tensors obtained from $h$ and $k$, respectively, by raising an index with $g$. Then $H$ and $K$ are $(1,1)$-tensors, or in other words endomorphisms of $T_x M$. So in particular, we can multiply them. Since they are symmetric, we can use (2.23) to get

$$g(x)(h, k) = g^{ij} h_{il} g^{lm} k_{mj} = g^{ij} h_{il} g^{lm} k_{mj} = H^i_j K^j_i = \text{tr}(HK) =: \text{tr}_g(hk).$$

The above expression is called the $g$-trace of $hk$. It is sometimes useful to write this in the notation of matrix multiplication, so that

$$\text{tr}_g(hk) = \text{tr}(g^{-1} hg^{-1} k),$$

which is of course only valid in local coordinates.

If $h \in S_x$, we can similarly define its $g$-trace to be

$$\text{tr}_g h = \text{tr} H = \text{tr}(g^{-1} h) = g^{ij} h_{ji}.$$
Remark 2.33. Note that we could have defined a scalar product on $S^2 T^* M$ more generally by
\[
\text{tr}_g(hk) + \alpha \text{tr}_g(h) \text{tr}_g(k)
\]
for any $\alpha \in \mathbb{R}$. These more general scalar products are studied in [46]. By setting $\alpha = 0$ we get the $L^2$ metric back, and for $\alpha = -1$ we get the metric used by DeWitt [10] mentioned in Section 1.3. The scalar product is positive definite if $\alpha \geq -1/n$; it is nondegenerate if $\alpha \neq -1/n$.

There are two reasons we have chosen to study the $L^2$ metric in particular. The first is that, as we have tried to show in this subsection, the $L^2$ metric arises canonically in the differential geometric context. The second is the connection to Teichmüller theory that was mentioned in Section 1.2 and which will be elucidated in Chapter 6.

We now want to define the $g$-trace of a section or a product of two sections of $S^2 T^* M$, which we can do by simply taking the $g$-trace at each point. We introduce the following notation for this:

Definition 2.34. Let $g$ be any Riemannian metric, and let $h$ and $k$ be elements of $S$. (We do not assume $g$, $h$ or $k$ to be smooth or even continuous.) We denote the $g$-trace of $hk$ by
\[
\langle h, k \rangle_g := \text{tr}_g(hk).
\]
For each fixed choice of $g$, $h$, and $k$, it is a function mapping $M \rightarrow \mathbb{R}$.

If it is necessary to explicitly denote at which point this expression is taken, we will write $\langle h(x), k(x) \rangle_{g(x)}$. Usually, though, the point $x$ will be clear from the context and omitted from the notation.

Lemma 2.35. For any fixed $g \in \mathcal{M}$ and $x \in M$, $\langle \cdot, \cdot \rangle_g$ is a positive definite scalar product on $S_x$. Furthermore, we can use it to define a smooth Riemannian metric $\langle \cdot, \cdot \rangle$ on the finite-dimensional manifold
\[
\mathcal{M}_x := \{ g \in S_x \mid g > 0 \} = \{ g(x) \mid g \in \mathcal{M} \}
\]
by using the scalar product $\langle \cdot, \cdot \rangle_g$ on each tangent space $T_x \mathcal{M}_x = S_x$. Of course, $g > 0$ indicates that $g$ defines a positive-definite scalar product on $T_x \mathcal{M}$.

Proof. We start with the proof that $\langle \cdot, \cdot \rangle_g$ is a positive-definite scalar product on $S_x$ for any fixed $g$. Bilinearity is clear, so we simply have to prove positive definiteness. If $h \in S_x$, then
\[
\langle h, h \rangle_g = \text{tr}(H^2)
\]
by \eqref{eq:2.24}. Let’s fix any arbitrary coordinates around $x$ and look at this expression locally. From elementary linear algebra, we know that the trace of any matrix is equal to the sum of its eigenvalues. Additionally, the eigenvalues of $H^2$ are the squares of the eigenvalues of $H$. Therefore, if $\lambda_1^H, \ldots, \lambda_n^H$ are the eigenvalues of $H$ and $h$ is nonzero,
\[
\text{tr}(H^2) = (\lambda_1^H)^2 + \cdots + (\lambda_n^H)^2 > 0.
\]
Of course, for this inequality to hold, we have to know that the eigenvalues of $H$ are real—but this was proved in Lemma 2.11. (Note that positive definiteness of $\langle \cdot, \cdot \rangle_g$ actually also follows easily from that of $g$ combined with \eqref{eq:2.22}. Nevertheless, we will use the facts stated here later, so it is worthwhile to mention them.)

As for the second statement, note first that $\mathcal{M}_x$ is indeed a finite-dimensional manifold, as it is an open set in the vector space $S_x$. (This also gives us the identification of $T_x \mathcal{M}_x$ with $S_x$.) Also, for any fixed $h, k \in S_x$, the function $g \mapsto \text{tr}_g(hk) = \text{tr}(g^{-1}hg^{-1}k)$ is clearly smooth over $\mathcal{M}_x$. Combined with the positive definiteness of $\langle \cdot, \cdot \rangle_g$ for fixed $g$, this completes the proof that $\langle \cdot, \cdot \rangle$ is a Riemannian metric. \qed
Since each tangent space of $\mathcal{M}$ is identified with $\mathcal{S}$, a Riemannian metric on $\mathcal{M}$ will give, for each Riemannian metric on $\mathcal{M}$, a positive scalar product on smooth sections of $S^2T^*\mathcal{M}$. We have just described a canonical positive definite scalar product on $S^2T^*_x\mathcal{M}$, and to pass to sections we do the obvious thing: we integrate it.

**Definition 2.36.** The $L^2$ metric on $\mathcal{M}$ is defined to be
\[
(h,k)_g := \int_M \text{tr}_g(hk)\,\mu_g \quad \text{for all } h,k \in \mathcal{S} \cong T_g\mathcal{M},
\]
where $\mu_g$ is the volume form induced by $g$.

For any given $g \in \mathcal{M}$, we denote by $\|\cdot\|_g$ the norm on $\mathcal{S}$ induced by $(\cdot,\cdot)_g$, that is,
\[
\|h\|_g := \sqrt{(h,h)_g} \quad \text{for all } h \in \mathcal{S}.
\]

Finally, we denote the distance function (a pseudometric) induced by $(\cdot,\cdot)$ simply by $d$.

The $L^2$ metric is indeed a smooth Riemannian metric—this is proved in [11] §4. (In fact, $(\cdot,\cdot)$ is smooth in the $H^s$ topology on $\mathcal{M}$ for any $s > n/2$.) We will not repeat the proof of smoothness here, but it is easy to see bilinearity and positive definiteness—the latter follows simply from positive definiteness of $(\cdot,\cdot)_g$ at each point of $\mathcal{M}$.

The name of the $L^2$ metric is not there just for fun. It is, in fact, a weak Riemannian metric inducing the $L^2$ topology on each tangent space, as the following theorem due to Palais [44] IX.2 shows. (We have already mentioned this theorem in Subsection 2.2.3, but we restate it here in this context and with an extra statement, the equivalence of the scalar products, which is implied by the proofs in the above reference.)

**Theorem 2.37.** Let $g_0, g_1 \in \mathcal{M}$. Then $(\cdot,\cdot)_{g_0}$ and $(\cdot,\cdot)_{g_1}$ are equivalent scalar products. In particular, they both induce the same topology on $\mathcal{S}$, the $L^2$ topology.

Ebin even pointed out in [11] §4 that the theorem still holds if $g_0$ and $g_1$ are only assumed to be continuous rather than smooth.

Let us make a brief technical note at this point. When we use the term “$L^2$ topology”, what we really mean is that we give this name to the topology induced from $(\cdot,\cdot)_g$ for some $g \in \mathcal{M}$. Of course, when we think of $L^2$ objects, we think of functions that are square integrable, so we might ask whether a similar interpretation holds for the completion of $\mathcal{S}$ with respect to $(\cdot,\cdot)_g$. In fact, looking at the coefficients of a tensor field as local functions, defined over a coordinate chart, we claim that elements of the completion of $\mathcal{S}$ with respect to $(\cdot,\cdot)_g$ are precisely those tensor fields with coefficients that are $L^2$-integrable over any chart.

The reason for this is that the proof of Theorem 2.37 is pointwise in character—that is, not only are $(\cdot,\cdot)_{g_0}$ and $(\cdot,\cdot)_{g_1}$ equivalent for any $g_0, g_1 \in \mathcal{M}$, but there are constants $C, C' > 0$ such that for all $x \in \mathcal{M}$ and $h,k \in \mathcal{S}_x$,
\[
\frac{1}{C} \text{tr}_{g_1}(h(x)k(x)) \leq \text{tr}_{g_0}(h(x)k(x)) \leq C \text{tr}_{g_1}(h(x)k(x))
\]
and
\[
\frac{1}{C'} \leq \left(\frac{\mu_{g_1}}{\mu_{g_0}}\right)(x) \leq C'.
\]
(For the reader who desires more details, the proof of Lemma 3.13 below will eventually make this clear.) Thus, $(\cdot,\cdot)_{g_0}$ and $(\cdot,\cdot)_{g_1}$ are equivalent not just on sections of $S^2T^*\mathcal{M}$ defined over all of $\mathcal{M}$, but also equivalent if we restrict them to sections defined only over some subset of $\mathcal{M}$.

Fix a coordinate chart $U$ and choose a metric $g^U$ with the property that $g^U_{ij} \equiv \delta_{ij}$ on $U$. Also fix an arbitrary metric $g \in \mathcal{M}$. On sections of $S^2T^*\mathcal{M}$ defined over $U$, $(\cdot,\cdot)_g$ is equivalent to $(\cdot,\cdot)_{g^U}$ by the arguments of the previous paragraph. But the completion
with respect to $(\cdot,\cdot)_g$ of the space of sections of $S^2T^*M$ over $U$ consists of exactly those sections with square integrable coefficients, since locally
\[
\text{tr}_{g'}(hk) = \delta_{il} \delta_{jm} k_{ij} = \sum_{i,j} h_{ij} k_{ij}.
\]

This shows that the topology (and the completion) of $S$ with respect to $(\cdot,\cdot)_g$ is the same as the “naive” $L^2$ topology coming from the (local) square integral of the coefficients of tensor fields. What this means is that we can use results on the $L^2$ topology for functions and apply them to $S$ with the topology given by $(\cdot,\cdot)_g$—always viewing the coefficients of tensor fields as local functions.

2.5.3. A product manifold structure for $M$. Let’s move on to studying the structure of $M$ with respect to the $L^2$ metric. The goal of this subsection is to define a splitting of $M$ as the product of the set of metrics inducing the same volume form and the set of volume forms on $M$.

Select any volume form $\mu \in \mathcal{V}$ and define
\[
\mathcal{M}_\mu := \{ g \in \mathcal{M} \mid \mu_g = \mu \},
\]
that is, $\mathcal{M}_\mu$ is the set of all metrics which induce the volume form $\mu$. Then $\mathcal{M}_\mu$ is a smooth submanifold of $\mathcal{M}$ (cf. [11] Lemma 8.8).

Consider the map $g \mapsto \mu_g$, mapping $\mathcal{M}$ to $\mathcal{V}$. We wish to compute the differential of this map, since this will help us to figure out what the tangent space to $\mathcal{M}_\mu$ at a point is. The result is given in the following lemma.

**Lemma 2.38.** Let $g \in \mathcal{M}$ and $h \in S$. We have
\[
D\mu_g[h] = \frac{1}{2} \text{tr}_g(h)\mu_g.
\]

**Proof.** We wish to compute
\[
D\mu_g[h] = \left. \frac{d}{dt} \right|_{t=0} \mu_{g+th}.
\]
If we write this is local coordinates $x^1, \ldots, x^n$ and let $I$ denote the $n \times n$ identity matrix, we have
\[
\left. \frac{d}{dt} \right|_{t=0} \mu_{g+th} = \left. \frac{d}{dt} \right|_{t=0} \sqrt{\det(g + th)} \; dx^1 \cdots dx^n
\]
\[
= \left( \left. \frac{d}{dt} \right|_{t=0} \sqrt{\det(I + tg^{-1}h)} \right) \left. \sqrt{\det g} \; dx^1 \cdots dx^n \right|_{t=0}
\]
\[
= \left( \left. \frac{d}{dt} \right|_{t=0} \sqrt{\det(I + tg^{-1}h)} \right) \mu_g.
\]

To compute the derivative term above, recall that for any square matrix $A$, $\exp \text{tr} A = \det \exp A$, where $\exp$ is the matrix exponential. Recall also that $\exp$ is a local diffeomorphism between a neighborhood $U$ of the zero matrix and a neighborhood $V$ of the identity matrix in the space of $n \times n$ matrices. So if $A_t$ is a one-parameter family of positive definite symmetric matrices with $A_0 = I$ and $A_t \in V$ then we can write $B_t = \log A_t$ uniquely. This allows us to compute
\[
\left. \frac{d}{dt} \right|_{t=0} \det A_t = \left. \frac{d}{dt} \right|_{t=0} \det \exp B_t = \left. \frac{d}{dt} \right|_{t=0} \exp \text{tr} B_t
\]
\[
= (\text{tr} B_t)' \exp \text{tr} B_t|_{t=0} = (\text{tr} B_t)|_{t=0},
\]
where the last equality follows from $B_0 = \log I = 0$. Now, note that $A \mapsto \tr A$ is a linear map, so its differential is given by the map itself again. Therefore

$$
\frac{d}{dt} \bigg|_{t=0} \det A_t = (\tr B_t)'|_{t=0} = \tr B_0'.
$$

We now claim that $B_0' = A_0'$. This follows from the fact that for any matrix $X$, 

$$
\frac{d}{dt} \bigg|_{t=0} \exp(tX) = X,
$$

If we define the notation $\Phi := \exp$ and $\Psi := \log$, we can write this another way:

$$
D\Phi(0)X = X = \Rightarrow D\Phi(0) = \text{id}.
$$

As $\Psi$ is the inverse function of $\Phi$ on the neighborhood $V$, $D\Psi(C) = D\Phi(\Psi(C))^{-1}$ for any $C \in V$. Since $\Psi(I) = 0$, we have that $D\Psi(I) = D\Phi(0)^{-1} = \text{id}$. This implies

$$
B_0' = \frac{d}{dt} \bigg|_{t=0} \log A_t = D\Psi(I)A_0' = A_0'.
$$

Substituting this into (2.28), we get

$$
\frac{d}{dt} \bigg|_{t=0} \det A_t = \tr A_0'.
$$

Using the above in (2.27) and making a straightforward computation finally gives the result.

Returning to $M_\mu$, since the map $g \mapsto \mu_g$ is constant over $M_\mu$, we see from the previous lemma that

$$
T_gM_\mu = \{ h \in S \mid \tr_g h = 0 \}.
$$

That is, the tangent space to $M_\mu$ at $g$ is given by the $g$-traceless tensors. Let us denote the set of $g$-traceless tensors by $S_g^T$.

Let $\mu \in \mathcal{V}$ be any smooth volume form on $M$. Then, as pointed out in Section 2.3 for any $\nu \in \mathcal{V}$ there exists a unique $C^\infty$ function, denoted $(\nu/\mu)$, such that

$$
(2.30) \quad \nu = \left( \frac{\nu}{\mu} \right) \mu.
$$

Furthermore, if $g \in \mathcal{M}$ and $f \in \mathcal{P}$, i.e., $f$ is a smooth positive function, then from the local expression for $\mu_g$ (cf. (2.16)) we see that

$$
(2.31) \quad \mu_{fg} = f^{n/2} \mu_g.
$$

From these facts, it is easy to see that if the metric $g$ induces the volume form $\mu$ and $\nu \in \mathcal{V}$, then the unique metric conformal to $g$ inducing the volume form $\nu$ is

$$
\tilde{g} := \left( \frac{\nu}{\mu} \right)^{2/n} g.
$$

This gives us the idea for a splitting of $\mathcal{M}$: by the considerations of the last paragraph, there is a bijection between $\mathcal{M}$ and $M_\mu \times \mathcal{P}$. In Section 2.3 we saw that $\mathcal{P}$ is diffeomorphic to $\mathcal{V}$, and so we can also say there is a bijection between $\mathcal{M}$ and $M_\mu \times \mathcal{V}$. This is more intuitive, as it basically says that choosing a metric from $\mathcal{M}$ is the same as choosing an element from $M_\mu$, which induces a fixed volume form, and then picking a volume form.

In concrete terms, we define a map

$$
(2.32) \quad i_\mu : M_\mu \times \mathcal{V} \to \mathcal{M},
$$

$$(g, \nu) \mapsto \left( \frac{\nu}{\mu} \right)^{2/n} g.$$
Thus, \( i_\mu \) maps \((g, \nu)\) to the unique metric conformal to \( g \) with volume form \( \nu \). It is straightforward to show that \( i_\mu \) is not only a bijection, but a diffeomorphism.

To compute the differential of \( i_\mu \), recall from Section 2.3 that \( T_\nu \mathcal{V} = \Omega^n(M) \). Also, since \( \alpha \mapsto (\alpha/\mu) \) is a linear map from \( \Omega^n(M) \) to \( C^\infty(M) \), its differential is again given by the map itself, i.e., \( \beta \mapsto (\beta/\mu) \). This gives us all we need to compute

\[
D i_\mu(g, \nu)[h, \alpha] = \frac{2}{n} \left( \frac{\nu}{\mu} \right)^{\frac{2}{n}-1} \left( \frac{\alpha}{\mu} \right) g + \left( \frac{\nu}{\mu} \right)^{2/n} h,
\]

where \( \alpha \in \Omega^n(M) = T_\nu \mathcal{V} \) and \( h \in S^T_g = T_g \mathcal{M}_\mu \).

As a submanifold of \( \mathcal{M} \), \( \mathcal{M}_\mu \) has a natural Riemannian metric induced from the \( L^2 \) metric of \( \mathcal{V} \). We can use the map \( i_\mu \) to define a Riemannian metric on \( \mathcal{V} \) as follows. For every \( g \in \mathcal{M}_\mu \), we can embed \( \mathcal{V} \) into \( \mathcal{M} \) via

\[
\mathcal{V} \cong \{ g \} \times \mathcal{V} \xrightarrow{i_\mu} \mathcal{M}.
\]

Let \( \langle \cdot, \cdot \rangle \) be the pullback of the \( L^2 \) metric \( \langle \cdot, \cdot \rangle \) along this embedding. To compute \( \langle \cdot, \cdot \rangle \), note that \( \mu_{i_\mu(g, \nu)} = \nu \), and for all \( \alpha \in \Omega^n(M) \) and \( \nu, \lambda \in \mathcal{V}, \)

\[
\left( \frac{\lambda}{\nu} \right)^{-1} = \left( \frac{\nu}{\lambda} \right) \quad \text{and} \quad \left( \frac{\alpha}{\nu} \right) \left( \frac{\nu}{\lambda} \right) = \left( \frac{\alpha}{\lambda} \right).
\]

Using this and (2.33), we can compute

\[
\langle \alpha, \beta \rangle_\nu = (D i_\mu(g, \nu)[0, \alpha], D i_\mu(g, \nu)[0, \beta])_{i_\mu(g, \nu)}
\]

\[
= \int_M \text{tr}_{(\nu/\mu)^{2/n}g} \left( \left( \frac{2}{n} \left( \frac{\nu}{\mu} \right)^{\frac{2}{n}-1} \left( \frac{\alpha}{\mu} \right) g \right) \left( \frac{2}{n} \left( \frac{\nu}{\mu} \right)^{\frac{2}{n}-1} \left( \frac{\beta}{\mu} \right) g \right) \right) \mu_{(\nu/\mu)^{2/n}g}
\]

\[
= \int_M \text{tr} \left( \frac{4}{n^2} \left( \frac{\nu}{\mu} \right)^{-2} \left( \frac{\alpha}{\mu} \right) \left( \frac{\beta}{\mu} \right) I \right) \nu
\]

\[
= \frac{4}{n} \int_M \left( \frac{\alpha}{\nu} \right) \left( \frac{\beta}{\nu} \right) \nu.
\]

Note that \( \langle \cdot, \cdot \rangle \) is actually independent of the elements \( g \) and \( \mu \) we chose to define the embedding \( \mathcal{V} \hookrightarrow \mathcal{M} \), so it is a natural object. In fact, \( \langle \cdot, \cdot \rangle \) is just the constant factor \( 4/n \times \) the most obvious Riemannian metric on \( \mathcal{V} \).

We know that \( \mathcal{V} \) is diffeomorphic to \( \mathcal{P} \). Furthermore, if \( g \in \mathcal{M} \) is any smooth metric, then the orbit of the conformal group \( \mathcal{P} \) through \( g \), \( \mathcal{P} \cdot g \), is also diffeomorphic to \( \mathcal{P} \). So composing diffeomorphisms appropriately, we can also see that \( \mathcal{M} \cong \mathcal{M}_\mu \times \mathcal{V} \cong \mathcal{M}_\mu \times \mathcal{P} \cdot g \). Each viewpoint may be useful, depending on the context.

The global splitting (2.32) also, of course, gives a splitting of the tangent space at each \( g \in \mathcal{M} \). Let’s describe this briefly.

Let \((g, \nu) \in \mathcal{M}_\mu \times \mathcal{V} \). From (2.33), it is easy to see that

\[
D i_\mu(g, \nu)[0, T_\nu \mathcal{V}] = C^\infty(M) \cdot \left( \frac{\nu}{\mu} \right)^{2/n} g = C^\infty(M) \cdot g
\]

In other words, the image of the tangent space of \( \mathcal{V} \) under the differential of \( i_\mu \) is the set of pure trace tensors. Let us denote the set of such tensors by \( S^T_g := C^\infty(M) \cdot g \).

(The superscript “c” stands for “conformal.”) Furthermore, we can also compute that if \( f := (\nu/\mu)^{n/2} \), then

\[
D i_\mu(g, \nu)[T_g \mathcal{M}_\mu, 0] = S^T_{fg} = S^T_{i_\mu(g, \nu)}.
\]

Note that this computation uses the fact that \( \text{tr}_{fg}(fh) = \text{tr}_g h \) for any \( h \in S \).
Let $g \in \mathcal{M}$ and $(\tilde{g}, \nu) := i_{\mu}^{-1}(g)$. By (2.34) and (2.35), the splitting (2.32) then implies that

\begin{equation}
T_g \mathcal{M} = (Di_{\mu}(\tilde{g}, \nu)[T_{\tilde{g}} \mathcal{M}_{\mu}, 0]) \oplus (Di_{\mu}(\tilde{g}, \nu)[0, T_{\nu} \mathcal{V}]) = S_T^g \oplus S_c^g.
\end{equation}

It is easy to see that this is, in fact, an orthogonal splitting of $T_g \mathcal{M}$ with respect to $(\cdot, \cdot)_g$. For if $h \in S_T^g$ and $k = fg \in S_c^g$, then we have

\begin{equation}
\text{tr}_g(hk) = \text{tr}(g^{-1}hg^{-1}(fg)) = f \cdot \text{tr}_g h = 0,
\end{equation}

since $\text{tr}_g h = 0$ by assumption.

The splitting given in this subsection plays an important role in the general theory of $\mathcal{M}$. In particular, as we will see in the next subsection, results on the curvature and geodesics of the $L^2$ metric can be nicely stated and more easily visualized using this product manifold structure.

### 2.5.4. The curvature of $\mathcal{M}$

The computation of the curvature (and, in the next section, of the geodesics of $\mathcal{M}$) is greatly simplified by a heuristic consideration. Namely, we can intuitively think of the $L^2$ metric on $\mathcal{M}$ as a product metric with an infinite number of factors, one for each $x \in \mathcal{M}$. “Summing up” the different terms in this product metric is then done by integration. Of course, this is only a formal construction, but it is a useful practical aid. More details about how this can be made rigorous, in a much more general context, are given in [19] Appendix. In a case like this, one often says that the computations are pointwise in nature.

To illustrate what this means, we take the example of a geodesic in $\mathcal{M}$. A path in $\mathcal{M}$ is a one-parameter family $g_t$ of Riemannian metrics on the base manifold $M$. Thus, for every $x \in \mathcal{M}$, $g_t(x)$ is a one-parameter family of positive definite elements of $S_c$ that glue together to a smooth metric over $M$ for each $t$. The geodesic $g_t$ is additionally completely determined by an initial metric $g_0$ and a tangent vector $g'_0 \in T_{g_0} \mathcal{M}$. When we say that the geodesic equation is pointwise, what we mean is that we can go one step further and say that the path the geodesic takes at a point, $g_t(x)$, is determined completely by the values $g_0(x)$ and $g'_0(x)$.

Now that we know what a pointwise computation is, we will keep these considerations in mind as we continue. However, before we can write down formulas for the curvature of $\mathcal{M}$, we need to take care of an issue that is technical in nature but central in its implications—namely the existence of the Levi-Civita connection of the $L^2$ metric.

Recall that in Subsection 2.4.3 we pointed out that the Levi-Civita connection of a weak Riemannian manifold does not necessarily exist. If it does exist, however, it is unique. The problem was that the Koszul formula (2.18) only guarantees the existence of the Levi-Civita covariant derivative of a vector field at a point as an element of the completion of the tangent space (with respect to the Riemannian metric), not of the tangent space itself.

Thus, given two vector fields $h$ and $k$ on $\mathcal{M}$ ($h$ and $k$ are, at each point of $\mathcal{M}$, smooth sections of $S^2 T^* M$), (2.18) only guarantees that the Levi-Civita covariant derivative $\nabla_h k|_g$ at a point $g \in \mathcal{M}$ is an element of $L^2(S^2 T^* M)$, since $(\cdot, \cdot)$ induces the $L^2$ topology on each tangent space.

To show that the Levi-Civita connection does indeed exist, i.e., that $\nabla_h k|_g$ is a smooth section of $S^2 T^* M$ for all vector fields $h, k \in C^\infty(TM)$ and all $g \in \mathcal{M}$, Ebin [11] §4 exhibited an explicit formula for $\nabla_h k|_g$ on $\mathcal{M}$ and showed that $\nabla_h k|_g$ is $H^s$ if $h, k,$ and $g$ are. Thus, it is also smooth if $h, k,$ and $g$ are all smooth. The precise formula is the following:

\begin{equation}
\nabla_h k|_g = \frac{d}{dt}
|_{t=0} [k(g+th(g)) - \frac{1}{2}(h g^{-1} k + k g^{-1} h) + \frac{1}{4}((\text{tr}_g k) h + (\text{tr}_g h) k - \text{tr}_g(hk)g),
\end{equation}

where $h(g) \in T_g \mathcal{M} \cong S$ is the value of the vector field $h$ at the basepoint $g$, and similarly $k(g+th(g)) \in S$ is the value of $k$ at $g + th(g)$ for small $t$. (Bear in mind that a smooth
vector field on \( \mathcal{M} \) is a smooth choice of an element of \( \mathcal{S} \) for each \( g \in \mathcal{M} \).) It is easily seen from (2.38) that \( \nabla_h k|_g \) is an \( H^\ast \) section of \( S^2 T^\ast M \) if \( h, k \), and \( g \) are (see [11] §4 for an explicit proof), and furthermore that this expression varies smoothly with \( g \).

Now that we know the Levi-Civita connection exists, we are assured that the curvature and geodesics of \( \mathcal{M} \) with its \( L^2 \) metric are defined. After some general discussion, the goal of this subsection is to take a look at the curvature of \( \mathcal{M} \) and, because it will play a role later, also that of \( \mathcal{V}, \mathcal{P}, \) and \( \mathcal{P} \cdot g \).

We now quote the theorem giving the curvature of \( \mathcal{M} \):

**Theorem 2.39 ([20] Prop. 2.6).** Let \( h, k, \ell \in T_g \mathcal{M} \), and let \( H = g^{-1} h, K = g^{-1} k \) and \( L := g^{-1} \ell \). We denote by \( I \) the section of the endomorphism bundle \( \text{End}(\mathcal{M}) \) that gives the identity map at each \( x \in \mathcal{M} \).

The Riemannian curvature tensor of \( \mathcal{M} \) with respect to the \( L^2 \) metric is given by

\[
g^{-1} R_g(h, k)\ell = -\frac{1}{4}[H, K], L] + \frac{n}{16}(\text{tr}(HL)K - \text{tr}(KL)H)
+ \frac{1}{16}(\text{tr}(K) \text{tr}(L)H - \text{tr}(H) \text{tr}(L)K)
+ \frac{1}{16}(\text{tr}(H) \text{tr}(KL) - \text{tr}(K) \text{tr}(HL))I.
\]

**Remark 2.40.** As is well known, in the literature on Riemannian geometry there are two conventions for defining the Riemannian curvature tensor. The convention we use is the following. If \( N \) is a Riemannian manifold with Levi-Civita connection \( \nabla \), we define

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
\]

for any vector fields \( X, Y \) and \( Z \) on \( N \).

The formula differs from the one in [20] by a negative sign, and is the same as the one used in [19].

Here we make the observation that \( R_g(h, k)\ell = 0 \) if any one of \( h, k \) or \( \ell \) is pure trace—i.e., of the form \( fg \) for \( f \in C^\infty(\mathcal{M}) \). This is readily checked using the above formula, but it can also be seen via more geometric arguments (as is done in [19]). Using this observation, it is possible to write the curvature in a more compact form, as well as give a clean expression for the sectional curvature of \( \mathcal{M} \). For the proof of the entire theorem we refer to the original source [19] Thm. 1.16 and Cor. 1.17.

**Corollary 2.41.** Let notation be as in Theorem 2.39. If any of \( h, k \), or \( \ell \) is pure trace, then \( R_g(h, k)\ell = 0 \). If \( h, k, \ell \in S_g T_g \mathcal{M} \), then we have

\[
g^{-1} R_g(h, k)\ell = -\frac{1}{4}[H, K], L] + \frac{n}{16}(\text{tr}(HL)K - \text{tr}(KL)H).
\]

By the splitting (2.36), this determines the Riemannian curvature tensor completely.

Furthermore, for \( h, k \in S_g T_g \mathcal{M} \), the sectional curvature of \( \mathcal{M} \) is given by

\[
K_g(h, k) = (R_g(h, k)k|_g) = \int_{\mathcal{M}} \left( \frac{1}{4} \text{tr}([H, K]^2) + \frac{n}{16}(\text{tr}(HK)^2 - \text{tr}(H^2) \text{tr}(K^2)) \right) \mu_g.
\]

If either of \( h \) or \( k \) is pure trace, then \( K_g(h, k) \) vanishes.

Finally, the above formula implies that \( K_g(h, k) \leq 0 \) for all \( h, k \in T_g \mathcal{M} \) and all \( g \in \mathcal{M} \).

Using a result of Freed and Groisser [19] Prop. 1.5, we can also prove the following:

**Proposition 2.42.** Equip \( \mathcal{V} \) with the weak Riemannian metric given by pullback along \( i_\mu \) (see (2.32)). Equip \( \mathcal{P} \) with the metric given by pullback along the diffeomorphism (2.15) with \( \mathcal{V} \). Finally, for \( g \in \mathcal{M} \), give the orbit \( \mathcal{P} \cdot g \) the metric it inherits as a submanifold of \( \mathcal{M} \).

Then \( \mathcal{V}, \mathcal{P}, \) and \( \mathcal{P} \cdot g \) are all isometric. Furthermore, they are flat, i.e., their Riemannian curvature vanishes.
By construction, it is clear that $\mathcal{P}$ is isometric to $\mathcal{V}$ with the given metrics.

As for $\mathcal{P} \cdot g$, since $(\nu/\mu)$ can be any positive function given an appropriate choice of a volume form $\nu$, the image of $i_\mu$ is exactly $\mathcal{P} \cdot g$. Since $\mathcal{V}$ with the pullback metric is isometric to its image under $i_\mu$ as a submanifold of $\mathcal{M}$, this shows that $\mathcal{P} \cdot g$ is isometric to $\mathcal{V}$. \end{proof}

### 2.5.5. Geodesics on $\mathcal{M}$

Now that we have given the curvature equation for $\mathcal{M}$, we’ll take a look at its geodesics. It turns out that the geodesic equation can be solved explicitly, and the result is the following (see [19 Thm. 2.3], [29 Thm. 3.2]):

**Theorem 2.43 (The geodesic equation of $\mathcal{M}$).** Let $g_0 \in \mathcal{M}$ and $h \in T_{g_0} \mathcal{M} = S$. Let $H := g_0^{-1} h$ and let $H^T$ be the traceless part of $H$. Define two one-parameter families $q_t$ and $r_t$ of functions on $\mathcal{M}$ as follows:

$$q_t(x) := 1 + \frac{t}{4} \operatorname{tr} H, \quad r_t(x) := \frac{t}{4} \sqrt{4 \operatorname{tr}(H^T)^2}.$$

Then the geodesic starting at $g_0$ with initial tangent $g_0' = h$ is given at each point $x \in \mathcal{M}$ by

$$g_t(x) = \begin{cases} (q_t^2(x) + r_t^2(x))^{\frac{2}{n}} g_0(x) \exp \left( \frac{4}{\sqrt{4 \operatorname{tr}(H^T)^2}} \arctan \left( \frac{r_t(x)}{q_t(x)} \right) H^T(x) \right), & H^T(x) \neq 0, \\ q_t(x)^{4/n} g_0(x), & H^T(x) = 0. \end{cases}$$

For precision, we specify the range of $\arctan$ in the above. At a point where $\operatorname{tr} H \geq 0$, it assumes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. At a point where $\operatorname{tr} H < 0$, $\arctan(r_t/q_t)$ assumes values as follows:

1. in $[0, \frac{\pi}{2})$ if $0 \leq t < -\frac{4}{\sqrt{\operatorname{tr} H}}$,
2. in $(\frac{\pi}{2}, \pi)$ if $-\frac{4}{\sqrt{\operatorname{tr} H}} < t < \infty$,

and we set $\arctan(r_t/q_t) = \frac{\pi}{2}$ if $t = -\frac{4}{\sqrt{\operatorname{tr} H}}$.

Finally, the geodesic is defined on the following domain. If there are points where $H^T = 0$ and $\operatorname{tr} H < 0$, then let $t_0$ be the minimum of $\operatorname{tr} H$ over the set of such points. In symbols,

$$t_0 := \inf \{ \operatorname{tr} H(x) \mid H^T(x) = 0 \text{ and } \operatorname{tr} H(x) < 0 \}.$$ Then the geodesic $g_t$ is defined for $t \in [0, -\frac{1}{t_0})$.

If there are no points where both $H^T = 0$ and $\operatorname{tr} H < 0$, then $g_t$ is defined on $[0, \infty)$.

**Remark 2.44.** The geodesic given in Theorem 2.43 is parametrized proportionally to arc length. That is, for each $\tau > 0$ such that $g_t$ is defined on $[0, \tau]$, we have

$$L(g_t|_{[0,\tau]}) = \tau \|h\|_{g_0}.$$ As for the distinguished submanifolds of $\mathcal{M}$ that we have studied, their geodesics are given in the following two propositions.

**Proposition 2.45 ([19 Prop. 2.1]).** If $g \in \mathcal{M}$, then $\mathcal{P} \cdot g$ is a totally geodesic submanifold. Therefore, the geodesic in $\mathcal{P} \cdot g$ starting at $g_0$ with initial tangent $\alpha g_0$ is given by

$$g_t = \left(1 + nt_0 \right)^{\frac{4}{n}} g_0.$$ As a result, the exponential mapping $\exp_{g_0}$ is a diffeomorphism from an open set $U \subset T_{g_0} (\mathcal{P} \cdot g)$ onto $\mathcal{P} \cdot g$. 

PROOF. Freed and Groisser prove that $\mathcal{V}$ with the given metric is flat, so if we can show that $\mathcal{V}$, $\mathcal{P}$, and $\mathcal{P} \cdot g$ are all isometric, then the statement is immediate.

By construction, it is clear that $\mathcal{P}$ is isometric to $\mathcal{V}$ with the given metrics.

As for $\mathcal{P} \cdot g$, since $(\nu/\mu)$ can be any positive function given an appropriate choice of a volume form $\nu$, the image of $i_\mu$ is exactly $\mathcal{P} \cdot g$. Since $\mathcal{V}$ with the pullback metric is isometric to its image under $i_\mu$ as a submanifold of $\mathcal{M}$, this shows that $\mathcal{P} \cdot g$ is isometric to $\mathcal{V}$. \end{proof}
Proposition 2.46 ([11] Thm. 8.9 and [19] Prop. 1.27, Prop. 2.2). The submanifold $\mathcal{M}_\mu$ is not totally geodesic. However, it is a globally symmetric space, and the geodesic starting at $g_0$ with initial tangent $g'_0 = h$ is given by

$$g_t = g_0 \exp(tH),$$

where $H := g_0^{-1}h$.

In particular, $\mathcal{M}_\mu$ is geodesically complete, and $\exp_g$ is a diffeomorphism from $T_g\mathcal{M}_\mu$ to $\mathcal{M}_\mu$ for any $g \in \mathcal{M}_\mu$.

Note a consequence of Theorem 2.43 that is very important to us. Our goal being the description of the completion of $\mathcal{M}$, the following corollary to Theorem 2.43 assures us that we actually have something to study.

Corollary 2.47. The manifold of metrics is incomplete (geodesically and as a metric space) with respect to its $L^2$ metric.

Proof. Choose any $g_0 \in \mathcal{M}$, and choose any $h \in S$ such that $\|h\|_{g_0} = 1$ and there is at least one point where $H^T = 0$ and $\text{tr} H < 0$. Then the geodesic $g_t$ starting at $g_0$ in the direction of $h$ has maximal domain of definition $[0, \frac{1}{H}]$, and its length over this domain is finite (in fact, equal to $-\frac{1}{H}$). $\square$

We are interested in studying the completion of $\mathcal{M}$, and we have just shown in Corollary 2.47 that it is incomplete. Furthermore, this is very simply expressed through the non-extensibility of a geodesic, for which we have an explicit formula. So it is worthwhile to take a closer look at why geodesics can fail to be extensible, and see what this does and does not tell us about the completion of $\mathcal{M}$.

First, by Theorem 2.43 a geodesic can fail to be forever extensible only if it has a point where $h(x) = g'_0(x)$ is pure trace, i.e., where $H^T(x) = 0$. Why is this? A quick look at the geodesic equation provides the answer: if $H^T(x) \neq 0$ over all of $M$, then $r_t(x) \neq 0$ as well. This is because $\text{tr}((H^T)^2) = 0$ if and only if $H^T = 0$, by Lemma 2.35. But then the scalar coefficient in front, $(q_t^2 + r_t^2)^{2/n}$, is always positive. Furthermore, since the matrix exponential maps symmetric matrices into positive definite matrices and $H$ is $g_0$-symmetric (i.e., $g_0H$ is symmetric), the exponential term does not destroy the positive-definiteness of $g_0$. Thus $g_t$ is positive-definite at all points of $M$ for all $t \in [0, \infty)$, and hence is a metric for all $t$.

Now, what can go wrong if $H^T(x) = 0$ for some $x \in M$? In this case, $r_t(x) = 0$ for all $t$, and the exponential term is absent in the geodesic equation. If we have $\text{tr} H(x) \geq 0$, then $g_t(x) > 0$ for all $t$, and so again $g_t(x)$ is positive definite for all $t \in [0, \infty)$. But if $\text{tr} H(x) < 0$, there is some $t'$ for which $r_{t'}(x) = 0$. Therefore, $g_{t'}(x) = 0$, and the geodesic has left the manifold of metrics. The geodesic can, however, be easily identified with its limit point in the $C^\infty$ topology of $S$, which is a semimetric, or a tensor field inducing a positive semidefinite scalar product at each point. However, only special kinds of semimetrics can be realized as limit points of geodesics, namely those that are either positive definite or zero at each point. But it is also easy to convince oneself that all such semimetrics can be realized as limit points of geodesics. Thus we have arrived at our first substantial piece of knowledge about the completion of $\mathcal{M}$. Instead of writing it down as a proposition, we’ll instead wait for more general and rigorous statements to be made later.

A semimetric that is nonzero but not positive definite cannot be realized in this way. An extremely simple example is the semimetric on the torus which is given in the standard chart by

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
Nevertheless, the question still remains as to whether such semimetrics might also be representatives, in some sense that has to be made precise, of points in the completion of $\mathcal{M}$. Answering this in the positive, in Section 5.3 will be one of our tasks in studying the completion and proving the main theorem.

Another question that presents itself at this point is whether a finite path (or a Cauchy sequence, according to the correspondence given in Section 2.1) in $\mathcal{M}$ can “develop infinities”, in the sense that one or more of its coefficients becomes unbounded (in a fixed coordinate chart) as we run through its domain of definition. (See Definition 2.58 below.) Certainly the coefficients of a geodesic always remain bounded on bounded $t$-intervals. Nevertheless, it will turn out that the metrics of a finite-length path can develop infinities, but that the infinities can also be neglected in a certain sense. We will explore this in Chapter 4.

At this point, however, it will be profitable to give the exponential mapping a somewhat closer inspection.

**Remark 2.48.** We have not yet shown that $\mathcal{M}$ is a metric space and have already remarked that a weak Riemannian manifold does not always have a metric space structure, so in a sense Corollary 2.47 could be seen as a bit of a non sequitur. However, in Section 3.1, we will prove that $\mathcal{M}$ is a metric space with the distance function coming from $(\cdot, \cdot)$. If we take this for granted, then the corollary makes sense.

**Remark 2.49.** It should be noted that even if all geodesics on $\mathcal{M}$ could be extended indefinitely, it would not imply that $\mathcal{M}$ is complete. This is because the Hopf-Rinow theorem does not hold for all infinite-dimensional manifolds. It does, in fact, hold for strong Riemannian Hilbert manifolds with nonpositive curvature (see [34] §1.H and [30] Cor. IX.3.9). However, even though $\mathcal{M}$ has nonpositive curvature, it is a weak Riemannian manifold and so this theorem does not apply.

### 2.5.6. A closer look at the exponential mapping.

In this subsection, we discuss the domain and range of the exponential mapping. The goal is to give an idea of why the exponential mapping is an insufficient tool for studying the completion of $\mathcal{M}$.

First, though, let us use a childishly simple example to demonstrate how the exponential mapping can sometimes be sufficient to describe the completion of a Riemannian manifold. Though this example is too simple to be interesting, it illustrates an important philosophical point and parallels the method we’ll use in Section 5.1.

If we take an open cylinder, say $N := S^1 \times (0, 1)$, with its standard flat metric $\gamma$, then the exponential mapping $\exp_x$ at any point $x \in N$ is an isometry from some open set $U \subseteq T_x N$ onto $N$. Therefore, we can identify the completion of $N$ with $\overline{U}$, the completion of $U$ with respect to $\gamma(x)$. Of course, this means that we can view the completion of $N$ as equivalence classes of geodesics emanating from $x$. If we consider $N$ as being embedded in the closed cylinder $N' := S^1 \times [0, 1]$, then two geodesics are equivalent if and only if they have the same limit points as curves on $N'$. This situation is depicted in Figure 3.

There are two essential aspects of the above example that made it so simple to treat. First, $S^1 \times (0, 1)$ is naturally embedded into a larger space, $S^1 \times [0, 1]$ (or even, if you like, $\mathbb{R}^3$) that contains its completion. Secondly, the exponential mapping is an isometry.

In our situation, studying the completion of $\mathcal{M}$, we luck out on the first point, as $\mathcal{M}$ can be viewed as sitting inside the vector space of all sections of $S^2 T^* M$—though of course we expect that this space is much larger than necessary to accommodate the completion of $\mathcal{M}$. The second point certainly does not hold in our case—the exponential mapping cannot be an isometry due to the fact that $\mathcal{M}$ has nonvanishing curvature. But things are even worse, as it turns out that the exponential mapping of $\mathcal{M}$ is highly nonsurjective. In a sense that we will see below, it is not even locally surjective—so even the fact that the completion of a metric space is very much a local concept does not help us here.
Figure 3. The open cylinder $S^1 \times (0,1)$. The colored lines represent geodesics emanating from the point $x$, and via an appropriate geodesic we can reach any point on $S^1 \times \{0,1\}$. The red and blue lines have different limit points and so are mutually inequivalent. The green line, however, has the same limit point as the red one and so is equivalent to it.

To see this, we can write down the domain and range of the exponential mapping explicitly. (This analysis is taken from [20 §3.3 and Thm. 3.4].) For $g \in \mathcal{M}$, define the open set

$$U^g_x := S_x \setminus \left\{ \lambda g(x) \mid -\infty < \lambda \leq -\frac{4}{n} \right\} \subset T_{g(x)}\mathcal{M}_x.$$ 

Furthermore, we define an open subbundle $U^g \subset S^2 T^{*} M$ by

$$U^g = \bigcup_{x \in M} U^g_x.$$ 

(Note that $U^g$ is a fiber bundle, not a vector bundle, so it is a subbundle of $S^2 T^* M$ when viewed as a fiber bundle.) Then it is not hard to see from Theorem 2.43 that the maximal domain of definition of $\exp^g$ consists of precisely the $C^\infty$ sections of $U^g$, i.e., those elements of $S$ with image lying in $U^g$. Let us denote this by $A^g := C^\infty(U^g)$.

Again from Theorem 2.43, one can compute what the range of the exponential mapping is, i.e., what $\exp^g(U^g)$ is. It turns out that this is given by

$$A^g := \left\{ g \exp H \mid H \in C^\infty(\text{End}(M)), \, \text{tr}((H^T)^2) < \frac{1}{n}(4\pi)^2 \right\} \subset \mathcal{M},$$

where “exp” in the above definition denotes the matrix exponential and $H^T$ denotes the traceless part of $H$.

Of course, the set $A^g$ omits many points of $\mathcal{M}$. A graphical illustration of this, which gives a very good impression of just how remarkably non-surjective $\exp^g$ is, can be found in [20 Fig. 1].

One thing that goes right in this setting is the following theorem:

**Theorem 2.50.** For each $g \in \mathcal{M}$, $\exp^g$ is a real analytic diffeomorphism from $U^g$ to $A^g$. Furthermore, if $\pi_M : T\mathcal{M} \to \mathcal{M}$ is the projection from the tangent bundle of $\mathcal{M}$ onto $\mathcal{M}$, then $(\pi_M, \exp) : T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is a real analytic diffeomorphism from an open neighborhood $U$ of the zero section to an open neighborhood $A$ of the diagonal. Explicitly,

$$U = \bigcup_{g \in \mathcal{M}} U^g \quad \text{and} \quad A = \bigcup_{g \in \mathcal{M}} A^g.$$ 

These sets are maximal domains of definition for the exponential mapping and its inverse.
This is a powerful theorem, and it certainly does not hold in general for weak Riemannian manifolds (even if real analyticity is dropped). However, its usefulness to us is limited. The reason is that at any point \( g \), the neighborhood \( U^g \) does not contain any \( L^2 \)-open (i.e., \( (\cdot,\cdot)_{g^0} \)-open) set. Therefore, we run into the problem described at the end of Subsection 2.4.3—we get no information from the exponential mapping about the distance between nearby points. Therefore, we will have to revert to more direct methods of proof in the coming chapters.

### 2.6. Conventions

Before we begin with the main body of the thesis, we will describe any nonstandard conventions that will be used throughout the text.

The first thing we do is fix a reference metric, with respect to which all standard concepts will be defined.

**Convention 2.51.** For the remainder of the thesis, we fix an element \( g \in \mathcal{M} \). Whenever we refer to the \( L^p \) norm, \( L^p \) topology, \( L^p \) convergence etc., we mean that induced by \( g \) unless we explicitly state otherwise. The designation nullset refers to Lebesgue measurable subsets of \( M \) that have zero measure with respect to \( \mu_g \). If we say that something holds almost everywhere, we mean that it holds off of a \( \mu_g \)-nullset.

If we have a tensor \( h \in \mathcal{S} \), we denote by the capital letter \( H \) the tensor obtained by raising an index with \( g \), i.e., locally \( H^j_j := g^{ik} h_{kj} \). Given a point \( x \in M \) and an element \( a \in \mathcal{M}_x \), the capital letter \( A \) means the same—i.e., we assume some coordinates and write \( A = g(x)^{-1} a \), though for readability we will generally omit \( x \) from the notation.

Next, we’ll fix an atlas of coordinates on \( M \) that is convenient to work with.

**Definition 2.52.** We call a finite atlas of coordinates \( \{(U_\alpha, \phi_\alpha)\} \) for \( M \) amenable if for each \( U_\alpha \), there exist a compact set \( K_\alpha \) and a different coordinate chart \( (V_\alpha, \psi_\alpha) \) (which does not necessarily belong to \( \{(U_\alpha, \phi_\alpha)\} \)) such that
\[
U_\alpha \subset K_\alpha \subset V_\alpha \quad \text{and} \quad \phi_\alpha = \psi_\alpha|U_\alpha.
\]

**Convention 2.53.** For the remainder of this thesis, we work over a fixed amenable coordinate atlas \( \{(U_\alpha, \phi_\alpha)\} \) for all computations and concepts that require local coordinates.

The next lemma we’ll prove shows one benefit of amenable coordinates: smooth (or even continuous) metrics satisfy some kind of upper and lower bounds in these coordinates. Intuitively, the lemma says the following: in amenable coordinates, the coordinate representations of a smooth metric are somehow “uniformly positive definite”. Additionally, the coefficients satisfy a uniform upper bound.

**Lemma 2.54.** For any metric \( \tilde{g} \in \mathcal{M} \), there exist constants \( \delta(\tilde{g}) > 0 \) and \( C(\tilde{g}) < \infty \), depending only on \( \tilde{g} \), with the property that for any \( \alpha \), any \( x \in U_\alpha \), and \( 1 \leq i, j \leq n \),
\[
|\tilde{g}_{ij}(x)| \leq C(\tilde{g}) \quad \text{and} \quad \lambda_{\min}^{\tilde{g}}(x) \geq \delta(\tilde{g}),
\]
where we of course mean the value of \( \tilde{g}_{ij}(x) \) in the chart \( (U_\alpha, \phi_\alpha) \).

**Proof.** The lower bound on the minimal eigenvalue follows directly from Lemma 2.11. The upper bound on the coefficients of \( \tilde{g} \) follows from the fact that \( |\tilde{g}_{ij}| \) is a continuous function in any given coordinate chart \( (U_\alpha, \phi_\alpha) \), and we have assumed that \( U_\alpha \) is contained in a compact set \( K_\alpha \), which in turn is contained in another chart \( (V_\alpha, \psi_\alpha) \) with \( \phi_\alpha = \psi_\alpha|U_\alpha \). Therefore, \( |\tilde{g}_{ij}| \) is defined on \( K_\alpha \) and assumes some maximum there—hence, it assumes
some maximum $C_\alpha$ on $U_\alpha$. Since there are only finitely many charts $U_\alpha$, we can take $C(\tilde{g}) := \max_\alpha C_\alpha$. \hfill \Box

**Remark 2.55.** The estimate $|\tilde{g}_{ij}(x)| \leq C(\tilde{g})$ also implies an upper bound in terms of $C(\tilde{g})$ on $\det \tilde{g}(x)$. This is clear from the fact that the determinant is a homogeneous polynomial in $\tilde{g}_{ij}(x)$ with $n!$ terms and coefficients $\pm 1$.

The main point of using an amenable coordinate atlas is the following: it gives us an easily understood and uniform—but nevertheless coordinate-dependent—notion of how “large” or “small” a metric is. Namely, we look at how large the absolute values of its entries are and how small its smallest eigenvalue is. The dependence of this notion on coordinates is perhaps somewhat dissatisfying at first glance, but it should be seen as merely an aid in our quest to prove statements that are, indeed, invariant in nature.

It is necessary to introduce somewhat more general objects than Riemannian metrics in this thesis:

**Definition 2.56.** Let $\tilde{g}$ be a section of $S^2T^*M$. Then $\tilde{g}$ is called a *(Riemannian)* semimetric if it induces a positive semidefinite scalar product on $T_xM$ for each $x \in M$.

To make the above idea of uniformly largeness or positive definiteness more precise for the case of a nonsmooth (semi)metric, we define two notions. The first is again some kind of “uniform positive definiteness”, and the second is a kind of uniform upper bound.

**Definition 2.57.** Let $\tilde{g}$ be a semimetric on $M$ (which we do not assume to be even measurable). Then $\tilde{g}$ is called inflated if there exists a constant $\delta > 0$ such that

$$\det \tilde{G}(x) \geq \delta$$

for a.e. $x \in M$. Otherwise $\tilde{g}$ is called deflated.

We define the set

$$X_{\tilde{g}} := \{x \in M \mid \tilde{g}(x) \text{ is not positive definite}\} \subset M,$$

which we call the deflated set of $\tilde{g}$.

We call $\tilde{g}$ bounded if there exists a constant $C$ such that

$$|\tilde{g}_{ij}(x)| \leq C$$

for a.e. $x \in M$ and all $1 \leq i, j \leq n$. Otherwise $\tilde{g}$ is called unbounded.

Since the study of the completion of $M$ boils down to the study of Cauchy sequences in $M$, it will turn out to be useful to define notions related to the above for a sequence of elements of $M$.

**Definition 2.58.** Let $\{g_k\} \subset M$ be any sequence. We define the sets

$$X_{\{g_k\}} := \{x \in M \mid \forall \delta > 0, \exists k \in \mathbb{N} \text{ s.t. } \det G_k(x) < \delta\},$$

$$S_{\{g_k\}} := \{x \in M \mid \forall C > 0, \exists k \in \mathbb{N} \text{ and } i, j \in \{1, \ldots, n\} \text{ s.t. } |(g_k)_{ij}(x)| > C\}.$$

We call $X_{\{g_k\}}$ the deflated set and $S_{\{g_k\}}$ the unbounded set of $\{g_k\}$.

We say the sequence $\{g_k\}$ deflates at $x$ if $x \in X_{\{g_k\}}$. We say it becomes unbounded at $x$ if $x \in S_{\{g_k\}}$.

The sequence $\{g_k\}$ is called inflated if $X_{\{g_k\}}$ is a nullset, and deflated otherwise. It is called $C^0$-bounded if $S_{\{g_k\}}$ is a nullset, and $C^0$-unbounded otherwise.

The last definition we need in this vein distinguishes elements of smooth metrics from (possibly nonsmooth) semimetrics.

**Definition 2.59.** A semimetric $\tilde{g}$ is called degenerate if $\tilde{g} \notin M$, and nondegenerate if $\tilde{g} \in M$. 
Note that by Remark 2.16, any measurable semimetric $\tilde{g}$ on $M$ induces a nonnegative measure on $M$ that is absolutely continuous with respect to the fixed volume form $\mu_g$.

A measurable Riemannian metric $\tilde{g}$ on $M$ gives rise to an “$L^2$ scalar product” on measurable functions in the following way. For any two functions $\rho$ and $\sigma$ on $M$, we define

$$(2.40) \quad (\rho, \sigma)_{\tilde{g}} = \int_M \rho \sigma \, \mu_{\tilde{g}}.$$ (We denote this by the same symbol as the $L^2$ scalar product on $S$; which is meant will always be clear from the context.) We put “$L^2$ scalar product” in quotation marks because unless we put specific conditions on $\rho$, $\sigma$, and $\tilde{g}$, (2.40) is not guaranteed to be finite. It suffices, for example, to demand that $\rho$ and $\sigma$ are continuous and that the total volume $\text{Vol}(M, \tilde{g}) = \int_M \mu_{\tilde{g}}$ of $\tilde{g}$ is finite. As in the case of the $L^2$ scalar product on $S$, if $g_0$ and $g_1$ are both continuous metrics, then $(\cdot, \cdot)_{g_0}$ and $(\cdot, \cdot)_{g_1}$ are equivalent scalar products on $C^\infty(M)$. Therefore they induce the same topology, which we call the $L^2$ topology.

Now, let $\tilde{g}$ be a measurable semimetric—we want to introduce a scalar product on functions induced from $\tilde{g}$ as well. As a semimetric, $\tilde{g}$ induces a nonnegative $n$-form in the same way that a metric induces a volume form. Locally, this is given by

$$\mu_{\tilde{g}} := \sqrt{\det \tilde{g}} \, dx^1 \cdots dx^n.$$ At points $x$ where $\tilde{g}(x)$ is not positive definite, we have $\det \tilde{g}(x) = 0$ by Proposition 2.9. Therefore $\mu_{\tilde{g}}(x) = 0$ as well, so $\mu_{\tilde{g}}$ is a volume form if and only if $\tilde{g}$ is a metric. Nevertheless, since it is measurable and nonnegative, $\mu_{\tilde{g}}$ induces a Lebesgue measure on $M$, and so we can define a positive semidefinite “$L^2$ scalar product” on functions via (2.40). (It is only positive semidefinite since if a function $\rho$ has the property that $\text{supp} \rho \cap M \setminus X_{\tilde{g}}$ has measure zero, then $(\rho, \rho)_{\tilde{g}} = 0$.) Again, if we want this to be a true (finite) scalar product, we should, e.g., restrict to continuous functions and finite-volume $\tilde{g}$ (those for which $\int_M \mu_{\tilde{g}} < \infty$).

We define one more piece of notation before we close this section.

**Definition 2.60.** By $\mathcal{M}_f$, we denote the space of measurable semimetrics on $M$ with finite volume, that is, semimetrics $\tilde{g}$ for which

$$\int_M \mu_{\tilde{g}} < \infty.$$
CHAPTER 3

First metric properties of $\mathcal{M}$

In this chapter, we study the most easily accessible properties of $\mathcal{M}$ as a metric space, which will form the basis for our continuing investigations in later chapters. Our first task, to be completed in Section 3.1, is to show that $(\mathcal{M}, d)$ has the structure of a metric space. As we demonstrated in Subsection 2.4.2, this is not automatic for weak Riemannian manifolds like $(\mathcal{M}, (\cdot, \cdot))$—the induced distance function is only guaranteed to be a pseudometric. Proving that $d$ is a metric will be done by finding a manifestly positive-definite metric (in the sense of metric spaces) on $\mathcal{M}$ that in some way bounds the $d$-distance between two points from below, implying that it is positive.

With this fact proved, we can move on to studying the completion of $\mathcal{M}$, with the reassurance that the answer will be interesting. (It’s of course of little interest to study the completion of a space in which all points have zero distance from one another, as in Subsection 2.4.2.) The strategy for obtaining the completion will be to study the completions first of simple subspaces and then of successively more complex subspaces of $\mathcal{M}$, until we have enough information to describe the completion of the full space.

To begin this program, in Section 3.2, we obtain the completion of any so-called amenable subset. Recall that in Definition 2.57 we have defined two separate “good” properties of nonsmooth metrics. The first is being bounded, heuristically not becoming too large at any points. The second is inflation, heuristically not becoming too small. Lemma 2.54 shows that smooth metrics are both inflated and bounded, but the constants of Lemma 2.54 depend on the metric in question. An amenable subset is one for which these constants can be chosen uniformly across the entire subset. These subsets have the nice property that the metric $d$ is equivalent to the metric induced from the $L^2$ norm $\|\cdot\|_g$, in the sense that their Cauchy sequences are the same. This allows us to identify the completion of an amenable subset with the $L^2$ completion of that subset. This is the first step in the strategy of bootstrapping our way to a description of the completion.

3.1. $\mathcal{M}$ is a metric space

As we have already remarked in Propositions 2.45 and 2.46, the exponential mappings of $\mathcal{P}^s$ and $\mathcal{M}^s_\mu$ are at each point diffeomorphims between an open neighborhood in the tangent space and the manifold itself. Therefore, they both satisfy the hypotheses of Theorem 2.31 and we immediately get the following two results.

**Theorem 3.1.** Let $\tilde{g} \in \mathcal{M}$. Then $(\mathcal{P} \cdot \tilde{g}, (\cdot, \cdot))$ is a metric space, where $(\cdot, \cdot)$ denotes the restriction of the $L^2$ metric on $\mathcal{M}$ to $\mathcal{P} \cdot \tilde{g}$.

**Theorem 3.2.** Let $\mu$ be any smooth volume form on $\mathcal{M}$. Then $(\mathcal{M}_\mu, (\cdot, \cdot))$ is a metric space, where $(\cdot, \cdot)$ denotes the restriction of the $L^2$ metric on $\mathcal{M}$ to $\mathcal{M}_\mu$.

As we remarked at the end of Subsection 2.5.6, we cannot infer any lower bounds on the distance between two points of $\mathcal{M}$ from the exponential mapping, so we will have to directly find these bounds. To do this, we will first show Lipschitz continuity of the function mapping a metric to the square root of its volume. This simple lemma will have far-reaching implications for our study. The first use of this lemma on the volume function is to aid us in obtaining the lower bound on the $d$-distance between two points that was
described in the introduction. This is, of course, after we introduce an appropriate metric to bound \(d\).

### 3.1.1. Lipschitz continuity of the square root of the volume

As just mentioned, we wish to show Lipschitz continuity of the square root of the volume on \(\mathcal{M}\). In fact, the following lemma shows that for any measurable \(Y \subseteq M\), the function defined by

\[
\tilde{g} \mapsto \sqrt{\text{Vol}(Y, \tilde{g})}
\]

is Lipschitz with respect to \(d\). Using this as a first step to proving that \(d\) is a metric takes its inspiration from \[30\] §3.3.

**Lemma 3.3.** Let \(g_0, g_1 \in \mathcal{M}\). Then for any measurable subset \(Y \subseteq M\),

\[
\left| \sqrt{\text{Vol}(Y, g_1)} - \sqrt{\text{Vol}(Y, g_0)} \right| \leq \frac{\sqrt{n}}{4} d(g_0, g_1).
\]

**Proof.** Let \(g_t, t \in [0, 1]\), be any path from \(g_0\) to \(g_1\), and define \(h_t := g_t^T - I\). We compute

\[
\partial_t \text{Vol}(Y, g_t) = \partial_t \int_Y \mu_{g_t} = \int_Y \partial_t \mu_{g_t} = \int_Y \frac{1}{2} \text{tr}_{g_t}(h_t) \mu_{g_t}
\]

\[
\leq \left( \int_Y \mu_{g_t} \right)^{1/2} \left( \frac{1}{4} \int_Y \text{tr}_{g_t}(h_t)^2 \mu_{g_t} \right)^{1/2}
\]

\[
\leq \frac{1}{2} \sqrt{\text{Vol}(Y, g_t)} \left( \int_M \text{tr}_{g_t}(h_t)^2 \mu_{g_t} \right)^{1/2},
\]

where the first line follows from Lemma [2.38], the second line follows from Hölder’s inequality, and the last line from the nonnegativity of \(\text{tr}_{g_t}(h_t)^2\). Now, let \(A\) and \(B\) be any \(n \times n\) matrices, and denote their traceless parts by \(A^T\) and \(B^T\), respectively. We then have the formula

\[
\text{tr}(AB) = \text{tr} \left( \left( A^T + \frac{1}{n} \text{tr}(A) I \right) \left( B^T + \frac{1}{n} \text{tr}(B) I \right) \right)
\]

\[
= \text{tr} \left( A^T B^T + \frac{1}{n} \text{tr}(A) \text{tr}(B) \right).
\]

The second line follows from the fact that traceless and pure trace matrices are orthogonal in the scalar product defined by \(\text{tr}(AB)\) (cf. [2.37]—the computation is still valid if the matrices in question are not symmetric). We have also used \(\text{tr} I = n\).

Using (3.2) with the \(g_t\)-trace and \(A = B = h_t\), we see that

\[
\text{tr}_{g_t}(h_t^2) = \text{tr}_{g_t} \left( (h_t^T)^2 \right) = \frac{1}{n} \text{tr}_{g_t}(h_t)^2,
\]

implying

\[
\text{tr}_{g_t}(h_t)^2 = n \left( \text{tr}_{g_t}(h_t^2) - \text{tr}_{g_t} \left( (h_t^T)^2 \right) \right) \leq n \text{tr}_{g_t}(h_t^2),
\]

since \(\text{tr}_{g_t} \left( (h_t^T)^2 \right) \geq 0\). Applying this to (3.1) gives

\[
\partial_t \text{Vol}(Y, g_t) \leq \frac{1}{2} \sqrt{\text{Vol}(Y, g_t)} \left( n \int_M \text{tr}_{g_t}(h_t^2) \mu_{g_t} \right)^{1/2}
\]

\[
\leq \frac{\sqrt{n}}{2} \sqrt{\text{Vol}(Y, g_t)} \|h_t\|_{g_t},
\]

We next compute

\[
\sqrt{\text{Vol}(Y, g_1)} - \sqrt{\text{Vol}(Y, g_0)} = \int_0^1 \partial_t \sqrt{\text{Vol}(Y, g_t)} \, dt = \int_0^1 \frac{1}{2} \partial_t \text{Vol}(Y, g_t) \, dt
\]

\[
\leq \int_0^1 \frac{\sqrt{n}}{4} \|h_t\|_{g_t} \, dt = \frac{\sqrt{n}}{4} L(g_t),
\]
where the inequality follows from (3.3). Since this holds for all paths from \( g_0 \) to \( g_1 \), and we can repeat the computation with \( g_0 \) and \( g_1 \) interchanged, it implies the result immediately. □

We note that Lemma 3.3 gives a positive lower bound on the distance between two metrics in \( M \) that have different total volumes—so we must now deal with the case where the two metrics have the same total volume.

### 3.1.2. A (positive definite) metric on \( M \)

Our strategy for proving that \( M \) is a metric space is to find a different metric (in the sense of metric spaces) on \( M \), the positive definiteness of which is apparent and which bounds \( d \) from below in some way. We do this in several steps. The first is to define a function on \( M \times M \) and show that it is indeed a metric.

**Definition 3.4.** Consider \( M_x = \{ \tilde{g} \in S_x \mid \tilde{g} > 0 \} \) (cf. (2.25)). Define a Riemannian metric \( \langle \cdot, \cdot \rangle^0 \) on \( M_x \) given by

\[
\langle h, k \rangle^0_\tilde{g} = \text{tr}_{\tilde{g}}(hk) \det g(x)^{-1} \tilde{g} \quad \forall h, k \in T_{\tilde{g}}M_x \cong S_x.
\]

(Recall that \( g \in M \) is our fixed reference element.) We denote by \( \theta^0_\tilde{g} \) the Riemannian distance function of \( \langle \cdot, \cdot \rangle^0 \).

Note that \( \theta^0_\tilde{g} \) is automatically positive definite, since it is the distance function of a Riemannian metric on a finite-dimensional manifold. By integrating it in \( x \), we can pass from a metric on \( M_x \) to a function on \( M \times M \) as follows:

**Definition 3.5.** For any measurable \( Y \subseteq M \), define a function \( \Theta_Y : M \times M \rightarrow \mathbb{R} \) by

\[
\Theta_Y(g_0, g_1) = \int_Y \theta^0_x(g_0(x), g_1(x)) \mu_g(x).
\]

We have omitted the metric \( g \) from the notation for \( \Theta_Y \). The next lemma justifies this choice.

**Lemma 3.6.** \( \Theta_Y \) does not depend on the choice of \( g \in M \) in the above definition. That is, if we choose any other \( \tilde{g} \in M \) and define \( \langle \cdot, \cdot \rangle^0 \) and \( \theta^0_{\tilde{g}} \) with respect to this new reference metric, then

\[
\int_Y \theta^0_x(g_0(x), g_1(x)) \mu_g(x) = \int_Y \theta^0_{\tilde{g}}(g_0(x), g_1(x)) \mu_{\tilde{g}}(x)
\]

**Proof.** Let \( \tilde{g} \in M \) be any other metric. Recall that \( \theta^0_{\tilde{g}} \) was the distance function associated to the Riemannian metric \( \langle \cdot, \cdot \rangle^0 \) on \( M_x \), and the metric \( g \) enters in the definition of this Riemannian metric. Take a path \( g_t(x) \) in \( M_x \). For now, let’s put \( g \) and \( \tilde{g} \) back in the notation, so that we can write formulas unambiguously. For example, if we use \( g \) to define \( \langle \cdot, \cdot \rangle^0 \), we write \( L_g(g_t(x)) \) for the length of \( g_t(x) \) w.r.t. \( \langle \cdot, \cdot \rangle^0 \); if we use \( \tilde{g} \) in the definition, we write \( L_{\tilde{g}}(g_t(x)) \) for the length; and similarly for other notation.
Using the definitions of $\Theta_Y^0$ and $\theta_x^2$, where infima are always taken over paths $g_t(x)$ from $g_0(x)$ to $g_1(x)$, and where $h_t(x) := g_t(x)'$, we can compute:

\[
\Theta_Y^0(g_0, g_1) = \int_Y \theta_x^2(g_0(x), g_1(x)) \mu_g(x)
\]

\[
= \int_Y (\inf L(g_t(x))) \mu_g(x)
\]

\[
= \int_Y (\inf \int_0^1 \sqrt{(g_t(x)')^2_{g_t(x)}} dt) \mu_g(x)
\]

\[
= \int_Y \left( \inf \int_0^1 \frac{\det g_t(x)}{\det g(x)} \sqrt{\det g(x)} dx \right) dx^1 \cdots dx^n
\]

\[
= \Theta_Y^0(g_0, g_1),
\]

where the last line follows from running the first lines of the computation through in reverse.

\[\blacklozenge\]

Lemma 3.7. Let any $Y \subseteq M$ be given. Then $\Theta_Y$ is a pseudometric on $M$, and $\Theta_M$ is a metric (in the sense of metric spaces).

Furthermore, if $Y_1 \subset Y_2$, then $\Theta_{Y_1}(g_0, g_1) \leq \Theta_{Y_2}(g_0, g_1)$ for all $g_0, g_1 \in M$.

Proof. Nonnegativity, vanishing distance for equal elements, symmetry and the triangle inequality are clear from the corresponding properties for $\theta_x^2$.

That $\Theta_M$ is positive definite is also not hard to prove. Since $\theta_x^2$ is a metric on $M_x$, $\theta_x^2(g_0(x), g_1(x)) > 0$ whenever $g_0(x) \neq g_1(x)$. But since $g_0$ and $g_1$ are smooth metrics, if they differ at a point, they differ over an open neighborhood of that point. Hence the integral of $\theta_x^2(g_0(x), g_1(x))$ must be positive.

The second statement follows immediately from nonnegativity of $\theta_x^2$.

\[\blacklozenge\]

3.1.3. Proof of the main result. We have set up everything we need to prove the main result of this section—that $d$ is a metric. To do this, we use Lemma 3.3 in order to control the volume of the metrics making up a path in terms of the length of that path, combined with a Hölder’s inequality argument, and show that the pseudometrics $\Theta_Y$ provide a lower bound for the distance between elements of $M$ as measured by $d$.

Proposition 3.8. For any $Y \subseteq M$ and $g_0, g_1 \in M$, we have the following inequality:

\[
\Theta_Y(g_0, g_1) \leq d(g_0, g_1) \left( \sqrt{n} d(g_0, g_1) + 2\sqrt{\text{Vol}(M, g_0)} \right).
\]

In particular, $\Theta_Y$ is a continuous pseudometric (w.r.t. $d$).

Proof. By Lemma 3.7 we need only prove the inequality for $Y = M$, and then it follows for any subset.

We can clearly find a path $g_t$ from $g_0$ to $g_1$ with $L(g_t) \leq 2d(g_0, g_1)$. Then for any $\tau \in [0, 1]$, we get

\[
2d(g_0, g_1) \geq L(g_t) \geq L(g_t|_{[0, \tau]}) \geq d(g_0, g_\tau) \geq \frac{4}{\sqrt{n}} \left| \sqrt{\text{Vol}(M, g_\tau)} - \sqrt{\text{Vol}(M, g_0)} \right|
\]
where the last inequality is Lemma 3.3. In particular, we get

\[ (3.5) \quad \sqrt{\text{Vol}(M, g_x)} \leq \sqrt{\text{Vol}(M, g_0)} + \frac{\sqrt{n}}{2} d(g_0, g_1) =: V \]

for all \( \tau \in [0, 1] \).

To find the length of \( g_t \), we first integrate \( \langle g'_t, g'_t \rangle \) over \( x \in M \), then take the square root, and finally integrate over \( t \). Ideally, we would wish to change the order of integration, so that we first integrate over \( t \), then over \( x \). We cannot do this exactly, but we can bound the computation of the length from below by an expression where we integrate in the opposite order, and this expression will involve \( \theta^2 \) and \( \Theta_M \). So let’s see how this works.

Let \( h_t := g'_t \). From Hölder’s inequality,

\[ \int_M \sqrt{\text{tr}_g(h_t^2)} \, d\mu_g \leq \left( \int_M d\mu_{g_t} \right)^{1/2} \left( \int_M \text{tr}_g(h_t^2) \, d\mu_{g_t} \right)^{1/2}, \]

which gives

\[ \| h_t \|_{g_t} = \left( \int_M \text{tr}_g(h_t^2) \, d\mu_{g_t} \right)^{1/2} \geq \frac{1}{\sqrt{\text{Vol}(M, g_t)}} \int_M \sqrt{\text{tr}_g(h_t^2)} \, d\mu_{g_t} \]

\[ \geq \frac{1}{V} \int_M \sqrt{\text{tr}_g(h_t^2)} \, d\mu_{g_t}, \]

where we have also used (3.5). To remove the \( t \)-dependence from the volume element, we use

\[ \mu_{g_t} = \frac{\sqrt{\det g_t}}{\sqrt{\det g}} \mu_g = \sqrt{\det G_t} \mu_g. \]

We then rewrite (3.6) as

\[ (3.7) \quad \| h_t \|_{g_t} \geq \frac{1}{V} \int_M \sqrt{\text{tr}_g(h_t^2) \det G_t} \, d\mu_g = \frac{1}{V} \int_M \sqrt{(h_t(x), h_t(x))^0_{g_t(x)}} \, \mu_g(x), \]

where we have used the Riemannian metric \( (\cdot, \cdot)^0 \) on \( M_x \) (cf. Definition 3.4).

Since we have removed the \( t \)-dependence from the measure above, we can change the order of integration in the calculation of the length of \( g_t \):

\[ (3.8) \quad L(g_t) = \int_0^1 \| h_t \|_{g_t} \, dt \geq \frac{1}{V} \int_0^1 \int_M \sqrt{(h_t(x), h_t(x))^0_{g_t(x)}} \, \mu_g(x) \, dt \]

\[ = \frac{1}{V} \int_M \int_0^1 \sqrt{(h_t(x), h_t(x))^0_{g_t(x)}} \, dt \, \mu_g(x). \]

Now we concentrate on the \( t \)-integral in the expression above. Since \( g_t(x) \) is a path in \( M_x \) from \( g_0(x) \) to \( g_1(x) \) with tangents \( h_t(x) \), the \( t \)-integral is actually the length of \( g_t(x) \) with respect to \( (\cdot, \cdot)^0 \). But by definition, this length is bounded from below by \( \theta^2(g_0(x), g_1(x)) \). Therefore, we can rewrite (3.8) as

\[ L(g_t) \geq \frac{1}{V} \int_M \theta^2(g_0(x), g_1(x)) \, \mu_g(x) = \frac{1}{V} \Theta_M(g_0, g_1). \]

But now the result is immediate given (3.5) and the fact that we have assumed \( L(g_t) \leq 2d(g_0, g_1) \).

The previous proposition allows us to achieve our goal for this section. Since \( \Theta_M \) is a (positive-definite) metric by Lemma 3.7, \( \Theta_M(g_0, g_1) > 0 \) for any \( g_0 \neq g_1 \). From this, Proposition 3.8 immediately implies that \( d(g_0, g_1) > 0 \) as well. Since we have already mentioned that the distance function induced by a weak Riemannian manifold is automatically a pseudometric, we have proved:

**Theorem 3.9.** \((M, d)\), where \( d \) is the distance function induced from the \( L^2 \) metric \((\cdot, \cdot)\), is a metric space.
3.2. The completion of an amenable subset

Now that we know that $\mathcal{M}$ is a metric space, we begin the study of its completion in this section. According to the plan of attack laid out at the beginning of the chapter, we will work on completing more and more general subsets of $\mathcal{M}$. This section is concerned with so-called amenable subsets, defined below, consisting of metrics that are somehow uniformly bounded and inflated. The main result of the section is that the completion of such a subset with respect to $d$ coincides with the completion with respect to the $L^2$ norm on $\mathcal{S}$, the vector space in which $\mathcal{M}$ resides. Note the difference to the case of a strong Riemannian manifold, where Theorem 2.19 guarantees that the topology induced by the Riemannian metric agrees with the manifold topology. Here, the weaker topology of the tangent spaces $T_{\tilde{g}}\mathcal{M}$ with the weak Riemannian metric $(\cdot, \cdot)$ is reflected in the weaker topology induced by the Riemannian distance function $d$ on an amenable subset.

3.2.1. Amenable subsets and their properties. Let’s make the above-mentioned notion of being uniformly bounded and inflated precise. Recall that we work over an amenable atlas (cf. Definition 2.52).

**Definition 3.10.** We call a subset $U \subset \mathcal{M}$ amenable if $U$ is convex and we can find constants $C, \delta > 0$ such that for all $\tilde{g} \in U$, $x \in \mathcal{M}$ and $1 \leq i, j \leq n$,

$$\lambda_{\min}^G(x) \geq \delta$$

(where we recall that $G = g^{-1}\tilde{g}$, with $g$ our fixed metric) and

$$|\tilde{g}_{ij}(x)| \leq C.$$

**Remark 3.11.** We make a few remarks about the definition:

1. Recall from Definition 2.57 that a semimetric $\tilde{g}$ is inflated if $\det G$ is bounded away from zero. Above, we have instead used the condition $\lambda_{\min}^G \geq \delta$, but this does indeed imply that the metrics of an amenable subset are uniformly inflated. This is because $\det G \geq (\lambda_{\min}^G)^n$, the determinant being the product of the eigenvalues.

2. We could also have defined an amenable subset using the $C^0$ topology on $\mathcal{M} \subset \mathcal{S}$. Namely, let $\text{cl}(U) \subset \mathcal{S}$ be the closure of $U$ in the $C^0$ topology of $\mathcal{S}$, and let $\partial \mathcal{M}$ be the boundary of $\mathcal{M}$ in this topology. ($\partial \mathcal{M}$ consists of semimetrics that fail to be positive definite and so have determinant $0$ at at least one point.) Then $U$ is amenable if and only if $U$ is bounded in the $C^0$ norm on $\mathcal{S}$ and $\text{cl}(U) \cap \partial \mathcal{M} = \emptyset$.

3. The requirement that $U$ is convex is technical, and is there to insure that we can consider simple, straight-line paths between points of $U$ to estimate the distance between them.

4. Recall that the function sending a matrix to its minimal eigenvalue is concave by Lemma 2.10. Also, the absolute value function on $\mathbb{R}$ is convex by the triangle inequality. Therefore, the two bounds given in Definition 3.10 are compatible with the requirement of convexity.

One useful property the metrics $\tilde{g}$ of an amenable subset have is that the Radon-Nikodym derivatives $(\mu_{\tilde{g}}/\mu_g)$, with respect to the reference volume form $\mu_g$, are bounded away from zero and infinity independently of $\tilde{g}$.

**Lemma 3.12.** Let $U$ be an amenable subset. Then there exists a constant $K > 0$ such that for all $\tilde{g} \in U$,

$$\frac{1}{K} \leq \left( \frac{\mu_{\tilde{g}}}{\mu_g} \right) \leq K$$
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Proof. First, we note that
\[
\begin{pmatrix}
\mu_{\tilde{g}} \\
\mu_g
\end{pmatrix} = \det \tilde{G} \quad \text{and} \quad \begin{pmatrix}
\mu_{\tilde{g}} \\
\mu_g
\end{pmatrix}^{-1} = \begin{pmatrix}
\mu_g \\
\mu_{\tilde{g}}
\end{pmatrix} = (\det \tilde{G})^{-1}.
\]

So the bounds (3.9) are equivalent to upper bounds on both \( \det \tilde{G} \) and \( (\det \tilde{G})^{-1} \).

Now, if the eigenvalues of \( \tilde{G} \) are \( \lambda_1^{\tilde{G}}, \ldots, \lambda_n^{\tilde{G}} \), then
\[
\det \tilde{G} = \lambda_1^{\tilde{G}} \cdots \lambda_n^{\tilde{G}} \geq \left( \lambda_{\min}^{\tilde{G}} \right)^n \geq \delta^n,
\]
where \( \delta \) is the constant guaranteed by the fact that \( \tilde{g} \in \mathcal{U} \). This allows us to bound \( (\det \tilde{G})^{-1} \) from above.

To bound \( \det \tilde{G} \) from above, it is sufficient to bound the absolute value of the coefficients of \( \tilde{G} = g^{-1} \tilde{g} \) from above. But bounds on the coefficients of \( \tilde{g} \) are already assured by the fact that \( \tilde{g} \in \mathcal{U} \), and bounds on the coefficients of \( g^{-1} \) are guaranteed by the fact that \( g^{-1} \) is a fixed, smooth metric on \( M \). So we are finished. \( \square \)

Amenable subsets guarantee good behavior of the norms on \( \mathcal{S} \) that are defined by their members—namely, the norms are in some sense “uniformly equivalent”. More precisely, we have:

Lemma 3.13. Let \( \mathcal{U} \subset \mathcal{M} \) be an amenable subset. Then there exists a constant \( K \) such that for all pairs \( g_0, g_1 \in \mathcal{U} \) and all \( h \in \mathcal{S} \),
\[
\frac{1}{K} ||h||_{g_1} \leq ||h||_{g_0} \leq K ||h||_{g_1}.
\]

Proof. Instead of showing that the norms of any two metrics \( g_0, g_1 \in \mathcal{U} \) are equivalent, we will show that the norm of any \( \tilde{g} \in \mathcal{U} \) is equivalent to that of our reference metric \( g \), i.e., there exists a constant \( K \) independent of \( \tilde{g} \) such that
\[
\frac{1}{K} ||h||_{\tilde{g}} \leq ||h||_{g} \leq K ||h||_{\tilde{g}}
\]
for all \( h \in \mathcal{S} \).

This is equivalent to the following statement. Let
\[
T_{\tilde{g}} : (S^2T^*M, \langle \cdot, \cdot \rangle_{\tilde{g}}) \to (S^2T^*M, \langle \cdot, \cdot \rangle_g)
\]
be the identity mapping on the level of sets, sending the bundle \( S^2T^*M \) with the Riemannian structure \( \langle \cdot, \cdot \rangle_{\tilde{g}} \) to itself with the Riemannian structure \( \langle \cdot, \cdot \rangle_g \). Let \( N(T_{\tilde{g}})(x) \) be the operator norm of \( T_{\tilde{g}}(x) : \mathcal{S}_x \to \mathcal{S}_x \), and let \( N(T_{\tilde{g}}^{-1})(x) \) be defined similarly. Then
\[
||h||^2_{\tilde{g}} = \int_M \langle T_{\tilde{g}}(x)h(x), T_{\tilde{g}}(x)h(x) \rangle_{g(x)} \mu_g(x)
\]
\[
\leq \int_M (N(T_{\tilde{g}})(x))^2 \langle h(x), h(x) \rangle_{\tilde{g}(x)} \left( \frac{\mu_g}{\mu_{\tilde{g}}} \right) (x) \mu_{\tilde{g}}(x)
\]
and similarly,
\[
||h||^2_g \leq \int_M (N(T_{\tilde{g}}^{-1})(x))^2 \langle h(x), h(x) \rangle_{\tilde{g}(x)} \left( \frac{\mu_{\tilde{g}}}{\mu_g} \right) \mu_g(x).
\]
So (3.10) holds if and only if there are constants \( K_0 \) and \( K_1 \) such that
\[
N(T_{\tilde{g}})(x)^2, N(T_{\tilde{g}}^{-1})(x)^2 \leq K_0 \quad \text{and} \quad \langle \mu_g/\mu_{\tilde{g}}, (\mu_{\tilde{g}}/\mu_g) \rangle \leq K_1.
\]

This last statement is the one we’ll prove. The existence of the constant \( K_1 \) is guaranteed by Lemma 3.12. So we need to show the existence of the constant \( K_0 \).

To do this, first note that \( N(T_{\tilde{g}}) \) and \( N(T_{\tilde{g}}^{-1}) \) are continuous functions on \( M \) for fixed \( \tilde{g} \). This follows immediately from the fact that \( g \) and \( \tilde{g} \) are smooth. (Of course, it would
even suffice for them to be continuous.) Secondly, we notice that \( N(T_{\tilde{g}})(x) \) and \( N(T_{\tilde{g}}^{-1})(x) \) depend only on the coordinate representations of \( \tilde{g}(x) \) and \( g(x) \).

Let \( SP_n \) denote the set of all positive definite scalar products on \( \mathbb{R}^n \), which we can identify with the set of all positive definite \( n \times n \) symmetric matrices. Let's define a function

\[
\tilde{N} : SP_n \times SP_n \rightarrow \mathbb{R}
\]

by setting \( \tilde{N}(a,b) \) to be equal to the operator norm of

\[
\text{id} : (\mathbb{R}^n,a) \rightarrow (\mathbb{R}^n,b).
\]

That is, \( \tilde{N}(a,b) \) is the smallest number such that

\[
b(v,v) \leq \tilde{N}(a,b) \cdot a(v,v)
\]

for all \( v \in \mathbb{R}^n \).

It is not hard to see that \( \tilde{N} \) is continuous in both of its arguments, with the topology on \( SP_n \) coming from its identification with the space of positive definite symmetric matrices. Furthermore, by the arguments above, we have

\[
N(T_{\tilde{g}})(x) = \tilde{N}(\tilde{g}(x),g(x)) \quad \text{and} \quad N(T_{\tilde{g}}^{-1})(x) = \tilde{N}(g(x),\tilde{g}(x)),
\]

where we of course define \( \tilde{N} \) in these cases using the coordinate representations of \( g(x) \) and \( \tilde{g}(x) \) in some chart around \( x \). (The value of \( \tilde{N} \) won’t depend on the chart.) Furthermore, by the bounds satisfied by metrics in an amenable subset and the continuity of \( \tilde{N} \), the set

\[
A := \{g(x) \mid x \in M\} \cup \{\tilde{g}(x) \mid x \in M, \ \tilde{g} \in \mathcal{U}\}
\]

is relatively compact when viewed as a subset of the space of positive definite symmetric matrices. Therefore \( \tilde{N}|_{A \times A} \) is bounded. But then \( (3.11) \) immediately implies the existence of the constant \( K_0 \).

Lemma 3.12 immediately implies that the function \( \tilde{g} \mapsto \text{Vol}(M,\tilde{g}) \) is bounded when restricted to any amenable subset. Recalling the form of the estimate in Proposition 3.8 then shows the following lemma.

**Lemma 3.14.** Let \( \mathcal{U} \) be an amenable subset and \( g \in \mathcal{M} \). Then there exists a constant \( V \) such that for any \( g_0, g_1 \in \mathcal{U} \) and \( Y \subset M \),

\[
\Theta_Y(g_0, g_1) \leq 2d(g_0, g_1) \left( \frac{2\sqrt{n}}{4}d(g_0, g_1) + \sqrt{V} \right).
\]

More precisely, \( V = \sup_{\tilde{g} \in \mathcal{U}} \text{Vol}(M,\tilde{g}) \), which is finite by the discussion preceding the lemma.

### 3.2.2. The completion of \( \mathcal{U} \) with respect to \( d \) and \( \| \cdot \|_g \).

We are now ready to prove a result that, in particular, implies equivalence of the topologies defined by \( d \) and \( \| \cdot \|_g \) on an amenable subset \( \mathcal{U} \).

**Theorem 3.15.** Consider the \( L^2 \) topology on \( \mathcal{M} \) induced from the scalar product \( (\cdot,\cdot)_g \) (where \( g \) is fixed). Let \( \mathcal{U} \subset \mathcal{M} \) be any amenable subset.

Then the \( L^2 \) topology on \( \mathcal{U} \) coincides with the topology induced from the restriction of the Riemannian distance function \( d \) of \( \mathcal{M} \) to \( \mathcal{U} \).

Additionally, the following holds:

1. There exists a constant \( K \) such that

\[
d(g_0, g_1) \leq K\|g_1 - g_0\|_g,
\]

for all \( g_0, g_1 \in \mathcal{U} \).

2. For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( d(g_0, g_1) < \delta \), then \( \|g_0 - g_1\|_g < \epsilon \).
for all $g, h, k$ for which runs from $S$ is the same as the manifold topology, which in turn is given by any norm on

Finally, we use a variant of a thick-thin decomposition of $M$ work over an amenable subset in order to show that our pointwise estimates are uniform.

It follows from the definition that

Proof. First, we show there is a constant $K$ such that

$$d(g_0, g_1) \leq K\|g_1 - g_0\|_g$$

for all $g_0, g_1 \in \mathcal{U}$. Consider the path

$$g_t := g_0 + th, \quad h := g_1 - g_0, \quad t \in [0, 1],$$

which runs from $g_0$ to $g_1$. Note that we can clearly find an amenable subset $\mathcal{U}'$ containing $\mathcal{U}$ and $g$. We then have

$$(3.12) \quad L(g_t) = \int_0^1 \| (g_t)' \|_g dt = \int_0^1 \| h \|_g dt \leq \int_0^1 K\| h \|_g dt = K\| g_1 - g_0\|_g$$

where $K$ is the constant associated to $\mathcal{U}'$ guaranteed by Lemma 3.13. Since $d(g_0, g_1) \leq L(g_t)$ and the constant $K$ depends only on the set $\mathcal{U}$, this inequality is shown.

We now turn to proving statement (2). Let $\epsilon > 0$ therefore be given. Our plan is to use the Riemannian metric $\langle \cdot, \cdot \rangle^0$ and its distance function $\theta_x^0$ to get pointwise bounds on $\| g_1 - g_0\|_g$ based on $d(g_0, g_1)$. We then use the bounds guaranteed by the fact that we work over an amenable subset in order to show that our pointwise estimates are uniform.

Finally, we use a variant of a thick-thin decomposition of $M$, where $\text{tr}_g((g_1 - g_0)^2)$ is small on the “thin” part and the “thick” part has volume bounded in terms of $d(g_0, g_1)$.

Since $\mathcal{M}_x$ is a finite-dimensional Riemannian manifold, the topology induced from $\theta_x^0$ is the same as the manifold topology, which in turn is given by any norm on $S_x$. For instance this norm is given by the scalar product $\langle \cdot, \cdot \rangle_{g(x)}$ on $S_x$, which we recall is given by

$$(3.13) \quad \langle h, k \rangle_{g(x)} = \text{tr}_g((h)k)$$

for $h, k \in S_x$. That these two topologies are the same implies, in particular, that for all $\zeta > 0$ and $\hat{g} \in \mathcal{M}_x$, we can find $\kappa > 0$ such that

$$B^{\theta_x^0}_{\hat{g}}(\zeta) \subset B^{\langle \cdot, \cdot \rangle_{g(x)}}_{\hat{g}}(\kappa),$$

where

$$B^{\langle \cdot, \cdot \rangle_{g(x)}}_{\hat{g}}(\kappa) := \left\{ \hat{g} \in \mathcal{M}_x \mid \sqrt{(\hat{g} - \hat{g}, \hat{g} - \hat{g})_{g(x)}} < \kappa \right\},$$

$$B^{\theta_x^0}_{\hat{g}}(\zeta) := \left\{ \hat{g} \in \mathcal{M}_x \mid \theta_x^0(\hat{g}, \hat{g}) < \zeta \right\}.$$

Now, for $x \in M$ and $\hat{g} \in \mathcal{M}$, we define a function $\eta_{x, \hat{g}}(\zeta)$ by

$$\eta_{x, \hat{g}}(\zeta) := \inf \left\{ \kappa \in \mathbb{R} \mid B^{\theta_x^0}_{\hat{g}(x)}(\zeta) \subset B^{\langle \cdot, \cdot \rangle_{g(x)}}_{\hat{g}(x)}(\kappa) \right\}$$

$$:= \inf \left\{ \kappa \in \mathbb{R} \mid \sqrt{(\hat{g} - g(x), \hat{g} - \hat{g}(x))_{g(x)}} < \kappa \quad \forall \hat{g} \text{ with } \theta_x^0(\hat{g}, g(x)) < \zeta \right\}.$$

Then, because of the smooth dependence of $\langle \cdot, \cdot \rangle^0$ and $\langle \cdot, \cdot \rangle_{g(x)}$ on $x$, $\eta_{x, \hat{g}}(\zeta)$ is continuous separately in $x$ and $\hat{g}$. If we define

$$\mathcal{U}_x := \{ \hat{g}(x) \mid \hat{g} \in \mathcal{U} \},$$

then $\mathcal{U}_x$ is a relatively compact subset of $\mathcal{M}_x$, since $\mathcal{U}$ is amenable. Since $M$ is also compact, for any fixed $\zeta > 0$, we can define a function

$$\eta(\zeta) := \sup_{x \in M} \eta_{x, \hat{g}}(\zeta) < \infty.$$  

It follows from the definition that $\eta(\zeta) \to 0$ for $\zeta \to 0$. 


Because of the relative compactness of $U_x$ for each $x \in M$, together with compactness of $M$, there exists a constant $C_0$ such that $\theta_x^\varphi(g_0(x), g_1(x)) \leq C_0$ for all $g_0, g_1 \in U$ and $x \in M$. This implies immediately that

$$\Theta_M(g_0, g_1) = \int_M \theta_x^\varphi(g_0(x), g_1(x)) \mu_g(x) \leq C_0 \text{Vol}(M, g).$$

Now, choose $\zeta > 0$ small enough that

$$\eta(\zeta) < \epsilon / \sqrt{2 \text{Vol}(M, g)}.$$

By Lemma 3.14 there exists a constant $V$ such that

(3.14) \quad $$\Theta_M(g_0, g_1) \leq 2d(g_0, g_1) \left( \frac{2\sqrt{n}}{4}d(g_0, g_1) + \sqrt{V} \right)$$

for all $g_0, g_1 \in U$.

Choose $\delta$ small enough that

$$2\delta \left( \frac{2\sqrt{n}}{4}\delta + \sqrt{V} \right) < \frac{\epsilon^2 \zeta}{2\eta(C_0)^2}.$$

We claim that $d(g_0, g_1) < \delta$ implies that $\|g_1 - g_0\|_g < \epsilon$. Note that the choices of $\zeta$ and $C_0$ were made independently of $g_0$ and $g_1$, hence $\delta$ is independent of $g_0$ and $g_1$, as required.

We define two closed subsets of $M$ by

$$M_+ := \{x \in M \mid \theta_x^\varphi(g_0(x), g_1(x)) \geq \zeta\},$$

$$M_- := \{x \in M \mid \theta_x^\varphi(g_0(x), g_1(x)) \leq \zeta\}.$$

From (3.14) and our choice of $\delta$, we have that

(3.15) \quad $$\int_M \theta_x^\varphi(g_0(x), g_1(x)) \mu_g(x) = \Theta_M(g_0, g_1) < \frac{\epsilon^2 \zeta}{2\eta(C_0)^2}.$$

This inequality also holds if we integrate over $M_+$ instead of all of $M$, so

$$\zeta \text{Vol}(M_+, g) = \zeta \int_{M_+} \mu_g \leq \int_{M_+} \theta_x^\varphi(g_0(x), g_1(x)) \mu_g(x) < \frac{\epsilon^2 \zeta}{2\eta(C_0)^2},$$

implying

$$\text{Vol}(M_+, g_0) < \frac{\epsilon^2}{2\eta(C_0)^2}.$$

From the definitions of $M_-$ and $\eta$, we have that

$$\sqrt{\langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)}} \leq \eta(\zeta)$$

on $M_-$. From $\theta_x^\varphi(g_0(x), g_1(x)) \leq C_0$, we have that

$$\sqrt{\langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)}} \leq \eta(C_0)$$
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on all of $M$, and in particular on $M_+$. Using this, we compute
\[
\|g_1 - g_0\|^2 = \int_M \langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)} \mu_g(x)
\]
\[
= \int_{M_-} \langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)} \mu_g(x)
\]
\[
+ \int_{M_+} \langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)} \mu_g(x)
\]
\[
\leq \eta(\zeta)^2 \int_{M_-} \mu_g + \eta(C_0)^2 \int_{M_+} \mu_g
\]
\[
< \eta(\zeta)^2 \text{Vol}(M, g) + \eta(C_0)^2 \frac{\epsilon^2}{2\eta(C_0)^2}
\]
\[
< \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2.
\]

This proves the second statement. $\square$

Let’s now equip $M$ with the $H^s$ topology for some fixed $s > n/2$. From Remark 3.11(2), we immediately get continuity (but not Lipschitz continuity) of the Riemannian distance function on all of $M$, not just amenable subsets.

**Corollary 3.16.** The Riemannian distance function $d$ of $(\cdot, \cdot)$ is continuous in the $H^s$ topology on $M$ for all fixed $s > n/2$.

**Proof.** Suppose we have $\tilde{g} \in M$ and a sequence $g_n \rightarrow_{H^s} \tilde{g}$. Since $s > n/2$, the Sobolev Embedding Theorem implies that $g_n \rightarrow_{C^0} \tilde{g}$. In particular, from Remark 3.11(2) we see that the set $\{g_n \mid n \in \mathbb{N}\} \cup \{\tilde{g}\}$ is contained in some amenable subset $U \subset M$. Therefore, Theorem 3.15 gives $d(g_n, \tilde{g}) \rightarrow 0$, showing continuity. $\square$

By Definition 2.4 of a Fréchet space, Corollary 3.16 then immediately implies:

**Corollary 3.17.** The Riemannian distance function $d$ of $(\cdot, \cdot)$ is continuous in the $C^\infty$ (manifold) topology on $M$.

Theorem 3.15 will give us our first result regarding the completion of $M$. First, though, we need to make some definitions and prove a statement about metric spaces.

**Definition 3.18.** We define
\[
\mathcal{S}^0 := H^0(S^2T^*M)
\]
\[
\mathcal{M}^0 := \{ g^0 \in \mathcal{S}^0 \mid g^0(x) > 0 \text{ for almost all } x \in M \}.
\]

That is, $\mathcal{S}^0$ consists of all $H^0$ (i.e., $L^2$) symmetric $(0, 2)$-tensor fields. $\mathcal{M}^0$ consists of the elements of $\mathcal{S}^0$ that induce a positive-definite scalar product on almost every tangent space of $M$. Thus, $\mathcal{S}^0$ and $\mathcal{M}^0$ are the completions of $\mathcal{S}$ and $\mathcal{M}$, respectively, with respect to the fixed norm $\| \cdot \|_g$. (At the moment, this has nothing to do with the completion of $M$ with respect to $d$.)

If $U \subset M$ is any subset, we define
\[
U^0 := \left\{ g^0 \in \mathcal{M}^0 \mid \exists g^0_n \in U, \ n \in \mathbb{N} : g^0_n \xrightarrow{L^2} g^0 \right\},
\]
that is, $U^0$ is the $L^2$-completion of $U$.

**Remark 3.19.** A couple of remarks on the definition.
(1) Note that \( M^0 \) is not open in the \( L^2 \) topology on \( S \). In fact, even more is true: the interior of \( M^0 \) is empty with respect to the \( L^2 \) topology. (We will prove this explicitly in Lemma 3.19.) This fundamental point implies that we cannot place a manifold structure on \( M^0 \) (or \( U^0 \)), at least not one with the natural model space \( S^0 \). Therefore, in light of Theorem 3.21 below, we will not be able to give a manifold structure to the completion of an amenable subset. That \( M^0 \) is not open is also related to the fact that the exponential mapping of \( M \) is not defined on any \( L^2 \)-open subset in any tangent space.

(2) Elements of \( U^0 \) satisfy the same bounds as elements of \( U \) at almost all \( x \in M \).

Let’s look back at Theorem 3.15 again. The first statement says that for any amenable subset \( U \) and any \( g \in M \), \( d \) is Lipschitz continuous with respect to \( \|\cdot\|_g \) when viewed as a function on \( U \times U \). The second statement says that \( \|\cdot\|_g \) is uniformly continuous on \( U \times U \) with respect to \( d \). To put this knowledge to good use, we will need the following lemma:

**Lemma 3.20.** Let \( X \) be a set, and let two metrics, \( d_1 \) and \( d_2 \), be defined on \( X \). Denote by \( \phi : (X, d_1) \rightarrow (X, d_2) \) the map which is the identity on the level of sets, i.e., \( \phi \) simply maps \( x \mapsto x \). Finally, denote by \( \overline{X}^1 \) and \( \overline{X}^2 \) the completions of \( X \) with respect to \( d_1 \) and \( d_2 \), respectively.

If both \( \phi \) and \( \phi^{-1} \) are uniformly continuous, then there is a natural homeomorphism between \( \overline{X}^1 \) and \( \overline{X}^2 \).

**Proof.** Recall the definition of the completion \( \overline{X}^i \) of the metric space \( (X, d_i) \), \( i = 1, 2 \), from Section 2.1. It is formed of the equivalence classes of Cauchy sequences \( \{x_n\} \), with metric (again denoted by \( d_i \)) given by

\[
d_i(\{x_n\}, \{y_n\}) = \lim d_i(x_n, y_n).
\]

Since a uniformly continuous function maps Cauchy sequences to Cauchy sequences, our assumptions on \( \phi \) and \( \phi^{-1} \) imply that \( d_1 \) and \( d_2 \) have the same Cauchy sequences. Thus, we only need to prove that the equivalence classes of these Cauchy sequences are the same in \( \overline{X}^1 \) and \( \overline{X}^2 \), that is,

\[
(3.16) \quad \lim d_1(\{x_n\}, \{y_n\}) = 0 \iff \lim d_2(\{x_n\}, \{y_n\}) = 0.
\]

But this is immediate from the uniform continuity of \( \phi \) and \( \phi^{-1} \).

The natural homeomorphism is of course given by the unique uniformly continuous extension of \( \phi \) to \( \overline{X}^1 \) (cf. statement (3) of Theorem 2.1). \( \square \)

We are now ready to state

**Theorem 3.21.** Let \( U \) be an amenable subset. Then we can identify \( \overline{U} \), the completion of \( U \) with respect to \( d \), with \( U^0 \), in the sense of Lemma 3.20 We can make the natural homeomorphism \( \overline{U} \rightarrow U^0 \) into an isometry by placing a metric on \( U^0 \) defined by

\[
d(g_0, g_1) = \lim_{k \rightarrow \infty} d(g^0_k, g^1_k),
\]

where \( \{g^0_k\} \) and \( \{g^1_k\} \) are any sequences in \( U \) that \( L^2 \)-converge to \( g_0 \) and \( g_1 \), respectively.

**Proof.** Denote by \( \hat{d} \) the metric induced from \( \|\cdot\|_g \) on \( U \) in the usual way for Hilbert spaces, i.e., \( \hat{d}(g_1, g_2) = \|g_1 - g_2\|_g \). As in Lemma 3.20, let \( \hat{\phi} : (U, \hat{d}) \rightarrow (U, \hat{d}) \) be the identity on the level of sets. Then Theorem 3.15 clearly implies that both \( \phi \) and \( \phi^{-1} \) are uniformly continuous \( (\phi^{-1} \) is even a Lipschitz map). Thus Lemma 3.20 gives the result. \( \square \)

We have thus found a nice description of the completion of very special subsets of \( M \). As already discussed, our plan now is to start removing the nice properties that allowed us to understand amenable subsets so clearly, advancing through the completions of ever larger and more generally defined subsets of \( M \).
To do this, however, we need to clear up our viewpoint and some technicalities. The issue is the following: it happens that one can find examples of Cauchy sequences in $\mathcal{M}$ (i.e., points of the precompletion) that do not $L^2$-converge to any point of $S^0$. (The skeptical reader can jump ahead to Section 5.1 for a proof of this fact, at least for the case when $\dim M = 1$, 2 or 3.) Nevertheless, we would like to somehow be able to unambiguously identify points of $\mathcal{M}$ with sections of $S^2T^*M$. Here, “unambiguous” means that each Cauchy sequence is identified with a unique section, and all equivalent Cauchy sequences are identified with the same section. If we could do this, we would have a bijection between $\mathcal{M}$ and some subset of the sections of $S^2T^*M$. Without a uniform, unambiguous notion of the “limit point” of a Cauchy sequence in $\mathcal{M}$, such an identification is not well-defined.

Thus, we will delay further study of the completion of $\mathcal{M}$ and its subsets until we see in exactly what way we can identify Cauchy sequences with sections of $S^2T^*M$. The goal of the next chapter is to resolve this with an appropriate convergence notion for sequences in $\mathcal{M}$. Then, in Chapter 5, we determine precisely what sections of $S^2T^*M$ actually do represent Cauchy sequences in $\mathcal{M}$, thus describing the bijection mentioned above.
CHAPTER 4

Cauchy sequences and $\omega$-convergence

In this chapter, we introduce and study a fundamental notion of convergence of our own invention for $d$-Cauchy sequences in $M$. We call this $\omega$-convergence, and its importance is made clear through two theorems we will prove, an existence and a uniqueness result. The existence result, proved in Section 4.1, says that every $d$-Cauchy sequence has a subsequence that $\omega$-converges to a measurable semimetric, which we will then show has finite total volume. The uniqueness result, proved in Section 4.3, is that two $\omega$-convergent Cauchy sequences in $M$ are equivalent (in the sense of (2.1)) if and only if they have the same $\omega$-limit. These results allow us to identify an equivalence class of $d$-Cauchy sequences with the unique $\omega$-limit that its representatives subconverge to, and thus give a meaning to points of $M$.

We might hope that our convergence notion for Cauchy sequences could at least imply pointwise convergence of the metrics of the sequence to some limit tensor field. However, we will have to back off of this hope somewhat, as it will turn out that one cannot demand that a $d$-Cauchy sequence converge in any pointwise sense at points where the metrics in the sequence deflate (cf. Definition 2.58). This is a consequence of the somewhat surprising result that one can bound $d(g_0, g_1)$, for any $g_0, g_1 \in M$, based only on the “intrinsic volumes” of the set on which $g_0$ and $g_1$ differ. Intrinsic means here that this volume is measured with respect to $g_0$ and $g_1$. In particular, the bound does not depend on how much $g_0$ and $g_1$ differ as tensors, say in a fixed coordinate system. Hence two sequences of metrics can be $d$-close and yet have very different pointwise limits (or no pointwise limits at all), provided the only differ on small-volume subsets. This will be made more precise in Section 4.1, where we define $\omega$-convergence.

We can demand that $\omega$-convergence imply pointwise convergence off of the deflated set. We can then use this to show that the volume forms $\mu_{g_k}$ of a Cauchy sequence $\{g_k\}$ converge pointwise almost everywhere. This will allow us to prove, in Section 4.2, that the volume of a subset of $M$ is continuous with respect to the topology of $\omega$-convergence.

4.1. Existence of the $\omega$-limit

We begin this section with an important estimate and some examples, followed by the definition of $\omega$-convergence and some of its basic properties. After that, we start on the existence proof by showing a pointwise version, i.e., an analogous result on $M_x$. Finally, we globalize this pointwise result to show the existence of an $\omega$-convergent subsequence for any Cauchy sequence in $M$.

4.1.1. Volume-based estimates on $d$ and examples. We have mentioned that $\omega$-convergence implies pointwise convergence only off the deflated set of a sequence of metrics. We also stated that this is forced upon us by a bound on the distance between two metrics that is based on the volume of the set on which they differ. So before we give the definition of $\omega$-convergence, let’s show this result. The proof is a bit technical, but the idea is very simple and is described at the beginning of the proof.

Proposition 4.1. Suppose that $g_0, g_1 \in M$, and let $E := \text{carr}(g_1 - g_0) = \{x \in M \mid g_0(x) \neq g_1(x)\}$. Then there exists a constant $C(n)$ depending only on $n = \dim M$ such
that

$$d(g_0, g_1) \leq C(n) \left( \sqrt{\text{Vol}(E, g_0)} + \sqrt{\text{Vol}(E, g_1)} \right).$$

In particular, we have

$$\text{diam} \left( \{ \tilde{g} \in M \mid \text{Vol}(M, \tilde{g}) \leq \delta \} \right) \leq 2C(n)\sqrt{\delta}.$$ 

**Proof.** The second statement follows immediately from the first, so we only prove the first.

The heuristic idea is the following. We want to construct a family of paths with three pieces, depending on a real parameter $s$, such that the metrics do not change on $M \setminus E$ as we travel along the paths. Therefore, we pretend that we can restrict all calculations to $E$. On $E$, the first piece of the path is the straight line from $g_0$ to $sg_0$ for some small positive number $s$. It is easy to compute a bound for the length of this path based on $\text{Vol}(E, g_0)$. The second piece is the straight line from $sg_0$ to $sg_1$, which, as we will see, has length approaching zero for $s \to 0$. The last piece is the straight line from $sg_1$ to $g_1$, which again has length bounded from above by an expression involving $\text{Vol}(E, g_1)$. This idea is illustrated in Figure 1.

Our job is to now take this heuristic picture, which uses paths of $L^2$ metrics, and construct a family of paths of smooth metrics that captures the essential properties.

For each $k \in \mathbb{N}$ and $s \in (0, 1]$, we define three families of metrics as follows. Choose closed sets $F_k \subseteq E$ and open sets $U_k$ containing $E$ such that $\text{Vol}(U_k, g_i) - \text{Vol}(F_k, g_i) \leq 1/k$ for $i = 0, 1$. (This is possible because the Lebesgue measure is regular.) Let $f_{k,s} \in C^\infty(M)$ be functions with the following properties:

1. $f_{k,s}(x) = s$ if $x \in F_k$,
2. $f_{k,s}(x) = 1$ if $x \notin U_k$ and
(3) \( s \leq f_{k,s}(x) \leq 1 \) for all \( x \in M \).

Now, for \( t \in [0, 1] \), define

\[
\hat{g}_{t}^{k,s} := ((1 - t) + tf_{k,s})g_{0} \\
g_{t}^{k,s} := f_{k,s}((1 - t)g_{0} + tg_{1}) \\
\hat{g}_{t}^{k,s} := ((1 - t) + tf_{k,s})g_{1}.
\]

We view these as paths in \( t \) depending on the family parameter \( s \). Furthermore, we define a concatenated path

\[
g_{t}^{k,s} := \hat{g}_{t}^{k,s} * g_{t}^{k,s} * (\hat{g}_{t}^{k,s})^{-1},
\]

where of course the inverse means we run through the path backwards. It is easy to see that \( g_{0}^{k,s} = g_{0} \) and \( g_{1}^{k,s} = g_{1} \) for all \( s \). Also note that each path making up \( g_{t}^{k,s} \) is just a straight-line path. The first is from \( g_{0} \) to \( f_{k,s}g_{0} \), the second is from \( f_{k,s}g_{0} \) to \( f_{k,s}g_{1} \), and the third is from \( f_{k,s}g_{1} \) to \( g_{1} \).

We now investigate the lengths of each piece of \( \hat{g}_{t}^{k,s} \). Recalling that by Convention 2.51 \( G_{0} = g^{-1}g_{0} \), we compute

\[
L(\hat{g}_{t}^{k,s}) = \int_{0}^{1} \| (\hat{g}_{t}^{k,s})' \|_{\hat{g}_{t}^{k,s}} \, dt
= \int_{0}^{1} \left( \int_{M} \text{tr}((1 - t) + tf_{k,s})g_{0} \left( ((f_{k,s} - 1)g_{0}) \right) \sqrt{\det(((1 - t) + tf_{k,s})G_{0})} \mu_{g} \right)^{1/2} \, dt
= \int_{0}^{1} \left( \int_{U_{k}} ((1 - t) + tf_{k,s})^{\frac{n}{2} - 2} \text{tr}g_{0} \left( ((1 - f_{k,s})g_{0}) \right) \sqrt{\det(G_{0})} \mu_{g} \right)^{1/2} \, dt.
\]

since \( \det(\lambda A) = \lambda^{n/2} \det A \) for any \( n \times n \)-matrix \( A \) and \( \lambda \in \mathbb{R} \). Note that in the last line, we only integrate over \( U_{k} \), which is justified by the fact that \( 1 - f_{k,s} = 0 \) on \( M \setminus U_{k} \). Since \( s > 0 \), it is easy to see that

\[
(1 - f_{k,s})^{2} \leq (1 - s)^{2} < 1,
\]

so that

\[
\text{tr}g_{0} \left( ((1 - f_{k,s})g_{0}) \right) = n(1 - f_{k,s})^{2} < n.
\]

This gives us the estimate

\[
L(\hat{g}_{t}^{k,s}) < \int_{0}^{1} \left( n \int_{U_{k}} ((1 - t) + tf_{k,s})^{\frac{n}{2} - 2} \mu_{g_{0}} \right)^{1/2} \, dt.
\]

Now, to estimate this, we note that for \( n \geq 4 \), \( \frac{n}{2} - 2 \geq 0 \) and therefore \( f_{k,s} \leq 1 \) implies that

\[
((1 - t) + tf_{k,s})^{\frac{n}{2} - 2} \leq 1.
\]

So in this case,

\[
(4.1) \quad L(\hat{g}_{t}^{k,s}) < \sqrt{n \text{Vol}(U_{k}, g_{0})}.
\]

For \( 1 \leq n < 3 \), \( \frac{n}{2} - 2 < 0 \) and therefore one can compute that \( f_{k,s} \geq s > 0 \) implies

\[
((1 - t) + tf_{k,s})^{\frac{n}{2} - 2} \leq (1 - t)^{\frac{n}{2} - 2}.
\]

In this case, then,

\[
(4.2) \quad L(\hat{g}_{t}^{k,s}) < \sqrt{n \text{Vol}(U_{k}, g_{0})} \int_{0}^{1} (1 - t)^{\frac{n}{2} - 1} \, dt,
\]

and the integral term is finite since \( \frac{n}{2} - 1 > -1 \). Furthermore, the value of this integral depends only on \( n \). Putting together (4.1) and (4.2) therefore gives

\[
(4.3) \quad L(\hat{g}_{t}^{k,s}) \leq C(n) \sqrt{\text{Vol}(U_{k}, g_{0})},
\]

where \( C(n) \) is a constant depending only on \( n \).
where \( C(n) \) is a constant depending only on \( n \).

In exact analogy, we can show that
\[
(4.4) \quad L(g_{t,k,s}^{k,s}) \leq C(n)\sqrt{\text{Vol}(U_k, g_1)},
\]
where we can even use the same constant \( C(n) \).

Next, we look at the second piece of \( g_{t,k,s}^{k,s} \). Here we have, using that \( g_1 - g_0 = 0 \) on \( M \setminus E \),
\[
\| (g_{t,k,s}^{k,s})' \|_{g_{t,k,s}^{k,s}}^2 = \int_M \text{tr}_{f_{k,s}}((1-t)g_{0}+tg_{1}) \left( (f_{k,s}(g_1 - g_0))^2 \right) \sqrt{\det((1-t)G_0 + tG_1)} \mu_g
\]
\[
= \int_{E \setminus F_k} \text{tr}_{(1-t)g_{0}+tg_{1}} \left( (g_1 - g_0)^2 \right) \sqrt{\det((1-t)G_0 + tG_1)} \mu_g.
\]

Note that in the last line above, we are only integrating over \( E \), and the factors of \( f_{k,s} \) in the trace term have canceled each other out. Also note that \( f_{k,s}(x) = s \) if \( x \in F_k \), and \( f_{k,s}(x) \leq 1 \) for all \( x \in M \). So it follows from the above that
\[
\| (g_{t,k,s}^{k,s})' \|_{g_{t,k,s}^{k,s}}^2 \leq s^{n/2} \int_{F_k} \text{tr}_{(1-t)g_{0}+tg_{1}} \left( (g_1 - g_0)^2 \right) \sqrt{\det((1-t)G_0 + tG_1)} \mu_g
\]
\[
+ \int_{E \setminus F_k} \text{tr}_{(1-t)g_{0}+tg_{1}} \left( (g_1 - g_0)^2 \right) \sqrt{\det((1-t)G_0 + tG_1)} \mu_g.
\]

For each fixed \( t \) and \( k \), the first term in the above clearly goes to zero as \( s \to 0 \). By our assumption on the sets \( F_k \), the second term goes to zero as \( k \to \infty \) for each fixed \( t \) (it does not depend on \( s \) at all). But since \( t \) only ranges over the compact interval \([0, 1]\) and all terms in the integrals depend smoothly on \( t \), both of these convergences are uniform in \( t \). From this, it is easy to see that
\[
(4.5) \quad \lim_{k \to \infty} \lim_{s \to 0} L(g_{t,k,s}^{k,s}) = 0.
\]

With all of this preparation, we can finally use \((4.3), (4.4)\) and \((4.5)\) to estimate
\[
d(g_0, g_1) \leq \inf_{k,s} L(g_{t,k,s}^{k,s}) \leq \lim_{k \to \infty} \lim_{s \to 0} L(g_{t,k,s}^{k,s}) \leq C(n) \left( \sqrt{\text{Vol}(E, g_0)} + \sqrt{\text{Vol}(E, g_1)} \right)
\]
by our assumptions on the sets \( U_k \). \(\square\)

Before we move on with general considerations, we give two simple examples that illustrate some important principles here. The first principle is, as we mentioned, that metrics that differ on a small-volume subset of \( M \) are close together, no matter how their coefficients differ individually. Thus, as the first example shows, a Cauchy sequence need not converge on a set with volume zero in the limit. The second example demonstrates that very different paths or sequences can be equivalent in \( \mathcal{M}^{\text{pre}} \), even if they become unbounded. It also hints at a principle that we’ll elaborate on in Subsection \((4.1.4)\) namely that we can essentially ignore that a Cauchy sequence in \( M \) becomes unbounded, taking a sequence or path of metrics that become unbounded at some points and replacing it with a sequence or path that remains bounded.

**Example 4.2 (A \( d \)-Cauchy sequence that does not converge pointwise).** Let our base manifold \( M \) now be the torus \( T^2 \). In the standard chart on the torus \(([0,1] \times [0,1] \text{ with edges identified})\), we define a sequence of metrics by
\[
g_k := \begin{pmatrix} \cos k & 0 \\ 0 & k^{-1} \end{pmatrix}.
\]

(These are, indeed, positive definite matrices, since \( \cos k \neq 0 \) for all \( k \in \mathbb{N} \).) On the one hand, this sequence does not converge pointwise, thanks to the oscillating \( \cos k \) coefficient.
On the other hand, since clearly
\[
\lim_{k \to \infty} \Vol(T^2, g_k) = \lim_{k \to \infty} \sqrt{\frac{\cos k}{k}} = 0,
\]
Proposition 4.1 allows us to see that \( \{g_k\} \) is indeed a Cauchy sequence.

Note that since \( |\cos k| \) is bounded, we can select a subsequence \( \{g_{k_n}\} \), equivalent to the original sequence, that does converge. This works for the example here, but as the next example shows, there are Cauchy sequences and finite paths with no convergent subsequence.

**Example 4.3 (Very different, but equivalent, finite paths, and an example of unboundedness).** We again let \( M = T^2 \), and we define a family of metrics by
\[
g^{r,s}_t := \begin{pmatrix} e^{rt} & 0 \\ 0 & e^{-st} \end{pmatrix}
\]
for \( t \in [1, \infty) \) and \( r, s > 0 \). We consider this to be a path depending on \( t \) for each fixed choice of \( r \) and \( s \). Each \( g^{r,s}_t \) has pointwise limit, as \( t \to \infty \), the “tensor”
\[
g_\infty = \begin{pmatrix} \infty & 0 \\ 0 & 0 \end{pmatrix}.
\]
Thus \( g^{r,s}_t \) becomes unbounded over the entire base manifold.

If we let \( h^{r,s}_t = (g^{r,s}_t)' \), then it is not hard to directly compute that
\[
\|h^{r,s}_t\|_{g^{r,s}_t} = \sqrt{r^2 + s^2 e^{(s-r)t}/4},
\]
which is integrable on \([1, \infty)\) if and only if \( s > r \), and therefore \( g^{r,s}_t \) is finite if \( s > r \). On the other hand, we also have that
\[
\lim_{t \to \infty} \Vol(T^2, g^{r,s}_t) = 0
\]
if and only if \( s > r \). (If \( s = r \), the volume is constant, and if \( s < r \), the volume diverges.) Thus, by Proposition 4.1 (or a direct computation, if one is so inclined), we have
\[
\lim_{t \to \infty} d(g^{r,s}_t, g^{a,b}_t) = 0
\]
whenever \( r > s > 0 \) and \( a > b > 0 \). In other words, though the coefficients of \( g^{r,s}_t \) and \( g^{a,b}_t \) can differ greatly, these finite paths are equivalent because the volume of the set on which they differ vanishes in the limit. In fact, any two paths \( g^1_t \) and \( g^2_t \) with
\[
\Vol(M, g^1_t) \to 0 \quad \text{and} \quad \Vol(M, g^2_t) \to 0
\]
are equivalent. Therefore, we can pick a representative from the equivalence class \( \{g^{r,s}_t\} \in \mathcal{M} \) that does not become unbounded, but rather converges to a true tensor (with coefficients assuming values in \( \mathbb{R} \)). A canonical choice might be a finite path with pointwise limit the zero section of \( S^2 T^* M \).

**4.1.2. \( \omega \)-convergence and its basic properties.** So we now clearly see that we have to back off from the demand that Cauchy sequences converge pointwise on their deflated sets. Nevertheless, we can expect other nice behavior of Cauchy sequences, and what we do expect is given in the next definition. The definition itself looks a bit technical, but is actually rather simple. Therefore, after stating it in full, we will explain each of its parts in more detail.

First, though, recall that we define general measure-theoretic notions (e.g., the notion of something holding almost everywhere, or a.e.) using the fixed reference metric \( g \) (cf. Convention 2.51). Furthermore, we need one definition before that of \( \omega \)-convergence.
DEFINITION 4.4. We denote by \( \mathcal{M}_m \) the set of all measurable semimetrics on \( M \). That is, \( \mathcal{M}_m \) is the set of all sections of \( \mathcal{S}^2T^*M \) that have measurable coefficients and that induce a positive semidefinite scalar product on \( T_xM \) for each \( x \in M \).

Define an equivalence relation \(~\) on \( \mathcal{M}_m \) by \( g_0 \sim g_1 \) if and only if

1. their deflated sets \( X_{g_0} \) and \( X_{g_1} \) differ at most by a nullset, and
2. \( g_0(x) = g_1(x) \) for a.e. \( x \in M \setminus (X_{g_0} \cup X_{g_1}) \).

We denote the quotient space of \( \mathcal{M}_m \) by \( \mathcal{M}_m/\sim \).

DEFINITION 4.5. Let \( \{g_k\} \) be a sequence in \( \mathcal{M} \), and let \( [g_{\infty}] \in \mathcal{M}_m/\sim \). Recall that we denote the deflated set of the sequence \( \{g_k\} \) by \( X_{\{g_k\}} \) and the deflated set of an individual semimetric \( \tilde{g} \) by \( X_{\tilde{g}} \) (cf. Definitions 2.57 and 2.58). We say that \( g_k \omega\)-converges to \( [g_{\infty}] \) if for every representative \( g_{\infty} \in [g_{\infty}] \), the following holds:

1. \( \{g_k\} \) is \( \det \)-Cauchy,
2. \( X_{g_{\infty}} \) and \( X_{\{g_k\}} \) differ at most by a nullset,
3. \( g_k(x) \to g_{\infty}(x) \) for a.e. \( x \in M \setminus X_{\{g_k\}} \), and
4. \( \sum_{k=1}^{\infty} \det(g_k, g_{k+1}) < \infty. \)

We call \( [g_{\infty}] \) the \( \omega \)-limit of the sequence \( \{g_k\} \) and write \( g_k \overset{\omega}{\longrightarrow} [g_{\infty}] \).

More generally, if \( \{g_k\} \) is a \( \det \)-Cauchy sequence containing a subsequence that \( \omega \)-converges to \( [g_{\infty}] \), then we say that \( \{g_k\} \omega \)-subconverges to \( [g_{\infty}] \).

So let’s go through the definition one part at a time.

Condition (1) is simply there for convenience, so we don’t have to repeatedly assume that a sequence is \( \omega \)-convergent and \( \det \)-Cauchy.

Condition (2) says that the limit metric is deflated at a point \( x \in M \) if and only if \( x \) is a point where \( \{g_k\} \) deflates (up to a nullset where this fails to hold).

Condition (3) says that \( \{g_k\} \) has a pointwise limit at almost every point off the deflated set. Note that this limit will necessarily be positive definite, since if \( x \in M \setminus X_{\{g_k\}} \), then there exists some \( \delta(x) > 0 \) such that

\[
\det g_k(x) \geq \delta(x)
\]

for all \( k \in \mathbb{N} \) and in every chart from the amenable atlas that contains \( x \).

Finally, condition (4) is technical and will aid us in proofs. Conceptually, it means that we can find paths \( \alpha_k \) connecting \( g_k \) to \( g_{k+1} \) such that the concatenated path \( \alpha_1 \ast \alpha_2 \ast \cdots \) has finite length (cf. the proof of Theorem 2.2). Given condition (1), we can always achieve this by passing to a subsequence. (We remark here, however, that these two conditions are not independent. In fact, (1) implies (1).)

Now that we have this definition out of the way, let’s move on to proving some properties of it. We first state an entirely trivial consequence of Definitions 4.4 and 4.5.

**Lemma 4.6.** Let \( [g_{\infty}] \in \mathcal{M} \), and let \( \{g_k\} \) be a sequence in \( \mathcal{M} \). Suppose that for one given representative \( g_{\infty} \in [g_{\infty}] \), \( \{g_k\} \) together with \( g_{\infty} \) satisfies conditions (1)-(4) of Definition 4.5. Then these conditions are also satisfied for \( \{g_k\} \) together with every other representative of \( [g_{\infty}] \).

Therefore, if we can verify these conditions for one representative of an equivalence class, this already implies \( \{g_k\} \overset{\omega}{\longrightarrow} [g_{\infty}] \).

We can thus consistently say that \( \{g_k\} \omega \)-converges to an individual semimetric \( g_{\infty} \in \mathcal{M}_m \) if the two together satisfy conditions (1)-(4) of Definition 4.5. By the lemma, this is completely synonymous with saying that \( \{g_k\} \omega \)-converges to the equivalence class \( [g_{\infty}] \in \mathcal{M}_m \). It is of course easier to show that \( \{g_k\} \omega \)-converges to one semimetric,
rather than a whole equivalence class. In the following we will generally simply prove \( \omega \)-convergence to the canonical choice of representative, the semimetric \( g_\infty \in [g_\infty] \) for which \( g_\infty(x) = 0 \) for all \( x \in X_{g_\infty} \).

The next property of \( \omega \)-convergence is also obvious.

**Lemma 4.7.** If \( \{g^0_k\} \) and \( \{g^1_k\} \) both \( \omega \)-converge to the same element \([g_\infty] \in M_m \), then \( \{g^0_k\} \) and \( \{g^1_k\} \) have the same deflated set, up to a nullset.

**Proof.** This is immediate from property (2) of Definition 4.3 as \( g_\infty(x) = 0 \) if and only if \( x \) is in the deflated set of both \( \{g^0_k\} \) and \( \{g^1_k\} \).

Recall that the main goal of this section is to show that each Cauchy sequence in \( M \) has an \( \omega \)-convergent subsequence. To do this, we will first prove a pointwise result in the following subsection.

### 4.1.3. (Riemannian) metrics on \( M_x \) revisited

In this subsection, we take a closer look at the Riemannian metrics \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle^0 \) (see Lemma 2.35 and Definition 3.4 respectively) that we have defined on the finite-dimensional manifold \( M_x \). Since the distance function \( \theta_x^2 \) is induced from \( \langle \cdot, \cdot \rangle \), and the metric \( \Theta_M \) on \( M \) is defined via \( \theta_x^2 \), we can get information on \( \Theta_M \) by studying \( \langle \cdot, \cdot \rangle^0 \). And information on \( \Theta_M \) yields information on \( d \) via the estimate of Lemma 3.8. Furthermore, by recalling the definitions of the two Riemannian metrics, we can see that \( \langle \cdot, \cdot \rangle^0 \) is intimately related to \( \langle \cdot, \cdot \rangle \):

\[
\langle h, k \rangle^0_g = \langle h, k \rangle_g \det \tilde{G} \quad \text{for all } \tilde{g} \in M_x \text{ and } h, k \in T_{\tilde{g}}M_x \cong S_x.
\]

Thus, we will first study the simpler Riemannian metric \( \langle \cdot, \cdot \rangle \) and find out what properties of \( \langle \cdot, \cdot \rangle^0 \) we can deduce in this way.

We first make the observation that \( \langle \cdot, \cdot \rangle \) differs very much in character from its integrated version \( \langle \cdot, \cdot \rangle \). In particular, \( M_x \) is complete with respect to \( \langle \cdot, \cdot \rangle \)!

This is not hard to see, as we can solve the geodesic equation of \( \langle \cdot, \cdot \rangle \) directly. Following the analogous computation for \( \langle \cdot, \cdot \rangle \) on \( M \) carried out in [19, Thm. 2.3], we first calculate the Christoffel symbols of \( \langle \cdot, \cdot \rangle \) and then use them to solve the geodesic equation. We note that our computation is basically just a simplified version of that in [19].

Before we start, let’s clear up some notation.

**Definition 4.8.** By \( d_x \), we denote the distance function induced on \( M_x \) by \( \langle \cdot, \cdot \rangle \). We denote the \( \langle \cdot, \cdot \rangle \)-length of a path \( a_t \) in \( M_x \) by \( L^{\langle \cdot, \cdot \rangle}(a_t) \) and the \( \langle \cdot, \cdot \rangle^0 \)-length by \( L^{\langle \cdot, \cdot \rangle^0}(a_t) \).

Now we compute the Christoffel symbols.

**Proposition 4.9.** Let \( h \) and \( k \) be constant vector fields on \( M_x \), and denote the Levi-Civita connection of \( \langle \cdot, \cdot \rangle \) by \( \nabla \). Then the Christoffel symbols of \( \langle \cdot, \cdot \rangle \) are given by

\[
\Gamma(h, k) = \nabla_h k|\tilde{g} = -\frac{1}{2} \left( h\tilde{g}^{-1}k + k\tilde{g}^{-1}h \right).
\]

**Proof.** All computations are done at the base point \( \tilde{g} \), which we will omit from the notation for convenience. Let \( \ell \) be any other constant vector field on \( M_x \). By the Koszul formula,

\[
2\langle \nabla_h k, \ell \rangle = h\langle k, \ell \rangle + k\langle \ell, h \rangle - \ell \langle h, k \rangle - \langle h, [k, \ell] \rangle - \langle k, [\ell, h] \rangle + \langle \ell, [h, k] \rangle.
\]

Notice, however, that the last three terms drop out, since \( h, k, \) and \( \ell \) are all constant, so their brackets with each other are zero. Therefore we have

\[
\Gamma(h, k) = \nabla_h k|\tilde{g} = -\frac{1}{2} \left( h\tilde{g}^{-1}k + k\tilde{g}^{-1}h \right).
\]

Now, it is well-known (and easy to verify by differentiating \( \tilde{g}^{-1} = 1 \)) that the derivative of the map \( \tilde{g} \mapsto \tilde{g}^{-1} \) at the point \( \tilde{g} \) is given by \( a \mapsto -\tilde{g}^{-1}a\tilde{g}^{-1} \). Using this, along
with the definition of $\langle \cdot, \cdot \rangle$ and the fact that $a, b \mapsto \text{tr}(ab)$ is bilinear, we get (denoting the derivative of a function $f$ in the direction $h$ by $h[f]$)

$$h(k, \ell) = h[\text{tr}_g(k\ell)] = h[\text{tr}((\tilde{g}^{-1}k)(\tilde{g}^{-1}\ell))]$$

$$\quad = -\text{tr}((\tilde{g}^{-1}h\tilde{g}^{-1}k)(\tilde{g}^{-1}\ell)) - \text{tr}((\tilde{g}^{-1}k)(\tilde{g}^{-1}h\tilde{g}^{-1}\ell)).$$

Repeating the same computation for the other permutations and substituting the results into (4.7) yields

$$2(\nabla_h k, \ell) = -\text{tr}((\tilde{g}^{-1}h\tilde{g}^{-1}k)(\tilde{g}^{-1}\ell)) - \text{tr}((\tilde{g}^{-1}k)(\tilde{g}^{-1}h\tilde{g}^{-1}\ell))$$

$$\quad \quad - \text{tr}((\tilde{g}^{-1}k\tilde{g}^{-1}\ell)(\tilde{g}^{-1}h))$$

$$\quad \quad + \text{tr}((\tilde{g}^{-1}\ell\tilde{g}^{-1}h)(\tilde{g}^{-1}k)) + \text{tr}((\tilde{g}^{-1}h)(\tilde{g}^{-1}k))$$

$$\quad = -\text{tr}(\tilde{g}^{-1}h\tilde{g}^{-1}k\ell) - \text{tr}(\tilde{g}^{-1}k\tilde{g}^{-1}h\ell)$$

$$\quad = -\langle h\tilde{g}^{-1}k, \ell \rangle - \langle k\tilde{g}^{-1}h, \ell \rangle,$$

where in the second-to-last line we have used the invariance of the trace under cyclic permutations. The result now follows directly. \hfill \Box

Using this, it is a relatively simple matter to solve the geodesic equation of $\langle \cdot, \cdot \rangle$.

**Proposition 4.10.** The geodesic $g_t$ in $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ with initial data $g_0$, $g'_0$ is given by

$$g_t = g_0 e^{t g_0^{-1} g'_0}.$$  

In particular, $(\mathcal{M}_x, d_x)$ is a complete metric space.

**Remark 4.11.** Note that this formula is exactly the same as the geodesic equation for $(\mathcal{M}_\mu, \langle \cdot, \cdot \rangle)$ (cf. Proposition 2.46). This is no accident, and occurs because the volume form is constant over $\mathcal{M}_\mu$. Therefore there is no contribution to the Christoffel symbols coming from the volume form. As a result, all of the dynamics come from the integrand of $\langle \cdot, \cdot \rangle$, and the integrand is exactly $\langle \cdot, \cdot \rangle$.

Nevertheless, since we haven’t derived the geodesic equation for $\mathcal{M}_\mu$, we prefer to prove Proposition 4.10 directly, without resorting to indirect arguments. The proof is not hard, anyway.

**Proof of the Proposition.** Let $a_t := g_t^{-1} a_t g_t$. Since $g_t$ is a geodesic, we have $\nabla_{a_t} a_t = 0$. Therefore

$$0 = a_t' + \Gamma(a_t, a_t) = a_t' - a_t g_t^{-1} a_t$$

by Proposition 4.9. Now, since $g_t' = a_t$, the $t$-derivative of $t \mapsto g_t^{-1}$ is the same as the derivative of $g_t \mapsto g_t^{-1}$ in the direction of $a_t$. Hence, $(g_t^{-1} a_t)' = g_t^{-1} a_t' - g_t^{-1} a_t g_t^{-1} a_t$. It is then easy to see that multiplying (4.8) on the left by $g_t^{-1}$ gives

$$(g_t^{-1} a_t)' = 0.$$

Thus $g_t^{-1} a_t'$ is constant, or $\log(g_t)' = g_t^{-1} a_t' \equiv g_0^{-1} g'_0$. The geodesic equation now follows, and it remains to show that $(\mathcal{M}_x, d_x)$ is complete.

Since $A \mapsto e^A$ maps symmetric matrices into positive definite matrices and $t g_0^{-1} g'_0$ is $g_0$-symmetric, $g_0 e^{t g_0^{-1} g'_0}$ is a positive definite matrix for all $t \in (-\infty, \infty)$. Thus $g_t$ is positive definite for all $t$, and so $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ is geodesically complete. Since $\mathcal{M}_x$ is finite-dimensional, the Hopf-Rinow theorem applies to show that $(\mathcal{M}_x, d_x)$ is complete. \hfill \Box
4.1. Existence of the \( \omega \)-Limit

First concrete step towards proving existence of the \( \omega \)-limit—it is necessary pointwise result.

**Lemma 4.12.** Let \( a_0, a_1 \in \mathcal{M}_x \). Then

\[
\left| \sqrt{\det A_1} - \sqrt{\det A_0} \right| \leq \frac{\sqrt{n}}{2} \theta^\omega_x(a_0, a_1).
\]

(Recall Convention 2.51 for the definitions of \( A_i \).)

**Proof.** Let \( a_t, t \in [0, 1] \), be any path from \( a_0 \) to \( a_1 \), and recall that \( A_t = g^{-1} a_t \) (cf. Convention 2.51). Following the proof of Lemma 3.3 we have

\[
\partial_t \sqrt{\det A_t} = \frac{1}{2} \left( \text{tr}_{a_t} \right) \sqrt{\det A_t} = \frac{1}{2} \left( \left( \text{tr}_{a_t} a'_t \right)^2 \det A_t \right)^{1/2}
\]

\[
\leq \frac{1}{2} \left( n \text{tr}(a'_t)^2 \right) \det A_t^{1/2} = \frac{\sqrt{n}}{2} \left( \sqrt{\det A_t} \right)^n,
\]

where the inequality follows, as in the proof of Lemma 3.3, from (3.2). This now implies

\[
\sqrt{\det A_1} - \sqrt{\det A_0} \leq \int_0^1 \partial_t \sqrt{\det A_t} dt \leq \frac{\sqrt{n}}{2} \int_0^1 \sqrt{\left( \right)^n} dt = \frac{\sqrt{n}}{2} L(a_t).
\]

Since this holds for all paths, we can replace the far right-hand side with \( \frac{\sqrt{n}}{2} \theta^\omega_x(a_0, a_1) \).

Now repeating the computation with \( \omega \) modified.

**Proposition 4.13.** Let \( a_k \) be a \( \theta^\omega_x \)-Cauchy sequence. Then either

1. \( \det A_k \to 0 \) for \( k \to \infty \), or
2. there exist constants \( C, \eta > 0 \) such that \( \| (a_k)_{ij} \| \leq C \) and \( \det A_k \geq \eta \) for all \( 1 \leq i, j \leq n \) and \( k \in \mathbb{N} \).

**Proof.** Keeping Lemma 4.12 in mind, it is more convenient to work with the square root of the determinant. This is, of course, completely equivalent for our purposes.

Now, by Lemma 4.12, the map \( a \to \sqrt{\det A} \) is \( \theta^\omega_x \)-Lipschitz. Since \( a_k \) is \( \theta^\omega_x \)-Cauchy, it is easy to see that \( \lim_{k \to \infty} \sqrt{\det A_k} \) exists, so let’s call this limit \( L \).

If for every \( \eta > 0 \), there exists \( k \) such that \( \sqrt{\det A_k} \leq \eta \), then clearly \( L = 0 \). It remains to show that if there exist \( i \) and \( j \) such that for all \( C > 0 \), there is a \( k \) such that \( |(a_k)_{ij}| > C \), then \( \sqrt{\det A_k} \to 0 \). We will assume that \( L > 0 \) and show a contradiction.

Let’s say that we are given \( b_0, b_1 \in \mathcal{M}_x \) with \( \det B_0, \det B_1 \geq \delta \). Let

\[
L_{-\delta} := \inf \left\{ L^{(c)}(b_t) \mid b_t \text{ is a path from } b_0 \text{ to } b_1 \text{ with } \det B_t \leq \delta/2 \text{ for some } t \in (0, 1) \right\},
\]

\[
L_{+\delta} := \inf \left\{ L^{(c)}(b_t) \mid b_t \text{ is a path from } b_0 \text{ to } b_1 \text{ with } \det B_t \geq \delta/2 \text{ for all } t \in [0, 1] \right\}.
\]

It is easy to see that \( \theta^\omega_x(b_0, b_1) = \min(L_{-\delta}, L_{+\delta}) \). Now let \( b_t \) be a path as in the definition of \( L_{-\delta} \), and assume \( \tau \in (0, 1) \) is such that \( \det B_\tau \leq \delta/2 \). Then using Lemma 4.12 we have

\[
L^{(c)}(b_t) = L^{(c)}(b_{1/2}) + L^{(c)}(b_{1/2})_\tau)
\]

\[
\geq \sqrt{n} \left| \sqrt{\det B_0} - \sqrt{\det B_\tau} \right| + \sqrt{n} \left| \sqrt{\det B_1} - \sqrt{\det B_\tau} \right|
\]

\[
\geq \sqrt{n} \left( \sqrt{\delta} - \sqrt{\frac{\delta}{2}} \right) = \sqrt{n} \left( 1 - \frac{1}{\sqrt{2}} \right) \sqrt{\delta}.
\]

Therefore \( L_{-\delta} \geq \sqrt{n}(1 - 1/\sqrt{2}) \delta \). Then, if \( b_t \) is a path as in the definition of \( L_{+\delta} \), we have

\[
L(b_t) = \int_0^1 \sqrt{\langle b_t, b_t' \rangle} dt = \int_0^1 \sqrt{\langle b_t, b_t' \rangle} \det B_t dt \geq \sqrt{\frac{\delta}{2}} \int_0^1 \sqrt{\langle b_t, b_t' \rangle} dt \geq \sqrt{\frac{\delta}{2}} \mathcal{D}_x(b_0, b_1).
\]
This gives $L_{+\delta} \geq \sqrt{\delta/2} d_x(b_0, b_1)$. Putting all of this together, we get that

\begin{equation}
\theta^2_x(b_0, b_1) \geq \min\{\sqrt{n}(1 - 1/\sqrt{2})\sqrt{\delta}, \sqrt{\delta/2} d_x(b_0, b_1)\}
\end{equation}

whenever $B_0, \det B_1 \geq \delta$.

Now, let’s apply the considerations of the last paragraph to the problem at hand. Let $i$ and $j$ be, as above, the indices for which $|(a_k)_{ij}|$ is unbounded, and choose a subsequence, which we again denote by $a_k$, such that $|(a_k)_{ij}| \geq k$ for all $k \in \mathbb{N}$. Passing to this subsequence does not change the limit $\lim_{k \to \infty} \sqrt{\det A_k}$.

Next, choose $K \in \mathbb{N}$ such that $k \geq K$ implies $\sqrt{\det A_k} \geq L/2$ and $k, l \geq K$ implies $\theta^2_x(a_k, a_l) \leq \frac{1}{2} \sqrt{n}(1 - 1/\sqrt{2})\sqrt{L/2}$. The latter assumption is possible since $a_k$ is Cauchy. By (4.9), if $k \geq K$, we also have

$$\theta^2_x(a_K, a_k) \geq \min\{\sqrt{n}(1 - 1/\sqrt{2})\sqrt{L/2}, \sqrt{L/4} d_x(a_K, a_k)\}.$$ 

But $\theta^2_x(a_K, a_k) \geq \sqrt{n}(1 - 1/\sqrt{2})\sqrt{L/2}$ violates our assumptions on $K$. Furthermore, $d_x(a_K, a_k) \to \infty$ since $|(a_k)_{ij}| \to \infty$ and $(M, \langle \cdot, \cdot \rangle)$ is complete. Therefore, if $\theta^2_x(a_K, a_k) \geq \sqrt{L/4} d_x(a_K, a_k)$ for all $k$, then our assumptions on $K$ are violated as well. Thus we have achieved the desired contradiction.

Since for every pair of constants $C, \eta > 0$, the set of elements $\tilde{g}$ of $\mathcal{M}_x$ with $|\tilde{g}_{ij}| \leq C$ and $\det \tilde{G} \geq \eta$ for all $1 \leq i, j \leq n$ is compact, we immediately get the following corollary of Proposition [4.13].

**Corollary 4.14.** Let $\{g_k\}$ be a $\theta^2_x$-Cauchy sequence. Then either

1. $\det G_k \to 0$ for $k \to \infty$, or
2. there exists an element $g_\infty \in \mathcal{M}_x$ such that $g_k \to g_\infty$, with convergence in the manifold topology of $\mathcal{M}_x$.

**Proof.** By the discussion preceding the corollary, if $\det G_k$ is bounded away from zero, then $\{g_k\}$ is contained within a compact subset of $\mathcal{M}_x$. Since a compact subset of a metric space is complete and $\{g_k\}$ is Cauchy, it $\theta^2_x$-converges to some limit $g_\infty$. Finally, since $\mathcal{M}_x$ is finite dimensional, the topology induced by $\theta^2_x$ coincides with the topology of $\mathcal{M}_x$ as a manifold (or open subset of $S_x$), so in fact $g_k \to g_\infty$.

This is essentially the pointwise equivalent of $\omega$-convergence. In the next subsection, we will globalize this result. Before we do that, though, we use this opportune moment to prove two last pointwise results, which will be useful in Section [4.3]. The first is the pointwise analog of Proposition [4.11].

**Proposition 4.15.** Let $\tilde{g}, \hat{g} \in \mathcal{M}_x$. Then there exists a constant $C'(n)$, depending only on $n$, such that

$$\theta^2_x(\tilde{g}, \hat{g}) \leq C'(n) \left( \sqrt{\det \tilde{G}} + \sqrt{\det \hat{G}} \right).$$

**Proof.** For this proof, we will denote the $\langle \cdot, \cdot \rangle^0$-length of a path simply by $L$. The metric $\langle \cdot, \cdot \rangle$ does not play a role here, and so there is no need to distinguish between the two lengths.

The proof goes very similarly to Proposition [4.11] but is simpler because we do not need to do any argument approximating paths of $L^2$ metrics by $C^\infty$ metrics. Since the ideas are the same, the proof can be safely skipped, but we include it here for completeness.

First, define paths $\tilde{g}_t^s$ and $\hat{g}_t^s$, for $0 < s \leq 1$ and $t \in [s, 1]$, by

$$\tilde{g}_t^s := t \tilde{g} \quad \text{and} \quad \hat{g}_t^s := t \hat{g}.$$ 

We consider these as a family of paths in the time variable $t$ with domain depending on the family parameter $s$. 
Second, define a family $g^s_t$ of paths in $t$ depending on the family parameter $s$ by

$$g^s_t := s \cdot ((1-t)\hat{g} + t\hat{g})$$

where again $0 < s \leq 1$ but this time $t \in [0, 1]$.

Then the concatenation $g^s_{t^*} := (g^s_t)^{-1} \ast g^s_{t^*}$ (here, $(g^s_t)^{-1}$ means we run through that path backwards) is, for each $s$, a path from $\hat{g}$ to $\hat{g}$. We will prove that

$$\lim_{s \to 0} L(g^s_t) \leq C'(n) \left( \sqrt{\det \hat{G}} + \sqrt{\det G} \right),$$

which will imply the result immediately.

First, note that $L(g^s_t) \leq \lim_{s \to 0} L(g^s_t)$ for all $s$. To compute the right-hand side, note that

$$\langle (g^s_t)', (g^s_t)' \rangle^0_{g^s_t} = \text{tr}_{\hat{g}}(\hat{g}^2) \det(t\hat{G}) = (n \det \hat{G}) t^{\frac{n}{2}-2}.$$

Therefore,

$$L(g^s_t) \leq \lim_{s \to 0} L(g^s_t) = \sqrt{n \det \hat{G}} \int_0^1 t^{\frac{n}{2}-1} dt.$$

Since $\frac{n}{2} - 1 > -1$, the above integral is finite, with a value depending only on $n$. Hence we have

$$L(g^s_t) \leq C'(n) \sqrt{\det \hat{G}}.$$

In exactly the same way, we can show

$$L(g^s_t) \leq C'(n) \sqrt{\det G},$$

even using the same constant.

Now, if we can show that $\lim_{s \to 0} L(g^s_t) = 0$, we will be finished. So we compute

$$\langle (g^s_t)', (g^s_t)' \rangle^0_{g^s_t} = \text{tr}_{s((1-t)\hat{g} + t\hat{g})} \left( s^2 (\hat{g} - \hat{g})^2 \right) \det(s((1-t)\hat{G} + t\hat{G}))$$

$$= s^{n/2} \text{tr}_{s((1-t)\hat{g} + t\hat{g})} \left( (\hat{g} - \hat{g})^2 \right) \det((1-t)\hat{G} + t\hat{G})$$

$$= s^{n/2} \langle (g^s_t)', (g^s_t)' \rangle^0_{g^s_t}.$$

This implies that

$$L(g^s_t) = s^{n/2} L(g^s_t),$$

from which $\lim_{s \to 0} L(g^s_t) = 0$ is immediate. This completes the proof.

The last pointwise result we need combines Corollary 4.14 and Proposition 4.15 to give a description of the completion of the metric space $(\mathcal{M}_x, \theta^0_x)$.

**Theorem 4.16.** For any given $x \in M$, let $\text{cl}(\mathcal{M}_x)$ denote the closure of $\mathcal{M}_x \subset S_x$ with regard to the natural topology. Then $\text{cl}(\mathcal{M}_x)$ consists of all positive semidefinite $(0,2)$-tensors at $x$. Let us denote the boundary of $\mathcal{M}_x$, as a subspace of $S_x$, by $\partial \mathcal{M}_x$.

Define an equivalence relation on $\text{cl}(\mathcal{M}_x)$ by $g_0 \sim g_1$ if and only if $g_0, g_1 \in \partial \mathcal{M}_x$. Thus, we simply identify the boundary of $\mathcal{M}_x$ together to a point.

Then the completion of $(\mathcal{M}_x, \theta^0_x)$ can be identified with the space $\text{cl}(\mathcal{M}_x)/\sim$. The distance function is given by

$$\theta^0_x(g_0, g_1) = \lim_{k \to \infty} \theta^0_x(g^0_k, g^1_k),$$

where $\{g^0_k\}$ and $\{g^1_k\}$ are any sequences in $\mathcal{M}_x$ converging (in the topology of $S_x$) to $g_0$ and $g_1$, respectively.

**Proof.** Note that $\hat{g} \mapsto \det \hat{G}$ is a continuous map from $S_x$ to the reals, that the map is positive when restricted to $\mathcal{M}_x$, and that it is constantly zero when restricted to $\partial \mathcal{M}_x$. The latter facts are implied by Proposition 2.9.

Let $\{g_k\}$ be any sequence in $\mathcal{M}_x$. By Corollary 4.14, if $\{g_k\}$ is Cauchy then either $g_k \to g_\infty \in \mathcal{M}_x$ (with convergence in the topology of $S_x$), or $\det G_k \to 0$. By Proposition
all sequences with $\det G_k \to 0$ are equivalent Cauchy sequences, and so they are identified in $(\mathcal{M}_x, \theta^2_x)$. Since the determinant is a continuous map, as noted above, we can thus identify such sequences with any given sequence converging to $\mathcal{M}$.

Finally, if $g_k \to g_\infty \in \mathcal{M}_x$ in the topology of $\mathcal{S}_x$, then we also have that $\theta^2_x(g_k, g_\infty) \to 0$, because $\mathcal{M}_x$ is finite dimensional and so the topology of $\theta^2_x$ coincides with the manifold topology. This implies that $\{g_k\}$ is Cauchy. By the same reasoning, we can show that if $\{\tilde{g}_k\}$ is a second sequence converging to $g_\infty$ in the topology of $\mathcal{S}_x$, then $\{g_k\}$ and $\{\tilde{g}_k\}$ are equivalent.

We have thus shown that $\{g_k\}$ is Cauchy if and only if either $g_k \to g_\infty \in \mathcal{M}_x$ or $\det G_k \to 0$ holds. We have also shown that all sequences with $\det G_k \to 0$ are equivalent, and that all sequences converging to the same element of $\mathcal{M}_x$ are equivalent. The statement of the theorem now follows. \qed

4.1.4. The existence proof. Corollary 4.14 gives us strong hints as to what to expect from Cauchy sequences in $\mathcal{M}$. Thinking heuristically, a $d$-Cauchy sequence $\{g_k\}$ in $\mathcal{M}$ should be a Cauchy sequence in $\theta^2_x$ for “most” points $x$ by the estimate of Proposition 3.8. Then we know that at “most” points $x$, either $\{g_k(x)\}$ converges or $\det G_k(x) \to 0$. That is, $\{g_k\}$ converges at “most” points outside the deflated set. The goal of this subsection is to make this heuristic idea precise and use it to prove existence of the $\omega$-limit.

**Lemma 4.17.** Let $\{g_k\}$ be a Cauchy sequence in $\mathcal{M}$. By passing to a subsequence if necessary, we can assume that

$$\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty.$$

Then the following holds:

$$\sum_{k=1}^{\infty} \Theta_M(g_k, g_{k+1}) < \infty.$$

Furthermore, define functions $\Omega$ and $\Omega_N$ for each $N \in \mathbb{N}$ by

$$\Omega_N := \sum_{k=1}^{N} \theta^2_x(g_k(x), g_{k+1}(x)), \quad \Omega := \sum_{k=1}^{\infty} \theta^2_x(g_k(x), g_{k+1}(x)).$$

Then $\Omega$ is a.e. finite, $\Omega \in L^1(M, g)$ and $\Omega_N \xrightarrow{L^1} \Omega$. Furthermore, by definition, $\Omega_N$ converges to $\Omega$ pointwise.

**Proof.** The first statement is clear, as is the statement that $\Omega_N \to \Omega$ pointwise. So we move on to the other statements.

Lemma 3.3 implies that $\sqrt{\text{Vol}(M, g_k)}$ is a Cauchy sequence in $\mathbb{R}$. Therefore it is bounded, and we can find a constant $V$ such that $\sqrt{\text{Vol}(M, g_k)} \leq V$ for all $k$. Thus, by Proposition 3.8

$$\Theta_M(g_k, g_{k+1}) \leq 2d(g_k, g_{k+1}) \left( \frac{\sqrt{n}}{2} d(g_k, g_{k+1}) + V \right).$$

But for large $k$, since $\{g_k\}$ is Cauchy, we must have $d(g_k, g_{k+1}) \leq 1$, so

$$\Theta_M(g_k, g_{k+1}) \leq \sqrt{n} d(g_k, g_{k+1})^2 + 2V d(g_k, g_{k+1}) \leq (\sqrt{n} + V) d(g_k, g_{k+1}) + 1.$$

The first statement is now immediate.

To prove the second statement, we recall the monotone convergence theorem of Lebesgue and Levi [4, Thm. 2.8.2]. Let $(X, \Sigma, \nu)$ be a measure space, and let $h_i$, for $i \in \mathbb{N}$, be measurable functions $X \to [0, +\infty]$. Suppose that $h_i \leq h_{i+1}$ for all $i \in \mathbb{N}$ and a.e. $x \in X$, then...
and furthermore that \( \sup_i \int h_i \, d\nu < \infty \). Then the function defined by \( h(x) := \lim_i h_i(x) \) is a.e. finite, and

\[
\lim_{i \to \infty} \int_X h_i \, d\nu = \int_X h \, d\nu.
\]

This theorem implies a criterion for exchanging infinite sums and integrals. In particular, let \( f_i \) be a sequence of nonnegative measurable functions on \( X \). Let \( F_N \) be the partial sum of the first \( N \) elements and define \( F := \lim_{N \to \infty} F_N \). Suppose that \( \sup_N \int F_N \, d\nu < \infty \). Then \( F \) and \( F_N \) clearly satisfy the requirements of the monotone convergence theorem, so we have

\[
\sum_{i=1}^{\infty} \int_X f_i \, d\nu = \lim_{N \to \infty} \sum_{i=1}^{N} \int_X f_i \, d\nu = \lim_{N \to \infty} \int_X F_N \, d\nu = \int_X F \, d\nu = \int_X \left( \sum_{i=1}^{\infty} f_i \right) \, d\nu.
\]

We can apply this to \( \Omega \) and \( \Omega_N \) to obtain

\[
\int_M \Omega \mu_g = \lim_{N \to \infty} \int_M \Omega_N \mu_g = \lim_{N \to \infty} \sum_{i=1}^{N} \int_M \theta^2_{\mu}(g_i, g_{i+1}) \mu_g = \sum_{k=1}^{\infty} \Theta_M(g_k, g_{k+1}) < \infty,
\]

where finiteness follows from the first part of the lemma. This proves that \( \Omega \) is a.e. finite and \( \Omega \in L^1(M, g) \). It remains to show that \( \Omega_N \rightharpoonup \Omega \). But this is now immediate from [47, Thm. 8.5.1], which states that if \( 1 \leq p < \infty \), \( f_i \rightharpoonup f \) a.e. and \( \|f_i\|_p \to \|f\|_p \), then \( f_i \rightharpoonup f \). □

Using this lemma, we can prove what we heuristically described before—that given a Cauchy sequence \( \{g_k\} \), we can find a subsequence \( \{g_{k_m}\} \) such that \( \{g_{k_m}(x)\} \) is \( \theta^2_{\mu} \)-Cauchy for “most” \( x \), allowing us to apply Proposition 4.13 at these points.

**Proposition 4.18.** Let \( \{g_k\} \) be a Cauchy sequence in \( \mathcal{M} \) such that

\[
\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty.
\]

Then \( \{g_k(x)\} \) is a \( \theta^2_{\mu} \)-Cauchy sequence for a.e. \( x \in M \).

**Proof.** By our assumption, all the conclusions of Lemma 4.17 hold. In particular, \( \Omega_N \rightharpoonup \Omega \) pointwise and \( \Omega \) is a.e. finite. Therefore, for a.e. \( x \in M \),

\[
(4.10) \quad \sum_{k=1}^{\infty} \theta^2_{\mu}(g_k, g_{k+1}) = \Omega(x) < \infty.
\]

It is then simple to show that \( \{g_k(x)\} \) is \( \theta^2_{\mu} \)-Cauchy at a point where (4.10) holds, for if \( l \leq m \),

\[
\theta^2_{\mu}(g_l, g_m) \leq \sum_{k=l}^{m} \theta^2_{\mu}(g_k, g_{k+1})
\]

by the triangle inequality. But (4.10) shows that the right-hand side of the above is small for \( l \) and \( m \) large, proving that \( \{g_k(x)\} \) is \( \theta^2_{\mu} \)-Cauchy. □

The previous proposition allows us to globalize Corollary 4.14. The precise statement is the following:

**Corollary 4.19.** Let \( \{g_k\} \) be a Cauchy sequence in \( \mathcal{M} \) such that

\[
\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty.
\]

Then for a.e. \( x \in M \), \( \{g_k(x)\} \) is \( \theta^2_{\mu} \)-Cauchy and either:

1. \( \det G_{tk}(x) \to 0 \) for \( k \to \infty \), or
that $X$ can show that $X$ measurable restricted to this set. Furthermore, Lemma 4.6 it therefore converges to $\{g_{\infty}\}$ given any two representatives $\mu$ (4.11) $\text{Vol}(\cdot, g)$. Thus fined as follows. At points $x$ defined as follows. At points $x$ is measurable, then for any representative $\mu$ it is clear from Definition 4.4 that $\{g_{\infty}\}$ is a convergent sequence in $\mathcal{M}_x$.

Furthermore, (1) holds for a.e. $x \in X_{(g_k)}$, and (2) holds for a.e. $x \in M \setminus X_{(g_k)}$.

**Proof.** By Proposition 4.18, $\{g_k(x)\}$ is $\theta_2^\omega$-Cauchy for a.e. $x$. Then Corollary 4.14 implies the result immediately. □

This corollary essentially delivers us the proof of the existence result.

**Theorem 4.20.** For every Cauchy sequence $\{g_k\}$, there exists an element $[g_{\infty}] \in \overline{\mathcal{M}}_m$ and a subsequence $\{g_{k_l}\}$ such that $\{g_{k_l}\}$ $\omega$-converges to $[g_{\infty}]$.

Explicitly, $[g_{\infty}]$ is the unique equivalence class containing the element $g_{\infty} \in \mathcal{M}_m$ defined as follows. At points $x \in M$ where $\{g_{k_l}(x)\}$ is $\theta_2^\omega$-Cauchy,

(1) $g_{\infty}(x) := 0$ for $x \in X_{(g_{k_l})}$ and

(2) $g_{\infty}(x) := \lim g_{k_l}(x)$ for $x \in M \setminus X_{(g_{k_l})}$.

At points $x \in M$ where $\{g_{k_l}(x)\}$ is not $\theta_2^\omega$-Cauchy, we set $g_{\infty}(x) := 0$.

**Proof.** Let $\{g_{k_l}\}$ be a subsequence of $\{g_k\}$ such that

$$\sum_{l=1}^{\infty} d(g_{k_l}, g_{k_{l+1}}) < \infty.$$ Then $\{g_{k_l}\}$ satisfies properties (1) and (4) of Definition 4.5 as well as the hypotheses of Corollary 4.19. Thus $\{g_{k_l}\}$ is a.e. $\theta_2^\omega$-Cauchy, and so $g_{\infty}$ is defined a.e. by the two conditions given above. From this, it is immediate that $\{g_{k_l}\}$ together with $g_{\infty}$ also satisfies properties (2) and (3) of Definition 4.5. Thus, $\{g_{k_l}\}$ $\omega$-converges to $g_{\infty}$, and by Lemma 4.6 it therefore converges to $[g_{\infty}]$—provided we can show that $g_{\infty} \in \mathcal{M}_m$.

Let’s prove this last fact. Clearly $g_{\infty}$ is a semimetric, so we must show that $g_{\infty}$ is measurable. Now, on $M \setminus X_{(g_{k_l})}$, $g_{\infty}$ is the a.e.-limit of measurable metrics, so it is measurable restricted to this set. Furthermore, $g_{\infty}(x) = 0$ for every $x \in X_{(g_{k_l})}$, so if we can show that $X_{(g_{k_l})}$ is measurable, then we are done. But the following formula shows that $X_{(g_{k_l})}$ can be built from countable unions and intersections of open sets:

$$X_{(g_{k_l})} = \{ x \in M \mid \forall \delta > 0, \exists \text{ s.t. } \det g_{k_l}(x) < \delta \} = \bigcap_{N \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} \left\{ x \in M \mid \det g_{k_l}(x) < \frac{1}{N} \right\}.$$ Knowing now that the $\omega$-limit of a Cauchy sequence of $\mathcal{M}$ exists (after passing to a subsequence), we go further into the properties of $\omega$-convergence.

### 4.2. $\omega$-convergence and the concept of volume

In this brief section, we wish to prove that the volumes of measurable subsets behave well under $\omega$-convergence. Specifically, we want to show that if $\{g_k\}$ $\omega$-converges to $[g_{\infty}]$ and $Y \subseteq M$ is measurable, then for any representative $g_{\infty} \in [g_{\infty}]$,

$$\text{Vol}(Y, g_k) \rightarrow \text{Vol}(Y, g_{\infty}).$$ (4.11)

To see that the above expression is well-defined, recall that a measurable semimetric $\tilde{g}$ on $M$ induces a nonnegative volume form and measure $\mu_{\tilde{g}}$ on $M$ (cf. Subsection 2.6) that is absolutely continuous with respect to the fixed volume form $\mu$. Furthermore, given any two representatives $g_{\infty}^0, g_{\infty}^1 \in [g_{\infty}]$, we have that $\mu_{g_{\infty}^0} = \mu_{g_{\infty}^1}$ as measures—it is clear from Definition 4.4 that $\mu_{g_{\infty}^0}$ and $\mu_{g_{\infty}^1}$ can differ at most on a nullset. Thus $\text{Vol}(Y, g_{\infty}^0) = \text{Vol}(Y, g_{\infty}^1)$. 


The proof of (4.11) is achieved via the Lebesgue dominated convergence theorem (cf. Theorem 2.14). So let \( \{g_k\} \) \( \omega \)-converge to \( g_\infty \), and let’s see what we need to do to apply this theorem. First, we need to show that \( (\mu_{g_k}/\mu_g) \xrightarrow{a.e.} (\mu_{g_\infty}/\mu_g) \). If we can also find a function \( f \in L^1(M, g) \) such that \( (\mu_{g_k}/\mu_g) \leq f \) a.e., then the Lebesgue dominated convergence theorem would imply that

\[
\text{Vol}(Y, g_\infty) = \int_Y \left( \frac{\mu_{g_\infty}}{\mu_g} \right) \mu_g = \lim_{k \to \infty} \int_Y \left( \frac{\mu_{g_k}}{\mu_g} \right) \mu_g = \lim_{k \to \infty} \text{Vol}(Y, g_k).
\]

We begin by showing a.e.-convergence.

**Lemma 4.21.** Let \( \{g_k\} \) \( \omega \)-converge to \( g_\infty \in \mathcal{M}_m \). Then

\[
\left( \frac{\mu_{g_k}}{\mu_g} \right) \xrightarrow{a.e.} \left( \frac{\mu_{g_\infty}}{\mu_g} \right).
\]

**Proof.** Recall that

\[
\left( \frac{\mu_{g_k}}{\mu_g} \right) = \sqrt{\det G_k} \quad \text{and} \quad \left( \frac{\mu_{g_\infty}}{\mu_g} \right) = \sqrt{\det G_\infty}.
\]

So we can prove the statement by working with the determinants above instead of the Radon-Nikodym derivatives.

We first prove that for a.e. \( x \in X_{(g_k)} \), \( \det G_k(x) \to 0 = \det G_\infty \) as \( k \to \infty \). By the definition of the deflated set, for every \( x \in X_{(g_k)} \) and \( \epsilon > 0 \), there exists \( k \in \mathbb{N} \) such that

\[
(4.12) \quad \det G_k(x) < \epsilon.
\]

But we also know from Proposition 4.18 and property 4 of Definition 4.5 that \( \{g_k(x)\} \) is \( \theta^2 \)-Cauchy for a.e. \( x \in M \). Hence, by Lemma 4.12, \( \left\{ \sqrt{\det G_k(x)} \right\} \) is a Cauchy sequence in \( \mathbb{R} \) at such points. Therefore it has a limit, and by (4.12) we know that this limit must be 0.

Now, for a.e. \( x \in M \setminus X_{(g_k)} \), \( g_k(x) \to g_\infty(x) \). Since the determinant is a continuous map from the space of \( n \times n \) matrices into \( \mathbb{R} \), this immediately implies that \( \det G_k(x) \to \det G_\infty(x) \) for a.e. \( x \in M \setminus X_{(g_k)} \). Combined with the last paragraph, this proves the desired result.

Our next task is to find an \( L^1 \) function that dominates \( (\mu_{g_k}/\mu_g) \).

**Lemma 4.22.** Let \( \{g_k\} \) be a Cauchy sequence such that

\[
\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty,
\]

and let \( \Omega \) be the function of Lemma 4.17. Then

\[
\left( \frac{\mu_{g_k}}{\mu_g} \right)(x) \leq \frac{\sqrt{n}}{2} \Omega(x) + \left( \frac{\mu_{g_1}}{\mu_g} \right)(x)
\]

for a.e. \( x \in M \) and all \( k \in \mathbb{N} \).

**Proof.** Fix some \( k \) for the moment. By Proposition 4.18, \( \{g_k(x)\} \) is \( \theta^2 \)-Cauchy for a.e. \( x \in M \). Let \( x \in M \) be a point where this holds. Then by Lemma 4.12, the triangle inequality, and the definitions of \( \Omega_N \) and \( \Omega_k \), we have

\[
\left| \sqrt{\det G_k} - \sqrt{\det G_1} \right| \leq \frac{\sqrt{n}}{2} \theta^2_x(g_k, g_1) \leq \frac{\sqrt{n}}{2} \sum_{m=1}^{k-1} \theta^2_x(g_m, g_{m+1}) = \frac{\sqrt{n}}{2} \Omega_{k-1}(x) \leq \frac{\sqrt{n}}{2} \Omega(x).
\]
In particular,
\[ \sqrt{\det G_k(x)} \leq \sqrt{\frac{n}{2}} \Omega(x) + \sqrt{\det G_1(x)}. \]
The result is now immediate. \[\square\]

Now, since \( \mu_{g_1} \) is smooth, it has finite volume, implying that \( (\mu_{g_1}/\mu_g) \in L^1(M, g) \). We have already seen in Lemma 4.17 that \( \Omega \in L^1(M, g) \). Therefore Lemma 4.22 gives the necessary function dominating \( (\mu_{g_1}/\mu_g) \), and we can apply the Lebesgue dominated convergence theorem as discussed before the lemmas to obtain:

**Theorem 4.23.** Let \( \{g_k\} \omega \)-converge to \( g_\infty \in M_f \), and let \( Y \subseteq M \) be any measurable subset. Then \( \text{Vol}(Y, g_k) \to \text{Vol}(Y, g_\infty) \).

An immediate corollary of this theorem is that the total volume of the \( \omega \)-limit is finite:

**Corollary 4.24.** If \( g_\infty \) is the \( \omega \)-limit of a sequence \( \{g_k\} \) in \( M \), then \( \text{Vol}(M, g_\infty) < \infty \). That is, \( g_{\infty} \in M_f \).

**Proof.** By Lemma 3.3, if \( \{g_k\} \) is a \( d \)-Cauchy sequence, then \( \{\text{Vol}(M, g_k)\} \) is a Cauchy sequence of positive real numbers. Therefore it converges to some finite nonnegative real number, and by Theorem 4.23 this number must be \( \text{Vol}(M, g_{\infty}) \). \[\square\]

Furthermore, as we might have suspected from the beginning, the volume of the deflated set of an \( \omega \)-convergent sequence vanishes in the limit.

**Corollary 4.25.** Let \( \{g_k\} \omega \)-converge to \( g_\infty \in M_f \). Then the deflated set \( X_{\{g_k\}} \) satisfies \( \text{Vol}(X_{\{g_k\}}, g_\infty) \to 0 \).

**Proof.** As noted in the proof of Theorem 4.20, \( X_{\{g_k\}} \) is measurable. Now, the definition of \( \omega \)-convergence implies that \( \text{Vol}(X_{\{g_k\}}, g_\infty) = 0 \), since \( g_\infty(x) = 0 \) for all \( x \in X_{\{g_k\}} \). So Theorem 4.23 gives the result. \[\square\]

Given Corollary 4.24, it behooves us to make the following definition, following which we refine the result of Theorem 4.20 using Corollary 4.24.

**Definition 4.26.** Let \( \hat{M}_f \subset \hat{M}_m \) denote the subset of those equivalence classes of semimetrics whose representatives are all elements of \( M_f \), i.e., finite-volume measurable semimetrics.

By the discussion at the beginning of the section, any two representatives of an equivalence class in \( \hat{M}_m \) have the same total volume. Therefore, if one representative of an equivalence class has finite volume, then all do. Moreover, for every \( \tilde{g} \in M_f \), \( [\tilde{g}] \in \hat{M}_f \).

The refinement of Theorem 4.20 is:

**Theorem 4.27.** For every Cauchy sequence \( \{g_k\} \), there exists an element \( [g_\infty] \in \hat{M}_f \) such that \( \{g_k\} \omega \)-subconverges to \( [g_\infty] \).

### 4.3. Uniqueness of the \( \omega \)-limit

The goal of this section is to prove the uniqueness of the \( \omega \)-limit in the sense mentioned in the introduction to the chapter: we will show that two \( \omega \)-convergent Cauchy sequences in \( M \) are equivalent if and only if they have the same \( \omega \)-limit.

We prove each direction in a separate subsection. After proving this uniqueness result, combining it with the existence result and the properties of \( \omega \)-convergence given above will show that for every equivalence class of Cauchy sequences in \( M \), there is a unique equivalence class of finite-volume, measurable semimetrics that each of its representatives subconverges to. Thus, \( \omega \)-convergence is a suitable convergence notion for choosing a limit point for Cauchy sequences in \( M \).
4.3.1. **First uniqueness result.** We first prove the statement that if two \( \omega \)-convergent Cauchy sequences are equivalent, then their \( \omega \)-limits agree. To do so, we will extend the pseudometric \( \Theta_Y \) (cf. Definition 3.5) to the precompletion of \( M \). For this, we need an easy lemma.

**Lemma 4.28.** Let \( Y \subseteq M \) be measurable. If \( \{g_k\} \) is a \( d \)-Cauchy sequence, then it is also \( \Theta_Y \)-Cauchy.

**Proof.** As noted in the proof of Lemma 4.17, since \( \{g_k\} \) is \( d \)-Cauchy, the sequence \( \sqrt{\text{Vol}(M, g_k)} \) in \( \mathbb{R} \) is bounded, say by a constant \( V \). Thus, by the estimate of Proposition 3.8, for any \( k, l \in \mathbb{N} \), we have

\[
\Theta_Y(g_k, g_l) \leq d(g_k, g_l) \left( \sqrt{n} d(g_k, g_l) + 2V \right) .
\]

From this the statement of the lemma is clear. \( \square \)

Now we give the extension of \( \Theta_Y \) mentioned above.

**Proposition 4.29.** Let \( Y \subseteq M \) be measurable. Then the pseudometric \( \Theta_Y \) on \( M \) can be extended to a pseudometric on \( \bar{M}^{\text{pre}} \), the precompletion of \( M \), via

\[
(4.13) \quad \Theta_Y(\{g_0^k\}, \{g_1^k\}) := \lim_{k \to \infty} \Theta_Y(g_0^k, g_1^k)
\]

This pseudometric is weaker than \( d \) in the sense that \( d(\{g_0^k\}, \{g_1^k\}) = 0 \) implies that \( \Theta_Y(\{g_0^k\}, \{g_1^k\}) = 0 \) for any Cauchy sequences \( \{g_0^k\} \) and \( \{g_1^k\} \). More precisely, we have

\[
(4.14) \quad \Theta_Y(\{g_0^k\}, \{g_1^k\}) \leq d(\{g_0^k\}, \{g_1^k\}) \left( \sqrt{n} d(\{g_0^k\}, \{g_1^k\}) + 2\sqrt{\text{Vol}(M, g_0)} \right) ,
\]

where \( g_0 \) is any element of \( M_f \) with \( g_0^0 \xrightarrow{\omega} g_0 \).

Furthermore, if \( \{g_0^k\} \) and \( \{g_1^k\} \) are sequences in \( M_V \) that \( \omega \)-converge to \( g_0 \) and \( g_1 \), respectively, then the formula

\[
(4.15) \quad \Theta_Y(\{g_0^1\}, \{g_1^1\}) = \int_Y \rho_\omega^2(g_0(x), g_1(x)) \mu(g(x))
\]

holds for all \( g_0, g_1 \in M_V \).

**Remark 4.30.** In (4.14), we choose any \( \omega \)-limit of \( \{g_0^k\} \). The existence of such a limit has already been proved, but not its uniqueness. On the other hand, if \( \bar{g}_0 \) is a different \( \omega \)-limit of \( \{g_0^0\} \), Theorem 4.23 guarantees that \( \text{Vol}(M, \bar{g}_0) = \text{Vol}(M, g_0) \). Therefore, the estimate (4.14) is independent of the choice of \( \omega \)-limit.

**Proof of Proposition 4.29.** The construction of a pseudometric on the precompletion of a metric space can be carried over to the case where we begin with a pseudometric space. Therefore, the limit in (4.13) is well-defined due to the fact that \( \{g_0^k\} \) and \( \{g_1^k\} \) are Cauchy sequences with respect to \( \Theta_Y \), and (4.13) indeed defines a pseudometric.

The inequality (4.14) is proved via the following simple computation, which uses (4.13), Proposition 3.8, and Theorem 4.23:

\[
\Theta_Y(\{g_0^k\}, \{g_1^k\}) = \lim_{k \to \infty} \Theta_Y(g_0^k, g_1^k)
\]

\[
\leq \lim_{k \to \infty} d(g_0^k, g_1^k) \left( \sqrt{n} d(g_0^k, g_1^k) + 2\sqrt{\text{Vol}(M, g_0^k)} \right)
\]

\[
= d(\{g_0^0\}, \{g_1^0\}) \left( \sqrt{n} d(\{g_0^0\}, \{g_1^0\}) + 2\sqrt{\text{Vol}(M, g_0)} \right) .
\]

As for the last statement, note first that \( \rho_\omega^2(g_0(x), g_1(x)) \) is well-defined by Theorem 4.16 since \( g_0 \) and \( g_1 \) are positive semidefinite tensors at each point \( x \in M \). To prove (4.15), we will first use Fatou’s Lemma (cf. Theorem 2.15) to show that \( \rho_\omega^2(g_0(x), g_1(x)) \) is integrable. We will then use this to apply the Lebesgue dominated convergence theorem.
So we start by letting \( \{g_k^0\} \) and \( \{g_k^1\} \) be sequences in \( \mathcal{M} \) \( \omega \)-converging to \( g_0 \) and \( g_1 \), respectively.

By Proposition 4.18, for a.e. \( x \in M \), \( \{g_k^0(x)\} \) and \( \{g_k^1(x)\} \) are \( \theta_x^2 \)-Cauchy. At such points, by definition,

\[
\theta_x^0(g_0(x), g_1(x)) = \lim_{k \to \infty} \theta_x^0(g_k^0(x), g_k^1(x)).
\]

So defining

\[ f_k(x) := \theta_x^0(g_k^0(x), g_k^1(x)), \quad f(x) := \theta_x^0(g_0(x), g_1(x)), \]

we have \( f_k \to f \) a.e.

Now, note that

\[ \Theta_Y(g_k^0, g_k^1) = \int_Y f_k(x) \mu_g(x). \]

We have already seen that \( \lim_{k \to \infty} \Theta_Y(g_k^0, g_k^1) \) exists, so \( \{\Theta_Y(g_k^0, g_k^1)\} \) is in particular a bounded sequence of real numbers. Thus

\[ \sup_k \int_Y f_k(x) \mu_g(x) = \sup_k \Theta_Y(g_k^0, g_k^1) < \infty, \]

where we have used Fatou’s lemma.

Now we wish to verify the assumptions of the Lebesgue dominated convergence theorem for \( f_k \) and \( f \). We note that for each \( l > k \), the triangle inequality gives

\[
f_k(x) = \theta_x^0(g_k^0(x), g_k^1(x)) \leq \sum_{m=k}^{l-1} \theta_x^0(g_m^0(x), g_{m+1}^0(x)) + \theta_x^0(g_l^0(x), g_l^1(x)) + \sum_{m=k}^{l-1} \theta_x^0(g_m^1(x), g_{m+1}(x)) \leq \sum_{m=1}^{l-1} \theta_x^0(g_m^0(x), g_{m+1}(x)) + \theta_x^0(g_l^0(x), g_l^1(x)) + \sum_{m=1}^{l-1} \theta_x^0(g_m^1(x), g_{m+1}(x)).
\]

Note that the only difference between the second and last lines is that the sums start at \( m = 1 \) instead of \( m = k \). Taking the limit \( l \to \infty \) of the above gives, for a.e. \( x \in M \),

\[
f_k(x) \leq \sum_{m=1}^{\infty} \theta_x^0(g_m^0(x), g_{m+1}(x)) + f(x) + \sum_{m=1}^{\infty} \theta_x^0(g_m^1(x), g_{m+1}(x)),
\]

where we have used (4.16). Now we claim that the right-hand side of the above inequality is \( L^1 \)-integrable. We already showed \( f \) is integrable using Fatou’s Lemma. As for the two infinite sums, they are each also integrable by Lemma 4.17 and \( \omega \)-convergence of \( g_k^i \), \( i = 0, 1 \) (specifically, property 4 of Definition 4.5 and Lemma 4.17). Thus each \( f_k \) is bounded a.e. by an \( L^1 \) function not depending on \( k \).

Now, knowing all of this, we can apply the Lebesgue dominated convergence theorem to show

\[ \Theta_Y(\{g_k^0\}, \{g_k^1\}) = \lim_{k \to \infty} \Theta_Y(g_k^0, g_k^1) = \lim_{k \to \infty} \int_Y f_k \mu_g = \int_Y f \mu_g = \int_Y \theta_x^0(g_0(x), g_1(x)) \mu_g(x), \]

which completes the proof. \( \square \)

With this proposition, proving the first uniqueness result becomes a relatively simple matter.

**Theorem 4.31.** Let two \( \omega \)-convergent sequences \( \{g_k^0\} \) and \( \{g_k^1\} \), with \( \omega \)-limits \( [g_0] \) and \( [g_1] \), respectively, be given. If \( g_k^0 \) and \( g_k^1 \) are equivalent, i.e., if

\[ \lim_{k \to \infty} \|g_k^0 - g_k^1\| = 0, \]

then \( [g_0] = [g_1] \).
Proof. Suppose the contrary; then for any representatives $g_0 \in [g_0]$ and $g_1 \in [g_1]$, one of two possibilities holds:

1. $X_{g_0}$ and $X_{g_1}$ differ by a set of positive measure, or
2. $X_{g_0} = X_{g_1}$, up to a nullset, but $g_0$ and $g_1$ differ on a set $E$ with $E \cap (X_{g_0} \cup X_{g_1}) = \emptyset$ and $\text{Vol}(E, g) > 0$, where $g$ is our fixed metric.

We will show that neither of these possibilities can actually occur.

To rule out (1), let $X_i := X_{g_i}$ denote the deformed set of the sequence $\{g_i\}$ for $i = 0, 1$. Then we claim $X_0 = X_1$, up to a nullset. If this is not true, then by swapping the two sequences if necessary, we see that $Y := (X_0 \setminus X_1)$ has positive volume with respect to $g_1$ and zero volume with respect to $g_0$. ($Y$ is simply the set on which $\{g_0^0\}$ deflates and $\{g_1^1\}$ doesn’t.) But then by Lemma 3.3,

$$\lim_{k \to \infty} d(g_0^0, g_k^1) \geq \lim_{k \to \infty} \sqrt{\text{Vol}(Y, g_k^1)} = \sqrt{\text{Vol}(Y, g_1)} > 0,$$

where we have used Theorem 4.23. This contradicts the assumptions of the theorem, so in fact $X_0 = X_1$ up to a nullset. Since by property (2) of Definition 4.5 $X_{g_i} = X_{g_i}$ up to a nullset as well, (1) cannot hold.

So suppose that (2) holds. Note that on $E$, $g_0$ and $g_1$ are both positive definite. Since $E$ has positive $g$-volume, we can conclude from Proposition 4.29 (specifically (4.15)) that $\Theta_E(\{g_0^0\}, \{g_1^1\}) > 0$. But then this and (4.14) also imply that

$$\lim_{k \to \infty} d(g_0^0, g_k^1) = d(\{g_0^0\}, \{g_1^1\}) > 0.$$  

This contradicts the assumptions of the theorem, and so (2) cannot hold either.  

\[\square\]

4.3.2. Second uniqueness result. Our goal in this subsection is to prove the following statement: up to equivalence, there is only one $d$-Cauchy sequence $\omega$-converging to a given element of $\hat{\mathcal{M}}_f$. That is, if we have two sequences $\{g_0^0\}, \{g_1^1\}$ that both $\omega$-converge to the same $[g_\infty] \in \hat{\mathcal{M}}_f$, then

$$d(\{g_0^0\}, \{g_1^1\}) = \lim_{k \to \infty} d(g_0^0, g_k^1) = 0.$$

After we’ve proved this statement, we combine it with the existence result from Section 4.1 and the results on volumes from Section 4.2, as mentioned in the introduction to this section.

We will first prove the above statement for sequences that remain within a given amenable subset $\mathcal{U}$, and will then use this to extend the proof to arbitrary sequences. Before any of this, though, we state a definition and a result from measure theory that we’ll need.

Definition 4.32 ([47, Dfn. 8.5.2]). Let $(X, \Sigma, \nu)$ be a measure space, and let $\mathcal{F}$ be a collection of measurable functions. We say that $\mathcal{F}$ is equicontinuous at $\emptyset$ if for any $\epsilon > 0$ and any sequence $\{E_k\}$ of measurable sets with

$$\bigcap_{k=1}^{\infty} E_k = \emptyset,$$

there exists $K_0$ such that

$$\int_{E_k} |f|\,d\nu < \epsilon$$

for all $f \in \mathcal{F}$ and $k > K_0$.

We note that in particular, if $\nu(X) < \infty$ and we are given a collection of functions $\mathcal{F}$ for which we can find some constant $C$ with

$$|f(x)| \leq C \quad \text{for a.e. } x \in X \text{ and every } f \in \mathcal{F},$$
then $\mathcal{F}$ is equicontinuous at $\emptyset$.

**Theorem 4.33** ([47] Thm. 8.5.14). Let $(X, \Sigma, \nu)$ be a measure space with $\nu(X) < \infty$, and let $f$ be a measurable function on $X$. Furthermore, let $f_k$ be a sequence of functions in $L^p(X, \nu)$. Then the following statements are equivalent.

1. $f_k \rightarrow f$ in $L^p(X, \nu)$.
2. $\{|f_k|^p | k \in \mathbb{N}\}$ is equicontinuous at $\emptyset$ and $f_k \rightarrow f$ in measure.

**Remark 4.34.** We make a couple of remarks about this theorem that we will need later:

1. By [47] Thm. 8.3.3, a.e. convergence implies convergence in measure. Therefore, Theorem 4.33 implies that if $\{|f_k|^p | k \in \mathbb{N}\}$ is equicontinuous at $\emptyset$ and $f_k \rightarrow f$ a.e., then $f_k \rightarrow f$ in $L^p(X, \nu)$.
2. By [47] Thm. 8.3.6, if $f_k \rightarrow f$ in measure, then there exists a subsequence $\{f_{k_l}\}$ such that $f_{k_l} \rightarrow f$ a.e. Combining this with Theorem 4.33 implies that if $f_k \rightarrow f$ in $L^p$, then there exists a subsequence $f_{k_l}$ such that $f_{k_l} \rightarrow f$ a.e.

We now state the second uniqueness result as confined to the context of amenable subsets.

**Proposition 4.35.** Let $\mathcal{U}$ be an amenable subset, and let $\mathcal{U}^0$ be the $L^2$-completion of $\mathcal{U}$. If two sequences $\{g_k^0\}$ and $\{g_k^1\}$ in $\mathcal{U}$ both $\omega$-converge to $[g_\infty] \in \tilde{\mathcal{M}}_f$, then $\{g_k^0\}$ and $\{g_k^1\}$ are equivalent. That is,

$$\lim_{k \rightarrow \infty} d(g_k^0, g_k^1) = 0.$$  

Furthermore, up to differences on a nullset, $[g_\infty]$ only contains one representative, $g_\infty$, and $\{g_k^0\}$ and $\{g_k^1\}$ both $L^2$-converge to $g_\infty$. In particular, $g_\infty \in \mathcal{U}^0$.

**Proof.** Note that Definition 3.10 of an amenable subset implies that the deflated sets of $\{g_k^0\}$ and $\{g_k^1\}$ are empty. Therefore, all representatives of $[g_\infty]$ differ at most by a nullset, and property [3] of Definition 4.45 implies that $g_k^0, g_k^1 \overset{a.e.}{\rightarrow} g_\infty$.

Since all $g_k^0$ and $g_k^1$ satisfy the same bounds a.e. in each coordinate chart, it is easy to see that the set

$$\{(g_k^0)_{ij} | 1 \leq i, j \leq n, k \in \mathbb{N}\}$$

is equicontinuous at $\emptyset$ in each coordinate chart for both $l = 0$ and $l = 1$. Therefore, Remark 4.34 gives that $\{g_k^0\}$ and $\{g_k^1\}$ converge in $L^2$ to $g_\infty$, proving the second statement. This also implies that

$$\lim_{k \rightarrow \infty} \|g_k^1 - g_k^0\|_g = 0.$$  

But now, invoking Theorem 3.15 gives

$$\lim_{k \rightarrow \infty} d(g_k^0, g_k^1) = 0.$$  

The next lemma establishes the strong correspondence between $L^2$- and $\omega$-convergence within amenable subsets.

**Lemma 4.36.** Let $\mathcal{U} \subset \mathcal{M}$ be amenable, and let $\tilde{g} \in \mathcal{U}^0$. Then for any sequence $\{g_k\}$ in $\mathcal{U}$ that $L^2$-converges to $\tilde{g}$, there exists a subsequence $\{g_{k_l}\}$ that $\omega$-converges to $\tilde{g}$.

In particular, for any element $\tilde{g} \in \mathcal{U}^0$, we can always find a sequence in $\mathcal{U}$ that both $L^2$- and $\omega$-converges to $\tilde{g}$.

**Proof.** Let $\{g_k\}$ be any sequence $L^2$-converging to $\tilde{g} \in \mathcal{U}^0$. Then $\tilde{g}$ together with any subsequence of $\{g_k\}$ already satisfies properties [1] and [2] of Definition 4.45. This is clear from Theorem 3.15 and Definition 3.10 of an amenable subset. (Property [2] is empty here, as $\{g_k\}$ has empty deflated set by the definition of an amenable subset.) Since $\{g_k\}$
is \(d\)-Cauchy by Theorem 3.15, it is also easy to see that there is a subsequence \(\{g_{k_m}\}\) of \(\{g_k\}\) satisfying property (1) of \(\omega\)-convergence.

To verify property (3), note that each equivalence class of Cauchy sequences as a metric subspace of \(\{g_k\}\) is nicely compatible with the metric \(d\)

Given the results that we have so far, we can give an alternative description of the completion of an amenable set using \(\omega\)-convergence instead of \(L^2\)-convergence.

**Proposition 4.37.** Let \(\mathcal{U} \subset \mathcal{M}\) be an amenable subset. Then the completion \(\overline{\mathcal{U}}\) of \(\mathcal{U}\) as a metric subspace of \(\mathcal{M}\) can be identified with \(\mathcal{U}^0\), the \(L^2\) completion of \(\mathcal{U}\), using \(\omega\)-convergence. That is, there is a natural bijection between \(\overline{\mathcal{U}}\) and \(\mathcal{U}^0\) given by identifying each equivalence class of Cauchy sequences \(\{(g_k)\}\) with the unique element of \(\mathcal{U}^0\) that they \(\omega\)-subconverge to.

**Proof.** The existence result—Theorem 4.20—the first uniqueness result—Theorem 4.31—and Proposition 4.35 together imply that for every equivalence class \(\{(g_k)\}\) of \(\omega\)-Cauchy sequences in \(\mathcal{U}\), there is a unique \(L^2\) metric \(g_\infty \in \mathcal{U}^0\) such that every representative of \(\{(g_k)\}\) \(\omega\)-converges to \(g_\infty\), and that the representatives of a different equivalence class cannot also \(\omega\)-subconverge to \(g_\infty\). This gives us the map from \(\overline{\mathcal{U}}\) to \(\mathcal{U}^0\) and shows that it is injective. Furthermore, by Lemma 4.36, there is a sequence in \(\mathcal{U}\) \(\omega\)-subconverging to every element of \(\mathcal{U}^0\). Thus, this map is also surjective.

With this identification, we can define a metric on \(\mathcal{U}^0\) by declaring the bijection of the previous proposition to be an isometry. The result is the following:

**Definition 4.38.** Let \(\mathcal{U}\) be an amenable subset. By \(d_{\mathcal{U}}\), we denote the metric on the completion of \(\mathcal{U}\), which we identify with the \(L^2\)-completion \(\mathcal{U}^0\) via Proposition 4.37. Thus, for \(g_0, g_1 \in \mathcal{U}^0\) and any sequences \(g_k^0 \xrightarrow{\omega} g_0\) and \(g_k^1 \xrightarrow{\omega} g_1\), we have

\[
\|d_{\mathcal{U}}(g_0, g_1) = \lim_{k \to \infty} d(g_k^0, g_k^1).
\]

Note that by the preceding results, we can equivalently define \(d_{\mathcal{U}}\) by assuming that \(\{g_k^0\}\) and \(\{g_k^1\}\) \(L^2\)-converge to \(g_0\) and \(g_1\), respectively.

The next lemma shows that the metric \(d_{\mathcal{U}}\) is nicely compatible with the metric \(d\).

**Lemma 4.39.** Let \(\mathcal{U} \subset \mathcal{M}\) be amenable, and suppose \(g_0, g_1 \in \mathcal{U}\) and \(g_2 \in \mathcal{U}^0\). Then

1. \(d(g_0, g_1) = d_{\mathcal{U}}(g_0, g_1)\), and
2. \(d(g_0, g_1) \leq d_{\mathcal{U}}(g_0, g_2) + d_{\mathcal{U}}(g_2, g_1)\).

**Proof.** Statement (1) is true simply by the definition of \(d_{\mathcal{U}}\). Statement (2) is proved by applying statement (1) and the triangle inequality for \(d_{\mathcal{U}}\).

With a little bit of effort, we can use previous results to extend Proposition 4.4 to a statement about \(\mathcal{M}\), to the completion of an amenable subset. We first prove a very special case in a lemma, followed by the full result.

**Lemma 4.40.** Let \(\mathcal{U}\) be any amenable subset and \(g^0, g^1 \in \mathcal{U}\). Let \(C(n)\) be the constant of Proposition 4.1, and let \(E \subseteq \mathcal{M}\) be measurable. Then

\[
d_{\mathcal{U}}(g^0, \chi(M \setminus E)g^0 + \chi(E)g^1) \leq C(n) \left(\sqrt{\text{Vol}(E, g^0)} + \sqrt{\text{Vol}(E, g^1)}\right)
\]

**Proof.** For each \(k \in \mathbb{N}\), choose closed subsets \(F_k\) and open subsets \(U_k\) such that \(F_k \subseteq E \subseteq U_k\) and \(\text{Vol}(U_k, g) - \text{Vol}(F_k, g) \leq 1/k\). Furthermore, choose functions \(f_k \in C^\infty(M)\) satisfying

1. \(0 \leq f_k(x) \leq 1\) for all \(x \in M\),
2. \(f_k(x) = 1\) for \(x \in F_k\) and
3. \(f_k(x) = 0\) for \(x \notin U_k\).
Then it is not hard to see that the sequence defined by
\[ g_k := (1 - f_k)g^0 + f_k g^1 \]
\(L^2\)-converges to \( \chi(M \setminus E)g^0 + \chi(E)g^1 \), so in particular
\[ d_\mathcal{U}(g^0, \chi(M \setminus E)g^0 + \chi(E)g^1) = \lim_{k \to \infty} d(g^0, g_k). \]
Furthermore, since \( g^0 \) and all \( g_k \) are smooth, Proposition 4.1 gives
\[ d(g^0, g_k) \leq C(n) \left( \sqrt{\text{Vol}(U_k, g^0)} + \sqrt{\text{Vol}(U_k, g_k)} \right). \]
By our assumptions on the sets \( U_k \), it is clear that \( \text{Vol}(U_k, g^0) \to \text{Vol}(E, g^0) \). So if we can show that \( \text{Vol}(U_k, g_k) \to \text{Vol}(E, g^1) \), then (4.17) and (4.18) combine to give the desired result.

Now, because \( g_k = g^1 \) on \( F_k \), we have
\[ \text{Vol}(U_k, g_k) = \int_{F_k} \mu_{g^1} + \int_{U_k \setminus F_k} \mu_{g_k}. \]
The first term converges to \( \text{Vol}(E, g^1) \) for \( k \to \infty \) by the definition of \( F_k \). We claim that the second term converges to zero. Note that since the bounds of Definition 3.10 are pointwise convex, we can enlarge \( \mathcal{U} \) to an amenable subset containing \( g_k \) for each \( k \in \mathbb{N} \).

Theorem 4.41. Let \( \mathcal{U} \) be any amenable subset with \( L^2\)-completion \( \mathcal{U}^0 \). Suppose that \( g_0, g_1 \in \mathcal{U}^0 \), and let \( E := \text{carr}(g_1 - g_0) = \{ x \in M \mid g_0(x) \neq g_1(x) \} \). Then there exists a constant \( C(n) \) depending only on \( n = \dim M \) such that
\[ d_\mathcal{U}(g_0, g_1) \leq C(n) \left( \sqrt{\text{Vol}(E, g_0)} + \sqrt{\text{Vol}(E, g_1)} \right). \]
In particular, we have
\[ \text{diam}_\mathcal{U} \left( \{ \tilde{g} \in \mathcal{U}^0 \mid \text{Vol}(M, \tilde{g}) \leq \delta \} \right) \leq 2C(n)\sqrt{\delta}. \]

Proof. Using Lemma 4.36, choose any two sequences \( \{g_k^0\} \) and \( \{g_k^1\} \) in \( \mathcal{U} \) that both \( L^2 \)- and \( \omega \)-converge to \( g_0 \) and \( g_1 \), respectively. Then by the triangle inequality and Lemma 4.39, for each \( k \in \mathbb{N} \),
\[ d_\mathcal{U}(g_0, g_1) \leq d_\mathcal{U}(g_0, g_k^0) + d(g_k^0, g_k^1) + d_\mathcal{U}(g_k^1, g_1). \]
By Theorem 3.10, the first and last terms above approach zero as \( k \to \infty \). Furthermore, we claim that the middle term satisfies
\[ \lim_{k \to \infty} d(g_k^0, g_k^1) \leq C(n) \left( \sqrt{\text{Vol}(E, g_0)} + \sqrt{\text{Vol}(E, g_1)} \right), \]
which would complete the proof.
By the triangle inequality (2) of Lemma 4.39, we have
\[ d(g_k^0, g_k^1) \leq d_\mathcal{U}(g_k^0, \chi(M \setminus E)g_k^0 + \chi(E)g_k^1) + d_\mathcal{U}(\chi(M \setminus E)g_k^0 + \chi(E)g_k^1, g_k^1). \]
By Lemma 4.40, the first term of the above satisfies
\[ d_\mathcal{U}(g_k^0, \chi(M \setminus E)g_k^0 + \chi(E)g_k^1) \leq C(n) \left( \sqrt{\text{Vol}(E, g_k^0)} + \sqrt{\text{Vol}(E, g_k^1)} \right). \]
Applying Theorem 4.23 allows us to conclude

\[ \lim_{k \to \infty} d_U(g_k^0, \chi(M \setminus E)g_k^0 + \chi(E)g_k^1) \leq C(n) \left( \sqrt{\operatorname{Vol}(E, g_0)} + \sqrt{\operatorname{Vol}(E, g_1)} \right). \]

Therefore, if we can show that the second term of (4.20) converges to zero as \( k \to \infty \), then we will have the desired result. But \( \{g_k^0\} \) \( L^2 \)-converges to \( g_0 \) and \( \{g_k^1\} \) \( L^2 \)-converges to \( g_1 \). Additionally, \( \chi(M \setminus E)g_0 = \chi(M \setminus E)g_1 = \lim_{k \to \infty} \chi(M \setminus E)g_k^1 \), where the limits are taken in the \( L^2 \) topology. This implies that, again in the \( L^2 \) topology,

\[ \lim_{k \to \infty} (\chi(M \setminus E)g_k^0 + \chi(E)g_k^1) = \lim_{k \to \infty} g_k^1. \]

By Definition 4.38, then,

\[ \lim_{k \to \infty} d_U(\chi(M \setminus E)g_k^0 + \chi(E)g_k^1, g_k^1) = 0, \]

which is what was to be shown. \( \square \)

Next, we need another technical result that will help us in extending the second uniqueness result from amenable subsets to all of \( \mathcal{M} \).

**Proposition 4.42.** Say \( g_0 \in \mathcal{M} \) and \( h \in \mathcal{S} \), and let \( E \subseteq \mathcal{M} \) be any open set. Define an \( L^2 \) tensor \( g_1 \in \mathcal{S}^0 \) by \( g_1 := g_0 + h^0 \), where \( h^0 := \chi(E)h \). Assume that we can find an amenable subset \( \mathcal{U} \) such that \( g_1 \in \mathcal{U}^0 \). Finally, define a path \( g_t \) of \( L^2 \) metrics by \( g_t := g_0 + th^t \), \( t \in [0, 1] \).

Then without loss of generality (by enlarging \( \mathcal{U} \) if necessary), \( g_t \in \mathcal{U}^0 \) for all \( t \), so in particular \( d_U(g_0, g_1) \) is well-defined. Furthermore,

\[ d_U(g_0, g_1) \leq L(g_t) := \int_0^1 \|h^0\|_{g_t} dt, \]

i.e., the length of \( g_t \), when measured in the naive way, bounds \( d_U(g_0, g_1) \) from above.

Lastly, suppose that on \( E \), the metrics \( g_t, t \in [0, 1] \), all satisfy the bounds

\[ |(g_t)_{ij}(x)| \leq C \quad \text{and} \quad \lambda^G_{\min}(x) \geq \delta \]

for some \( C, \delta > 0 \), all \( 1 \leq i, j \leq n \) and a.e. \( x \in E \). (That this is satisfied for some \( C \) and \( \delta \) is guaranteed by \( g_t \in \mathcal{U}^0 \).) Then there is a constant \( K = K(C, \delta) \) such that

\[ d_U(g_0, g_1) \leq K\|h^0\|_g. \]

**Proof.** The existence of the enlarged amenable subset \( \mathcal{U} \) is clear from the construction of \( g_t \). So we turn to the proof of (4.21).

Let any \( \epsilon > 0 \) be given. By Theorem 3.15, we can choose \( \delta > 0 \) such that for any \( \tilde{g}_0, \tilde{g}_1 \in \mathcal{U} \), \( \|\tilde{g}_1 - \tilde{g}_0\|_g < \delta \) implies \( d(\tilde{g}_0, \tilde{g}_1) < \epsilon \).

Next, for each \( k \in \mathbb{N} \), we choose closed sets \( F_k \subseteq E \) and open sets \( U_k \supseteq E \) with the property that \( \operatorname{Vol}(U_k, g) - \operatorname{Vol}(F_k, g) < 1/k \). Given this, let’s even restrict ourselves to \( k \) large enough that

\[ \|\chi(U_k \setminus F_k)h\|_g < \min\{\delta, \epsilon\}. \]

We then choose \( f_k \in C^\infty(M) \) satisfying

1. \( f_k(x) = 1 \) if \( x \in F_k \),
2. \( f_k(x) = 0 \) if \( x \notin U_k \) and
3. \( 0 \leq f_k(x) \leq 1 \) for all \( x \in M \),
The first consequence of our assumptions above is
\begin{equation}
\|g_1 - (g_0 + f_k h)\|_g \leq \|\chi(U_k \setminus F_k)h\|_g < \delta.
\end{equation}

The second inequality is \((4.22)\), and the first inequality holds for two reasons. First, on both \(F_k\) and \(M \setminus U_k\), \(g_0 + f_k h = g_0 + \chi(F_k)h = g_1\). Second, on \(U_k \setminus F_k\), \(g_1 - (g_0 + f_k h) = (1 - f_k)h\), and by our third assumption on \(f_k\), \(0 \leq 1 - f_k \leq 1\). Now, inequality \((4.23)\) allows us to conclude, by our assumption on \(\delta\), that
\begin{equation}
d_d(g_0 + f_k h, g_1) < \epsilon.
\end{equation}

Since by the triangle inequality
\[
d_d(g_0, g_1) \leq d_d(g_0, g_0 + f_k h) + d_d(g_0 + f_k h, g_1) < d_d(g_0, g_0 + f_k h) + \epsilon,
\]
we must now get some estimates on \(d_d(g_0, g_0 + f_k h)\) to prove \((4.21)\).

To do this, define a path \(g_t^k\) in \(\mathcal{M}\), for \(t \in [0, 1]\), by \(g_t^k := g_0 + tf_k h\). Then we have, as is easy to see,
\begin{equation}
d_d(g_0, g_0 + f_k h) \leq L(g_t^k) = \int_0^1 \|f_k h\|_{g_t^k} dt.
\end{equation}

This is almost what we want, but we first have to replace \(f_k h\) with \(h^0 = \chi(E)h\). Also note that the \(L^2\) norm in \((4.25)\) is that of \(g_t^k\). To put this in a form useful for proving \((4.21)\), we therefore also have to replace \(g_t^k\) with \(g_t\).

Using the facts that on \(F_k\), \(f_k h = h^0\) and \(g_t^k = g_t\), as well as that \(f_k = 0\) on \(M \setminus U_k\), we can write
\begin{equation}
\|f_k h\|_{g_t^k}^2 = \int_M \text{tr}_{g_t^k} ((f_k h)^2) \mu_{g_t^k} = \int_{F_k} \text{tr}_{g_t^k} ((h^0)^2) \mu_{g_t^k} + \int_{U_k \setminus F_k} \text{tr}_{g_t^k} ((f_k h)^2) \mu_{g_t^k}.
\end{equation}

For the first term above, we clearly have
\begin{equation}
\int_{F_k} \text{tr}_{g_t^k} ((h^0)^2) \mu_{g_t^k} \leq \|h^0\|_{g_t^k}^2.
\end{equation}

As for the second term, it can be rewritten and estimated by
\[
\int_{U_k \setminus F_k} \text{tr}_{g_t^k} ((f_k h)^2) \mu_{g_t^k} = \|\chi(U_k \setminus F_k) f_k h\|_{g_t^k} \leq \|\chi(U_k \setminus F_k) h\|_{g_t^k},
\]
where the inequality follows from our third assumption on \(f_k\) above. Now, recall that \(g_t\) is contained within an amenable subset \(\mathcal{U}\). It is possible to enlarge \(\mathcal{U}\), without changing the property of being amenable, so that \(\mathcal{U}\) contains \(g_t^k\) for all \(t \in [0, 1]\) and all \(k \in \mathbb{N}\). That the enlarged subset satisfies bounds as in Definition \(3.10\) is clear from the corresponding bounds on \(g_0\) and \(g_t\), and the fact that they are convex, pointwise conditions—cf. part \(4\) of Remark \(3.11\). Therefore, by Lemma \(3.13\), there exists a constant \(K' = K'(g_0, g_1)\)—i.e., \(K'\) does not depend on \(k\)—such that
\[
\|\chi(U_k \setminus F_k) h\|_{g_t^k} \leq K' \|\chi(U_k \setminus F_k) h\|_g.
\]

But by \((4.22)\), we have that \(\|\chi(U_k \setminus F_k) h\|_g < \epsilon\). Combining this with \((4.26)\) and \((4.27)\), we therefore get
\[
\|f_k h\|_{g_t^k}^2 \leq \|h^0\|_{g_t^k} + K' \epsilon.
\]

The above inequality, substituted into \((4.25)\), gives
\[
d_d(g_0, g_t^k) \leq \int_0^1 (\|h^0\|_{g_t^k} + K' \epsilon) \ dt = L(g_t) + K' \epsilon.
\]
The final step in the proof is then to estimate, using the above inequality and (4.24), that
\[ d_U(g_0, g_1) \leq d(g_0, g_0^1) + d_U(g_0^1, g_1) < L(g_1) + (1 + K')\epsilon. \]
Since \( \epsilon \) was arbitrary and \( K' \) is independent of \( k \), we are finished with the proof of (4.21).

Finally, the third statement follows from the following estimate, which is proved in exactly the same way as Lemma 3.13
\[ \|h^0\|_{g_1} = \left( \int_E \text{tr}_{g_1}(h^2) \mu_{g_1} \right)^{1/2} \leq K(C, \delta)\|h^0\|_{g_i}. \]

With Theorem 4.41 and Proposition 4.42 as part of our toolbox, we are now ready to take on the proof of the second uniqueness result in its full generality.

So let two \( d \)-Cauchy sequences \( \{g_k^0\} \) and \( \{g_k^1\} \), as well as some \( g_{\infty} \in \mathcal{M}_f \), be given. Suppose further that \( \{g_k^0\} \) and \( \{g_k^1\} \) both \( \omega \)-converge to \( g_{\infty} \) for \( k \rightarrow \infty \). We will prove that
\[ \lim_{k \rightarrow \infty} d(g_k^0, g_k^1) = 0. \]
The heuristic idea of our proof is very simple, which is belied by the rather technical nature of the rigorous proof. The point, though, is essentially that for all \( l \in \mathbb{N} \), we break \( M \) up into two sets, \( E_l \) and \( M \setminus E_l \). The set \( E_l \) has positive volume with respect to \( g_\infty \), but \( \{g_k^0\} \) and \( \{g_k^1\} \) \( L^2 \)-converge to \( g_\infty \) on \( E_l \), so the contribution of \( E_l \) to \( d(g_k^0, g_k^1) \) vanishes in the limit \( k \rightarrow \infty \). The set \( M \setminus E_l \) contains the deflated sets of \( \{g_k^0\} \) and \( \{g_k^1\} \), so the sequences need not converge on \( M \setminus E_l \). However, we choose things such that \( \text{Vol}(M \setminus E_l, g_\infty) \) vanishes in the limit \( l \rightarrow \infty \), so that Proposition 4.1 implies that the contribution of \( M \setminus E_l \) to \( d(g_k^0, g_k^1) \) vanishes after taking the limits \( k \rightarrow \infty \) and \( l \rightarrow \infty \) in succession.

The rigorous proof is achieved in three basic steps, which we will describe after some brief preparation.

For each \( l \in \mathbb{N} \), let
\[ E_l := \left\{ x \in M \mid \det g_k^i(x) > \frac{1}{l}, \ |(g_k^i)_{rs}(x)| < l \ \forall i = 0, 1; \ k \in \mathbb{N}; \ 1 \leq r, s \leq n \right\}, \]
where these local notions are of course defined with respect to our fixed amenable atlas (cf. Convention 2.53), and the inequalities in the definition should hold in each chart containing the point \( x \) in question. Thus, \( E_l \) is a set over which the sequences \( g_k^i \) neither deflate nor become unbounded. We first note that for each \( k \in \mathbb{N} \), there exists an amenable subset \( U_k \) such that the metrics \( g_k^0, g_k^1 \) and \( g_k^0 + \chi(E_l)(g_k^1 - g_k^0) \) are contained in \( U_k \). This is possible due to smoothness of \( g_k^0 \) and \( g_k^1 \), as well as pointwise convexity of the bounds of Definition 3.10.

The steps in our proof are the following. We will show first that
\[ \lim_{k \rightarrow \infty} d_{U_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) = 0 \]
for all fixed \( l \in \mathbb{N} \). Second,
\[ \lim_{k \rightarrow \infty} d_{U_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \leq 2C(n)\sqrt{\text{Vol}(M \setminus E_l, g_\infty)} \]
for all fixed \( k \in \mathbb{N} \) (where \( C(n) \) is the constant from Theorem 4.41). And third,
\[ \lim_{l \rightarrow \infty} \text{Vol}(E_l, g_\infty) = \text{Vol}(M, g_\infty). \]
Since the triangle inequality of Lemma 4.39(2) implies that
\[ d(g_k^0, g_k^1) \leq d_{U_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) + d_{U_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \]
for all \( l \in \mathbb{N} \), taking the limits \( k \to \infty \) followed by \( l \to \infty \) of both sides then gives (4.28).

We now prove each of (4.30), (4.31) and (4.32) in its own lemma.

**Lemma 4.43.**

\[
\lim_{k \to \infty} d_{U_k}(g_k^0, g_k^0 + \chi(E_i)(g_k^1 - g_k^0)) = 0
\]

**Proof.** We know that

\[
g_k^0, g_k^0 + \chi(E_i)(g_k^1 - g_k^0) \in U_k,
\]

where \( U_k \) is an amenable subset. Therefore, for each fixed \( k \in \mathbb{N} \), Proposition 4.42 applies to give

\[
\left| d_{U_k}(g_k^0, g_k^0 + \chi(E_i)(g_k^1 - g_k^0)) \right| \leq K_l \| \chi(E_i)(g_k^1 - g_k^0) \|_g,
\]

where \( K_l \) is some constant depending only on \( l \). (That the constant only depends on \( l \) is the result of the fact that \( g_k^0 \) and \( g_k^1 \) satisfy the bounds given in (4.29) on \( E_l \), which only depend on \( l \).)

Now, recalling the definition (4.29) of \( E_l \), we note that for all \( 1 \leq i, j \leq n \) and all \( k \in \mathbb{N} \), we have \( |(g_k^0)_{ij}(x) - (g_k^0)_{ij}(x')| \leq 4l^2 \) for \( x \in E_i \), and hence the family of (local) functions

\[
\{ \chi(E_i)((g_k^0)_{ij} - (g_k^0)_{ij}) \mid 1 \leq i, j \leq n, k \in \mathbb{N} \}
\]
is equicontinuous at \( \emptyset \). Furthermore, since property (3) of Definition 4.5 implies that \( \chi(E_i)g_k^h \to \chi(E_l)g_\infty \) a.e. for \( a = 0, 1 \), we have that \( \chi(E_i)(g_k^1 - g_k^0) \to 0 \) a.e. Therefore, Remark 4.34 implies that

\[
\| \chi(E_i)(g_k^1 - g_k^0) \|_g \to 0
\]

for \( k \to \infty \). Together with (4.33), this implies the result immediately. \( \square \)

**Lemma 4.44.**

\[
\lim_{k \to \infty} d_{U_k}(g_k^0 + \chi(E_i)(g_k^1 - g_k^0), g_k^1) \leq 2C(n) \sqrt{\text{Vol}(M \setminus E_i, g_\infty)}
\]

**Proof.** First note that \( g_k^0 = g_k^0 + \chi(E_i)(g_k^1 - g_k^0) \) on \( E_i \). Therefore, by Theorem 4.41

\[
d_{U_k}(g_k^0 + \chi(E_i)(g_k^1 - g_k^0), g_k^1) \leq C(n) \left( \sqrt{\text{Vol}(M \setminus E_i, g_k^0)} + \sqrt{\text{Vol}(M \setminus E_i, g_k^0)} \right).
\]

But now the result follows immediately from Theorem 4.23 since \( \text{Vol}(M \setminus E_i, g_k^0) \to \text{Vol}(M \setminus E_i, g_\infty) \) for \( i = 0, 1 \). \( \square \)

**Lemma 4.45.**

\[
\lim_{l \to \infty} \text{Vol}(E_l, g_\infty) = \text{Vol}(M, g_\infty).
\]

**Proof.** Recall that \( X_{g_\infty} \subseteq M \) denotes the deflated set of \( g_\infty \), i.e., the set where \( g_\infty \) is not positive definite. This set has volume zero w.r.t. \( g_\infty \), since \( \mu_{g_\infty} = 0 \) a.e. on \( X_{g_\infty} \). Therefore \( \text{Vol}(M, g_\infty) = \text{Vol}(M \setminus X_{g_\infty}, g_\infty) \).

We note that \( \chi(E_l) \) converges a.e. to \( \chi(M \setminus X_{g_\infty}) \) and that \( \chi(E_l)(x) \leq 1 \) for all \( x \in M \). Since \( g_\infty \) has finite volume, the constant function 1 is integrable w.r.t. \( \mu_{g_\infty} \), and therefore the Lebesgue dominated convergence theorem (Theorem 2.14) implies that

\[
\lim_{l \to \infty} \text{Vol}(E_l, g_\infty) = \lim_{l \to \infty} \int_M \chi(E_l) \mu_{g_\infty} = \int_M \chi(M \setminus X_{g_\infty}) \mu_{g_\infty} = \text{Vol}(M \setminus X_{g_\infty}, g_\infty).
\]

\( \square \)

As already noted, Lemmas 4.43, 4.44 and 4.45 combine to give the desired result. We summarize what we have just proved in a theorem.
4.3. UNIQUENESS OF THE $\omega$-LIMIT

**Theorem 4.46.** Let $[g_{\infty}] \in \widehat{M}_f$. Suppose we have two sequences $\{g_k^0\}$ and $\{g_k^1\}$ with $g_k^0, g_k^1 \xrightarrow{\omega} [g_{\infty}]$ for $k \to \infty$. Then

$$\lim_{k \to \infty} d(g_k^0, g_k^1) = 0,$$

that is, $\{g_k^0\}$ and $\{g_k^1\}$ are equivalent in the precompletion $\widehat{M}^\text{pre}$ of $M$.

As we have already discussed, combining this theorem with the existence result (Theorem 4.27) and the first uniqueness result (Theorem 4.31) gives us an identification of $M$ with a subset of $\widehat{M}_f$. We summarize this in a theorem:

**Theorem 4.47.** There is a natural identification of $M$, the completion of $M$, with a subset of $\widehat{M}_f$, the measurable semimetrics with finite volume on $M$ modulo the equivalence given in Definition 4.4.

This identification is given by an injection $\Omega : \widehat{M} \hookrightarrow \widehat{M}_f$, where we map an equivalence class $\left([g_k]\right)$ of $d$-Cauchy sequences to the unique element of $\widehat{M}_f$ that all of its members $\omega$-subconverge to. This map is an isometry onto its image if we give $\Omega(M)$ the metric $\bar{d}$ defined by

$$\bar{d}([g_0], [g_1]) := \lim_{k \to \infty} d(g_k^0, g_k^1)$$

where $\{g_k^0\}$ and $\{g_k^1\}$ are any sequences in $M$ $\omega$-converging to $[g_0]$ and $[g_1]$, respectively.

This is an extremely useful theorem, as it allows us to drop the distinction between an $\omega$-convergent sequence and the element of $\widehat{M}_f$ that it converges to. By Lemma 4.6, we can even identify an $\omega$-convergent sequence with any representative of the equivalence class in $\widehat{M}_f$ that it converges to. From now on, we will employ this trick to simplify formulas and proofs.

Our job in the next chapter will be to show that the identification described in Theorem 4.47 is actually a surjection. This will allow us to identify $\widehat{M}$ with the space $\widehat{M}_f$ itself, instead of just a subset thereof. In doing so, we will prove the main result of this thesis.
CHAPTER 5

The completion of $\mathcal{M}$

In this chapter, our previous efforts come to fruition and we are able to complete our description of $\mathcal{M}$ by proving, in Section 5.4, that the map $\Omega : \mathcal{M} \rightarrow \hat{\mathcal{M}}_f$ defined in the previous chapter is a bijection.

To prepare ourselves for this proof, Section 5.1 first looks at a simpler example of a completion, namely that of the orbit space of the conformal group—a submanifold of $\mathcal{M}$ that we first encountered in Section 2.5. This example is not just illustrative of our situation—formally it is extremely similar to our proof of the surjectivity of $\Omega$, though the latter is, of course, significantly more challenging technically. Nevertheless, the computations of this example will be directly employed in the surjectivity proof.

Section 5.2 provides some necessary preparation for the surjectivity proof by going into more depth on the behavior of volume forms under $\omega$-convergence. After this, Section 5.3 presents a partial result on the image of $\Omega$. Namely, we show that all equivalence classes of measurable, bounded semimetrics (cf. Definition 2.57) are contained in $\Omega(\mathcal{M})$. This marks the final preparation we need to prove the main result.

5.1. Completion of the orbit space of $\mathcal{P}$

For our fixed but arbitrary metric $g \in \mathcal{M}$, consider the orbit space $\mathcal{P} \cdot g$. (Later we will consider this space for other metrics $\tilde{g} \in \mathcal{M}$ rather than just our fixed $g$. But since $g$ was chosen arbitrarily, anything we prove about $\mathcal{P} \cdot g$ will hold for $\mathcal{P} \cdot \tilde{g}$ as well.) Recalling that $\mathcal{P}$ is the Fréchet Lie group of smooth, positive functions on $\mathcal{M}$, we see that the orbit consists of metrics of the form $\rho g$, where $\rho$ is a positive $C^\infty$ function. As we have already seen in Subsection 2.3.3, since $\mathcal{P}$ is an open subset of $C^\infty(\mathcal{M})$, each tangent space to $\mathcal{P} \cdot g$ is canonically identified with $C^\infty(\mathcal{M}) \cdot g$, the set of what we called pure trace tensors.

By Proposition 2.45, there is an open set $U \subset C^\infty(\mathcal{M})$ with the property that the exponential mapping $\exp_g$ is a diffeomorphism between the set $U \cdot g$ and $\mathcal{P} \cdot g$. For convenience, we define a mapping

$$\psi : U \xrightarrow{\cong} \mathcal{P} \cdot g \quad \lambda \mapsto \exp_g(\lambda g) = \left(1 + \frac{n}{4} \lambda\right)^\frac{1}{2} g.$$  

Note that $\psi$ is not an isometry on radial geodesics, since the $L^2$ norm induced by $g$ on functions is a non-unit scalar multiple of the $L^2$ norm induced by $g$ on pure trace tensors:

$$(\kappa g, \lambda g)_g = \int_M \text{tr}_g((\kappa g)(\lambda g)) \mu_g = \int_M \kappa \lambda \text{tr}(I) \mu_g = n \int_M \kappa \lambda \mu_g = n(\kappa, \lambda)_g,$$

where we have denoted the $n \times n$ identity matrix by $I$. By the above, if we define a radial geodesic by $g_t = \psi(t\lambda)$ for $t \in [0, 1]$, then we get $L(g_t) = \|\lambda g\|_g = \sqrt{n} \|\lambda\|_g$.

We can even determine the set $U$ explicitly. Let $\lambda \in C^\infty(M)$. Algebraically, we could define $\psi$ for any such $\lambda$, but if we want $\psi(\lambda)$ to be a metric, we must at least require that $\lambda(x) \neq -\frac{4}{n}$ for all $x \in M$. Furthermore, since $\psi$ is defined using $\exp_g$, we should have that if $\psi(\lambda)$ is defined, then $\psi(t\lambda)$ is defined (and is a metric) for $t \in [0, 1]$. This rules out the
possibility that \( \lambda(x) < -\frac{4}{n} \) at some point \( x \in M \), so we see that

\[
U = \left\{ \lambda \in C^\infty(M) \mid \lambda(x) > -\frac{4}{n} \text{ for all } x \in M \right\}.
\]

By Proposition 2.42, \( P \cdot g \) is flat. In the case of a strong Riemannian Hilbert manifold, as in the case of a finite-dimensional manifold, this would imply that \( \exp_g \) is an isometry, and hence that \( \psi \) is an isometry up to a scalar factor. But since \( P \cdot g \) is a weak Riemannian manifold, to make this conclusion we would first have to prove such a general result. This is not necessary, however, as we can show directly that the desired conclusion holds in our case.

**Proposition 5.1.** Up to a scalar factor of \( \sqrt{n} \), \( \psi \) is an isometry. More precisely, if \( d_{P \cdot g} \) is the distance function induced on \( P \cdot g \) as a submanifold of \( (M, \langle \cdot, \cdot \rangle) \), we have

\[
d_{P \cdot g}(\psi(\kappa), \psi(\lambda)) = \sqrt{n}||\lambda - \kappa||_g
\]

for all \( \kappa, \lambda \in U \).

**Proof.** We first note that \( \psi(x) \) is the distance function induced on \( g \) as a submanifold of \( (M, \langle \cdot, \cdot \rangle) \), this would imply that \( \exp_g \) is a diffeomorphism.

Now Proposition 2.29 implies that \( d_{P \cdot g}(\psi(\kappa), \psi(\lambda)) = \sqrt{n}||\varphi^{-1}\psi(\lambda)||_g \), since the shortest path between \( \psi(\kappa) \) and \( \psi(\lambda) \) is the unique radial geodesic emanating from \( \psi(\kappa) \) and ending at \( \psi(\lambda) \). Therefore, we must prove that \( ||\varphi^{-1}\psi(\lambda)||_g = ||\lambda - \kappa||_g \).

We define \( \sigma := \varphi^{-1}\psi(\lambda) \). Then \( \varphi(\sigma) = \psi(\lambda) \) implies, by (5.1) and (5.3), that

\[
\left(1 + \frac{n}{4} \sigma \right)^{\frac{4}{n}} \left(1 + \frac{n}{4} \kappa \right)^{\frac{4}{n}} g = \left(1 + \frac{n}{4} \lambda \right)^{\frac{4}{n}} g.
\]

Solving for \( \sigma \) gives

\[
\varphi^{-1}\psi(\lambda) = \sigma = \frac{4}{n} \left(\left(1 + \frac{n}{4} \lambda \right)^{-1} \left(1 + \frac{n}{4} \kappa \right)^{-1} - 1 \right).
\]

Now, since \( \mu_{\rho g} = \rho^{n/2} \mu_g \) for any \( \rho \in P \), we have

\[
\mu_{\psi(\kappa)} = \left(1 + \frac{n}{4} \kappa \right)^2 \mu_g.
\]

Using these two equations, we finish the proof with a computation:

\[
\begin{align*}
||\varphi^{-1}\psi(\lambda)||_g^2 &= \frac{16}{n^2} \int_M \left( \left(1 + \frac{n}{4} \lambda \right)^{-1} - 1 \right)^2 \mu_{\psi(\kappa)} \\
&= \frac{16}{n^2} \int_M \left( \left(1 + \frac{n}{4} \lambda \right)^{-1} - 1 \right)^2 \left(1 + \frac{n}{4} \kappa \right)^2 \mu_g \\
&= \frac{16}{n^2} \int_M \left( \left(1 + \frac{n}{4} \lambda \right) - \left(1 + \frac{n}{4} \kappa \right) \right)^2 \mu_g \\
&= \int_M (\lambda - \kappa)^2 \mu_g = ||\lambda - \kappa||_g^2.
\end{align*}
\]

\( \Box \)

Using this proposition, we can immediately determine \( \overline{P \cdot g} \), the completion of an orbit of the conformal group.
\textbf{Theorem 5.2.} $\rho_\cdot g$ is isometric to the set of tensors of the form $\rho g$ with $\rho$ measurable, $\rho(x) \geq 0$ a.e., and $\text{Vol}(M, \rho g) < \infty$. Equivalently, this set is those metrics $\rho g$ where $\rho \in L^{n/2}(M)$, i.e., $\int \rho^{n/2} \mu_g < \infty$, and $\rho(x) \geq 0$ a.e.

The distance function on $\rho_\cdot g$ is given by $d_{\rho_\cdot g}(\rho_1 g, \rho_2 g) = \sqrt{n} \| \psi^{-1}(\rho_2 g) - \psi^{-1}(\rho_2 g) \|_g$.

\textbf{Remark 5.3.} Although $L^{n/2}(M)$ is not a normed space for $n = 1$, we simply define it as the set of measurable functions with integrable square root.

\textbf{Proof of Theorem 5.2.} Let’s look at the first statement. The equivalence of the two formulations in the theorem is clear from the fact that $\mu_g = \rho^{n/2} \mu_g$, so $\text{Vol}(M, \rho g) = \int \mu_g = \int \rho^{n/2} \mu_g$. We will therefore show only the second statement.

Since $\psi$ is an isometry, up to a scalar factor, it is clear that $\rho_\cdot g \cong \psi(U)$, where $U$ is the open neighborhood on which $\psi$ is a diffeomorphism. But from (5.2) and the fact that $\| \cdot \|_g$ is the $L^2$ norm on functions, we immediately see that

$$ U = \left\{ \lambda \in L^2(M) \mid \lambda(x) \geq -\frac{4}{n} \text{ a.e.} \right\}, $$

and therefore

$$ \rho_\cdot g = \psi(U) = \left\{ \left(1 + \frac{n}{4} \lambda \right) \frac{4}{n} g \mid \lambda \in L^2(M), \lambda(x) \geq -\frac{4}{n} \text{ a.e.} \right\}. $$

If we define $\rho = \rho(\lambda) := \left(1 + \frac{n}{4} \lambda \right) \frac{4}{n} g$, then it remains to prove that $\rho \in L^{n/2}(M)$ for any $\lambda \in L^2(M)$. But

$$ \int_M \rho^{n/2} \mu_g = \int_M \left(1 + \frac{n}{4} \lambda \right)^2 \mu_g = \int_M \mu_g + \frac{n}{2} \int_M \lambda \mu_g + \frac{n^2}{16} \int_M \lambda^2 \mu_g. $$

The first term in the above expression is finite by compactness of $M$, and the third is finite since $\lambda \in L^2(M)$. Using this, one can then see the second term is finite by H"{o}lder’s inequality.

As for the statement about the distance function, this follows from the fact that $\psi$ extends uniquely to an isometry (up to the scalar factor $\sqrt{n}$) from $U$ to $\rho_\cdot g$. This is thanks to statement (3) of Theorem 2.1.

This theorem immediately tells us what the completion of $\mathcal{M}$ is when $M$ is one-dimensional—of course, there is only one diffeomorphism class of compact one-dimensional manifolds, so in this case $M = S^1$. The theorem gives us complete information here because any smooth metric on $S^1$ can be obtained from the standard metric $g$ by multiplication with a smooth function. Therefore $\mathcal{M} = \rho \cdot g$, and Theorem 5.2 immediately implies:

\textbf{Corollary 5.4.} We work over a one-dimensional base manifold $M$, so that $M = S^1$. Let $g$ be the standard metric on $S^1$. Then

$$ \overline{\mathcal{M}} \cong \left\{ \rho g \mid \sqrt{\rho} \in L^1(M, g), \rho(x) \geq 0 \text{ a.e.} \right\}. $$

Of course, we still have an infinite number of cases left to deal with if we want to find the completion of $\mathcal{M}$ in arbitrary dimension. We need a few preliminary results in order to proceed.

\section*{5.2. Measures induced by measurable semimetrics}

For use in Section 5.4, we need to record a couple of properties of the measure $\mu_{\tilde{g}}$ induced by an element $\tilde{g} \in \mathcal{M}_f$. 
5.2.1. Weak convergence of measures. The first property we wish to prove is the following. Suppose \( \tilde{g} \in \mathcal{M}_f \) and \( \{g_k\} \) is a sequence \( \omega \)-converging to \( \tilde{g} \). Furthermore, let \( \rho \in C^0(\mathcal{M}) \) be any continuous function. Then we claim that

\[
\lim_{k \to \infty} \|\rho\|_{g_k} = \|\rho\|_{\tilde{g}},
\]

where we recall that for any measurable semimetric \( \tilde{g} \) and any function \( \sigma \) on \( \mathcal{M} \),

\[
\|\sigma\|_{\tilde{g}} = \left( \int_{\mathcal{M}} \sigma^2 \mu_{\tilde{g}} \right)^{1/2}.
\]

To prove this, we need to introduce the notion of weak convergence (sometimes also called weak-* convergence) of Borel measures. We do this in the general setting before we apply it to our situation. So let \( X \) be a topological space, and denote by \( \mathfrak{M}(X) \) the set of nonnegative, totally finite measures on the Borel algebra of \( X \). (Recall that a totally finite measure is one for which every measurable set has finite measure.) Suppose that our space \( X \) is completely regular. By this we mean that points and closed sets are separated by continuous functions, i.e., given any closed set \( F \subset X \) and any point \( x \not\in F \), there exists a continuous function \( f : X \to \mathbb{R} \) with \( f(x) = 0 \) and \( f(y) = 1 \) for all \( y \in F \). Most common spaces satisfy this condition; in particular, every topological manifold (and hence our base manifold \( \mathcal{M} \)) is completely regular. In this setting, we can make the following definition.

**Definition 5.5.** The sequence \( \{\nu_k\} \subset \mathfrak{M}(X) \) is said to converge weakly to \( \nu \in \mathfrak{M}(X) \) if for every bounded continuous function \( f \) on \( X \),

\[
\int_X f \, d\nu_k \to \int_X f \, d\nu.
\]

To prove that \( \omega \)-convergence of metrics implies weak convergence of the induced measures, we need the Portmanteau theorem [52 Thm. 8.1], a portion of which we quote here:

**Theorem 5.6 (Portmanteau theorem).** Let \( \nu \) be a measure in \( \mathfrak{M}(X) \), and let \( \{\nu_k\} \) be a sequence in \( \mathfrak{M}(X) \). Then the following conditions are equivalent:

1. \( \nu_k \) converges weakly to \( \nu \),
2. \( \limsup \nu_k(F) = \nu(F) \) for all closed sets \( F \subset X \),

With this theorem at hand, it is a simple matter to prove the claim from above.

**Lemma 5.7.** Let \( \tilde{g} \in \mathcal{M}_f \), and let \( \rho \in C^0(\mathcal{M}) \) be any continuous function. If the sequence \( \{g_k\} \) \( \omega \)-converges to \( \tilde{g} \), then \( \mu_{g_k} \) converges weakly to \( \mu_{\tilde{g}} \), so in particular

\[
\lim_{k \to \infty} \|\rho\|_{g_k} = \|\rho\|_{\tilde{g}}.
\]

**Proof.** We wish to apply Theorem 5.6, which refers to Borel measures. According to our conventions, the measures \( \mu_{g_k} \) and \( \mu_{\tilde{g}} \) are considered as measures on the Lebesgue algebra of \( \mathcal{M} \), but since the Borel algebra is a subalgebra of the Lebesgue algebra, we can use Theorem 5.6 by simply restricting these measures to the Borel algebra.

By Theorem 4.23, condition (2) of Theorem 5.6 holds. Therefore, \( \mu_{g_k} \) converges weakly to \( \mu_{\tilde{g}} \), implying the lemma immediately.

5.2.2. \( L^p \) spaces. We now move on to the next fact we need. In this subsection, we prove that if \( \tilde{g} \in \mathcal{M}_f \), i.e., \( \tilde{g} \) is a measurable, finite-volume semimetric, then the set of \( C^\infty \) functions is dense in \( L^p(\mathcal{M}, \tilde{g}) \) for \( 1 \leq p < \infty \), just as in the case of a smooth volume form. (Of course, by \( L^p(\mathcal{M}, \tilde{g}) \) we mean those functions on \( \mathcal{M} \) whose absolute value to the \( p \)-th power is integrable with respect to \( \mu_{\tilde{g}} \).)

To prove this claim, we first prove a statement about measures on \( \mathbb{R}^n \) that is proved almost identically to [4 Cor. 4.2.2], where the statement is made for Borel measures. To prove it for Lebesgue measures, only one tiny modification is necessary.
Theorem 5.8. Let a nonnegative measure \( \nu \) on the algebra of Lebesgue sets in \( \mathbb{R}^n \) be bounded on bounded sets. Then the class \( C_0^\infty(\mathbb{R}^n) \) of smooth functions with bounded support is dense in \( L^p(\mathbb{R}^n, \nu) \), \( 1 \leq p < \infty \).

Proof. By the proof of [4, Cor. 4.2.2], if \( F \) is any Borel measurable set with \( \nu(F) < \infty \), then \( F \) can be approximated to arbitrary accuracy by sets from the algebra \( \mathcal{A} \) generated by cubes with edges parallel to the coordinate axes. (By this we mean that for any given \( \epsilon > 0 \), we can find a set \( A \subseteq \mathcal{C} \) such that \( \nu(F \setminus A) + \nu(A \setminus F) < \epsilon \).

Now, say that \( E \) is a Lebesgue measurable set with \( \nu(E) < \infty \). Then by Lemma 2.13 \( E = F \cup G \), where \( F \) is Borel measurable and \( \nu(G) = 0 \). By approximating \( F \) with sets from \( \mathcal{C} \), we can therefore approximate \( E \) with sets from \( \mathcal{C} \).

This means that linear combinations of the characteristic functions of sets in \( \mathcal{C} \) are dense in \( L^p(\mathbb{R}^n, \nu) \). But we can easily approximate such functions by smooth functions with compact support—it suffices to be able to approximate any open cube, which is easily done.

Now, since any \( \tilde{g} \in \mathcal{M}_f \) has finite volume, its induced measure \( \mu_{\tilde{g}} \) clearly satisfies the hypotheses of the theorem in any coordinate chart. Therefore, we have:

Corollary 5.9. If \( \tilde{g} \in \mathcal{M}_f \), then \( C^\infty(\mathcal{M}) \) is dense in \( L^p(\mathcal{M}, \tilde{g}) \).

5.3. Bounded semimetrics

In this section, we go one step further in our understanding of the injection \( \Omega : \mathcal{M} \to \mathcal{M}_f \) that was introduced in Theorem 4.47. Specifically, we want to see that the image \( \Omega(\mathcal{M}) \) contains all equivalence classes of bounded, measurable semimetrics (cf. Definition 5.10).

Our strategy for proving this is to first prove the fact for smooth semimetrics by showing that for any smooth semimetric \( g_0 \), there is a finite path \( g_t \), \( t \in (0,1] \), in \( \mathcal{M} \) with \( \lim_{t \to 0} g_t = g_0 \) (where we take the limit in the \( C^\infty \) topology of \( \mathcal{S} \)). Similarly to the constructions in Section 2.1, it is then simple to construct a sequence \( \{g_{t_k}\} \) from \( g_t \) such that \( g_{t_k} \omega \to g_0 \) for \( k \to \infty \). If we simply let \( t_k \) be any monotonically decreasing sequence converging to zero, then it is trivial to show \( \omega \)-convergence of this sequence.

5.3.1. Paths to the boundary. Before we get into the proofs, we put ourselves in the proper setting, for which we first need to introduce the notion of a quasi-amenable subset. These are defined by weakening the requirements for an amenable subset (cf. Definition 3.10), giving up the condition of being “uniformly inflated”:

Definition 5.10. We call a subset \( \mathcal{U} \subseteq \mathcal{M} quasi-amenable if \( \mathcal{U} \) is convex and we can find a constant \( C \) such that for all \( \tilde{g} \in \mathcal{U} \), \( x \in \mathcal{M} \) and \( 1 \leq i,j \leq n \),

\[
|\tilde{g}_{ij}(x)| \leq C.
\]

Quasi-amenable subsets are bounded subsets of \( \mathcal{S} \), but they can run right up to the boundary of \( \mathcal{M} \) as a topological subset of \( \mathcal{S} \). We denote this boundary by \( \partial \mathcal{M} \). Since \( \mathcal{M} \) consists of all smooth \( (0,2) \)-tensor fields on \( \mathcal{M} \) which induce positive definite scalar products on \( T_x \mathcal{M} \) at all \( x \in \mathcal{M} \), we have that each tensor field in \( \partial \mathcal{M} \) induces a smooth, positive semidefinite scalar product at each point of \( \mathcal{M} \). That is,

\[
\partial \mathcal{M} = \{ h \in \mathcal{S} : h \notin \mathcal{M} \text{ and } h(x)(X,X) \geq 0 \text{ for all } X \in T_x \mathcal{M} \}.
\]

So \( \partial \mathcal{M} \) consists of all smooth semimetrics that somewhere fail to be positive definite.

Let \( \mathcal{U} \) be any quasi-amenable subset, and denote by \( \text{cl}(\mathcal{U}) \) the closure of \( \mathcal{U} \) in the \( C^\infty \) topology of \( \mathcal{S} \). Thus, \( \text{cl}(\mathcal{U}) \) may contain some smooth semimetrics.
Now, suppose some \( g_0 \in \text{cl}(U) \cap \partial M \) is given, and let \( g_1 \in U \) have the property that \( h := g_1 - g_0 \in M \), i.e., that \( h \) is positive definite. Exploiting the linear structure of \( M \), we define the simplest path imaginable from \( g_0 \) to \( g_1 \):

\[
g_t := g_0 + th.
\]

Then by the convexity of \( U \), \( g_t \) is a path \((0, 1] \to U\) with limit (in the topology of \( S \)) as \( t \to 0 \) equal to \( g_0 \).

**Remark 5.11.** We make two remarks about this setup:

1. Requiring that \( h > 0 \) is a technical assumption that we will use later; we do not believe it to be essential to the end result.
2. It is not hard to see that any \( g_0 \in \partial M \) is contained in \( \text{cl}(U) \) for an appropriate quasi-amenable subset \( U \).

Recall that the length of \( g_t \) is given by

\[
L(g_t) = \int_0^1 \|g_t\| \, dt = \int_0^1 \left( \int_M \text{tr}_{g_t}((g_t')^2) \mu_{g_t} \right)^{1/2} \, dt
\]

To prove that \( g_t \) is a finite path, we must therefore estimate the integrand,

\[
\text{tr}_{g_t}(h^2) \sqrt{\det(g^{-1}g_t)}.
\]

This will follow from pointwise estimates combined with a compactness/continuity argument.

**5.3.2. Pointwise estimates.** Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be real, symmetric \( n \times n \) matrices, with \( A_t := A + tB \) for \( t \in (0, 1] \). We will assume that \( B > 0 \) and that \( A \geq 0 \). (In this scheme, \( A \) and \( B \) play the role of \( g_0(x) \) and \( h(x) \), respectively, at some point \( x \in M \).) Furthermore, we fix an arbitrary matrix \( C \) that is invertible and symmetric (this plays the role of \( g(x) \)).

Therefore, to get a pointwise estimate on \( \text{tr}_{g_t}(h^2) \sqrt{\det(g^{-1}g_t)} \), we need to estimate \( \text{tr}_{A_t}(B^2) \sqrt{\det(C^{-1}A_t)} \). We prove the desired estimate in three lemmas.

For any symmetric matrix \( D \), let \( \lambda_D^\text{min} = \lambda_D^1 \leq \cdots \leq \lambda_D^n = \lambda_D^\text{max} \) be its eigenvalues numbered in increasing order.

**Lemma 5.12.**

\[
\begin{align*}
\lambda_{\text{min}}^A &\geq \lambda_{\text{min}}^A + t\lambda_{\text{min}}^B, \\
\lambda_{\text{max}}^A &\leq \lambda_{\text{max}}^A + t\lambda_{\text{max}}^B.
\end{align*}
\]

**Proof.** By Lemma 2.10, the function mapping a self-adjoint matrix to its minimal (resp. maximal) eigenvalue is concave (resp. convex). This, combined with the facts that \( \lambda_B^\text{max} > 0 \) (since \( B > 0 \)) and \( t \leq 1 \), gives the result immediately. \( \square \)

**Lemma 5.13.**

\[
\text{tr}_{A_t}(B^2) \sqrt{\det C^{-1}A_t} \leq \frac{n (\lambda_{\text{max}}^B)^2 (\lambda_{\text{max}}^A)^{n-1}}{\sqrt{\det C} (\lambda_{\text{min}}^A + t\lambda_{\text{min}}^B)^{3/2}}.
\]

**Proof.** We focus on the trace term first. Note

\[
\text{tr}_{A_t}(B^2) = \text{tr} \left((A_t^{-1}B)^2\right).
\]
Since $B$ is a symmetric matrix, there exists a basis for which $B$ is diagonal, so that $B = \text{diag}(\lambda_1^B, \ldots, \lambda_n^B)$. In this basis, if we denote $A_t^{-1} = (a_{ij}^t)$, then we have
\[
\text{tr} \left( \left( A_t^{-1} B \right)^2 \right) = \sum_{ij} a_{ij}^t \lambda_i^B a_{ji}^t \lambda_j^B \\
= \sum_{ij} \left( a_{ij}^t \right)^2 \lambda_i^B \lambda_j^B \\
\leq \left( \lambda_{\text{max}}^B \right)^2 \sum_{ij} \left( a_{ij}^t \right)^2 \\
= \left( \lambda_{\text{max}}^B \right)^2 \text{tr} \left( A_t^{-2} \right),
\]
(5.8)
where the second line follows from symmetry of $A_t^{-1}$ and the last line follows from
\[
\text{tr} \left( A_t^{-2} \right) = \sum_{ij} a_{ij}^t a_{ji}^t = \sum_{ij} \left( a_{ij}^t \right)^2.
\]

Now, recall from the discussion in the proof of Lemma 2.35 that the trace of the square of a matrix is given by the sum of the squares of its eigenvalues. Therefore,
\[
\text{tr} \left( A_t^{-2} \right) = \sum_i \left( \lambda_i^{A_t} \right)^{-2} \leq n \left( \lambda_{\text{min}}^{A_t} \right)^{-2}.
\]
(5.9)
This takes care of the trace term.

For the determinant term, we clearly have
\[
\det A_t = \lambda_1^{A_t} \cdots \lambda_n^{A_t} \leq \lambda_{\text{min}}^{A_t} \left( \lambda_{\text{max}}^{A_t} \right)^{n-1}.
\]
Combining equations (5.8), (5.9) and (5.10) with the estimate of Lemma 5.12 now immediately yields the result.

Since $A \geq 0$, we know that $\lambda_{\text{min}}^A \geq 0$. Therefore we can also immediately write the estimate of Lemma 5.14 in a weaker, “worst-case” form:

**Lemma 5.14.**
\[
\text{tr} \left( A_t \right) \left( B^2 \right) \sqrt{\det C^{-1} A_t} \leq \frac{n \left( \lambda_{\text{max}}^B \right)^2 \left( \lambda_{\text{max}}^{A_t} \right)^{n-1}}{\sqrt{\det C \left( \lambda_{\text{max}}^B \right)^{3/2}}} \frac{1}{t^{3/2}}.
\]

**5.3.3. Finiteness of $L(g_t)$.** We want to use the pointwise estimate of Lemma 5.14 to prove the main result of the section.

It is clear that to pass from the pointwise result of Lemma 5.14 to a global result, we will have to estimate the maximum and minimum eigenvalues of $h$, as well as the maximum eigenvalue of $g_t$. We begin by noting that since we work over an amenable coordinate atlas (cf. Definition 2.52), all coefficients of $h$, $g$ and $g_0$ are bounded in absolute value. Therefore, so are their determinants. In particular, since $g > 0$ and $h > 0$, we can assume that $\det g \geq C_0$ and $C_1 \geq \det h \geq C_2$ over each chart of the amenable atlas for some constants $C_0, C_1, C_2 > 0$.

**Lemma 5.15.** The quantities $\lambda_{\text{max}}^h$ and $\lambda_{\text{max}}^g$, as local functions on each coordinate chart, are uniformly bounded, say $\lambda_{\text{max}}^h(x) \leq C_3$ and $\lambda_{\text{max}}^g(x) \leq C_4$ for all $x$ and $t$.

**Proof.** Recall the formula (2.13) for the maximal eigenvalue of a symmetric matrix. If $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in a chart around the point $x \in M$, then
\[
\lambda_{\text{max}}^h(x) = \max_{v \in T_x M} \frac{\langle v, h(x)v \rangle}{\langle v, v \rangle} \quad \text{and} \quad \lambda_{\text{max}}^g(x) = \max_{v \in T_x M} \frac{\langle v, g_t(x)v \rangle}{\langle v, v \rangle},
\]
and
\[
\lambda_{\text{max}}^h(x) = \max_{v \in T_x M} \frac{\langle v, g(x)v \rangle}{\langle v, v \rangle}.
\]
Keep in mind that we work over an amenable atlas and that the unit sphere in each $T_x M$ (with respect to the Euclidean scalar product $\langle \cdot, \cdot \rangle$) is compact. Since $h$ is continuous and $g_t \in \mathcal{U}$ for all $t \in (0,1]$ we can find some constant that bounds $|h_{ij}(x)|$ and $|(g_t)_{ij}(x)|$ uniformly for all $x \in M$, all $1 \leq i, j \leq n$, and all $t \in (0,1]$. From this uniform bound, it is easy to see that there are constants $C_3$ and $C_4$ such that

$$\langle v, h(x)v \rangle \leq C_3, \quad \langle v, g_t(x)v \rangle \leq C_4$$

for all $x \in M$, $t \in (0,1]$ and $v \in T_x M$ with $\langle v, v \rangle = 1$. Since passing to the maximum preserves these inequalities, we get the desired bounds on the eigenvalues. □

Lemma 5.16. The quantity $\lambda_{\text{min}}^h$, as a function over each coordinate chart, is uniformly bounded away from $0$, say $\lambda_{\text{min}}^h \geq C_5 > 0$.

Proof. Letting as usual $\lambda_1^h(x) \leq \cdots \leq \lambda_n^h(x)$ be the eigenvalues of $h(x)$ listed in increasing order, we have

$$\det h(x) = \lambda_1^h(x) \cdots \lambda_n^h(x) \leq \lambda_{\text{min}}^h(x) \lambda_{\text{max}}^h(x)^{n-1}.$$ 

Therefore, by Lemma 5.15 and the discussion before it,

$$\lambda_{\text{min}}^h(x) \geq \lambda_{\text{max}}^h(x)^{1-n} \det h(x) \geq C_3^{1-n} C_2 =: C_5.$$

□

Theorem 5.17. Define a path $g_t$ as in (5.10). Then

$$L(g_t) < \infty.$$ 

Proof. At each point $x \in M$ we have

$$\text{tr}_{g_t(x)}(h(x)^2) \sqrt{\det(g(x)^{-1}g_t(x))} \leq \frac{n \left( \lambda_{\text{max}}^h(x) \right)^2 \left( \lambda_{\text{min}}^g(x) \right)^{n-1} t^{3/2}}{\sqrt{\det g(x)} \left( \lambda_{\text{min}}^h(x) \right)^{3/2}} \leq \frac{1}{\sqrt{C_0}} \frac{C_2^2}{C_5^{3/2}} C_4^{n-1} \frac{1}{t^{3/2}} =: C_6 t^{3/2},$$

where the first inequality follows from Lemma 5.14 and the second line follows from the discussion before Lemma 5.15 as well as Lemma 5.15 itself and Lemma 5.16.

By the integrability of $t^{-3/4}$, then,

$$L(g_t) = \int_0^1 \left( \int_M \text{tr}_{g_t(x)}(h(x)^2) \sqrt{\det(g(x)^{-1}g_t(x))} \mu_g \right)^{1/2} dt \leq \int_0^1 \left( \int_M C_6 t^{3/2} \mu_g \right)^{1/2} dt = \sqrt{C_6 \text{Vol}(M, g)} \int_0^1 \frac{1}{t^{3/2}} dt < \infty.$$ 

□

5.3.4. Bounded, nonsmooth semimetrics. We now move on to showing that the equivalence class of any bounded semimetric, not just smooth ones, is contained in $\Omega(M)$. The results we’ve just proved will come in handy.

Let’s review what we already know about the image of $\Omega$. From Proposition 4.37 we know that the completion of an amenable subset $\mathcal{U}$ can be identified with its $L^2$-completion $\mathcal{U}^\omega$. So the equivalence class of any measurable metric that can be obtained as the $L^2$ limit of a sequence of metrics from an amenable subset belongs to $\Omega(M)$. Furthermore, as we noted in the introduction to this section, it is easy to see that Theorem 5.17 implies that for any smooth semimetric $\tilde{g}$, there exists a sequence in $\mathcal{M}$ that $\omega$-converges to $\tilde{g}$. Thus $\tilde{g}$ also belongs to $\Omega(M)$.

Recall that by the discussion following Theorem 4.47, it is not necessary to distinguish between equivalence classes in $\Omega(M)$ (or individual semimetrics that represent them)
and sequences in \( \mathcal{M} \) that \( \omega \)-converge to them—i.e., points of \( \overline{\mathcal{M}} \). Thus, for the types of (semi)metrics listed in the last paragraph, we will continue to drop this distinction in what follows—expressions like \( d(g_0, g_1) \) are well-defined even when \( g_0 \) and \( g_1 \) are not smooth metrics, as long as we have \([g_0], [g_1] \in \Omega(\overline{\mathcal{M}})\).

We will achieve our goal in this section essentially through studying the completion of a quasi-amenable subset (cf. Definition 5.10) analogously to the methods we used for amenable subsets in Section 3.2.

To begin with, we want to prove a result about quasi-amenable subsets that is a generalization of Theorem 3.15. That result was for amenable subsets, and so we expect the result for quasi-amenable subsets to be weaker. This is indeed the case, but before we can prove the larger result, we first need to prove a couple of lemmas.

**Lemma 5.18.** Let \( \mathcal{U} \subset \mathcal{M} \) be quasi-amenable. Recall that we denote the closure of \( \mathcal{U} \) in the \( C^\infty \) topology of \( \mathcal{S} \) by \( \text{cl}(\mathcal{U}) \), and we denote the boundary of \( \mathcal{M} \) in the \( C^\infty \) topology of \( \mathcal{S} \) by \( \partial \mathcal{M} \). Then for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( d(g_0, g_0 + \delta g) < \epsilon \) for all \( g_0 \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \).

**Proof.** For any \( g_0 \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \), we consider the path \( g_t := g_0 + th \), where \( h := \delta g \) and \( t \in (0, 1] \). The proof consists of reexamining the estimates of Theorem 5.17 and showing that they only depend on upper bounds on the entries of \( g_0 \) (and \( g \), but we get these automatically when we work over an amenable atlas), and that the bound on the length of \( g_t \) goes to zero as \( \delta \to 0 \).

So, recall the main estimate (5.11) of Theorem 5.17:

\[
\text{tr}_{g_t}(h(x)^2)\sqrt{\det(g(x)^{-1}g_t(x))} \leq \frac{n \left( \lambda^h_{\max}(x) \right)^2 \left( \lambda^g_{\max}(x) \right)^{n-1}}{\sqrt{\det g(x)}} \left( \lambda^h_{\min}(x) \right)^{3/2} \frac{1}{\sqrt{\delta^3}},
\]

Since \( \det g(x) \) is constant w.r.t. \( \delta \), we ignore this term. By Lemma 5.12

\[
\lambda^g_{\max}(x) \leq \lambda^g_{\max}(x) + \lambda^h_{\max}(x) = \lambda^g_{\max}(x) + \delta \lambda^g_{\max}(x),
\]

where the final inequality follows since the eigenvalues of \( \delta g(x) \) are clearly just \( \delta \) times the eigenvalues of \( g(x) \). Therefore, using the same arguments as in Lemma 5.15, \( \lambda^g_{\max}(x) \) is bounded from above, uniformly in \( x \) and \( t \), by a constant that decreases as \( \delta \) decreases. Furthermore, this constant does not depend on our choice of \( g_0 \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \), since the proof of Lemma 5.15 depended only on uniform upper bounds on the entries of \( g_0 \), and we are guaranteed the same upper bounds on all elements of \( \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \) since \( \mathcal{U} \) is quasi-amenable.

We now focus our attention on the term

\[
\frac{\left( \lambda^h_{\max}(x) \right)^2}{\left( \lambda^h_{\min}(x) \right)^{3/2}} = \frac{\left( \delta \lambda^g_{\max}(x) \right)^2}{\left( \delta \lambda^g_{\min}(x) \right)^{3/2}} = \frac{\left( \lambda^g_{\max}(x) \right)^2}{\left( \lambda^g_{\min}(x) \right)^{3/2}} \sqrt{\delta}.
\]

This expression clearly goes to zero as \( \delta \to 0 \). Therefore, we have shown that the constant \( C_6 \) in the estimate (5.11) depends only on the choice of \( \mathcal{U} \) and \( \delta \), and that \( C_6 \to 0 \) as \( \delta \to 0 \). The result now follows.

The next lemma implies, in particular, that \( \partial \mathcal{M} \) is not closed in the \( L^2 \) topology of \( \mathcal{S} \), nor is it in the topology of \( d \) on \( \Omega(\overline{\mathcal{M}}) \). It also implies that around any point in \( \mathcal{M} \), there exists no \( L^2 \)- or \( d \)-open neighborhood.

**Lemma 5.19.** Let \( \mathcal{U} \subset \mathcal{M} \) be any quasi-amenable subset. Then for all \( \epsilon > 0 \), there exists a function \( \rho_\epsilon \in C^\infty(\mathcal{M}) \) with the properties that for all \( g_1 \in \mathcal{U} \),

1. \( \rho_\epsilon g_1 \in \partial \mathcal{M} \),
2. \( \rho_\epsilon(x) \leq 1 \) for all \( x \in \mathcal{M} \),
3. \( \|g_1 - \rho_\epsilon g_1\|_g < \delta \) and
(4) \( d(g_1, \rho \circ g_1) < \epsilon. \)

**Proof.** Let \( x_0 \in M \) be any point, and for each \( n \in \mathbb{N} \), choose a function \( \rho_n \in C^\infty(M) \) satisfying

1. \( \rho_n(x_0) = 0, \)
2. \( 0 \leq \rho_n(x) \leq 1 \) for all \( x \in M \) and
3. \( \rho_n \equiv 1 \) outside an open set \( Z_n \) with \( \text{Vol}(Z_n, g) \leq 1/n. \)

Then clearly \( \|g_1 - \rho_n g_1\|_g \to 0 \) as \( n \to \infty \), and this convergence is uniform in \( g_1 \) because of the upper bounds guaranteed by the fact that \( g_1 \in \mathcal{U}. \)

Furthermore, if we estimate the length of the path \( g^n_i := \rho_n g_1 + t(g_1 - \rho_n g_1) \), there will be no contribution to the integral from points of \( M \setminus Z_n \), and on \( Z_n \), we can find a constant \( C_6 \) as in (5.11) that does not depend on \( n \), simply by assuming the worst case that \( \rho_n \equiv 0 \) on \( Z_n \) for all \( n \). Furthermore, \( C_6 \) does not depend on \( g_1 \), just on the choice of \( \mathcal{U} \), by the same arguments as in the proof of Lemma 5.18.

Therefore we get that

\[
L(g^n_i) \leq \sqrt{C_6 \text{Vol}(Z_n, g)} \int_0^1 \frac{1}{t^{3/4}} \, dt \leq \sqrt{\frac{C_6}{n}} \int_0^1 \frac{1}{t^{3/4}} \, dt,
\]

which converges to zero as \( n \to \infty \). Choosing \( n \) large enough completes the proof. \( \square \)

The next theorem is the desired analog of Theorem 3.15. Note that only one half of Theorem 5.19 holds in this case, and even this is proved only in a weaker form.

**Theorem 5.20.** Let \( \mathcal{U} \subset \mathcal{M} \) be quasi-amenable. Then for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( g_0, g_1 \in \text{cl}(\mathcal{U}) \) with \( \|g_0 - g_1\|_g < \delta \), then \( d(g_0, g_1) < \epsilon \).

In particular, \( d \) is uniformly continuous on \( \mathcal{M} \) restricted to \( \text{cl}(\mathcal{U}) \), and if \( \phi : (\text{cl}(\mathcal{U}), \| \cdot \|_g) \to (\text{cl}(\mathcal{U}), d) \) is the identity mapping on the level of sets (i.e., \( \phi(g) = g \)), then \( \phi \) is uniformly continuous.

**Proof.** First, we enlarge \( \mathcal{U} \) if necessary to include all metrics satisfying the bound given in Definition 5.10. This enlarged \( \mathcal{U} \) is then clearly convex by the triangle inequality for the absolute value, and hence it is still a quasi-amenable subset.

Now, let \( \epsilon > 0 \) be given. We prove the statement first for \( g_0, g_1 \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \), then use this to prove the general case.

By Lemma 5.18 we can choose \( \delta_1 > 0 \) such that \( d(\tilde{g}, \tilde{g} + \delta_1 g) < \epsilon/3 \) for all \( \tilde{g} \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \). We define an amenable subset of \( \mathcal{M} \) by

\[
\mathcal{U} : = \{ \tilde{g} + \delta_1 g \mid \tilde{g} \in \mathcal{U} \}.
\]

This set is, indeed, amenable, since for each \( x \in M \), Lemma 2.10 implies that

\[
\lambda_{\text{min}}^{\tilde{g} + \delta_1 g}(x) \geq \lambda_{\text{min}}^{\tilde{g}}(x) + \lambda_{\text{min}}^{\delta_1 g}(x) \geq \delta_1 \lambda_{\text{min}}^{\tilde{g}}(x).
\]

Now, by Theorem 3.15 there exists \( \delta > 0 \) such that if \( \tilde{g}_0, \tilde{g}_1 \in \mathcal{U} \) with \( \|\tilde{g}_0 - \tilde{g}_1\|_g < \delta \), then \( d(\tilde{g}_0, \tilde{g}_1) < \epsilon/3 \). Let \( g_0, g_1 \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \) be such that \( \|g_0 - g_1\|_g < \delta \). If we define \( \tilde{g}_i := g_i + \delta_1 g \) for \( i = 1, 2 \), then it is clear that \( \|\tilde{g}_0 - \tilde{g}_1\|_g = \|g_0 - g_1\|_g < \delta \). Given this and the definition of \( \delta_1 \), we have

\[
d(g_0, g_1) \leq d(g_0, \tilde{g}_0) + d(\tilde{g}_0, \tilde{g}_1) + d(\tilde{g}_1, g_1) < \epsilon.
\]

Now we prove the general case. Let \( \epsilon > 0 \) be given. By the special case we just proved, we can choose \( \delta > 0 \) such that if \( g_0, g_1 \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M} \) with \( \|g_0 - g_1\|_g < \delta \), then \( d(g_0, g_1) < \epsilon/3 \). Let \( g_0, g_1 \in \mathcal{U} \) be any elements with \( \|g_0 - g_1\|_g < \delta \). By Lemma 5.19 and our enlargement of \( \mathcal{U} \), we can choose a function \( \rho \in C^\infty(M) \) such that for \( i = 0, 1, \)

1. \( \rho g_i \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M}, \)
2. \( \rho(x) \leq 1 \) for all \( x \in M \), and
3. \( d(g_i, \rho g_i) < \epsilon/3. \)
(If $g_i \in \text{cl}(\mathcal{U}) \cap \partial \mathcal{M}$ for both $i = 1$ and $2$, we might as well just choose $\rho \equiv 1$.) In particular, the second property of $\rho$ implies that
\[
\|\rho g_1 - \rho g_0\|_g \leq \|g_1 - g_0\|_g < \delta.
\]
Then we immediately get
\[
d(g_0, g_1) \leq d(g_0, \rho g_0) + d(\rho g_0, \rho g_1) + d(\rho g_1, g_1) < \epsilon.
\]
This proves the general case and thus the theorem.

Remark 5.21. Let’s take a brief moment to discuss why only one half of Theorem 3.15 holds for quasi-amenable subsets. The problem is that the determinants of elements of a quasi-amenable subset need not satisfy any uniform lower bounds. Hence two metrics $g_0$ and $g_1$ from a quasi-amenable subset can differ greatly with respect to $\| \cdot \|_g$, yet do so only on a subset of $M$ that has small volume with respect to $g_0$ and $g_1$ themselves. In this situation, Proposition 4.1 implies that $d(g_0, g_1)$ will also be small. So we cannot say that $\|g_1 - g_0\|_g$ is small whenever $d(g_0, g_1)$ is, and something like statement (2) of Theorem 3.15 cannot hold for quasi-amenable subsets.

With the above theorem at hand, it is now possible to obtain the information on $\Omega(M)$ that we desired. First notice that a bounded semimetric is precisely a semimetric that can be obtained as the $L^2$ limit of a sequence of metrics contained within some quasi-amenable subset.

Using the relationship between $d$ and $\| \cdot \|_g$ determined in Theorem 5.20, we can prove the following.

Proposition 5.22. Let $[\tilde{g}] \in \hat{M}_f$ be an equivalence class containing at least one bounded, measurable semimetric. Then for any bounded representative $\tilde{g} \in [\tilde{g}]$, there exists a sequence $\{g_k\}$ in $M$ that both $L^2$- and $\omega$-converges to $\tilde{g}$. Thus $[\tilde{g}] \in \Omega(M).$

Moreover, suppose $\tilde{g} \in U^0$ for some quasi-amenable subset $U \subset M$. Then for any sequence $\{g_t\}$ in $U$ that $L^2$-converges to $\tilde{g}$, $\{g_t\}$ is $d$-Cauchy and there exists a subsequence $\{g_k\}$ that also $\omega$-converges to $\tilde{g}$.

Proof. By the discussion preceding the proposition, for every representative $\tilde{g} \in [\tilde{g}]$, we can find a quasi-amenable subset $U \subset M$ such that $\tilde{g} \in U^0$. (Recall that $U^0$ denotes the completion of $U$ w.r.t. $\| \cdot \|_g$, cf. Definition 4.18.) Thus, there exists a sequence $\{g_t\}$ that $L^2$-converges to $\tilde{g}$. It is $d$-Cauchy by Theorem 5.20. We wish to show that it contains a subsequence $\{g_k\}$ that also $\omega$-converges to $\tilde{g}$, so we still need to verify properties (2)–(4) of Definition 4.5 (we just noted that $\{g_t\}$ is $d$-Cauchy, so property (1) holds).

By passing to a subsequence, we can assume that property (4) is satisfied for $\{g_t\}$. We verify property (3) in the same way as in the proof of Lemma 4.36. That is, $L^2$-convergence of $\{g_t\}$ implies by Remark 4.34 that there exists a subsequence $\{g_k\}$ of $\{g_t\}$ that converges to $\tilde{g}$ a.e. Finally, a.e.-convergence of $\{g_k\}$ to $\tilde{g}$ and continuity of the determinant function imply that property (2) holds.

Thus, like we did for more restricted types of metrics before, this proposition allows us to cease to distinguish between bounded semimetrics and sequences $\omega$-converging to them.

5.4. Unbounded metrics and the proof of the main result

Up to this point, we have an injection $\Omega : \hat{M} \to \hat{M}_f$, and we have determined that the image $\Omega(\hat{M})$ contains all equivalence classes containing bounded semimetrics. In this section, we prove that $\Omega$ is surjective. We will make good use of what we already know about $\Omega(\hat{M})$ in order to do so.
The following theorem is the surjectivity statement. It is proved using the same philosophy as in the construction of the completion of $P \cdot g$ that was given in Section 5.1. We simply need to adapt the arguments given there to our situation.

**Theorem 5.23.** Let any $[\hat{g}] \in \overline{M}_f$ be given. Then there exists a sequence $\{g_k\}$ in $M$ such that

$$g_k \overset{\omega}{\rightarrow} [\hat{g}].$$

Thus, $\Omega : \overline{M} \rightarrow \overline{M}_f$ is surjective.

**Proof.** In view of Proposition 5.22, it remains only to prove this for the equivalence class of a measurable, unbounded semimetric $\tilde{g} \in M_f$.

Given any element $\hat{g} \in M_f$, we can define $\exp_\lambda$ on tensors of the form $\sigma \tilde{g}$, where $\sigma$ is any function, purely algebraically. We simply set

$$\exp_\lambda(\sigma \tilde{g}) := \left(1 + \frac{n}{4} \sigma\right)^{4/n} \tilde{g},$$

so that the expression coincides with the usual one if $\sigma \in C^\infty(M)$ with $\sigma > -\frac{4}{n}$ (cf. (5.2)). If $\sigma$ is additionally measurable, then $\exp_\lambda(\sigma \tilde{g})$ will also be measurable.

Now, let $\hat{g} \in M_f$. Then we can find a measurable, positive function $\xi$ on $M$ such that $g_0 := \xi \hat{g}$ is a bounded semimetric. The same calculation as in the proof of Theorem 5.2 shows that finite volume of $\hat{g}$ implies $\rho := \xi^{-1} \in L^{n/2}(M, g_0)$.

Define the map $\psi$ by $\psi(\sigma) := \exp_{g_0}(\sigma g_0)$, and let

$$\lambda := \frac{4}{n} \left(\rho^{n/4} - 1\right).$$

Then clearly $\psi(\lambda) = \rho g_0 = \hat{g}$. Moreover, we claim that $\lambda \in L^2(M, g_0)$ and hence, by Corollary 5.9, we can find a sequence $\{\lambda_k\}$ of smooth functions that converge in $L^2(M, g_0)$ to $\lambda$. That $\lambda \in L^2(M, g_0)$ follows from two facts. First, $\rho \in L^{n/2}(M, g_0)$, implying that $\rho^{n/4} \in L^2(M, g_0)$. Second, finite volume of $g_0$ implies that the constant function $1 \in L^2(M, g_0)$ as well.

Since $\lambda_k \rightarrow \lambda$ in $L^2(M, g_0)$, Remark 4.34 implies that by passing to a subsequence, we can also assume that $\lambda_k \rightarrow \lambda$ pointwise a.e., where we note that here, “almost everywhere” means with respect to $\mu_{g_0}$. With respect to the fixed, smooth, strictly positive volume form $\mu_g$, this actually means that $\lambda_k(x) \rightarrow \lambda(x)$ for a.e. $x \in M \setminus X_{g_0}$, since $X_{g_0}$ is a nullset with respect to $\mu_{g_0}$. Note also that $X_{g_0} = X_{\hat{g}}$, since we assumed that the function $\xi$ is positive. Therefore $\lambda_k(x) \rightarrow \lambda(x)$ for a.e. $x \in M \setminus X_{\hat{g}}$.

Furthermore, since from (5.13) and positivity of $\xi$ it is clear that $\lambda > -\frac{4}{n}$, we can choose the sequence $\{\lambda_k\}$ such that $\lambda_k > -\frac{4}{n}$ for all $k \in \mathbb{N}$. This implies, in particular, that $X_{\psi(\lambda_k)} = X_{g_0} = X_{\hat{g}}$, which is easily seen from (5.12).

We make one last assumption on the sequence $\{\lambda_k\}$. Namely, by passing to a subsequence, we can assume that

$$\sum_{k=1}^{\infty} ||\lambda_{k+1} - \lambda_k||_{g_0} < \infty.$$  

Using a limiting argument, we can show a statement analogous to, but weaker than, Proposition 5.1. Namely, if $d$ is the metric on $\Omega(\overline{M})$ defined in Theorem 4.47, then

$$d(\psi(\sigma), \psi(\tau)) \leq \sqrt{n} ||\tau - \sigma||_{g_0}$$

for all $\sigma, \tau \in C^\infty(M)$ with $\sigma, \tau > -\frac{2}{n}$. We delay the proof of this statement to Lemma 5.24 below, though, and first finish the proof of the theorem.

We wish to construct a sequence that $\omega$-converges to $\hat{g}$ using the sequence $\{\psi(\lambda_k)\}$. We can’t use $\{\psi(\lambda_k)\}$ directly, since it is a sequence in $\Omega(\overline{M})$, not $M$ itself. So we first verify the properties of $\omega$-convergence for $\{\psi(\lambda_k)\}$ and then construct a sequence
in \( \mathcal{M} \) that approximates \( \{ \psi(\lambda_k) \} \) well enough that it still satisfies all the conditions for \( \omega \)-convergence.

Since the sequence \( \{ \lambda_k \} \) is convergent in \( L^2(M, g_0) \), it is also Cauchy in \( L^2(M, g_0) \). Using the inequality (5.15), it is then immediate that \( \{ \psi(\lambda_k) \} \) is a Cauchy sequence in \( (\Omega(\mathcal{M}), d) \). This verifies property [1] of \( \omega \)-convergence (cf. Definition 4.4).

We next verify property [2]. Note that \( X_{\tilde{g}} \supseteq X_{\psi(\lambda_k)} \), since we have already shown that \( X_{\psi(\lambda_k)} = X_{\tilde{g}} \). (Keep in mind here the subtle point that \( X_{\psi(\lambda_k)} \) is the deflated set of the \textit{individual} semimetric \( \psi(\lambda_k) \), while \( X_{\psi(\lambda_k)} \) is the deflated set of the sequence \( \{ \psi(\lambda_k) \} \). Refer to Definitions 2.57 and 2.58 for details.) The inclusion implies that

\[
M \setminus X_{\psi(\lambda_k)} \subseteq M \setminus X_{\tilde{g}},
\]

so it suffices to show that \( \psi(\lambda_k)(x) \to \tilde{g}(x) \) for a.e. \( x \in M \setminus X_{\tilde{g}} \). But this is clear from the definition of \( \psi \) and the fact, proved above, that \( \lambda_k(x) \to \lambda(x) \) for a.e. \( x \in M \setminus X_{\tilde{g}} \).

To verify property [3], we claim that \( X_{\psi(\lambda_k)} = X_{\tilde{g}} \), up to a nullset. In the previous paragraph, we already showed that \( X_{\tilde{g}} \subseteq X_{\psi(\lambda_k)} \). Furthermore, for a.e. \( x \in M \setminus X_{\tilde{g}} \), \( \{ \psi(\lambda_k)(x) \} \) converges to \( \tilde{g}(x) \), which is positive definite, so for a.e. \( x \in M \setminus X_{\tilde{g}} \), \( \lim \det \psi(\lambda_k) > 0 \). This immediately implies that \( X_{\psi(\lambda_k)} \subseteq X_{\tilde{g}} \), up to a nullset.

The last property to verify is [4]. But this is immediate from (5.14) and (5.15).

So we have shown that \( \{ \psi(\lambda_k) \} \) satisfies the properties of \( \omega \)-convergence, save that it is a sequence of measurable semimetrics, rather than a sequence of smooth metrics as required. To get a sequence in \( \mathcal{M} \) that \( \omega \)-converges to \( \tilde{g} \), recall that each of the functions \( \lambda_k \) is smooth and therefore bounded, and also that \( g_0 \) is a bounded, measurable semimetric. Therefore, for each fixed \( k \in \mathbb{N} \), \( \psi(\lambda_k) \) is also a bounded, measurable semimetric, and so by Proposition 5.22 we can find a sequence \( \{ g^k \} \) in \( \mathcal{M} \) that \( \omega \)-converges to \( \psi(\lambda_k) \) for \( l \to \infty \). By a standard diagonal argument, it is then possible to select \( l_k \in \mathbb{N} \) for each \( k \in \mathbb{N} \) such that the sequence \( \{ g^{l_k} \} \) \( \omega \)-converges to \( \tilde{g} \) for \( k \to \infty \). Thus we have found the desired sequence.

It still remains to prove (5.15). The following lemma does this and thus completes the proof of the theorem.

**Lemma 5.24.** If \( \sigma, \tau \in C^\infty(M) \) satisfy \( \sigma, \tau > -\frac{4}{\pi} \), then

\[
d(\psi(\sigma), \psi(\tau)) \leq \sqrt{n}||\tau - \sigma||_{g_0}.
\]

**Proof.** Since \( g_0 \) is bounded, we can find a quasi-amenable subset \( \mathcal{U} \) such that \( g_0 \in \mathcal{U}^0 \), i.e., such that \( g_0 \) belongs to the completion of \( \mathcal{U} \) with respect to \( \| \cdot \|_g \). Using Proposition 5.22 choose a sequence \( \{ g_k \} \) in \( \mathcal{U} \) that both \( L^2 \)- and \( \omega \)-converges to \( g_0 \). For each \( k \in \mathbb{N} \), define a map \( \psi_k \) by \( \psi_k(\kappa) := \exp_{g_k} (\kappa g_k) \).

By the triangle inequality, we have

(5.16) \[ d(\psi(\sigma), \psi(\tau)) \leq d(\psi(\sigma), \psi_k(\sigma)) + d(\psi_k(\sigma), \psi_k(\tau)) + d(\psi_k(\tau), \psi(\tau)) \]

for each \( k \). But since \( g_k \in \mathcal{M} \), Proposition 5.1 applies to give

(5.17) \[ d(\psi_k(\sigma), \psi_k(\tau)) \leq \sqrt{n}||\tau - \sigma||_{g_k} \]

\[ \to 0 \quad \text{and} \quad d(\psi_k(\tau), \psi(\tau)) \to 0, \]

then we are finished. In fact, if it holds for one, then it clearly holds for the other, so we prove it only for \( \sigma \).
Since $g_k, g_0 \in \mathcal{U}$, it suffices by Proposition 5.22 to show that $\psi_k(\sigma)$ $L^2$-converges to $\psi(\sigma)$. But this is simple, for if we set

$$\alpha := \left(1 + \frac{n}{4}\sigma\right)^{4/n},$$

then $\psi_k(\sigma) = \alpha g_k$ and $\psi(\sigma) = \alpha g_0$. Thus

$$\|\psi(\sigma) - \psi_k(\sigma)\|_g = \|\alpha g_0 - \alpha g_k\|_g \leq \max |\alpha| \cdot \|g_0 - g_k\|_g \to 0,$$

where the convergence follows from our assumptions on the sequence $g_k$. □

From Theorem 4.47, we already know that the map $\Omega : \mathcal{M} \to \widehat{\mathcal{M}}_f$ is an injection. Theorem 5.23 now states that this map is a surjection as well. Thus, we have already proved the main result of this thesis, which we state again here in full detail.

**Theorem 5.25.** There is a natural identification of $\mathcal{M}$, the completion of $\mathcal{M}$ with respect to the $L^2$ metric, with $\widehat{\mathcal{M}}_f$, the set of measurable semimetrics with finite volume on $\mathcal{M}$ modulo the equivalence given in Definition 4.4.

This identification is given by a bijection $\Omega : \mathcal{M} \to \widehat{\mathcal{M}}_f$, where we map an equivalence class $[\{g_k\}]$ of $d$-Cauchy sequences to the unique element of $\widehat{\mathcal{M}}_f$ that all of its members $\omega$-subconverge to. This map is an isometry if we give $\widehat{\mathcal{M}}_f$ the metric $\bar{d}$ defined by

$$\bar{d}([g_0], [g_1]) := \lim_{k \to \infty} d(g_0^k, g_1^k)$$

where $\{g_0^k\}$ and $\{g_1^k\}$ are any sequences in $\mathcal{M}$ $\omega$-subconverging to $[g_0]$ and $[g_1]$, respectively.

As an end to this chapter, before we describe our application of this theorem, we briefly discuss the geometry of elements of $\widehat{\mathcal{M}}_f$. In fact, an element of $\widehat{\mathcal{M}}_f$ does not define a geometry in the usual sense, since the metric space structure does not agree between different representatives of one equivalence class. To illustrate this, let’s again take our favorite example $\mathcal{M} = T^2$, and consider the two equivalent semimetrics

$$g_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

As metric spaces, $g_0$ is just a point (the torus is completely collapsed) and $g_1$ is a round circle (one dimension of the torus has collapsed).

On the other hand, since representatives of a given equivalence class in $\widehat{\mathcal{M}}_f$ all have equal induced measures, things like $L^p$ spaces of functions are well-defined for an equivalence class, as they are the same across all representatives. But even more is true—$C^k$ and $H^s$ spaces of sections of fiber bundles can again be defined as in Subsection 2.2.3 since not only are the measures induced by two representatives equal, but the representatives themselves are equal almost everywhere with respect to their common measure. Therefore, an equivalence class doesn’t induce a well-defined scalar product on a vector bundle at any individual point, but the integral of the scalar product does not depend on the chosen representative.

So while one must be careful about regarding an element of $\widehat{\mathcal{M}}_f$ as defining a geometry, there are indeed many geometric concepts that are well-defined for elements of $\widehat{\mathcal{M}}_f$. 
CHAPTER 6

Application to Teichmüller theory

In this chapter, we describe an application of our main theorem to the theory of Teichmüller space. Teichmüller space was historically defined in the context of complex manifolds, but the work of Fischer and Tromba translates this original picture into the context of Riemannian geometry, using the manifold of metrics. (See, in particular, the papers [16] and [17], as well as the related [15], [18], [53] and [54].)

In Section 6.1, we describe Teichmüller space according to Fischer and Tromba’s picture. Along with discussing some properties of Teichmüller space, we also describe a much-studied Riemannian metric on it, the so-called Weil-Petersson metric. The Weil-Petersson metric arises very naturally in this context, and there is also a very natural way to generalize it, which we give in Section 6.2. It is in this section that our application is given.

The book [55] is an excellent presentation of Fischer and Tromba’s approach to Teichmüller space. It is essentially a self-contained work incorporating the references listed above. We will use it as the standard reference in this chapter—any facts that are not directly cited or proved can be found in this book.

6.1. Teichmüller space

Convention 6.1. For the entirety of this chapter, let our base manifold $M$ be a smooth, closed, oriented, two-dimensional manifold of genus $p \geq 2$.

Convention 6.2. In this chapter, we abandon Convention 2.51. That is, when we write $g$ for a metric in $M$, we no longer assume that this is fixed, but allow $g$ to vary arbitrarily.

6.1.1. The definition of Teichmüller space. Since the group $P$ of positive $C^\infty$ functions acts on $M$ by pointwise multiplication, we can define the quotient space by this action, $M/P$. It is not hard to see that this action is smooth, free and proper, from which one can show that the quotient space $M/P$ is a smooth Fréchet manifold. This is called the manifold of conformal classes on $M$. The name comes from that of a conformal class $[g]$, which is the set of all metrics of the form $\rho g$, where $\rho$ is a smooth, positive function.

We cannot use a conformal class to give a well-defined notion of the length of vectors in a tangent space, since this varies among representatives of the class. However, the angle between two vectors is the same for all representatives, so this notion is well-defined for conformal classes. This is analogous to the way that a conformal mapping preserves angles—in fact, the identity mapping $(M,g) \to (M,\rho g)$ is obviously conformal for any $g \in M$ and $\rho \in P$.

Let $D$ denote the Fréchet Lie group of smooth, orientation-preserving diffeomorphisms of $M$ (cf. Remark 2.7). There is an action of $D$ on $M$ given by pull-back. Actually, we can even define the action on $S$: for all $\phi \in D$, $h \in S$, $x \in M$, and $v, w \in T_x M$, the explicit formula is

$$\phi^* h(x)(v, w) = h(\phi(x))(D\phi(x)v, D\phi(x)w).$$

This action is compatible with the $P$-action on $M$ in the sense that if $g_0$ and $g_1$ are equivalent under the $P$-action, say $g_1 = \rho g_0$, then $\phi^* g_0$ and $\phi^* g_1$ are also equivalent under
the $\mathcal{P}$-action, since $\varphi^* g_1 = (\rho \circ \varphi) \varphi^* g_0$. In other words, there is a natural action of $\mathcal{D}$ on $\mathcal{M}/\mathcal{P}$ that makes the projection $\tilde{\pi} : \mathcal{M} \to \mathcal{M}/\mathcal{P}$ $\mathcal{D}$-equivariant. Thus, we can define the quotient space

$$\mathcal{R} := (\mathcal{M}/\mathcal{P})/\mathcal{D},$$

which is called the Riemann moduli space of $\mathcal{M}$, or usually just moduli space.

**Remark 6.3.** As we mentioned above, the Riemann moduli space (and Teichmüller space, which we’ll meet later) was originally defined in terms of complex structures on $\mathcal{M}$, not metrics. It turns out that the manifold $\mathcal{M}/\mathcal{P}$ is, in a sense, diffeomorphic to the manifold of complex structures on $\mathcal{M}$, which gives the connection to the original theory. Since we don’t need this connection for our purposes, however, we omit it and instead refer the reader again to [55] for details. The approach we take here may be less familiar, but is more economic given our previous preparations.

Moduli space has a somewhat technically challenging structure. Since the action of $\mathcal{D}$ on $\mathcal{M}$ has a fixed point at any metric with a nontrivial isometry group, moduli space has singularities. It turns out that these are not very difficult to deal with, as they are only orbifold singularities—this follows from the fact that the isometry group of a Riemann surface with genus $p \geq 2$ is necessarily finite (see Lemma 6.7 below). Yet one might still prefer to work with a smooth manifold. Teichmüller space is a manifold that can be seen as a sort of intermediate space between the manifold of conformal classes and moduli space. We define this now.

Let $\mathcal{D}_0 \subset \mathcal{D}$ be the subset of diffeomorphisms that are homotopic to the identity. It turns out that the action of $\mathcal{D}_0$ on $\mathcal{M}$ is free and, though the proof is quite involved, one can show that the quotient space

$$\mathcal{T} := (\mathcal{M}/\mathcal{P})/\mathcal{D}_0$$

is a smooth manifold. (This is not done directly, but rather using the intermediate step of identifying $\mathcal{M}/\mathcal{P}$ with the space of hyperbolic metrics on $\mathcal{M}$; see below.) This quotient space is the Teichmüller space of $\mathcal{M}$, or simply Teichmüller space.

Not only is Teichmüller space a smooth manifold, it is finite-dimensional. This allows us to avoid the many difficulties that arise when dealing with infinite-dimensional spaces like $\mathcal{M}$.

The mapping class group of $\mathcal{M}$ is defined to be $\text{MCG} := \mathcal{D}/\mathcal{D}_0$. This group acts on Teichmüller space, and we have

$$\mathcal{R} = \mathcal{T}/\text{MCG}.$$ 

The general philosophy to keep in mind in this setup is that natural objects on Teichmüller space should be $\text{MCG}$-invariant so that they descend to moduli space. This corresponds to ensuring that objects defined on $\mathcal{M}/\mathcal{P}$ are $\mathcal{D}$-invariant and not just $\mathcal{D}_0$-invariant.

**6.1.2. The Weil-Petersson metric on Teichmüller space.** Before we can define the Weil-Petersson metric, we need to discuss hyperbolic metrics on $\mathcal{M}$. In particular, the following theorem allows us to identify the quotient space $\mathcal{M}/\mathcal{P}$ with the set of hyperbolic metrics on $\mathcal{M}$. We define a hyperbolic metric as one that has constant scalar curvature $-1$. (Other authors may use the sectional curvature or Gaussian curvature, which differs from the scalar curvature simply by a constant factor. We stick here to the convention of [55] for simplicity.)

**Theorem 6.4 (Poincaré uniformization theorem).** Let $g \in \mathcal{M}$ be any Riemannian metric on the closed, oriented, smooth surface $M$ of genus $p \geq 2$. Then there exists a unique $\lambda(g) \in \mathcal{P}$ such that $\lambda(g)g$ is hyperbolic.

Additionally, it can be shown that the assignment $g \mapsto \lambda(g)$ is smooth.
Let \( \mathcal{M}_{-1} \subset \mathcal{M} \) denote the subset of hyperbolic metrics on \( M \). Theorem 6.4 implies that there is a bijection between \( \mathcal{M}/\mathcal{P} \) and \( \mathcal{M}_{-1} \). It can be shown that in fact, \( \mathcal{M}_{-1} \) is a smooth submanifold of \( \mathcal{M} \) and this bijection is actually a diffeomorphism.

Furthermore, we can easily show that \( \mathcal{M}_{-1} \) is \( \mathcal{D} \)-invariant. Denote the scalar curvature of a metric \( g \in \mathcal{M} \) by \( R(g) \)—this is a function on \( M \), and \( g \in \mathcal{M}_{-1} \) if and only if \( R(g) = -1 \). But for all \( x \in M \) and \( g \in \mathcal{M}_{-1} \),

\[
R(\varphi^* g)(x) = R(g)(\varphi(x)) = -1.
\]

Therefore \( \varphi^* g \in \mathcal{M}_{-1} \) as well.

Using the statements of the last two paragraphs, we can diffeomorphically identify Teichmüller space with the space of hyperbolic metrics modulo diffeomorphisms homotopic to the identity. That is,

\[
\mathcal{T} = (\mathcal{M}/\mathcal{P})/\mathcal{D}_0 \cong \mathcal{M}_{-1}/\mathcal{D}_0.
\]

This is the model of Teichmüller space that we will use from here on. We furthermore denote the projection by

\[
\pi : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0.
\]

Since \( \mathcal{M}_{-1} \) is a submanifold of \( \mathcal{M} \), the \( L^2 \) metric \((\cdot,\cdot)\) on \( \mathcal{M} \) induces a weak Riemannian metric on \( \mathcal{M}_{-1} \) by restriction. We again denote this metric by \((\cdot,\cdot)\), and we claim that \( \mathcal{D} \) acts by isometries on \((\cdot,\cdot)\). To see this, we first denote the pull-back action by

\[
(6.2) \quad A : \mathcal{S} \times \mathcal{D} \to \mathcal{S},
\]

and for any \( \varphi \in \mathcal{D} \), we define a map

\[
(6.3) \quad A_{\varphi} : \mathcal{S} \to \mathcal{S}
\]

\[
\quad h \mapsto A(h,\varphi) = \varphi^* h.
\]

Note from the definition (6.1) of the pull-back that \( A_{\varphi} \) is a linear map. Therefore, its differential at each point is equal to the map itself. From this, we see that for any \( g \in \mathcal{M} \) and any \( h,k \in T_g\mathcal{M} \cong \mathcal{S} \),

\[
(DA_{\varphi}(g) h, DA_{\varphi}(g) k)_{A_{\varphi}(g)} = \int_M tr_{\varphi^* g}((\varphi^* h)(\varphi^* k)) \mu_{\varphi^* g}.
\]

Let’s define \( f := \text{tr}_g(hk) \), so that \( f \) is a function on \( M \). It’s not hard to convince oneself that

\[
tr_{\varphi^* g}((\varphi^* h)(\varphi^* k)) = f \circ \varphi = \varphi^* f,
\]

as well as that \( \mu_{\varphi^* g} = \varphi^* \mu_g \). Therefore,

\[
(DA_{\varphi}(g) h, DA_{\varphi}(g) k)_{A_{\varphi}(g)} = \int_M (\varphi^* f) \varphi^* \mu_g = \int_M \varphi^* (f \mu_g) = \int_M f \mu_g = (h,k)_g.
\]

This shows that \((\cdot,\cdot)\) is \( \mathcal{D} \)-invariant.

\( \mathcal{D} \)-invariance of \((\cdot,\cdot)\) implies that it descends to an \( \text{MCG} \)-invariant Riemannian metric, also denoted \((\cdot,\cdot)\), on the quotient \( \mathcal{M}_{-1}/\mathcal{D}_0 \). This metric is called the Weil-Petersson metric.

The Weil-Petersson metric is an extremely interesting and intensely studied object. Some of its most important properties are the following. There is a natural complex structure on Teichmüller space, which we won’t describe here, and with respect to this structure the Weil-Petersson metric is Kähler. It has strictly negative sectional curvature and strictly negative holomorphic sectional curvature. The Weil-Petersson metric is incomplete, and its completion leads to interesting connections with the so-called Deligne-Mumford compactification of moduli space. We will explore this metric some more in Subsection 6.2.1.

For the moment, though, we leave the Weil-Petersson metric and move on to some other properties of Teichmüller space that we will need.
6.1.3. The fiber bundle structure of Teichmüller space. It is clear that Teichmüller space is the base space of a principal \( \mathcal{D}_0 \)-bundle with total space \( \mathcal{M}_{-1} \rightarrow \mathcal{M}_{-1}/\mathcal{D}_0 \). If we put the \( L^2 \) metric on \( \mathcal{M}_{-1} \) and the Weil-Petersson metric on \( T \), then this bundle forms what is called a weak Riemannian principal \( \mathcal{D}_0 \)-bundle. That is, it is a principal \( \mathcal{D}_0 \)-bundle with a weak Riemannian metric on each of the base and total spaces, and the differential of the projection is an isometry when restricted to the horizontal space. In other words, for all \( g \in \mathcal{M}_{-1} \), \( D\pi_g |_{\mathcal{H}_g} \) is an isometry. Here, \( \mathcal{H}_g \) is the horizontal tangent space, i.e., the tangent space to the orbit \( \mathcal{D}_0 \cdot g \). Then \( \mathcal{H}_g = V_g \). Of course, the tangent space decomposes as \( T_{\mathcal{M}_{-1}} = \mathcal{H}_g \oplus V_g \).

We can easily determine what the vertical tangent space \( V_g \) is. If \( \varphi_t \), for \( t \in (-\epsilon, \epsilon) \), is a one-parameter family of diffeomorphisms for which \( \varphi_0 = \text{id} \), then the differential of \( \varphi_t \) at \( t = 0 \) is a vector field. That is, if we denote the set of \( C^\infty \) vector fields on \( \mathcal{M} \) by \( \mathcal{X}(\mathcal{M}) \), then there is some \( X \in \mathcal{X}(\mathcal{M}) \) for which

\[
\frac{d}{dt} \bigg|_{t=0} \varphi_t = X.
\]

Every \( X \in \mathcal{X}(\mathcal{M}) \) arises in this way. Furthermore, if \( g \in \mathcal{M} \) is any metric, then by definition,

\[
\frac{d}{dt} \bigg|_{t=0} \varphi^*_t g = L_X g,
\]

where \( L_X g \) is the Lie derivative of \( g \) with respect to \( X \). Therefore we have

\[
(6.4) \quad V_g = \{ L_X g \mid X \in \mathcal{X}(\mathcal{M}) \}.
\]

It is also possible to explicitly describe the horizontal tangent space \( \mathcal{H}_g \), though we will not derive this description here. Define the divergence of an element \( h \in \mathcal{S} \) to be the one-form given locally by

\[
<\delta g h>_i = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} \left( g^{ik} h_{ki} \sqrt{\det g} \right) - \frac{1}{2} g^{kl} g^{jm} h_{ml} \frac{\partial g_{jk}}{\partial x^l}.
\]

Then we have

\[
\mathcal{H}_g = S^T_g := \{ h \in \mathcal{S} \mid \text{tr}_g h = 0, \ \delta g h = 0 \}.
\]

Horizontal lifts of paths exist for the bundle \( \pi : \mathcal{M}_{-1} \rightarrow \mathcal{M}_{-1}/\mathcal{D}_0 \). That is, given a path \( \gamma : [0,1] \rightarrow \mathcal{M}_{-1}/\mathcal{D}_0 \) and an element \( g \in \pi^{-1}(\gamma(0)) \), there exists a unique path \( \tilde{\gamma} : [0,1] \rightarrow \mathcal{M}_{-1} \) such that \( \pi \circ \tilde{\gamma} = \gamma \), \( \tilde{\gamma}(0) = g \), and \( \tilde{\gamma}(t) \in H_{\gamma(t)}(t) \) for all \( t \). Note that this fact does not hold in general for weak Riemannian principal bundles. There are a number of ways to see that it does hold here—we will now present a proof that relies on the existence of a slice for the \( \mathcal{D}_0 \)-action and the ability to take any path in \( \mathcal{M} \) and construct a horizontal path from it.

The existence of a slice is given by the following theorem.

**Theorem 6.5** (155 Thms. 2.4.2 and 2.4.5). Let \( g \in \mathcal{M}_{-1} \) be arbitrary. Then there exists a local smooth submanifold \( \mathcal{X}_g \subset \mathcal{M}_{-1} \) passing through \( g \) such that each point of \( \mathcal{X}_g \) corresponds to exactly one orbit of \( \mathcal{D}_0 \). That is, if \( \varphi \in \mathcal{D}_0 \), \( \tilde{g} \in \mathcal{X}_g \) and \( \varphi^* \tilde{g} \in \mathcal{X}_g \), then \( \varphi = \text{id} \). Furthermore, the local submanifolds \( \mathcal{X}_g \) form the (nonlinear) charts of an atlas for \( \mathcal{M}_{-1}/\mathcal{D}_0 \).

Now, we want to take a given path in \( \mathcal{M} \) and construct a horizontal path from it. We note that the horizontal space for the \( \mathcal{D} \)-action on \( \mathcal{M} \), i.e., the vectors tangent to the \( \mathcal{D} \)-orbits, is given by 19 §3

\[
\tilde{\mathcal{H}}_g = \{ h \in \mathcal{S} \mid \delta g h = 0 \}.
\]
The vertical tangent space is again given by (6.4), since we showed (6.4) for any \( g \in \mathcal{M} \), not just \( g \in \mathcal{M}(-1) \). Again we have a decomposition \( T_{g} \mathcal{M} = \tilde{H}_{g} \oplus V_{g} \).

Let’s denote the projection of \( \mathcal{M} \) onto the \( D_{0} \)-orbit space by

\[
\pi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}/D_{0}.
\]

We simply view this as a mapping of sets, since we have not considered any particular structure on \( \mathcal{M}/D_{0} \) (and don’t need to).

The statement we need is the following.

**Lemma 6.6.** Let \( g_{t}, \ t \in [0, 1] \), be any piecewise \( C^{1} \) path in \( \mathcal{M} \). Then there exists a unique piecewise \( C^{1} \) path \( \tilde{g}_{t}, \ t \in [0, 1] \), with the properties that \( \tilde{g}_{0} = g_{0}, \tilde{g}_{t} \) is horizontal wherever \( g_{t} \) is differentiable, and \( \tilde{g}_{t} \) is equivalent to \( g_{t} \) under the \( D_{0} \)-action on \( \mathcal{M} \). That is, \( \tilde{g}_{0} = g_{0}, \tilde{g}_{t}' \in H_{\tilde{g}_{t}} \) for all \( t \) for which \( g_{t}' \) exists, and \( \pi_{\mathcal{M}}(g_{t}) = \pi_{\mathcal{M}}(\tilde{g}_{t}) \) for all \( t \in [0, 1] \).

Furthermore, \( \tilde{g}_{t} \) is of minimal length among the class of all paths equivalent to \( g_{t} \) (though it is of course not the unique minimizer).

**Proof.** Without loss of generality, we assume that \( g_{t} \) is actually \( C^{1} \) on its entire domain. (Otherwise just apply the proof to each segment on which it is \( C^{1} \).) By the decomposition shown above, for each \( t \in [0, 1] \), there exist \( h_{t} \in \tilde{H}_{g} \) and \( X_{t} \in \mathfrak{X}(M) \) such that

\[
g_{t}' = h_{t} + L_{X_{t}}g_{t}.
\]

Now, as is well known (or easily looked up, say in [31] Thm. 17.15), since \( M \) is compact, we can integrate the time-dependent vector field \( -X_{t} \) to get a one-parameter family of diffeomorphisms \( \varphi_{t} \) for which \( \varphi_{0} = \text{id} \) and

\[
\frac{d}{dt} \varphi_{t} = -X_{t} \circ \varphi_{t}
\]

for all \( t \in [0, 1] \).

We then define \( \tilde{g}_{t} := \varphi_{t}^{*}g_{t} \) and claim that this is the desired path. Clearly \( \pi_{\mathcal{M}}(\tilde{g}_{t}) = \pi_{\mathcal{M}}(g_{t}) \). To show that \( \tilde{g}_{t}' \in H_{\tilde{g}_{t}} \), we recall that \( A \) denotes the action of \( D \) on \( S \) (cf. (6.2)) and compute

\[
\tilde{g}_{t}' = \frac{d}{dt}(\varphi_{t}^{*}g_{t}) = \frac{d}{dt}A(g_{t}, \varphi_{t}) = DA(g_{t}, \varphi_{t})[g_{t}', -X_{t} \circ \varphi_{t}],
\]

since the \( t \)-derivative of \( \varphi_{t} \) is \( -X_{t} \circ \varphi_{t} \).

Now, denote the partial derivatives of \( A \) in the first and second arguments by \( D_{1}A \) and \( D_{2}A \), respectively. We have

\[
D_{1}A(g_{t}, \varphi_{t})g_{t}' = DA_{\varphi_{t}}(g_{t})g_{t}' = \varphi_{t}^{*}g_{t}' = \varphi_{t}^{*}(h_{t} + L_{X_{t}}g_{t}).
\]

Recall that \( A_{\varphi_{t}} = A(\cdot, \varphi_{t}) \) by definition (cf. (6.3)), and the second equality follows because, as mentioned above, \( A_{\varphi_{t}} \) is a linear map.

Next, we compute

\[
D_{2}A(g_{t}, \varphi_{t})[-X_{t} \circ \varphi_{t}] = \frac{d}{ds}\bigg|_{s=0} (\varphi_{t+s}^{*}g_{t}) = \varphi_{t}^{*}\left( \frac{d}{ds}\bigg|_{s=0} (\varphi_{t+s} \circ \varphi_{t}^{-1})^{*}g_{t} \right)
\]

\[
= -\varphi_{t}^{*}(L_{X_{t}}g_{t}).
\]

Since \( DA = D_{1}A + D_{2}A \), inserting (6.6) and (6.7) into (6.5) gives

\[
\tilde{g}_{t}' = \varphi_{t}^{*}h_{t} =: \tilde{h}_{t}.
\]

But since \( \delta_{\varphi_{t}}h_{t} = 0 \), we also have \( \delta_{\varphi_{t}}\tilde{h}_{t} = \delta_{\varphi_{t}}(\varphi_{t}^{*}h_{t}) = 0 \). Thus we have shown that \( \tilde{g}_{t}' \in H_{\tilde{g}_{t}} \), and so \( \tilde{g}_{t} \) is horizontal as desired.

Uniqueness of \( \tilde{g}_{t} \) with the desired properties follows from the fact that on a Riemann surface of genus \( p \geq 2 \), there are no Killing fields—we prove this in Lemma 6.7 immediately.
following the proof of this lemma. Thus, the family \( \varphi_t \) above is the only one for which \( \varphi_0 = \operatorname{id} \) and \( \psi_t^* g_0 \) is horizontal.

To show that \( \tilde{g}_t \) is of minimal length among all paths equivalent to \( g_t \), let \( \tilde{g}_t \) be another path with \( \pi_{\mathcal{M}}(\tilde{g}_t) = \pi_{\mathcal{M}}(g_t) \), and let \( \psi_t \) be the unique one-parameter family of diffeomorphisms from \( \mathcal{D}_0 \) such that \( \tilde{g}_t = \psi_t^* \tilde{g}_t \). Just as above, we can compute that

\[
\tilde{g}_t' = \frac{d}{dt} (\psi_t^* \tilde{g}_t) = \psi_t^* \tilde{h}_t + \psi_t^* (L_{Y_t} \tilde{g}_t),
\]

where

\[
Y_t := \left( \frac{d}{dt} \psi_t \right) \circ \psi_t^{-1} \in \mathfrak{X}(M).
\]

and we recall that \( \tilde{h}_t = \tilde{g}_t' \). By the orthogonality of horizontal and vertical vectors,

\[
L(\tilde{g}_t) = \int_0^1 \|\tilde{g}_t'\|_{\tilde{g}_t} dt = \int_0^1 \|\psi_t^* \tilde{h}_t\|_{\psi_t^* \tilde{g}_t} dt + \int_0^1 \|\psi_t^* (L_{Y_t} \tilde{g}_t)\|_{\psi_t^* \tilde{g}_t} dt
\]

\[
= \int_0^1 \|\tilde{h}_t\|_{\tilde{g}_t} dt + \int_0^1 \|\psi_t^* (L_{Y_t} \tilde{g}_t)\|_{\psi_t^* \tilde{g}_t} dt \geq L(g_t),
\]

where we have used the \( \mathcal{D} \)-invariance of \( \langle \cdot, \cdot \rangle \) in the second line. \( \square \)

**Lemma 6.7.** Let \( g \in \mathcal{M} \) be any Riemannian metric on the genus \( p \) surface \( M \). Then \( g \) has finite isometry group. In particular, \( g \) admits no Killing fields.

**Proof.** By the Poincare uniformization theorem (Theorem 6.4), there exists a function \( \rho \in C^\infty(M) \) and a metric \( \bar{g} \in \mathcal{M}_{-1} \) such that \( g = \rho \bar{g} \). Our goal is to show that every isometry of \( g \) is also an isometry of \( \bar{g} \), which then implies that the isometry group of \( g \) is finite, since by Hurwitz’s theorem [13] p. 258 the isometry group of \( \bar{g} \) is finite.

So let \( \varphi \in \mathcal{D} \) be an isometry of \( g \). We then have that \( \varphi^* g = g \), so

\[
(\varphi^* \rho)^* \bar{g} = \varphi^* g = g = \rho \bar{g}.
\]

Thus \( \varphi^* \bar{g} \) and \( \bar{g} \) are conformally equivalent. Furthermore, since the space of hyperbolic metrics is \( \mathcal{D} \)-invariant, these two metrics are both hyperbolic. But since the Poincaré uniformization theorem says that there is exactly one hyperbolic metric in each conformal class of metrics, we must have \( \varphi^* \bar{g} = \bar{g} \). Thus \( \varphi \) is an isometry of \( \bar{g} \), as was to be shown. \( \square \)

**Remark 6.8.** Note the following astounding fact, implied by the proof of Lemma 6.7. We have just shown that the unique hyperbolic metric in a conformal class is, in a very strong sense, the most symmetric metric in that class. Namely, *any isometry of any metric* in that class is also an isometry of the hyperbolic metric.

Using the results above, we can prove the existence of horizontal lifts.

**Theorem 6.9.** For any \( C^1 \) path \( \gamma : [0, 1] \to \mathcal{M}_{-1}/\mathcal{D}_0 \) and any \( g \in \pi^{-1}(\gamma(0)) \), there exists a unique horizontal lift \( \tilde{\gamma} : [0, 1] \to \mathcal{M}_{-1}/\mathcal{D}_0 \) with \( \tilde{\gamma}(0) = g \). In particular, \( \tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)} \) for all \( t \in [0, 1] \).

Furthermore, \( L(\tilde{\gamma}) = L(\gamma) \) and \( \tilde{\gamma} \) has minimal length among the class of curves whose image projects to \( \gamma \) under \( \pi \).

**Proof.** Recall that \( \mathcal{X}_g \) denotes the slice around \( g \) guaranteed by Theorem 6.5. By the compactness of the interval \([0, 1]\), we can choose a finite set \( \{g_1, \ldots, g_m\} \subset \mathcal{M}_{-1} \) such that the collection \( \{\pi(\mathcal{X}_{g_1}), \ldots, \pi(\mathcal{X}_{g_m})\} \) covers \( \gamma \); furthermore, we choose this collection such that all the sets in it have nonempty intersection with \( \gamma \). Let the collection further be chosen such that the intersection \( \pi(\mathcal{X}_{g_k}) \cap \gamma \) is equal to \( \gamma(J_k) \) for some interval \( J_k \subset [0, 1] \).

To achieve this condition, we simply shrink the slices if necessary. Finally, we assume that the numbering is done such that the initial points of the intervals \( J_k \) are in increasing order—again, we may have to shrink the slices to achieve this. In particular, this assures us that \( 0 \in J_1 \).
Let $\gamma_k$ denote the lift of $\gamma|_{k}$ to $X_{g_k}$, and let $\hat{\gamma}_k$ be the horizontal path equivalent to $\gamma_k$ guaranteed by Lemma 6.6. The path $\hat{\gamma}_k$ is a horizontal lift of $\gamma|_{k}$. Let $\varphi_1 \in D_0$ be the unique element such that $\varphi_1^* \gamma_1(0) = g$, and define $\tilde{\gamma}_1 := \varphi_1^* \gamma_1$. Note that $\tilde{\gamma}_1$ is still a horizontal lift of $\gamma|_{k}$, and that $\tilde{\gamma}_1(0) = g$. Let $a_2 \in J_1 \cap J_2$, and let $\varphi_2 \in D_0$ be the unique element such that $\varphi_2^* \tilde{\gamma}_2(a_2) = \tilde{\gamma}_1(a_2)$. Define $\tilde{\gamma}_2 := \varphi_2^* \tilde{\gamma}_2$. By repeating this procedure, we get a path $\tilde{\gamma}_k$ that is a horizontal lift of $\gamma|_{k}$, such that $\tilde{\gamma}_k$ intersects $\tilde{\gamma}_{k+1}$ in at least one point, for each $k = 1, \ldots, m$.

Using the uniqueness of the horizontal paths of Lemma 6.6, we see that since the paths $\tilde{\gamma}_k$ and $\tilde{\gamma}_{k+1}$ intersect in one point, they intersect over the entire range where they are equivalent under $D_0$. Therefore, we can glue the paths $\tilde{\gamma}_k$ together to a differentiable path $\tilde{\gamma}$ that is a horizontal lift of $\gamma$—and since $\tilde{\gamma}_1(0) = g$, we clearly have $\tilde{\gamma}(0) = g$, as desired.

The minimality of $\tilde{\gamma}$ follows from Lemma 6.6. To show that $L(\tilde{\gamma}) = L(\gamma)$, recall that $\pi : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/D_0$ is a weak Riemannian principal $D_0$-bundle, so $D\pi(g)|_{H_g}$ is an isometry for every $g \in \mathcal{M}_{-1}/D_0$. Therefore

$$L(\tilde{\gamma}) = \int_0^1 \|\tilde{\gamma}'(t)\|_{\tilde{\gamma}(t)} \, dt = \int_0^1 \|D\pi(\tilde{\gamma}(t))\tilde{\gamma}'(t)\|_{\pi(\tilde{\gamma}(t))} \, dt = \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \, dt = L(\gamma).$$

The structures we’ve described in this section will all be put to work for us in the next section.

6.2. Metrics arising from submanifolds of $\mathcal{M}$

In this section, our goal is to define an entire class of metrics that includes the Weil-Petersson metric, and to use the main result of the thesis, Theorem 5.25, to prove a fact about the completions of Teichmüller space with respect to such metrics.

Before we do this in Subsection 6.2.2 we will go into some more detail on the completion of the Weil-Petersson metric. This discussion will motivate our considerations in Subsection 6.2.2.

6.2.1. Completing Teichmüller space with respect to the Weil-Petersson metric. It has long been known that the Weil-Petersson metric is incomplete—this was initially and independently proved in [56] and [7]. The proof shows that there are Weil-Petersson geodesics that, in finite time, leave Teichmüller space.

The limit points of such geodesics can be given a meaning as Riemann surfaces themselves, which we would like to describe heuristically here. We will not justify anything, but instead suggest to the reader the various references given in this chapter.

First, we note that for a hyperbolic metric on a compact surface, there is a unique geodesic in each free homotopy class, and this is the shortest curve in the class [27] Lem. 2.4.4.

Let a Weil-Petersson geodesic $\gamma : [0,1] \to T$ be such that it cannot be continuously extended to the domain $[0,1]$, and let $\tilde{\gamma} : (0,1]$ be a horizontal lift of $\gamma$. Then there exist $r$ disjoint, nonhomotopic, noncontractible simple closed curves $\eta_{1}^{r}, \ldots, \eta_{r}$ on $M$, with $1 \leq r \leq 3p - 3$, such that the following holds. For each $i = 1, \ldots, r$, let $\eta_{i}^{r}$ denote the unique $\tilde{\gamma}(t)$-geodesic in the free homotopy class of $\eta_{i}^{r}$. Then the length of each $\eta_{i}^{r}$ with respect to $\tilde{\gamma}(t)$ converges to zero for $t \to 0$. In this case, we say that the curves $\eta_{1}^{r}, \ldots, \eta_{r}$ are pinched along $\tilde{\gamma}$, since geometrically the curves shrink to points.

Thanks to the so-called collar lemma (see, e.g., [48] or, for surfaces of variable curvature, [5]), around each geodesic on a hyperbolic surface there exists a neighborhood that is diffeomorphic to an open-ended cylinder. Furthermore, the width of this neighborhood increases to infinity as the length of the hyperbolic geodesic decreases to zero. Thus, around
Figure 1. The formation of cusps when pinching a simple closed hyperbolic geodesic. On the left, the blue, homologically trivial, curve is pinched and the resulting limit surface is disconnected, with two cusps. On the right, the red, homologically nontrivial curve is pinched—the limit surface has lower genus than the original one, and has two cusps.

each of the curves $\eta^i$, two so-called cusps develop as $t \to 0$, meaning that in the limit, two infinitely long cylinders extend out from the surface, and the width of these cylinders approaches zero at infinity. Figure 1 illustrates this in case of the two basic possibilities here. By pinching a homologically trivial curve, we end up with a disconnected surface of the same total genus (heuristically, the same number of “donut holes”). When pinching a homologically nontrivial curve, the limit surface stays connected but is of lower genus. By combining these two pictures for all pinched curves, one can imagine a general limit surface. By adding in all such limit surfaces, we obtain the completion of Teichmüller space with respect to the Weil-Petersson metric, which we will denote by $\overline{T}$.

Let’s translate this discussion into the language that we’ve been using throughout the rest of the thesis. What essentially happens is that the family of metrics $\tilde{\gamma}(t)$ is equivalent (under the $D_0$-action on $\mathcal{M}_{-1}$) to a family $g_t$ of metrics that becomes unbounded and deflates along the curves $\eta^1, \ldots, \eta^r$ as $t \to 0$. With respect to the metrics $g_t$, the lengths of vectors tangent to each $\eta^i$ converge to zero, and the lengths of vectors perpendicular to each $\eta^i$ become infinite. Thus, if we define $g_0$ to be equal to the pointwise limit of $g_t$ off of $\eta^1, \ldots, \eta^r$ and, say, zero on $\eta^1, \ldots, \eta^r$, then we get a limit metric on $M$ that is measurable. Furthermore, by [13], p. 233, the volume of the limit surface is finite, as we would expect from our main result, Theorem 5.25.

Thus, the completion of Teichmüller space fits very nicely into the setting that we have established in this thesis. Of course, since we are dealing only with a special type of metric on a special type of base manifold, and we only consider horizontal paths—i.e., there are no limit metrics that arise from families of degenerating diffeomorphisms—the limit metrics that are possible make up only a small subset, with very nice properties, of the limit metrics that we get when considering the completion of all of $\mathcal{M}$.

Before we leave this subsection, let’s just mention a couple of the rich properties of the completion of Teichmüller space with respect to the Weil-Petersson metric.

As described above, $\overline{T}$ is closely related to a compactification of moduli space. As in the case of $\overline{\mathcal{M}}$ (and the completion of any metric space), the distance function of the Weil-Petersson metric extends to the completion $\overline{T}$. Here, however, more is true. In a certain sense, the Weil-Petersson Riemannian metric (the scalar product) also extends to $\overline{T}$. This is proved, and given precise meaning, in [33].

The action of the mapping class group $\text{MCG}$ extends to $\overline{T}$ [1], and the action of any individual element of $\text{MCG}$ is an isometry of the extended Weil-Petersson metric on $\overline{T}$.
Therefore, we can form the quotient
\[ \mathcal{R} := \mathcal{T}/\text{MCG}, \]
and the Weil-Petersson distance function projects to a complete metric (in the sense of metric spaces) on \( \mathcal{R} \).

It turns out that \( \mathcal{R} \) is a compactification of moduli space. Moreover, this compactification agrees with the famous Deligne-Mumford compactification \([9]\), which arises via very different considerations in algebraic geometry. Thus, the Weil-Petersson metric forms the bridge between two very important aspects of Riemannian geometry and algebraic geometry.

Hopefully this has provided sufficient motivation to convince the reader that the Weil-Petersson metric and generalizations thereof are worthwhile objects of study.

### 6.2.2. Generalizations of the Weil-Petersson metric.

The natural way to generalize the Weil-Petersson metric within this context is to take also non-hyperbolic (variable curvature) representatives for each conformal class in \( \mathcal{M}/\mathcal{P} \), giving us some submanifold of \( \mathcal{M} \) which differs from \( \mathcal{M}_{-1} \) but still contains exactly one representative of each conformal class. The goal of this subsection is to describe this idea rigorously.

This idea is directly inspired by \([22]\) and \([23]\), where metrics on Teichmüller space were also defined using variable curvature metrics in place of the hyperbolic metric. These metrics differ from the ones considered here, however. After we define our own generalization, we remark on the differences. Unfortunately, completely describing the concepts necessary to understand the differences is outside the scope of this thesis, so we must regrettably do this in a way that will be helpful only to those “in the know.”

By the Poincaré uniformization theorem, Theorem 6.4, the principal \( \mathcal{P} \)-bundle \( \mathcal{M} \to \mathcal{M}/\mathcal{P} \) is trivial, and \( \mathcal{M}_{-1} \) is a section of this bundle. (Of course, we could have already deduced from the product structure \( \mathcal{M} \cong \mathcal{P} \times \mathcal{M}_{\mu} \) given in Subsection 2.5.3 that the bundle is trivial.) The idea now is to select a different section of \( \mathcal{M} \to \mathcal{M}/\mathcal{P} \). In fact, we will simultaneously consider all smooth sections \( \mathcal{N} \) with the property that they are \( \mathcal{D} \)-invariant, which we require so that we still have diffeomorphisms \( \mathcal{T} \cong \mathcal{N}/\mathcal{D}_0 \) and \( \mathcal{R} \cong \mathcal{N}/\mathcal{D} \).

**Definition 6.10.** We call a smooth, \( \mathcal{D} \)-invariant section of \( \mathcal{M} \to \mathcal{M}/\mathcal{P} \) a **modular section**. Given a modular section \( \mathcal{N} \subset \mathcal{M} \), we call the quotients \( \mathcal{N}/\mathcal{D}_0 \) and \( \mathcal{N}/\mathcal{D} \) the \( \mathcal{N} \)-model of Teichmüller space and the \( \mathcal{N} \)-model of moduli space, respectively.

The proof of the next lemma is obvious from the decomposition \( \mathcal{M} \cong \mathcal{P} \times \mathcal{M}_{-1} \) implied by the Poincaré uniformization theorem.

**Lemma 6.11.** For all \( g \in \mathcal{M}_{-1} \), choose \( \rho_g \in \mathcal{P} \) such that

1. the assignment \( g \mapsto \rho_g \) is smooth and
2. \( \rho_{\varphi^* g} = \varphi^* \rho_g \) for all \( g \in \mathcal{M}_{-1} \).

Then the set
\[ \mathcal{N} := \{ \rho_g g \mid g \in \mathcal{M}_{-1} \} \]
is a modular section. Furthermore, every modular section arises in this way.

Modular sections other than \( \mathcal{M}_{-1} \) of course exist. Let us mention just one important example, that of the space of Bergman metrics on \( \mathcal{M} \). It requires a few facts about Riemann surfaces that we won’t prove, and can be safely skipped.

**Example 6.12.** As is well-known and proved, for example, in \([55]\), in two dimensions complex structures are in one-to-one correspondence with conformal structures—so each element of \( \mathcal{M}/\mathcal{P} \) determines complex structure on \( \mathcal{M} \). So for this example, we work with complex instead of conformal structures.
Let \( c \) be a complex structure on \( M \). Then the space of holomorphic one-forms on \((M, c)\) has complex dimension \( p \), the genus of \( M \) [13 Prop. III.2.7]. Let \( \theta_1, \ldots, \theta_p \) be an \( L^2 \)-orthonormal basis of this space. That is,
\[
\frac{i}{2} \int_M \theta_j \wedge \overline{\theta_k} = \delta_{jk}.
\]
The Bergman metric is defined by
\[
g_B := \frac{1}{p} \sum_{i=1}^p \theta_i \overline{\theta_j}.
\]
It is clear from this construction that the set of all Bergman metrics is indeed a modular section.

The Bergman metric can also be seen as the pull-back of the flat metric on the Jacobian of \( M \) via the Albanese period map. It arises, for example, in arithmetic geometry [6].

Let us now return to our general considerations.

CONVENTION 6.13. For the remainder of this chapter, let \( \mathcal{N} \) be a fixed but arbitrary modular section.

The next proposition tells us that we are, from the differential topological point of view, justified in calling \( \mathcal{N}/\mathcal{D}_0 \) the \( \mathcal{N} \)-model of Teichmüller space.

**Proposition 6.14.** The quotient \( \mathcal{N}/\mathcal{D}_0 \) is a smooth, finite-dimensional manifold. Furthermore, we have diffeomorphisms \( \Psi : \mathcal{M}_{-1} \to \mathcal{N} \) and \( \Phi : \mathcal{M}_{-1}/\mathcal{D}_0 \to \mathcal{N}/\mathcal{D}_0 \). The diffeomorphism \( \Psi \) is \( \mathcal{D} \)-equivariant.

**Proof.** Let \( \rho_g \) be the assignment that gives \( \mathcal{N} \) from \( \mathcal{M}_{-1} \), as in Lemma 6.11. Then it is clear that the following map is a diffeomorphism:
\[
\Psi : \mathcal{M}_{-1} \to \mathcal{N},
\]
\[
g \mapsto \rho_g g.
\]
The rest of the claims follow from the fact that \( \Psi \) is \( \mathcal{D} \)-equivariant:
\[
\Psi(\varphi^* g) = \rho_{\varphi^* g}(\varphi^* g) = (\varphi^* \rho_g)(\varphi^* g) = \varphi^*(\rho_g g) = \varphi^* \Psi(g)
\]
by the assumptions on \( \rho_g \). Thus, the manifold structure on \( \mathcal{N}/\mathcal{D}_0 \) is given by the bijection with \( \mathcal{M}_{-1}/\mathcal{D}_0 \) induced by \( \Psi \).

As in the case of the section \( \mathcal{M}_{-1} \), the \( L^2 \) metric on \( \mathcal{M} \) restricts to \( \mathcal{N} \), and its \( \mathcal{D} \)-invariance implies that it projects to an \( \text{MCG} \)-invariant metric on \( \mathcal{N}/\mathcal{D}_0 \). We call this metric, as well as the metric it induces on \( \mathcal{T} \sim \mathcal{M}_{-1} \) via the diffeomorphism of Proposition 6.14, the *generalized Weil-Petersson metric* on the \( \mathcal{N} \)-model of Teichmüller space. As in the case of the bundle \( \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0 \), these metrics turn the bundle \( \mathcal{N} \to \mathcal{N}/\mathcal{D}_0 \) into a weak Riemannian principal \( \mathcal{D}_0 \)-bundle.

**Remark 6.15.** The following remark requires some basic knowledge about Teichmüller theory. For those lacking this, it can be safely skipped.

For those with this background, we note here the difference between the metrics of Habermann and Jost (cf. [22], [23]) and the generalized Weil-Petersson metrics we have just introduced.

Of course, Teichmüller space was historically defined in complex analysis as the space of complex structures on \( M \) modulo \( \mathcal{D}_0 \). (See, for example, [26].) The correspondence between complex structures and conformal classes is given by the existence of local isothermal (or conformal) coordinates on any two-dimensional manifold.

Now, recall that the cotangent space of Teichmüller space, when defined in the complex analytic way, is given by the space of holomorphic quadratic differentials on \( M \) with respect
to the given complex structure. The correspondence between these and horizontal vectors of \( \mathcal{M}_{-1} \) is given by the fact that a traceless, divergence-free element of \( S \) is the real part of a holomorphic quadratic differential.

Habermann and Jost generalize the Weil-Petersson metric on the complex analytic version of Teichmüller space as follows. From this point of view, a point \( \tau \in T \) represents an equivalence class of complex structures on \( \mathcal{M} \). Let’s choose representatives \( \Sigma_\tau \) of the equivalence classes \( \tau \in T \)—thus, \( \Sigma_\tau \) is a complex manifold with one complex dimension—in a smooth manner (we will have to be vague about what this means for reasons of space).

For each fixed \( \tau \in T \), choose complex coordinates \( z \) on \( \Sigma_\tau \) and a Hermitian metric \( \lambda^2_\tau dzd\bar{z} \).

If \( \lambda^2_\tau dzd\bar{z} \) is chosen such that it varies smoothly with \( \tau \), then we get a Riemannian cometric on \( T \) by defining, for each \( \tau \in T \) and any two holomorphic quadratic differentials on \( \Sigma_\tau \) locally given by \( \psi_0(z) dzd\bar{z} \) and \( \psi_1(z) dzd\bar{z} \),

\[
(6.8) \quad \langle \psi_0, \psi_1 \rangle_\tau := \frac{i}{2} \int_{\Sigma_\tau} \frac{\psi_0(z)\bar{\psi}_1(z)}{\lambda^2_\tau(z)} \, dz \wedge d\bar{z}.
\]

If \( \lambda^2_\tau dzd\bar{z} \) is the hyperbolic metric on \( \Sigma_\tau \) for each \( \tau \in T \), then \( \langle \cdot, \cdot \rangle \) is the Weil-Petersson cometric on \( T \). Otherwise we get some generalization of it.

The difference between these generalizations and the ones we study here is that a horizontal tangent vector to \( N \) is divergence-free, but need no longer be traceless. Thus, in contrast to the case where \( N = \mathcal{M}_{-1} \), a horizontal tangent vector to \( N \) need not in general be the real part of a holomorphic quadratic differential, and so some extra terms will enter into (6.8) if we try to view our generalized Weil-Petersson metric through the lens of the complex analytic theory of Teichmüller space. In essence, the objects \( \psi_0 \) and \( \psi_1 \) on which (6.8) is evaluated are natural tangent vectors when we view \( \Sigma_\tau \) as an element of the moduli space of one-dimensional complex manifolds, but not when we view \( (\Sigma, \lambda^2_\tau dzd\bar{z}) \) as an element of the section \( N \).

Our next goal is to establish the existence of horizontal lifts for the bundle \( N \to N/D_0 \). Thanks to our previous work on \( \mathcal{M}_{-1} \), this is not difficult.

Let us define some notation before stating the result. We denote the bundle projection by

\[ \pi_N : N \to N/D_0, \]

and we denote the horizontal tangent space of this bundle at \( g \in N \) by

\[ H^N_g := V^\perp_g \subset T_g N. \]

By Proposition 6.14 we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_{-1} & \xrightarrow{\Psi} & N \\
\pi \downarrow & & \downarrow \pi_N \\
\mathcal{M}_{-1}/D_0 & \xrightarrow{\Phi} & N/D_0,
\end{array}
\]

where the horizontal arrows are diffeomorphisms and the vertical arrows are projections.

**Theorem 6.16.** For any \( C^1 \) path \( \gamma : [0, 1] \to N/D_0 \) and any \( g \in \pi^{-1}_N(\gamma(0)) \), there exists a unique horizontal lift \( \tilde{\gamma} : [0, 1] \to N \) with \( \tilde{\gamma}(0) = g \). In particular, \( \tilde{\gamma}'(t) \in H^N_{\tilde{\gamma}(t)} \) for all \( t \in [0, 1] \).

Furthermore, \( L(\tilde{\gamma}) = L(\gamma) \) and \( \tilde{\gamma} \) has minimal length among the class of curves whose image projects to \( \gamma \) under \( \pi_N \).
6. APPLICATION TO TEICHMULLER THEORY

Proof. Let \( \tilde{\gamma} \) be the horizontal lift of \( \Phi^{-1} \circ \gamma \) to \( \mathcal{M}_{-1} \) with initial point \( \Psi^{-1}(g) \), and let \( \tilde{\gamma} := \Psi \circ \tilde{\gamma} \). This is a path in \( \mathcal{N} \). Finally, we let \( \hat{\gamma} \) be the horizontal path equivalent to \( \tilde{\gamma} \) guaranteed by Lemma 6.6.

We claim that \( \hat{\gamma} \) is the desired lift. It is a path in \( \mathcal{N} \) by \( D \)-invariance of \( \mathcal{N} \), and it is clearly horizontal. By construction, we see that \( \hat{\gamma}(0) = g \).

Finally, it is easily seen from commutativity of \( \text{(6.9)} \) and the fact that \( \pi \circ \tilde{\gamma} = \Phi^{-1} \circ \gamma \) that \( \pi_N \circ \hat{\gamma} = \gamma \). Uniqueness of \( \hat{\gamma} \) with the given properties follows from uniqueness of the paths of Lemma 6.6.

The remainder of the theorem is proved precisely as in Theorem 6.3. \( \square \)

This theorem allows us to prove the application of the thesis’ main result that we have in mind for the generalized Weil-Petersson metric. In the following, we denote the distance function of \( (\mathcal{N}, (\cdot, \cdot)) \) by \( d_{\mathcal{N}} \).

Theorem 6.17. Let \( \{ [g_k] \} \) be a Cauchy sequence in the \( \mathcal{N} \)-model of Teichmüller space, \( \mathcal{N}/D_0 \), with respect to the generalized Weil-Petersson metric. Then there exist representatives \( \tilde{g}_k \in [g_k] \) and an element \( [g_\infty] \in \mathcal{M}_f \) such that \( \{ \tilde{g}_k \} \) is a \( d_{\mathcal{N}} \)-Cauchy sequence that \( \omega \)-subconverges to \( [g_\infty] \).

Furthermore, if \( \{ [g_k^0] \} \) and \( \{ [g_k^1] \} \) are equivalent Cauchy sequences in \( \mathcal{N}/D_0 \), then there exist representatives \( \tilde{g}_k^0 \in [g_k^0] \) and \( \tilde{g}_k^1 \in [g_k^1] \), as well as an element \( [g_\infty] \in \mathcal{M}_f \), such that \( \{ \tilde{g}_k^0 \} \) and \( \{ \tilde{g}_k^1 \} \) are \( d_{\mathcal{N}} \)-Cauchy sequences that both \( \omega \)-subconverge to \( [g_\infty] \).

Finally, if \( \{ [g_k^0] \} \) and \( \{ [g_k^1] \} \) are inequivalent Cauchy sequences in \( \mathcal{N}/D_0 \), then there exists no choice of representatives \( \tilde{g}_k^0 \in [g_k^0] \) and \( \tilde{g}_k^1 \in [g_k^1] \) such that \( \{ \tilde{g}_k^0 \} \) and \( \{ \tilde{g}_k^1 \} \) \( \omega \)-subconverge to the same element of \( \mathcal{M}_f \).

Proof. The first claim would follow from Theorem 5.25 if we could show that there are representatives \( \tilde{g}_k \in [g_k] \) such that \( \{ \tilde{g}_k \} \) is a \( d_{\mathcal{N}} \)-Cauchy sequence, since this implies that it is also a \( d \)-Cauchy sequence. So this is what we will show.

Let’s denote the distance function induced by the generalized Weil-Petersson metric on \( \mathcal{N}/D_0 \) by \( \delta \). For each \( k \in \mathbb{N} \), let \( \gamma_k : [0, 1] \to \mathcal{N}/D_0 \) be any path from \( [g_k] \) to \( [g_{k+1}] \) such that

\[
L(\gamma_k) \leq 2\delta([g_k], [g_{k+1}]).
\]

For any \( \tilde{g}_1 \in \pi_{\mathcal{N}}^{-1}([g_1]) \), let \( \tilde{\gamma}_1 \) be the horizontal lift of \( \gamma_1 \) to \( \mathcal{N} \) with \( \tilde{\gamma}_1(0) = \tilde{g}_1 \) which is guaranteed by Theorem 6.16. Then clearly \( \tilde{g}_2 := \gamma_2(1) \in \pi_{\mathcal{N}}^{-1}([g_2]) \). Furthermore,

\[
d_{\mathcal{N}}(\tilde{g}_1, \tilde{g}_2) \leq L(\tilde{\gamma}_1) = L(\gamma_1) \leq 2\delta([g_1], [g_2]).
\]

We repeat this process, i.e., let \( \tilde{\gamma}_2 \) be the unique horizontal lift of \( \gamma_2 \) with \( \tilde{\gamma}_2(0) = \tilde{g}_2 \), and set \( \tilde{g}_3 := \tilde{\gamma}_2(1) \), etc. We again get

\[
d_{\mathcal{N}}(\tilde{g}_2, \tilde{g}_3) \leq 2\delta([g_2], [g_3]).
\]

By continuing, we get a sequence of representatives \( \tilde{g}_k \in [g_k] \) such that for each \( k \in \mathbb{N} \),

\[
d_{\mathcal{N}}(\tilde{g}_k, \tilde{g}_{k+1}) \leq 2\delta([g_k], [g_{k+1}]).
\]

Thus, since \( \{ [g_k] \} \) is a Cauchy sequence, \( \{ \tilde{g}_k \} \) is a \( d_{\mathcal{N}} \)-Cauchy sequence, as was to be shown.

To prove the second statement, it suffices by Theorem 5.25 to show that we can find representatives \( \tilde{g}_k^0 \in [g_k^0] \) and \( \tilde{g}_k^1 \in [g_k^1] \) such that \( \{ \tilde{g}_k^0 \} \) and \( \{ \tilde{g}_k^1 \} \) are equivalent \( d_{\mathcal{N}} \)-Cauchy sequences.

To do this, select representatives \( \tilde{g}_k^0 \in [g_k^0] \) as guaranteed by the first statement of the proof, so that in particular \( \{ \tilde{g}_k^0 \} \) is \( d_{\mathcal{N}} \)-Cauchy. Next, for each \( k \in \mathbb{N} \), choose a path \( \gamma_k \) in \( \mathcal{N}/D_0 \) from \( [g_k^0] \) to \( [g_k^1] \) such that

\[
L(\gamma_k) \leq 2\delta([g_k^0], [g_k^1]).
\]
Let $\tilde{\gamma}_k$ be the horizontal lift of $\gamma_k$ to $\mathcal{N}$ with $\tilde{\gamma}_k(0) = \tilde{g}_k^0$, and define $\tilde{g}_k^1 := \tilde{\gamma}_k(1) \in [g_k^1]$. Then
$$d_{\mathcal{N}}(\tilde{g}_k^0, \tilde{g}_k^1) \leq L(\tilde{\gamma}_k) = L(\gamma_k) \leq 2\delta([g_k^0], [g_k^1]).$$
From the above inequality, the fact that $\{\tilde{g}_k^0\}$ is $d_{\mathcal{N}}$-Cauchy, and the fact that $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are equivalent Cauchy sequences, it is easy to see that $\{\tilde{g}_k^1\}$ is $d_{\mathcal{N}}$-Cauchy and that $\{\tilde{g}_k^0\}$ and $\{\tilde{g}_k^1\}$ are equivalent $d_{\mathcal{N}}$-Cauchy sequences.

To prove the last statement, note that since $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are inequivalent, we have
$$\epsilon := \lim_{k \to \infty} \delta([g_k^0], [g_k^1]) > 0.$$
By definition, we also have
$$\delta([g_k^0], [g_k^1]) = \inf \{d_{\mathcal{N}}(\tilde{g}_k^0, \tilde{g}_k^1) \mid \tilde{g}_k^0 \in [g_k^0], \tilde{g}_k^1 \in [g_k^1]\}.$$ Thus, no matter what representatives $\tilde{g}_k^0 \in [g_k^0]$ and $\tilde{g}_k^1 \in [g_k^1]$ we choose,
$$\lim_{k \to \infty} d_{\mathcal{N}}(\tilde{g}_k^0, \tilde{g}_k^1) \geq \epsilon > 0.$$ So Theorem 5.25 implies the statement immediately. \[\square\]

We have thus given one interesting application of our main result. We close the thesis with some brief comments about the above theorem. Of course, this theorem is considerably weaker than the picture for hyperbolic metrics in two regards. Firstly, the convergence notion that one takes for hyperbolic metrics (which we have not given explicitly here) is stronger than $\omega$-convergence. Secondly, the class of limit metrics is very bad—our results do not rule out that a Cauchy sequence of metrics degenerates anywhere on the surface $M$, whereas a Cauchy sequence of hyperbolic metrics can degenerate only on a finite set of simple closed curves, as we saw above. We hope that by exploiting knowledge about the conformal structure induced by a sequence of metrics in $\mathcal{N}$, one should be able to constrain these degenerations in the limit of a Cauchy sequence—ideally, for well-behaved $\mathcal{N}$, restricting degeneration to the “nodes,” as the limits of these closed curves are known in Teichmüller theory. Limitations on degenerations also arise from the $\mathcal{D}$-invariance of $\mathcal{N}$ and the fact that only horizontal paths in $\mathcal{N}$—and not vertical paths, coming from families of diffeomorphisms—matter for the quotient $\mathcal{N}/\mathcal{D}_0$. However, going deeper into these aspects is beyond the scope of this thesis and must be regarded as a future direction for study.

Despite the shortcomings of the above result, we see it as quite useful, as it gives relatively strong information about a new class of metrics on Teichmüller space—namely, that their completions can consist only of finite-volume metrics. Furthermore, it illustrates the potential for applications of our main theorem and provides a starting point for further investigations.
Metrics and convergence notions

Riemannian metrics and the distance functions associated to them

Riemannian metrics and the distance functions associated to them

\[ \mathcal{M} \langle \cdot, \cdot \rangle \quad d^{\mathcal{M}} \]
\[ \mathcal{U} \langle \cdot, \cdot \rangle \quad d^{\mathcal{U}} \]
\[ \mathcal{N} \langle \cdot, \cdot \rangle \quad d^{\mathcal{N}} \]
\[ \mathcal{M}_x \langle \cdot, \cdot \rangle \quad d_x \]
\[ \mathcal{M}_x \langle \cdot, \cdot \rangle^0 \quad \theta_x^{\mathcal{U}} \]

† Here, \( \mathcal{U} \) represents an amenable subset and \( \mathcal{N} \) represents a modular section.

Relations between various notions of convergence and Cauchy sequences

In the following chart, we illustrate the relationships between the different notions of Cauchy and convergent sequences on \( \mathcal{M} \). We let \( \{ g_k \} \) be a sequence in \( \mathcal{M} \) and \( \tilde{g} \in \mathcal{M}_f \). A double arrow ("\( \Rightarrow \)"") between two statements means that the one implies the other. A single arrow ("\( \rightarrow \)"") means that one statement implies the other, assuming the condition that is listed below the chart.

(1) After passing to a subsequence
(2) If there exists an amenable subset \( \mathcal{U} \) such that \( \{ g_k \} \subset \mathcal{U} \), then there exists some \( \tilde{g} \in \mathcal{U}^0 \) such that the implication holds
(3) If there exists a quasi-amenable subset \( \mathcal{U} \) such that \( \{ g_k \} \subset \mathcal{U} \)
(4) After passing to a subsequence, there exists some \( \tilde{g} \in \mathcal{M}_f \) such that the implication holds
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<th>Symbol</th>
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<td>$A$</td>
<td>The pull-back action $S \times D \to S$</td>
<td>p. 6.1.2</td>
</tr>
<tr>
<td>$A_\varphi$</td>
<td>The linear map $A(\cdot, \varphi) : S \to S$</td>
<td>p. 6.1.2</td>
</tr>
<tr>
<td>$\text{cl}(\mathcal{U})$</td>
<td>The closure of the subset $\mathcal{U} \subseteq \mathcal{M}$ in the $C^\infty$ topology of $S$</td>
<td>p. 101</td>
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<tr>
<td>$d$</td>
<td>The Riemannian distance function of $(\mathcal{M}, \langle \cdot, \cdot \rangle)$</td>
<td>Definition 2.36, p. 41</td>
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<tr>
<td>$d_\mathcal{U}$</td>
<td>The distance function induced by $d$ on the completion of an amenable subset $\mathcal{U}$</td>
<td>Definition 4.38, p. 89</td>
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<tr>
<td>$d_x$</td>
<td>The Riemannian distance function of $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$</td>
<td>Definition 4.8, p. 75</td>
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<tr>
<td>$\text{End}(\mathcal{M})$</td>
<td>The endomorphism bundle of the manifold $\mathcal{M}$</td>
<td>p. 23</td>
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<tr>
<td>$g$</td>
<td>From Section 2.6 of Chapter 2 onwards, a fixed, smooth reference metric</td>
<td>Convention 2.51, p. 51</td>
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<tr>
<td>$H_g$</td>
<td>The horizontal tangent space for the bundle $\mathcal{M}<em>{-1} \to \mathcal{M}</em>{-1}/\mathcal{D}_0$ at $g$</td>
<td>p. 114</td>
</tr>
<tr>
<td>$\tilde{H}_g$</td>
<td>The horizontal tangent space for the bundle $\mathcal{M} \to \mathcal{M}/\mathcal{D}_0$ at $g$</td>
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<tr>
<td>$H^N_g$</td>
<td>The horizontal tangent space for the bundle $\mathcal{N} \to \mathcal{N}/\mathcal{D}_0$ at $g$</td>
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<tr>
<td>$i_\mu$</td>
<td>A diffeomorphism $\mathcal{M}_\mu \times \mathcal{P} \to \mathcal{M}$</td>
<td>Equation 2.32, p. 43</td>
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<tr>
<td>$L^{\langle \cdot, \cdot \rangle}(a_t)$</td>
<td>The length of the path $a_t$ in $\mathcal{M}_x$ with respect to the Riemannian metric $\langle \cdot, \cdot \rangle$</td>
<td>Definition 4.8, p. 75</td>
</tr>
<tr>
<td>$L^{\langle \cdot, \cdot \rangle^0}(a_t)$</td>
<td>The length of the path $a_t$ in $\mathcal{M}_x$ with respect to the Riemannian metric $\langle \cdot, \cdot \rangle^0$</td>
<td>Definition 4.8, p. 75</td>
</tr>
<tr>
<td>$M$</td>
<td>The base manifold, a smooth, closed, finite-dimensional manifold.</td>
<td>Convention 2.8, p. 23</td>
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<tr>
<td>$\mathcal{M}$</td>
<td>The Fréchet manifold of smooth Riemannian metrics on $M$</td>
<td>p. 38</td>
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<tr>
<td>$\mathcal{M}^s$</td>
<td>The Hilbert manifold of Riemannian metrics on $M$ with $H^s$ coefficients (for $s &gt; n/2$)</td>
<td>p. 38</td>
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<tr>
<td>$\mathcal{M}^0$</td>
<td>The set of $L^2$-sections of $S^2T^*M$ that are a.e. positive definite</td>
<td>Definition 3.18, p. 65</td>
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<tr>
<td>$\mathcal{M}_f$</td>
<td>The set of measurable semimetrics on $M$ with finite volume</td>
<td>Definition 2.60, p. 2.60</td>
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<tr>
<td>$\hat{M}_f$</td>
<td>The quotient of $M_f$ formed by identifying semimetrics that differ only on their degenerate sets and a nullset</td>
<td>Definition 4.26, p. 84</td>
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<tr>
<td>$\mathcal{M}_m$</td>
<td>The set of measurable semimetrics on $M$</td>
<td>Definition 4.4, p. 74</td>
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<tr>
<td>$\hat{\mathcal{M}}_m$</td>
<td>The quotient of $\mathcal{M}_m$ formed by identifying semimetrics that differ only on their degenerate sets and a nullset</td>
<td>Definition 4.4, p. 74</td>
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<tr>
<td>$\mathcal{M}_\mu$</td>
<td>The Fréchet manifold of metrics inducing the volume form $\mu$</td>
<td>Equation (2.26), p. 42</td>
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<td>$\mathcal{M}_x$</td>
<td>The manifold of positive-definite symmetric $(0,2)$-tensors at $x \in M$</td>
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<td>In Chapter 6 the manifold of hyperbolic metrics on $M$</td>
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<td>$MCG$</td>
<td>In Chapter 6 the mapping class group $\mathcal{D}/\mathcal{D}_0$ of $M$</td>
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<td>$n$</td>
<td>The dimension of the base manifold $M$</td>
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<td>$\mathcal{N}$</td>
<td>A fixed modular section</td>
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<td>In Chapter 6 the genus of the Riemann surface $M$</td>
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<tr>
<td>$\mathcal{P}$</td>
<td>The Fréchet manifold of smooth, positive functions on $M$</td>
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<tr>
<td>$\mathcal{R}$</td>
<td>The moduli space of a Riemann surface of genus $p \geq 2$</td>
<td>p. 112</td>
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<td>$\mathcal{S}$</td>
<td>The Fréchet space of smooth, symmetric $(0,2)$-tensor fields on $M$</td>
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<tr>
<td>$\mathcal{S}^s$</td>
<td>The Hilbert space of symmetric $(0,2)$-tensor fields on $M$ with $H^s$ coefficients</td>
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<tr>
<td>$\mathcal{S}_x$</td>
<td>The vector space of symmetric $(0,2)$-tensors at $x \in M$</td>
<td>p. 2.5.2</td>
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<tr>
<td>$S{g_k}$</td>
<td>The singular set of a sequence ${g_k} \subset \mathcal{M}$</td>
<td>Definition 2.58, p. 52</td>
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<td>$\mathcal{S}^c_g$</td>
<td>The set of pure trace tensors (w.r.t. $g$)</td>
<td>p. 44</td>
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<tr>
<td>$\mathcal{S}^T_g$</td>
<td>The set of $g$-traceless tensors</td>
<td>p. 43</td>
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<tr>
<td>$\mathcal{S}^{TT}_g$</td>
<td>The set of traceless, divergence-free tensors (w.r.t. $g$)</td>
<td>p. 113</td>
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<tr>
<td>$T$</td>
<td>The Teichmüller space of a Riemann surface of genus $p \geq 2$</td>
<td>p. 112</td>
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<tr>
<td>$\text{tr}_{\tilde{g}}$</td>
<td>The $\tilde{g}$-trace of a tensor or product of tensors</td>
<td>Definition 2.34, p. 40</td>
</tr>
<tr>
<td>$\mathcal{U}$</td>
<td>Usually denotes an amenable or quasi-amenable subset of $\mathcal{M}$</td>
<td>Definition 3.10, p. 60</td>
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<tr>
<td>$\mathcal{U}^0$</td>
<td>The $L^2$-completion of the set $\mathcal{U} \subset \mathcal{M}$ (i.e., the completion with respect to $|\cdot|_g$)</td>
<td>Definition 3.18, p. 65</td>
</tr>
<tr>
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Bibliography


List of Corrections

The following is a list of the changes that have been made from the version that was submitted in September 2008 to the Mathematical Institute of the University of Leipzig.

We have not listed the corrections of minor typos that did not affect the mathematical consistency of the text.

p. 25: Corrected typo in the set notation for the maximal atlas.

p. 51: Added condition that $\phi_\alpha = \psi_\alpha|U_\alpha$ to definition of amenable atlas; adjusted proof of Lemma 2.54 to reflect this.

p. 74: Added remark on dependence of conditions for $\omega$-convergence.

p. 82: Corrected typos in second paragraph of proof of Theorem 4.20—all appearances of $X_{g_{k_t}}$ changed to $X_{(g_{k_t})}$.

p. 85ff: Added Lemma 4.28 and Remark 4.30; improved statement and corrected proof of Proposition 4.29.

p. 86: Corrected typo in proof of Proposition 4.29 in second to last paragraph, $g_{k}^i$ changed to $g_{i}^k$.

p. 107: Changed statement to reflect that an element of $\hat{M}_f$ may have both bounded and unbounded representatives.

p. 108: Changed $\psi(\lambda_k)$ and $\psi(\lambda_{k+1})$ in (5.14) to $\lambda_k$ and $\lambda_{k+1}$, respectively.

p. 120: Corrected definition of “$L^2$-orthonomal” in Example 6.12.