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BROWNIAN INTERSECTION LOCAL TIME:
SOME GLIMPSES

by

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INTRODUCTION

To get a complete understanding of the subject, it is aesthetically alluring to trace back a little bit of history behind it. Of course, this is not supposed to be a long stroll through mathematical history and in course of it, significant contributions of many mathematicians for more than a couple of decades can hardly be touched. Sincere apologies are offered to them.

Having undergone sparkling mathematical research spanning more than two decades in the realm of stochastic processes by means of devicing their construction and unraveling their path properties, many better minds of the probability world started investigating questions of "intersection" nature. Needless to say, Brownian motion, perhaps the most appealing of all random processes, was of interest. For example, starting with two independent Brownian motions, can one expect them to meet at some point in the future? However, answering those apparently simple looking questions turned out to be harder than expected. In a paper ([DE00a] and also in the subsequent ones, [DE00b] and [DE00c]) by Dvoretzky, Erdös, Kakutani and Taylor during the 1950's, it was shown that arbitrarily many Brownian motions intersect with positive probability in dimension two whereas in three dimension, at most two motions meet each other. In higher dimensions, no non-trivial intersection is possible. These classical results tempted the mathematicians to dive deeper into the realm of this subject although it turned out that the intersection set is of extremely high complexity and understanding the geometry or the topology of this set was far from obvious. However, subsequent research due to Taylor (see [Ta64]) and Fristedt (see [Fr67]) led to many interesting facts about the size of the intersection set and its Hausdorff dimension. The real upsurge in such activities came when it was possible to define a natural measure supported on the intersection set which somehow computed the "amount of intersection" of the Brownian paths. This random measure was referred to as the "Brownian intersection local time". Historically, the notion of this object was motivated by problems of physics. However, in appendix to a paper by Symanzik ([Sy69]), Varadhan gave a construction of a similar object for the case of planar Brownian bridge. Later, Dynkin ([Dy81]) gave a general construction of additive functionals of Markov processes which included the special case of intersection local time. The precise construction and the properties of this measure was derived by Geman, Horowitz and Rosen ([GHR84]) using the Gaussian character of Brownian motion and results on potential theory. But it was Le Gall around 1986 ([LG86]) who first propounded the idea of considering intersections of "Wiener Sausages" and taking the asymptotic limit of the normalized Lebesgue measure on that to get a random limiting measure. Couple of years later, it was again Le Gall who found out a correct gauge function giving rise to a natural Hausdorff measure on the intersection set. Remarkably, the confluence of these apparently three disparate approaches led to the same notion of Brownian intersection local time. Further research was motivated by problems from geometric measure theory and in particular, understanding the behavior of random fractals. Many long standing open problems were addressed (and also solved) by work of Dembo, Peres, Rosen and Zeitouni (see [DP00a] for results on occupation measure for Brownian paths in dimension

exceeding two and [DP00b] for dimension two and also [DP00d] for intersections of paths in the plane) in connection to the so called "thick points" for the Brownian motion, which are the random regions of the ambient space where the mass of intersection local time is locally extremely dense with probability one. Recent work of König and Mörters ([KM02] and [KM05]) studied the upper tails of the intersection local time in terms of a large deviation principle which pushed the research a bit further by realizing the "size" of the set of thick points in terms of its Hausdorff dimension spectrum.

Of course, this was not entirely an aimless pursuit as quantum physics had always been a steady source of problems of "intersection" nature owing to the overwhelming flow of ideas from the intersection properties of random processes and those of more complicated models in non-equilibrium statistical mechanics. There seems to be a plethora of proposals on the way to pursue research in this branch. Some trends can be found in the penultimate section of this thesis.

Now we will take a quick glance on the contents of each section in this write-up. In section 1, we introduce the Brownian intersection set and recall some properties of that. In section 2, we briefly review how one can think of the intersection local time as a Lebesgue density of the occupation measure for the so called "confluent" Brownian motion. In section 3, we entirely dedicate ourselves to Le Gall's construction of intersection local time via Wiener sausages. Here we are rather precise in spelling out all the details scrupulously since this approach seems to be spearheading the trend of modern research. In section 3, we again briefly review the Hausdorff measure construction due to Le Gall. In section 4, we study the upper tail asymptotic result by König and Mörters. Section 5, 6 and 7 are essentially devoted to the proof of the above mentioned result and in the process, based on a result by Trashorras, we find out a simpler proof of the same result. Section 8 is an informal cruise through the set of open problems in this premises. In section 9, we append some basic facts about large deviations, Hausdorff measures and Sobolev spaces useful for the beginners in the subject.

Finally, one confession should be made. This write-up is a brief overview of the existing work carried out by some mathematicians and inspite of the staggering number of articles in this area, only the tip of the iceberg can be seen. However, an attempt has been made to provide this thesis with copious number of references and hopefully it would account for the lack of exhaustiveness. One important final remark before we start striding the mountains: this whole text is redolent of an ambient probability space hanging around behind the screen.

1. Brownian Intersections

Let us consider p independent Brownian motions $W_1, W_2, ..., W_p$ running in $\mathbb{R}^d$ where $d \geq 2$ until their first exit times $T_1, T_2, ..., T_p$ from a fixed open ball $B(0, R)$, or in the transient case, until infinity (i.e. $R = \infty$). We are interested in the random set of points in the space where the paths of these motions intersect. More precisely, we focus on the set

$$S = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^d | x = W_i(t_i) \text{ for some } t_i \in [0, T_i)\}. \tag{1.1}$$

In other words, S is the intersection of the Brownian paths, the set of space points that are hit by all the motions before their individual exit time. S is a random set with extremely high complexity and was the object of study of many mathematicians. One of the classical results concerning this set is due to Dvoretzky, Erdös, Kakutani and Taylor (see [DE00a], [DE00b] and [DE00c]) which says that with probability one, S has points different from the starting point if and only if

$$p < \frac{d}{d-2}.$$

In other words, the intersection set is non-trivial if and only if either $d = 2$, $p \in \mathbb{N}$, or $d = 3$, $p = 2$. Since having only one point (namely, the starting point of all the motions) in S is not interesting for our purpose, we will restrict our discussion to the above two cases, i.e. we will consider either arbitrarily many motions in $\mathbb{R}^2$ or only two motions in $\mathbb{R}^3$. At times, we might also refer to the case of only one motion running in any dimension $d \geq 2$.

Subsequent research due to Taylor (see [Ta64]) and Fristedt (see [Fr67]) showed that

- S is a set of Lebesgue measure zero in $d \geq 2$ almost surely.
-

$$\dim(S) = \begin{cases} 2 & \text{for } d = 2 \text{ and } p \in \mathbb{N} \\ 1 & \text{for } d = 3 \text{ and } p = 2 \\ 2 & \text{for } d \geq 2 \text{ and } p = 1. \end{cases} \tag{1.2}$$

Although the result in the last case is quite well-known (the Hausdorff dimension of the image of the Brownian curve in any dimension is 2), we include it for completeness.

Remarks :

- Heuristically thinking, these results are not incredibly hard to believe. We know that a Brownian curve in the plane almost fills out the space since it has Hausdorff dimension 2. Therefore it can be roughly thought of as a plane in any dimension. Now even if we have arbirarily many planes in the two dimensional space, they are likely to intersect and if that is the case, they intersect again along a plane, which is of dimension two. Again, in the three dimensional case, two planes are more likely to have an intersection compared to three planes and again if two planes meet in the space, they do so along a line, which is of dimension one. Therefore, although the proofs of these results are technically hard, they are not intuitively abstruse.
- As we mentioned earlier, understanding the topology of the set S is quite non-trivial. For an example, it is intuitively easy to conjecture that S is totally disconnected. However, proving this fact still remains to be an open.

There are many more finer results pertaining to the path of a single Brownian motion. But that would be outside the purview of our discussion as we have an "intersecting" epithet in our title. Hence we turn to our object of interest, namely, the Brownian intersection local times.

2. Confluent Brownian motion and intersection local time

As asserted in the introduction, we want to construct a measure on S which in some sense computes the amount of intersection of the Brownian paths. By work of Geman, Horowitz and Rosen (see [GHR84]), the random set S can be equipped with a natural finite measure ℓ, the (projected) intersection local time, which can be symbolically described by the formula

$$\ell(A) = \int_A dy \ \prod_{j=1}^p \int_0^{T_j} ds \ \delta_y(W_j(s)) \text{ for every } A \subset \mathbb{R}^d \text{ Borel.}$$

It is clear from the above formula that ℓ can be thought of as a 'uniform' measure on S which gives the 'spatial' amount of intersection of the paths in a given set. Of course, this formula has to be justified since the map $y \mapsto \int_0^{T_i} ds \delta_y(W_j(s))$ is not well-defined (we know that the Brownian occupation measure does not have a density). Now we briefly review this approach of Geman, Horowitz and Rosen.

We know that the Brownian occupation measure does not possess a Lebesgue density in $d \geq 2$ (that is, it is not absolutely continuous with respect to the Lebesgue measure). But it turns out that the so called confluent Brownian motion whose zero set, by definition, corresponds to the time points where the confluences take place, has a Lebesgue density. Keeping up with the notion of a local time, this (projected) density is called Brownian intersection local time. To outline the complete picture, we start of with the definition of a local time.

Let us fix two natural numbers N and D. Let $X \colon \mathbb{R}_+^N \to \mathbb{R}^D$ be a Borel function. Fix a Borel set A in $\mathbb{R}_+^N$. Then we can define the **occupation measure** of X relative to A by

$$\mu_A(B) = \lambda_N(A \cap X^{-1}(B)) \ \ \forall B \in \mathcal{B}(\mathbb{R}^D).$$

If $\mu_A \ll \lambda_d$, we write

$$\alpha(y, A) = \frac{d\mu_A}{d\lambda_D}(y) \ \ \forall y \in \mathbb{R}^D.$$

for the corresponding Radon-Nikodym derivative. This function $\alpha(y, A)$ is called the **occupation density** or the **local time** on A with respect to the Borel function X. If there is an occupation density for each A then we may choose $\alpha(y, A)$ to be a kernel (i.e., measurable in y and a finite measure in A).

Now the above general set up can be applied to a particular situation, namely, the **confluent Brownian motion** $W \colon \mathbb{R}_+^p \to \mathbb{R}^{d(p-1)}$ which is defined by

$$W(s_1, s_2, ..., s_p) = \big(W_1(s_1) - W_2(s_2), W_2(s_2) - W_3(s_3), ..., W_{p-1}(s_{p-1}) - W_p(s_p)\big). \qquad (2.1)$$

Now we know that the occupation measure of a single Brownian path in $d \geq 2$ does not have a density. But it was proved by the authors that the occupation measure corresponding to the confluent Brownian motions does have a density. In other words, with probabiltity one, the occupation density $\alpha(y, A)$ for the confluent Brownian motion process W exists for every Borel set A in $\mathbb{R}_+^p$ and may be chosen so that $(y, t) \mapsto \alpha(y, Q_t)$ is jointly continuous, where $Q_t = \prod_{i=1}^p [0, t_i]$.

This implies that, with probability one, there is a family $\{\mu_y \colon y \in (\mathbb{R}^{d(p-1)})\}$ of finite measures on $\prod_{i=1}^{p}[0, T_i)$ such that

(i) The mapping $y \mapsto \mu_y$ is continuous with respect to the vague topology on the space $\mathbb{M}(\mathbb{R}^p)$ of locally finite measures on $\mathbb{R}^p$.

(ii) For all Borel functions $g \colon \mathbb{R}^{d(p-1)} \to [0, \infty]$ and $f \colon \Pi_{i=1}^{p}[0, T_i) \to [0, \infty]$

$$\int g(y)\langle f, \mu_y \rangle \, dy = \int_{\prod_{i=1}^{p}[0,T_i)} f \cdot g(W) \, ds_p...ds_1.$$

It follows from the above two properties that, for each y, the measure μ_y is supported by the level set

$$M_y = \{(s_1, s_2, ..., s_p) \in \prod_{i=1}^{p}[0, T_i) \colon W(s_1, s_2, ..., s_p) = y\}.$$

Note that M_0 is the set of time vectors at which the p motions coincide, which is the set we are interested in. Now we consider the mapping $T \colon \prod_{i=1}^{p}[0, T_i) \to \mathbb{R}^d$ defined by $T(t_1, t_2, ..., t_p) = W_1(t_1)$. Then

$$T(M_0) = S.$$

Now for every Borel set B in S, define

$$\ell(B) = \mu_0(T^{-1}(B)),$$

i.e., ℓ is the image measure of μ_0 under T. The measure ℓ on S defined above is called the *Brownian intersection local time* of the p Brownian motions.

3. Wiener Sausages and intersection local time.

A much simpler and nicer construction of the intersection local time was carried out by Le Gall (see [LG86]) using Wiener sausages, which by definition, is a tubular neighborhood around the Brownian path (as the name suggests). Then we look at the renormalized Lebesgue measure on the intersection of the independent sausages and then let the intersection of the sausages shrink to the intersection of the paths. The random limit thus obtained coincides with the object named as Brownian intersection local times by Geman, Horowitz and Rosen.

We formulate the above heuristic discussion in a precise form. For every $\epsilon > 0$, we define the **Wiener sausage** around each W_i by

$$S_\epsilon^i = \{x \in \mathbb{R}^d \colon \text{there is } t \in [0, T_i) \text{ with } |x - W_i(t)| < \epsilon\} \quad \text{for } i = 1..., p$$

and take their intersection

$$S_\epsilon = \bigcap_{i=1}^p S_\epsilon^i.$$

We observe that $S = \bigcap_{\epsilon > 0} S_\epsilon$. Now, for every $\epsilon > 0$, we define the normalised Lebesgue measure ℓ_ϵ on $\mathbb{R}^d$ by

$$d\ell_\epsilon(y) = s_d(\epsilon) \cdot 1_{S_\epsilon}(y) \, dy$$

where

$$s_d(\epsilon) = \begin{cases} \pi^{-p} \log^p(\frac{1}{\epsilon}) & \text{if } d = 2 \\ (2\pi\epsilon)^{-2} & \text{if } d = 3 \text{ and } p = 2 \\ \frac{2}{\omega_d(d-2)} \epsilon^{2-d} & \text{if } d \geq 3 \text{ and } p = 1. \end{cases}$$

Then it turns out that the limit $\epsilon \downarrow 0$ yields the Brownian intersection local time. More precisely, For every $A \subset \mathbb{R}^d$ that is almost surely an ℓ-continuity set,

$$\lim_{\epsilon \to 0} \ell_\epsilon(A) = \ell(A) \text{ in } L^q(\mathbb{P}) \text{ for any } q \in [1, \infty), \tag{3.1}$$

where ℓ is the (projected) intersection local time measure defined in the previous approach of Geman, Horowitz and Rosen.

Example : Let us pause here for a moment and analyse the above result with the help of a simpler example. Suppose we have a smooth curve $\gamma : [0, 1] \to \mathbb{R}^2$. An elementary result in analysis says that

$$\lim_{\epsilon \to 0} \frac{\lambda(\gamma_\epsilon)}{2\epsilon} = L(\gamma) \tag{3.2}$$

where,

$$\gamma_\epsilon = \{y \in \mathbb{R}^2 \: : \: |y - \gamma(t)| \leq \epsilon \text{ for some } t \in [0, 1]\}$$

That is γ_ϵ is the tubular ϵ-neighborhood of the image of the curve and $L(\gamma)$ is the length of the curve.

We compare the above fact with the one obtained by Le Gall. For simplicity, we take $d = 2$ and $p = 1$. Then 3.1 says that

$$\lim_{\epsilon \to 0} \frac{\lambda(S_\epsilon \cap A)}{(\frac{\pi}{\log \epsilon})} = \ell(A). \tag{3.3}$$

A glance on 3.2 and 3.3 implies that ℓ can be thought of as the "size" of the Brownian curve (note that the only difference in the above two results is made by the normalizing constants). In other words, the intersection local time is nothing but a mesaure of the intersection of the Brownian paths.

The reasons behind getting a rather simple looking normalizing constant in the first case than the second one can be explained on a heuristic level as follows.

- The first curve is differentiable everywhere and the second one is not differentiable anywhere
- The co-dimension of the image of the first curve is $(2-1) = 1$ which appears as the exponent of ϵ in the denomenator. Whereas, in the second case, the co-dimension of the Brownian curve is $(2-2) = 0$.

As proposed, we go through the proof of the above construction of the intersection local time as this approach turns out to be most useful for our purpose. We need some results on the hitting time of a Brownian motion. We state and prove them step by step.

First, let us start with a Brownian motion B with values in $\mathbb{R}^d$. Let ζ be an exponential random time with parameter $\lambda > 0$. We assume that ζ is independent of B. We are interested to work with the process B killed at time ζ, which is a symmetric Markov process with **Green's function** defined as:

$$G_\lambda(x,y) = \int_0^\infty e^{-\lambda s} p_s(x,y)\, ds \quad \text{for each } x \text{ and } y \text{ in } \mathbb{R}^d \tag{3.4}$$

where the Brownian transition probability function is defined as

$$p_s(x,y) = (2\pi s)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2s}}.$$

We observe that, $G_\lambda(x,y) = \infty$ if $x = y$ (since $d \geq 2$) and $G_\lambda(x,y) < \infty$ otherwise. Moreover, $G_\lambda(x,y)$ is bounded away from 0 on compact sets. We can also easily find out the asymptotic behavior of the Green's function as x comes closer to y. More precisely:

$$\text{as } |y-x| \to 0 \quad \begin{cases} G_\lambda(x,y) \sim \ C_d\, |y-x|^{2-d} & \text{if } d \geq 3 \\ G_\lambda(x,y) \sim \ \frac{1}{\pi}\, \log \frac{1}{|y-x|} & \text{if } d = 2 \end{cases} \tag{3.5}$$

where C_d is the volume of the d- dimensional unit ball.

Now, for $y \in \mathbb{R}^d$ and $\epsilon > 0$, we set :

$$T_\epsilon(y) \ = \ \inf\{t : |B_t - y| \leq \epsilon\}.$$

In other words, $T_\epsilon(y)$ is the random time when the Brownian motion hits the ball of radius ϵ around y for the first time.

Lemma 3.1.   (i) *For every $y \neq 0$ we have,*

$$\begin{cases} \lim_{\epsilon \to 0} \ (log\frac{1}{\epsilon})\ \mathbb{P}[T_\epsilon(y) < \zeta] = \pi\ G_\lambda(0,y) & \text{for } d = 2 \\ \lim_{\epsilon \to 0} \ \epsilon^{2-d}\ \mathbb{P}[T_\epsilon(y) < \zeta] = \left(\frac{d}{2} - 1\right)\ C_d\ G_\lambda(0,y) & \text{for } d \geq 3 \end{cases} \tag{3.6}$$

where C_d is the volume of the d-dimensional unit ball.
   (ii) *There exists a constant $C_{\lambda,d}$ such that, for any $\epsilon \in (0, \frac{1}{2})$ and $y \in \mathbb{R}^2$, we have:*

$$\mathbb{P}\left[T_\epsilon(y) < \zeta\right] \ \leq \ \begin{cases} C_{\lambda,d}\ G_\lambda(0, \frac{y}{2}) \times \log\left(\frac{1}{\epsilon}\right)^{-1} & \text{for } d = 2 \\ C_{\lambda,d}\ G_\lambda(0, \frac{y}{2}) \times \epsilon^{d-2} & \text{for } d \geq 3 \end{cases} \tag{3.7}$$

Proof : We will prove the above lemma for the case $d = 2$ only (since the proof is same for the case $d \geq 3$). Now we observe that

$$\mathbb{E}\left[\int_0^\zeta 1_{(|B_s - y| \leq \epsilon)}\, ds\right] = \int_0^\infty ds\ e^{-\lambda s} \int_{|z-y|\leq\epsilon} dz\ p_s(0,z) \ \overset{\epsilon \to 0}{\sim}\ \pi\ \epsilon^2\ G_\lambda(0,y). \tag{3.8}$$

The last asymptotic limit results from the dominated convergence theorem.

On the other hand, assuming that $|y| \geq \epsilon$ we get :

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\zeta 1_{(|B_s-y|\leq\epsilon)}\,ds\right] &= \mathbb{E}_\zeta \otimes \mathbb{E}_B\left[\int_0^\zeta 1_{B_s\in K_\epsilon(y)}\,ds\right] \\
&= \mathbb{E}_\zeta \otimes \mathbb{E}_B\left[\int_{T_\epsilon(y)}^\zeta 1_{B_s\in K_\epsilon(y)}\,ds \cdot 1_{T_\epsilon(y)<\zeta}\right] \\
&= \mathbb{E}_B\left[\mathbb{E}_{\zeta_2}\left[\int_{T_\epsilon(y)}^{\zeta_2} 1_{B_s\in K_\epsilon(y)}\,ds\Big|T_\epsilon(y)<\zeta_2\right]\mathbb{P}_{\zeta_1}\left[T_\epsilon(y)<\zeta_1\right]\right] \\
&= \mathbb{E}_B\left[\mathbb{E}_{\zeta_2}\left[\int_{T_\epsilon(y)}^{T_\epsilon(y)+\zeta_2} 1_{B_s\in K_\epsilon(y)}\,ds\right]\mathbb{P}_{\zeta_1}\left[T_\epsilon(y)<\zeta_1\right]\right] \\
&= \mathbb{P}_\zeta\left[T_\epsilon(y)<\zeta\right]\cdot \mathbb{E}_B\left[\mathbb{E}_\zeta\left[\int_{T_\epsilon(y)}^{T_\epsilon(y)+\zeta} 1_{B_s\in K_\epsilon(y)}\,ds\right]\right] \\
&= \mathbb{P}_\zeta\left[T_\epsilon(y)<\zeta\right]\cdot \mathbb{E}_\zeta\left[\mathbb{E}_B\left[\int_{T_\epsilon(y)}^{T_\epsilon(y)+\zeta} 1_{B_s\in K_\epsilon(y)}\,ds\right]\right] \\
&= \mathbb{P}_\zeta\left[T_\epsilon(y)<\zeta\right]\cdot \mathbb{E}_\zeta\left[\mathbb{E}_B^{y_\epsilon}\left[\int_0^\zeta 1_{B_s\in K_\epsilon(y)}\,ds\right]\right],
\end{aligned}
$$

where $|y_\epsilon - y| = \epsilon$. The third equality follows from Fubini's theorem. The fourth one is a consequence of the fact that $\mathfrak{D}(\zeta) = \mathfrak{D}(\zeta - t|\zeta > t)$ for all $t > 0$, where for a random variable X, $\mathfrak{D}(X)$ denotes its distribution. The fifth one results from the strong Markov property at time $T_\epsilon(y)$ which enforces the independence of ζ and B. The sixth equality again follows from Fubini's theorem and the last one is obtained by using the strong Markov property of B at time $T_\epsilon(y)$. Now:

$$
\begin{aligned}
\mathbb{E}_\zeta \otimes \mathbb{E}_B^{y_\epsilon}\left[\int_0^\zeta 1_{B_s\in K_\epsilon(y)}\,ds\right] &= \mathbb{E}_B^{y_\epsilon} \otimes \mathbb{E}_\zeta\left[\int_0^\zeta 1_{B_s\in K_\epsilon(y)}\,ds\right] \\
&= \int_0^\infty ds\ e^{-\lambda s}\ \mathbb{P}_B^{y_\epsilon}\left(B_s \in K_\epsilon(y)\right) \\
&= \int_{|z-y|\leq\epsilon} dz\ G_\lambda(y_\epsilon, z) \\
&\overset{\epsilon\to0}{\sim}\ \epsilon^2 \log\frac{1}{\epsilon}.
\end{aligned}
\tag{3.9}
$$

We combine the above equation with (3.8) to conclude the proof of the lemma. $\square$.

Lemma 3.2. *Let $t > 0$ and $y \neq 0$. Then:*

(i) *for $d = 2$:*

$$
\lim_{\epsilon\to0}\left(\log\frac{1}{\epsilon}\right)\mathbb{P}\left[T_\epsilon(y)\leq t\right] = \pi\int_0^t p_s(0,y)\,ds
$$

(ii) *for $d \geq 3$*

$$
\lim_{\epsilon\to0}\left(\log\frac{1}{\epsilon}\right)\mathbb{P}\left[T_\epsilon(y)\leq t\right] = \left(\frac{d}{2}-1\right)C_d\int_0^t p_s(0,y)\,ds
$$

Proof : Let us again prove this lemma only for the case $d = 2$ (since the other case can be similarly treated). We denote by $\gamma_\epsilon(ds)$ the law of $T_\epsilon(y)$. By Lemma 3.1,

$$\lim_{\epsilon \to 0} \left(\log \frac{1}{\epsilon} \right) \int_0^\infty e^{-\lambda s} \gamma_\epsilon(ds) = \pi \, G_\lambda(0, y) = \pi \int_0^\infty e^{-\lambda s} \, p_s(0, y) \, ds. \tag{3.10}$$

Now the above result holds for every $\lambda > 0$. In other words, the Laplace transform of the measures $|\log \frac{1}{\epsilon}|\gamma_\epsilon(ds)$ converges to the Laplace transform of the measure $\pi \, p_s(0, y) \, ds$. Hence it follows that the measures $\left| \log \frac{1}{\epsilon} \right| \gamma_\epsilon(ds)$ converge weakly to the measure $\pi \, p_s(0, y) \, ds$. In particular,

$$\lim_{\epsilon \to 0} \left(\log \frac{1}{\epsilon} \right) \gamma_\epsilon([0, t]) = \pi \int_0^t p_s(0, y) \, ds. \qquad \square \tag{3.11}$$

Remark : It is worth making a comment that all the above facts also hold true in a more general set up. To be more explicit, in the above two lemmas our general idea was to compute the expected time spent in the closed ball of radius ϵ around y, namely $K_\epsilon(y) = y - \epsilon K_1(0)$ where $K_1(0)$ is the closed unit ball of $\mathbb{R}^d$. Now, similar statements (like that of Lemma 3.1 and Lemma 3.2) can be made if the closed unit ball $K_1(0)$ is replaced by an arbitrary non-polar compact subset K of $\mathbb{R}^d$. The idea of the proof comes from basic results of probabilistic potential theory which gives the hitting probability of K for the process B killed at time ζ :

$$\mathbb{P}_y[T_K < \zeta] = \int_K G_\lambda(y, z) \, \mu_K^\lambda(dz) \tag{3.12}$$

where $T_K = \inf\{t\colon B_t \in K\}$ and $\mu_K^\lambda(dz)$ is a finite measure supported on K, the $\lambda -$ *equilibrium measure* of K. The total mass of $\mu_K^\lambda(dz)$ is denoted by $C_\lambda(K)$ and called the $\lambda - capacity$ of K. The non-polarity of K is equivalent to the fact that $C_\lambda(K) > 0$. Again, a basic formula of probabilistic potential theory gives :

$$C_\lambda(K) = \left[\inf\{ \int_K \mu(dy)\mu(dz) \, G_\lambda(y, z) \; : \mu \in \mathfrak{M}_1(K)\} \right]^{-1} \tag{3.13}$$

where $\mathfrak{M}_1(K)$ denotes the set of all probability measures supported on K. However, for our purpose, it is not necessary to implement the sophisticated tools of potential theory for arbitrary non-polar compact subset of $\mathbb{R}^d$. Looking at the closed unit ball suffices to derive the construction of our object of study, the intersection local time.

Let us now turn to the first step of the construction of intersection local time via **Wiener Sausages**.

Let $p \geq 2$ be an integer and $B^1, ..., B^p$ denote p independent Brownian motions running in $\mathbb{R}^2$ starting at $x^1, ..., x^p$ respectively. We recall from the previous construction of Geman, Horowitz and Rosen that the intersection local time $\alpha(ds_1, ..., ds_p)$ is a random measure on $(\mathbb{R}_+)^p$ supported on $\{(t_1, ..., t_p) \in \mathbb{R}_+^p \; : \; B_{t_1}^1 = B_{t_2}^2 = ... = B_{t_p}^p\}$ and it is symbolically given by:

$$\alpha(ds_1 ... ds_p) = \delta_0 \left(B_{s_1}^1 - B_{s_2}^2 \right) ... \delta_0 \left(B_{s_{p-1}}^{p-1} - B_{s_p}^p \right) ds_1 .... ds_p \tag{3.14}$$

where δ_0 denotes the Dirac measure at 0. Equivalently,

$$\alpha(ds_1 ... ds_p) = \int_{\mathbb{R}^2} \delta_y \left(B_{s_1}^1 \right) ... \delta_y \left(B_{s_p}^p \right) ds_1 .... ds_p. \tag{3.15}$$

The idea of this approach is to replace the Dirac measure at y in the latter symbolical expression by a suitable approximation. We set:

$$\delta^\epsilon_{(y)}(z) = (\pi\epsilon^2)^{-1} \, 1_{K_\epsilon(y)}(z) \tag{3.16}$$

and

$$\phi_\epsilon(z_1, ..., z_p) = \int_{\mathbb{R}^2} \prod_{j=1}^{p} \delta^\epsilon_{(y)}(z_j) \, dy \tag{3.17}$$

and

$$\alpha_\epsilon(\, ds_1 ... \, ds_p) = \phi_\epsilon\left(B^1_{s_1}, ..., B^p_{s_p}\right) \, ds_1 ... \, ds_p \tag{3.18}$$

Note that ϕ_ϵ is translation invariant i.e. $\phi_\epsilon(z_1, ..., z_p) = \phi_\epsilon(z_1 + x, ..., z_p + x)$ for every $x \in \mathbb{R}^2$ (since Lebesgue measure is translation invariant).

Proposition 3.3. *With probability one, there exists a random measure $\alpha(\, ds_1, ..., \, ds_p)$ on $(\mathbb{R}_+)^p$ such that for any bounded Borel sets $A^1, ..., A^p$ of $(\mathbb{R}_+)$,*

(i)

$$\lim_{\epsilon \to 0} \alpha_\epsilon(A^1 \times ... \times A^p) = \alpha(A^1 \times ... \times A^p)$$

in the L^n-norm, for any $n < \infty$.

(ii) *the measure $\alpha(\cdot)$ is almost surely supported by*

$$\{(s_1, ..., s_p) \; : \; B^1_{s_1} = ... = B^p_{s_p}\}.$$

(iii) *With probability one, for any $1 \le j \le p$ and any $t \ge 0$,*

$$\alpha\left(\{s_j = t\}\right) = 0.$$

(iv) (**Le Gall's moment formula**) *Finally, for every $n \in \mathbb{N}$ and every $A^1, ..., A^p$, we have:*

$$\mathbb{E}\left[\alpha\left(A^1 \times \cdots \times A^p\right)^n\right] = \int_{(\mathbb{R}^2)^n} dy_1 \ldots dy_n$$
$$\times \prod_{j=1}^{p}\left[\int_{(A^j)^n_<} ds_1 \ldots ds_n \sum_{\sigma \in \Sigma_n}\left(p_{s_1}(x^j, y_{\sigma(1)}) \times \prod_{k=2}^{n} p_{s_k - s_{k-1}}\left(y_{\sigma(k-1)}, y_{\sigma(k)}\right)\right)\right] \tag{3.19}$$

where Σ_n is the symmetric group of permutations of $\{1, \ldots n\}$ and

$$(A^j)^n_< = \{(s_1, \ldots, s_n) \in (A^j)^n \; : \; 0 \le s_1 < \cdots < s_n\}.$$

Proof : We shall prove the above proposition in several steps.

Step 1 : We first check the L^2 convergence of $\alpha_\epsilon(A^1 \times \cdots \times A^p)$. For that, it suffices to prove that

$$\lim_{\epsilon, \epsilon' \to 0} \mathbb{E}\left[\alpha_\epsilon(A^1 \times \cdots \times A^p) \, \alpha'_\epsilon(A^1 \times \cdots \times A^p)\right] \tag{3.20}$$

exists and is finite. Now, we have

$$\mathbb{E}\left[\alpha_\epsilon(A^1 \times \cdots \times A^p)\,\alpha'_\epsilon(A^1 \times \cdots \times A^p)\right] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} dy\; dy' \prod_{j=1}^p \left(\int_{(A^j)^2} ds\, ds'\; \mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\,\delta^{\epsilon'}_{(y')}(B^j_{s'})\right]\right)$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} dy\; dy' \prod_{j=1}^p \int_{(A^j)^2_<} ds\, ds'$$

$$\times\, \mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\,\delta^{\epsilon'}_{(y')}(B^j_{s'}) + \delta^{\epsilon'}_{(y')}(B^j_s)\,\delta^\epsilon_{(y)}(B^j_{s'})\right].$$

Here the first equality follows from Fubini's theorem.

Now we claim that, for $(s, s') \in (A^j)^2_<$,

$$\lim_{\epsilon,\epsilon'\to 0} \mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\,\delta^{\epsilon'}_{(y')}(B^j_{s'})\right] = p_s(x^j, y)\, p_{s'-s}(y, y').$$

To see this, we observe:

$$\lim_{\epsilon,\epsilon'\to 0} \mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\,\delta^{\epsilon'}_{(y')}(B^j_{s'})\right] = \lim_{\epsilon,\epsilon'\to 0} \frac{1}{\pi\epsilon^2}\frac{1}{\pi\epsilon'^2} \mathbb{P}\left[B_s \in K_\epsilon(y),\; B_{s'-s} \in K_{\epsilon'}(y'-y)\right]$$

$$= \lim_{\epsilon,\epsilon'\to 0} \frac{1}{\pi\epsilon^2}\frac{1}{\pi\epsilon'^2} \int_{K_\epsilon(y)} p_s(x^j, z)\, dz \int_{K_{\epsilon'}(y'-y)} p_{s'-s}(0, y'-y)\, dz$$

$$= \frac{1}{\pi\epsilon^2} \int_{K_\epsilon(y)} p_s(x^j, y)\, dz\; \frac{1}{\pi\epsilon'^2} \int_{K_{\epsilon'}(y'-y)} p_{s'-s}(0, y'-y)\, dz$$

$$= p_s(x^j, y)\, p_{s'-s}(y, y').$$

The first equality follows from the Markov property and the rest is routine.

Now we want to use the dominated convergence theorem. For that, we will find a function $\phi(y, y', s, s')$ such that for every $M > 0$,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} dy\, dy' \left(\int_{[0,M]^2_<} ds\, ds'\; \phi(y, y', s, s')\right)^p < \infty \tag{3.21}$$

and for any $y, y' \in \mathbb{R}^2, 0 < s < s' < \infty$, $\epsilon, \epsilon' \in (0, 1)$,

$$\mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\,\delta^{\epsilon'}_{(y')}(B^j_{s'})\right] \leq \phi(y - x^j, y', s, s'). \tag{3.22}$$

The existence of such a function justifies the passage to the limit under the integral sign.

Without entailing any loss of generality, we assume that all the Brownian motions start from the origin, which means $x^j = 0$ and drop the superscript j henceforth. We have two cases. When $|y| \geq 2\epsilon$, then:

$$\mathbb{E}[\delta_{(y)}^\epsilon(B_s)] = \frac{1}{\pi\epsilon^2}\mathbb{P}[\,B_s \in K_\epsilon(y)]$$

$$= \frac{1}{\pi\epsilon^2}\int_{|z-y|\leq\epsilon} \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{|z|^2}{2}}\; dz$$

$$\leq \frac{1}{\pi\epsilon^2}\frac{1}{\sqrt{2\pi s}} \int_{K_\epsilon(y)} e^{-\frac{y^2}{4}}\, dz \tag{3.23}$$

$$= p_s(0, \frac{y}{2}).$$

For deriving the third inequality in the above expression we have used the fact that the map $|z| \mapsto e^{-\frac{|z|^2}{2}}$ is decreasing.

Now when $|y| < 2\epsilon$, then of course,

$$\mathbb{E}[\delta^\epsilon_{(y)}(B_s)] \leq (\pi\epsilon^2)^{-1} \vee (2\pi s)^{-1} \leq 4(|y|^{-2} \wedge s^{-1}).$$

Therefore:

$$\mathbb{E}[\delta^\epsilon_{(y)}(B_s)] \leq \psi(y,s)$$

where

$$\psi(y,s) = 4\, 1_{(|y|<2\epsilon)}\, \big[|y|^{-2} \wedge s^{-1}\big] + p_s\left(0, \frac{y}{2}\right)$$

We note that:

$$\int_0^M \psi(y,s)\, ds \leq C_M\, G_1(0,y) \tag{3.24}$$

where

$$G_1(0,y) = \int_0^\infty \mathrm{e}^{-s}\, p_s(x,y)\, ds.$$

We now bound $\mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\, \delta^{\epsilon'}_{(y')}(B^j_{s'})\right]$. The easy case is when $|y - y'| \geq 2(\epsilon + \epsilon')$. Then the strong Markov property at time s gives :

$$\mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\, \delta^{\epsilon'}_{(y')}(B^j_{s'})\right] \leq \mathbb{E}\left[\delta^\epsilon_{(y)}(B_s)\right]\, p_{s'-s}(0, \frac{y'-y}{2}) \leq \psi(y,s)\, p_{s'-s}(0, \frac{y'-y}{2}).$$

Suppose now that $|y' - y| < 2(\epsilon + \epsilon') \leq 4$. If $\epsilon \leq \epsilon'$, then again the Markov property gives:

$$\mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\, \delta^{\epsilon'}_{(y')}(B^j_{s'})\right] \leq \mathbb{E}\left[\delta^\epsilon_{(y)}(B_s)\right] \left((\pi\epsilon^2)^{-1} \wedge (2\pi(s'-s))^{-1}\right)$$

$$\leq 16\, \psi(y,s)(\, |y'-y|^{-2} \wedge (s's)^{-1}).$$

If $\epsilon' < \epsilon$, then we have to consider each of the cases $s' - s > |y' - y|^2$ and $s' - s \leq |y' - y|^2$. For the first case, we have:

$$\mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\, \delta^{\epsilon'}_{(y')}(B^j_{s'})\right] \leq \mathbb{E}\left[\delta^\epsilon_{(y)}(B_s)\right]\, (2\pi(s'-s))^{-1} \leq \psi(y,s)\, (|y'-y|^{-2} \wedge (s'-s)^{-1}).$$

Secondly, if $s' - s \leq |y' - y|^2$ and $\epsilon' < \epsilon$, then :

$$\mathbb{E}\left[\delta^\epsilon_{(y)}(B^j_s)\, \delta^{\epsilon'}_{(y')}(B_{s'})\right] \leq (\pi\epsilon^2)^{-1}\mathbb{E}\left[\delta^{\epsilon'}_{(y')}(B_{s'})\right]$$

$$\leq 16\, |y-y'|^{-2}\, \psi(y',s')$$

$$\leq 16\, \left(|y'-y|^{-2} \wedge (s'-s)^{-1}\right) \psi(y',s').$$

We combine all the previous estimates to obtain:

$$\mathbb{E}\left[\delta^\epsilon_{(y)}(B_s)\, \delta^{\epsilon'}_{(y')}(B_{s'})\right] \leq \phi(y,y',s,s')$$

with

$$\phi(y,y',s,s') = (\psi(y,s) + \psi(y',s')) \left(p_{s'-s}(0, \frac{y'-y}{2}) + 16(|y'-y|^{-2} \wedge (s'-s)^{-1})\right).$$

we note that:

$$\int_{[0,M]^2_<} ds\, ds'\, \phi(y,y',s,s') \leq C'_M \left(G_1(0, \frac{y}{2}) + G_1(0, \frac{y'}{2})\right) G_1(0, \frac{y'-y}{2}).$$

Now we know that the function $y \mapsto G_1(0, y)$ is in L^p for all $p < \infty$. Therefore we can use the dominated convergence theorem to get the desired result. Summarizing we write:

$$\tilde{\alpha}(A^1 \times \cdots \times A^p) = \lim_{\epsilon \to 0} \alpha(A^1 \times \cdots \times A^p)$$

in L^2. Note that $\tilde{\alpha}$ a priori depends on $A^1, \ldots, A^p$.

Step 2 : Now we have to check that the convergence holds in L^n for every $n \in \mathbb{N}$ and that the n-th moment of $\tilde{\alpha}(A^1 \times \cdots \times A^p)$ is the right hand side of (3.19). For that, we check the convergence of :

$$
\begin{aligned}
\mathbb{E}\left[\alpha \ (A^1 \times \cdots \times A^p)\right] &= \int_{\mathbb{R}^2} dy_1 \ldots dy_n \prod_{j=1}^{p} \left(\int_{(A^j)^n} ds_1 \ldots ds_n \mathbb{E}\left[\prod_{k=1}^{n} \delta^{\epsilon}_{(y_k)}(B^j_{s_k})\right] \right) \\
&= \int_{\mathbb{R}^2} dy_1 \ldots dy_n \prod_{j=1}^{p} \left(\sum_{\sigma \in \Sigma_n} (\int_{(A^j)^n_<} ds_1 \ldots ds_n \mathbb{E}\left[\prod_{k=1}^{n} \delta^{\epsilon}_{y_{\sigma(k)}}(B^j_{s_k})\right] \right).
\end{aligned}
$$
(3.25)

Now we again use the same arguments used in Step-1 to obtain:

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\prod_{j=1}^{p} \delta_{(y_\sigma(k))}(B^j_{s_k}) \right] = p_{s_1}(x^j, y_{\sigma(1)}) \prod_{k=2}^{n} p_{s_k - s_{k-1}}(y_{\sigma(k-1)}, y_{\sigma(k)}).$$

Again, like the previous step, we have to use the dominated convergence theorem and we use the same techniques used before. We consider separately the cases $|y_{\sigma(k)} - y_{\sigma(k-1)}| \geq 4\epsilon$ and $|y_{\sigma(k)} - y_{\sigma(k-1)}| < 4\epsilon$ and use the Markov property at times $s_{n-1}, s_{n-2}, \ldots, s_1$ to obtain :

$$\mathbb{E}\left[\prod_{j=1}^{p} \delta_{(y_\sigma(k))}(B^j_{s_k}) \right] \leq 4 \, \psi(y_{\sigma(1)} - x^j, s_1) \times \cdots \times 4\psi(y_{\sigma(n)} - y_{\sigma(n-1)}, s_n - s_{n-1})$$

where the function ψ is the same one defined in the previous step. Then the bound of (3.24) justifies the passage to the limit in the right hand side of (3.25).

Step 3 : Now we will construct a random measure $\alpha(.)$ such that for any $A^1, \ldots, A^p$, we have $\alpha(A^1, \ldots, A^p) = \tilde{\alpha}(A^1, \ldots A^p)$ almost surely. We first consider the case when $A^j = [a_j, b_j]$ where $a_j \leq b_j \leq M$. Now, by the previous step, we know that $\mathbb{E}\left[\tilde{\alpha}(A^1, \ldots, A^p)^n\right]$ is the right hand side of (3.19). Then we apply generalized Hölder's inequality to the right hand side of (3.19) to obtain:

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\alpha}(A_1 \times \ldots A_p)^n\right] &\leq (n!)^p \prod_{j=1}^{p} \left(\int_{(\mathbb{R}^2)^n} dy_1 \ldots dy_n \left(\int_{(A^j)^n_<} ds_1 \ldots ds_n \, p_{s_1}(x^j, y_1) \prod_{k=2}^{n} p_{s_k - s_{k-1}}(y_{k-1}, y_k) \right)^p \right)^{\frac{1}{p}} \\
&\leq (n!)^p \prod_{j=1}^{p} \left(\int_{(\mathbb{R}^2)^n} dy_1 \ldots dy_n G^{a_j, b_j}(x^j, y_1)^p \prod_{k=2}^{n} G^{0, b_j - a_j}(y_{k-1}, y_k)^p \right)^{\frac{1}{p}}
\end{aligned}
$$

where

$$G^{u,v}(x, y) = \int_u^v ds \, p_s(x, y) \text{ for all } x, y, v, u \in \mathbb{R}.$$

Now we use Jensen's inequality to check that

$$\int dy\, G^{u,v}(x,y)^p \leq C_p(v-u)$$

for some constant C_p. We conclude that

$$\mathbb{E}\left[\tilde{\alpha}(A^1 \times \cdots \times A^p)\right] \leq (C_p)^n\, (n!)^p \prod_{j=1}^{p}(b_j - a_j)^{\frac{n}{p}}.$$

It follows from the above inequality and the multidimensional version of the Kolmogorov-Chentsov theorem that the mapping

$$(a_1, b_1, a_2, b_2, \ldots, a_p, b_p) \mapsto \tilde{\alpha}\left([a_1, b_1] \times \cdots \times [a_p, b_p]\right)$$

has a continuous version, denoted by $\alpha([a_1, b_1] \times \cdots \times [a_p, b_p])$. It easily follows by the definition of $\tilde{\alpha}$ that $\alpha([a_1, b_1] \times \cdots \times [a_p, b_p])$ is a finitely additive function of $[a_1, b_1] \times \cdots \times [a_p, b_p]$ (First we consider the case when a_j, b_j are rationals and then generalize it to obtain finite additivity) But the family of sets of the form $[a_1, b_1] \times \cdots \times [a_p, b_p]$ with $a_j, b_j \geq 0$ form a semi-algebra of sets and hence, by Kolmogorov's extension theorem, the finitely additive set function α can be extended to a Radon measure on the σ-algebra generated by the semi-algebra.

But now, we want to extend our result to arbitrary Borel sets A_i of $\mathbb{R}_+$ so that α and $\tilde{\alpha}$ agree on the product $A_1 \times \cdots \times A_p$.

Suppose we have a sequence of sets $A^{(n)} = A_1^{(n)} \times A_2^{(n)} \cdots \times A_p^{(n)}$ which increase (or respectively decrease) to the set $A = A_1 \times A_2 \times \cdots \times A_p$, then we have the following two facts:

(i) By (3.19), $\tilde{\alpha}(A^{(n)})$ converges in L^2 to $\tilde{\alpha}(A)$.

(ii) α being almost surely a Radon measure, $\alpha(A^{(n)})$ converges to $\alpha(A)$ with probability one.

The above two facts and the Monotone class theorem implies that

$$\alpha(A_1 \times \cdots \times A_p) = \tilde{\alpha}(A_1 \times \cdots \times A_p) \tag{3.26}$$

for every bounded Borel subset A_i of $\mathbb{R}_+$.

Step 4 : It remains to show that the measure α has the desired properties.

Now the fact that for every $t \geq 0$, $\alpha(s_j = t) = 0$ is trivial by the continuity of the mapping $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p) \mapsto \alpha\left([a_1, b_1] \times \cdots \times [a_p, b_p]\right)$.

Finally, if $A = [a_1, b_1] \times \cdots \times [a_p, b_p]$ is a closed rectangle with rational co-ordinates and $C = \{\, \omega \,:\, A \cap \{(s_1, \ldots s_p) \,:\, B_{s_1}^1(\omega) = \cdots = B_{s_p}^p(\omega)\} = \emptyset\}$ then it follows that $\alpha_\epsilon(A) = 0$ for small ϵ on C and hence $\alpha(A) = 0$ almost surely on C. Since this is true with probability one for all rectangles with rational co-ordinates, it follows that

$$\mathrm{supp}(\alpha) = \{(s_1, \ldots s_p) \,:\, B_{s_1}^1 = \cdots = B_{s_p}^p\}. \tag{3.27}$$

The proof of the proposition is now complete. $\square$

Remarks

- Although we have restricted our discussion to the two dimensional case (the intersection of arbitrarily many Brownian motions in $\mathbb{R}^2$), it is quite evident that a similar result (with exactly the same proof) can be stated in three or even higher dimensional cases by setting

$$\delta^\epsilon_{(y)}(z) \ = \ [\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}\epsilon^d]^{-1}1_{D(y,\epsilon)}(z).$$

  But then of course, we have to restrict our discussion to at most two motions in $\mathbb{R}^3$ or only one motion in higher dimension, in order to get a non-trivial intersection set.

- It was convenient in the previous proof to assume that the starting point of each of the motions is non-random. However, it is clear that the proposition still holds in more general situation where the initial points are non-deterministic. The right hand side of (3.19) should then be integrated with respect to $\mu^1(dx_1),\dots,\mu^p(dx_p)$ where μ^j denotes the initial distribution of B^j.

Corollary 3.4. *Suppose that $B^1_0 = \cdots = B^p_0$. Then for any $t \geq 0$ and $\lambda > 0$, we have*

$$\alpha([0,\lambda t]^p) \ =^{(d)} \ \lambda\,\alpha([0,t]^p)$$

where for two random variables X and Y, the notation $X =^{(d)} Y$ means X and Y have the same distribution.

Proof : Without loss of generality, we may assume that $B^1_0 = \cdots = B^p_0 = 0$. We shall use a scaling argument. Set:

$$\tilde{B}^j_t \ = \ \lambda^{-\frac{1}{2}}B^j_{\lambda t}.$$

. Then $(\tilde{B}^j_t)_{t\geq 0}$ has the same distribution as $(B^j_t)_{t\geq 0}$. Furthermore, for any $\epsilon > 0$, we have:

$$\tilde{\alpha}_\epsilon([0,t]^p) \ := \ \int_{[0,t]^p} \left(\int_{\mathbb{R}^2} \prod_{j=1}^{p}(\pi\epsilon^2)^{-1}\,1_{D(y,\epsilon)}(\tilde{B}^j_{s_j}) \right)\,ds_1\dots ds_p = \lambda^{-1}\alpha_{\lambda^{\frac{1}{2}}\epsilon}([0,\lambda t]^p)$$

where $K(y,\epsilon)$ is the disc of radius ϵ around y and α_ϵ and $\tilde{\alpha}_\epsilon$ is defined as in (3.18). Also note that α_ϵ and $\tilde{\alpha}_\epsilon$ are same in distribution. It follows that:

$$\tilde{\alpha}([0,t]^p) \ = \ \lambda^{-1}\alpha([0,\lambda t]^p) \quad \text{a.s} \ . \quad \square$$

Corollary 3.5. *If $B^1_0 = \cdots = B^p_0$, then for every $t > 0$, $\alpha([0,t]^p) > 0$ almost surely.*

Proof : Note that the events $\{\alpha([0,t]^p) > 0\}$ decrease as t decreases. It follows from the previous corollary that

$$\mathbb{P}\left[\alpha([0,1]^p) > 0\right] = \mathbb{P}\left[\alpha([0,t]^p) > 0\right]$$

$$= \mathbb{P}\left[\bigcap_{s>0}\{\alpha([0,s]^p) > 0\}\right].$$

Since $\mathbb{E}\left[\alpha([0,1]^p)\right] > 0$ (by the moment formula (3.19)), it follows that

$$\mathbb{P}\left[\alpha([0,1]^p) > 0\right] \ > \ 0.$$

The Kolmogorov 0-1 law gives

$$\mathbb{P}\left[\alpha([0,1]^p) > 0\right] = 1 \qquad \square$$

Now we can recover the celebrated old result (see [DE00a], [DE00b] and [DE00c]) about the existence of intersection points:

Corollary 3.6. *The paths of $B^1, \ldots, B^p$ have a common point different from their starting point.*

Proof : The previous corollary and the fact that α is supported on $\{B^1_{s_1} = \cdots = B^p_{s_p}\}$ implies that, provided $B^1_0 = \cdots = B^p_0$, for any $\epsilon > 0$, there exists $t_1, \ldots, t_p \in (0, \epsilon)$ such that $B^1_{t_1} = \cdots = B^p_{t_p}$. $\square$

So far, we have described the Brownian intersection local time as a uniform measure supported on the intersection of the Brownian paths. Now we provide an alternative characterization of the intersection local time via approximation of the renormalized Lebesgue measure on the intersection of the Wiener sausages.

Theorem 3.7. *We have:*

$$\lim_{\epsilon \to 0} \left(\log \frac{1}{\epsilon}\right)^p \lambda \left(\bigcap_{i=1}^{p} S^i_\epsilon(0, t)\right) = \pi^p \, \alpha([0,t]^p) \tag{3.28}$$

in the L^2-norm, where:

$$S^i_\epsilon(0, t) = \{x \in \mathbb{R}^d : |x - B^i_s| < \epsilon \text{ for some } s \in (0, t)\}. \tag{3.29}$$

In other words, $S^i_\epsilon(0, t)$ is the ϵ-sausage around the i-th Brownian path observed from time 0 upto time t.

Remark: The convergence in the above theorem is true in L^n for any $n < \infty$. However, we restrict our attention only to the L^2 convergence.

Proof : For simplicity, we again assume that the starting points of $B^1_0 = x^1, B^2_0 = x^2, \ldots, B^p_0 = x^p$ are all non-random. we fix $t > 0$.
Then

$$\alpha_\epsilon([0,t]^p) = \int_{[0,t]^p} \left(\int_{\mathbb{R}^2} \prod_{j=1}^{p} \delta^\epsilon_{(y)}(B^j_{s_j}) \, dy\right) ds_1 \ldots ds_p = \int_{\mathbb{R}^2} \left(\prod_{j=1}^{p} \int_0^t \delta^\epsilon_{(y)}(B^j_s) \, ds\right) dy$$

$$= \int_{\mathbb{R}^2} \prod_{j=1}^{p} X^j_\epsilon(y) \, dy \tag{3.30}$$

where

$$X^j_\epsilon(y) = \int_0^t \delta^\epsilon_{(y)}(B^j_s) \, ds$$

and

$$X_\epsilon(y) = \prod_{j=1}^{p} X_\epsilon^j(y).$$

Similarly, we can write:

$$\pi^{-p} \left(\log \frac{1}{\epsilon} \right)^p \lambda \left(\bigcap_{i=1}^{p} S_\epsilon^i(0,t) \right) = \int_{\mathbb{R}^2} \prod_{j=1}^{p} Y_\epsilon^j(y) \, dy$$

where

$$Y_\epsilon^j(y) = \pi^{-1} \left(\log \frac{1}{\epsilon} \right) I(y \in S_\epsilon^j(0,t))$$

and

$$Y_\epsilon(y) = \prod_{j=1}^{p} Y_\epsilon^j(y).$$

Hence the assertion of the theorem is equivalent to

$$\lim_{\epsilon \to 0} \mathbb{E} \left[\left(\int_{\mathbb{R}^2} dy (X_\epsilon(y) - Y_\epsilon(y)) \right)^2 \right] = 0. \tag{3.31}$$

We have:

$$\mathbb{E} \left[\left(\int_{\mathbb{R}^2} dy (X_\epsilon(y) - Y_\epsilon(y)) \right)^2 \right] = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} dy \, dz \left[\prod_{j=1}^{p} \mathbb{E}[X_\epsilon^j(y) X_\epsilon^j(z)] - 2 \prod_{j=1}^{p} \mathbb{E}[X_\epsilon^j(y) Y_\epsilon^j(z)] + \prod_{j=1}^{p} \mathbb{E}[Y_\epsilon^j(y) Y_\epsilon^j(z)] \right]. \tag{3.32}$$

Now we shall investigate the limiting behavior of each term of the right hand side in the above equation. Henceforth we assume that $y \neq z$ and $y, z \neq x^j$. Then we have:

$$\begin{aligned}
\lim_{\epsilon \to 0} \mathbb{E}[X_\epsilon^j(y) X_\epsilon^j(z)] &= \lim_{\epsilon \to 0} \mathbb{E} \left[\int_0^t ds \delta_{(y)}^\epsilon(B_s^j) \right] \\
&= \lim_{\epsilon \to 0} \mathbb{E} \left[\int_0^t ds \left(\int_s^t \delta_{(y)}^\epsilon(B_{s'}^j) \delta_{(z)}^\epsilon(B_{s'}^j) \, ds' \right) \right] \\
&= \int_0^t ds \int_s^t ds' \left[p_s(x^j, y) p_{s'-s}(y, z) + p_s(x^j, z) p_{s'-s}(z, y) \right] \\
&=: F_t(x^j, y, z).
\end{aligned} \tag{3.33}$$

The passage to the limit of the third equality is justified by the same reasoning used in proving the Step 1 of Proposition 3.3. Moreover, the upper bounds obtained in the Step-1 of Proposition 3.3 give

$$\mathbb{E}[X_\epsilon^j(y) X_\epsilon^j(z)] \leq C \left(G_1(0, \frac{y}{2}) + G_1(0, \frac{z}{2}) \right) G_1 \left(0, \frac{(z-y)}{2} \right). \tag{3.34}$$

Next we have:

$$\begin{aligned}
\mathbb{E} \left[X_\epsilon^j(y) Y_\epsilon^j(z) \right] &= \mathbb{E} \left[\int_0^t ds \delta_{(y)}^\epsilon(B_s^j) . \pi^{-1} (\log \frac{1}{\epsilon}) I(z \in S_\epsilon^j(0,t)) \right] \\
&= \pi^{-2} \epsilon^{-2} (\log \frac{1}{\epsilon}) \mathbb{E} \left[\left(\int_0^t ds \, 1_{K(y,\epsilon)}(B_s^j) \right) I(z \in S_\epsilon^j(0,t)) \right].
\end{aligned}$$

Now define

$$T_\epsilon^j(z) = \inf\{t \geq 0 : B_t^j \in K_\epsilon(z)\}.$$

Then, we have

$$\mathbb{E}\left[I(z \in S_\epsilon^j(0,t)) \int_0^t ds \, 1_{D(y,\epsilon)}(B_s^j)\right] = \mathbb{E}\left[I(T_\epsilon^j(z) \le t) \int_{T_\epsilon^j(z)}^t 1_{D(y,\epsilon)}(B_s^j) \, ds\right]$$

$$+ \mathbb{E}\left[\int_0^t 1_{D(y,\epsilon)}(B_s^j) \, I[s < T_\epsilon^j(z) \le t]\right].$$

Now we apply the strong Markov property at time $T_\epsilon^j(z)$ and then pass to the limit as $\epsilon \to 0$ and use Lemma 3.2 to obtain:

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\pi^{-1}(\log(\frac{1}{\epsilon}))I(T_\epsilon^j(z) \le t) \, \pi_{-1}\epsilon^{-2} \int_{T_\epsilon^j(z)}^t 1_{D(y,\epsilon)}(B_s^j) \, ds\right] = \int_0^t ds' \, p_{s'}(x^j,z) \int_{s'}^t ds \, p_{s'-s}(z,y).$$

Now :

$$\mathbb{E}\left[I(T_\epsilon^j(z) \le t) \int_{T_\epsilon^j(z)}^t 1_{D(y,\epsilon)}(B_s^j) \, ds\right] = \mathbb{E}\left[\int_0^t ds \, 1_{D(y,\epsilon)}(B_s^j) \, I(z \in S_\epsilon^j(s,t))\right]$$

$$- \mathbb{E}\left[\int_0^t ds \, 1_{D(y,\epsilon)}(B_s^j) \, I(z \in S_\epsilon^j(0,s) \cap S_\epsilon^j(s,t))\right].$$

Now, by the result of Lemma (3.2), we have:

$$\lim_{\epsilon \to 0} \pi^{-2}\epsilon^{-2}(\log \frac{1}{\epsilon}) \, \mathbb{E}\left[\int_0^t ds \, 1_{D(y,\epsilon)}(B_s^j) \, I(z \in S_\epsilon^j(s,t))\right] = \int_0^t (\pi^{-1}\epsilon^{-2}) \lim_{\epsilon \to 0} \int_{D(y,\epsilon)} p_s(x^j,y') \, dy'$$

$$\times \lim_{\epsilon \to 0} \pi^{-1}\left(\log \frac{1}{\epsilon}\right) \, \left[I(z \in S_\epsilon^j(s,t))\right]$$

$$= \int_0^t ds \, p_s(x^j,y) \int_s^t ds' \, p_{s'-s}(y,z).$$

Again the strong Markov property at time $T_\epsilon(z)$ and the bounds of the third part of the Lemma 3.1 give

$$\epsilon^{-2}(\log \frac{1}{\epsilon})\mathbb{E}\left[\int_0^t ds \, 1_{D(y,\epsilon)}(B_s^j) \, I(z \in S_\epsilon^j(0,s) \cap S_\epsilon^j(s,t))\right] = o\left((\log \frac{1}{\epsilon})^{-1}\right)$$

as ϵ goes to 0. Then, we see that

$$\lim_{\epsilon \to 0} \left[X_\epsilon^j(y)Y_\epsilon^j(z)\right] = \int_0^t ds' \, p_s'(x^j,z) \int_s'^t ds \, p_{s'-s}(z,y) \; + \int_0^t ds p_s(x^j,y) \int_s^t p_{s'-s}(y,z) \; - 0 \tag{3.35}$$

$$= F_t(x^j,y,z).$$

Moreover, Lemma 3.1 and the previous arguments show that $\mathbb{E}\left[X_\epsilon^j(y)Y_\epsilon^j(z)\right]$ satisfies the same upper bound as $\mathbb{E}\left[X_\epsilon^j(y)X_\epsilon^j(z)\right]$.

Finally, we consider,

$$\left[Y_\epsilon^j(y)Y_\epsilon^j(z)\right] = \pi^{-2}\left(\log \frac{1}{\epsilon}\right)^2 \mathbb{P}\left[T_\epsilon^j(y) \le t, \, T_\epsilon^j(z) \le t\right].$$

Now we observe that:

$$\mathbb{P}\left[T_\epsilon^j(y) \le t, \, T_\epsilon^j(z) \le t\right] \le \mathbb{P}\left[T_\epsilon^j(y) \le t, \, z \in S_\epsilon^j(T_\epsilon^j(y),t)\right]$$

$$+ \mathbb{P}\left[T_\epsilon^j(z) \le t, \, y \in S_\epsilon^j(T_\epsilon^j(z),t)\right].$$

It follows that $\mathbb{E}[Y_\epsilon^j(y)Y_\epsilon^j(z)]$ has the same upper bound as $\mathbb{E}\left[X_\epsilon^j(y)X_\epsilon^j(z)\right]$.

Again, we use the strong Markov property and Lemma 3.2 (same arguments as before) to conclude that

$$\limsup_{\epsilon \to 0} \mathbb{E}\left[Y_\epsilon^j(y)Y_\epsilon^j(z)\right] \leq F_t(x^j, y, z). \tag{3.36}$$

Now passage to the limit in the right hand side of (3.32) is justified by (3.33) , (3.35), (3.36) and the subsequent use of the dominated convergence theorem with the help of (3.34) (and the corresponding upper bounds for the other terms too) and the fact that $y \mapsto G_1(0, y)$ is in L^n for any $n < \infty$.

It follows that

$$\limsup_{\epsilon \to 0} \mathbb{E}\left[\left(\int_{\mathbb{R}^2} dy \ (X_\epsilon(y) - Y_\epsilon(z))\right)^2\right] \leq 0.$$

This completes the proof of 3.31 and hence the proof of the theorem.   □

Remark : Although we have dealt with the two dimensional case so far, it is not difficult to see that a similar result with the same proof can be obtained for higher dimensional cases. For example, the intersection local time of two Brownian motions moving in $\mathbb{R}^3$ can be approximated in the same manner by the renormalized Lebesgue measure on the intersection of the corresponding Wiener sausages. More precisely,

$$\lim_{\epsilon \to 0} (2\pi\epsilon)^{-2} \ \lambda\left(S_\epsilon^1(0, t) \cap S_\epsilon^2(0, t)\right) = \alpha([0, t]^2).$$

We note the difference of the normalizing constants. For dimensions $d > 3$ and only one motion, we have:

$$\lim_{\epsilon \to 0} \left(\epsilon^{2-d} \ \Gamma\left(\frac{d}{2}\right) \ \frac{1}{d-2} \pi^{-\frac{d}{2}}\right) \ \lambda\left(S_\epsilon(0, t)\right) = \alpha([0, t]).$$

4. Brownian intersection local time as a Hausdorff measure:

The third construction of the intersection local time was also carried out by Le Gall (see [LG87]). It turned out that this measure coincides with a constant multiple of a Hausdorff measure induced by a suitable gauge function. Look at the appendix (subsection 10.2) for the definition and examples of Hausdorff measure and dimension.

This approach was motivated by a problem concerning the size of the Brownian intersection set. Recall that the Hausdorff dimension of the intersection of any number of motions in $\mathbb{R}^2$ is 2. But it is intuitively clear that the size of the intersection of p motions should be bigger than that of $p + 1$ motions. This heuristic observation leads to the consideration of a Hausdorff measure on the intersection set induced by a suitable gauge function. More precisely if $g_r(x) = x^2(\log \frac{1}{x})^r$ for any $r \in \mathbb{R}$, then in $d = 2$ we have

$$\mu_{g_r}(S) = \begin{cases} 0 & \text{if } r < p \\ \infty & \text{if } r > p \end{cases} \tag{4.1}$$

where μ_{g_r} is the g_r- Hausdorff measure. This result was conjectured by Taylor in [?] and proved by Le Gall in [LG86]. The techniques used in [LG86] yields a similar result in $d = 3$. If $f_r(x) = x \left(\log \frac{1}{x}\right)^r$,

then

$$\mu_{f_r}(S) = \begin{cases} 0 & \text{if } r \leq 0 \\ \infty & \text{if } r > 0. \end{cases} \tag{4.2}$$

It is worth making a comment about the case $p = 1$. The Hausdorff measure of the image of a single Brownian path was studied by many authors, including Levy (see [Le53]) and Ray (see [Ra63]). But the correct gauge function was determined by Ciesielski and Taylor (see see [CT62]) for $d \geq 3$ and Taylor (see [Ta66]) for $d = 2$. The function is given by:

$$h(x) = \begin{cases} x^2 \left(\log \frac{1}{x} \log \, \log \, \log \frac{1}{x} \right) & \text{if } d = 2 \\ x^2 \left(\log \, \log \frac{1}{x} \right) & \text{if } d \geq 3. \end{cases}$$

However, we shall focus on the case of more than one motion in appropriate dimension. The results (4.1) and (4.2) were improved by the same author in [LG87] by computing a "correct" gauge function g such that the g-measure of S is positive and the measure is σ-finite. In fact, it was proved that for some positive constants C and C'

$$C \, \ell(A) \leq \mu_{g_p}(A \cap S) \leq C' \, \ell(A) \tag{4.3}$$

where

$$g_p(x) = \begin{cases} x^2 \left(\log \, \log \, \log \, \log \frac{1}{x} \right)^p & \text{if } d = 2 , \, p \in \mathbb{N} \\ x \left(\log \, \log \frac{1}{x} \right)^2 & \text{if } d = 3, \, p = 2 \end{cases}$$

and ℓ is the (projected) intersection local time of the Brownian paths.

The key argument leading to the above result is Le Gall's moment formula (3.19) and the existence of two postive constants M and M' (depending only on the d and p) such that for $d = 2$,

$$M^k \, (\log R)^{pk} \, (k!)^p \leq \mathbb{E}\left[(\ell(B(0;1)))^k \right] \leq M'^k \, (\log R)^{pk} \, (k!)^p \text{ for } k \in \mathbb{N}$$

where it is assumed that the Brownian motions run until their first individual exit time from a fixed ball of radius R with $2 \leq R < \infty$. For $d = 3$ and $p = 2$,

$$M^k (k!)^2 \, m \leq \mathbb{E}\left[(\ell(B(0;1)))^k \right] \leq M'^k (k!)^2 \text{ for } k \in \mathbb{N}.$$

The above result was farther sharpened by the same author in [LG87(II)] by showing that with probability one, the intersection local time ℓ is exactly equal to a constant multiple of the g_p-Hausdorff measure on S:

$$\ell(A) = C_p \, \mu_{g_p}(A \cap S) \quad \text{for every } A \in \mathbb{B}(\mathbb{R}^2)$$

for some constant C_p.

Remarks :

- From the above result it follows that the Hausdorff dimension of S is 2 for $d = 2$ and $p \in \mathbb{N}$ since the exponent of x in the gauge function g_p is also 2 (of course, it contains some log terms too, but they do not influence the dimension). The same argument accounts for a similar result in the three dimensional case with two motions.
- It is worth observing that ℓ is a random object which is equal to a Hausdorff measure induced by a suitable gauge function. Remarkably, the gauge function is non-random and depends on p and d in a rather simple manner.

5. Upper tail asymptotics

Let us turn to the following question : What is the behavior of the Brownian paths if they are forced to produce an extremely large amount of intersection with each other? More precisely, we look at the sample paths for which all the p motions have especially much interaction. This question concerns the extremely "thick" parts of the space. Now ℓ being a measure of intersection of the Brownian paths, the above question boils down to studying the random regions of the space where the mass of ℓ is locally extremely dense with probability one. We would also like to ask "how many" such thick points exist in the space. Ofcourse, the expression "how many" is a bit vague at the moment. But we shall come back with a precise formulation of this question a bit later.

For answering these questions, it is necessary to understand the "upper tails" of the random variables $\ell(U)$ for compact sets U, say, for balls $U \subset \mathbb{R}^d$. In other words, we study the logarithmic decay rate of the probability of the event $\{\ell(U) > a\}$ as $a \to \infty$. This work has been done by König and Mörters in 2002 (see [KM02]) and was refined by the same authors in 2005 (see [KM05]). In this section we describe that result briefly. First, we need some technical stuff.

Let $B \subset \mathbb{R}^d$ be the domain of the Brownian motions (i.e we let the motions run until their first individual exit time from the ball B). We assume that B is an open ball, possibly equal to $\mathbb{R}^d$ for $d \geq 3$ and for $d = 2$ we assume that B is bounded (Recall that a Brownian motion is recurrent in $d \leq 2$ and transient in higher dimensions). The following function space is the buliding block for our analysis:

$$\mathcal{D}(B) = \begin{cases} H_0^1(B) & \text{if } B \text{ is bounded} \\ D^1(\mathbb{R}^d) & \text{if } B = \mathbb{R}^d \end{cases} \tag{5.1}$$

Where

$$D^1(\mathbb{R}^d) = \{f \in L^1_{loc}(\mathbb{R}^d) \;:\; f \text{ vanishes at infinity and the distributional gradient of } f \text{ is in } L^2(\mathbb{R}^d)\}.$$

Now we fix an open bounded set U in $\mathbb{R}^d$ such that U is compactly contained in B (i.e., $\overline{U} \subset B$). Then the upper tails of $\ell(U)$ are identified as follows:

Theorem 5.1. *(König/Mörters'2002)*

$$\lim_{a \to \infty} \frac{1}{a} \log \mathbb{P}\left[\ell(U) > a^p\right] = -\Theta(U) \tag{5.2}$$

where

$$\Theta(U) = \inf\{\frac{p}{2} \|\nabla\psi\|_2^2 \;:\; \psi \in \mathcal{D}(B), \|1_U \psi\|_{2p}^2 = 1\}. \tag{5.3}$$

Remarks

- **Minimizer of of the variational formula** : The minimizer ψ in the variational formula (5.3) exists and it is a solution to the following Euler-Lagrange equation for the Laplace operator in U:

$$\triangle\psi(x) = -\frac{2}{p}\Theta(U)\psi^{2p-1}(x) \, 1_U(x) \text{ for } x \in B \backslash \partial U \tag{5.4}$$

  It is worth noting that the minimizing functions ψ are harmonic outside U. Moreover, for $p = 1$, the above equation is a linear eigenvalue problem and its uniqueness properties are known. However, for $p \geq 2$, the non-linearity triggers off a chain of problems. For example, in these non-linear cases, it is not known how many minimizers there are for the variational formula, to the best of our knowledge.

- **Exponential moments** From the above theorem, one easily defers a necessary and sufficient condition for the integrability of the random variable $\exp(c\ell(U)^{\frac{1}{p}})$ for any $c > 0$. More precisely,

$$\int_\Omega e^{\ell(U)^{\frac{1}{p}}} \, d\mathbb{P} \begin{cases} < \infty & \text{if } \Theta(U) > 1 \\ = \infty & \text{if } \Theta(U) < 1. \end{cases}$$

To see this, put $g = \ell(U)^{\frac{1}{p}}$. Then (5.2) says that

$$\mathbb{P}[g > a] \simeq e^{-a\Theta(U)}$$

Again,

$$\int_\Omega e^g > \int_{g>a} e^g > e^a \mathbb{P}[g > a] \simeq e^{a(1-\Theta(U))}$$

Which implies that $\int_\Omega e^g = \infty$ if $\Theta(U) < 1$.
However, if $\Theta(U) > 1$, then

$$\int_\Omega e^g \leq \int_0^\infty \mathbb{P}(g > b) e^b \, db \simeq \int_0^\infty e^{-b\Theta(U)} e^b \, db = \int_0^\infty e^{b(1-\Theta(U))} \, db < \infty.$$

This question was first answered for the case $p = 1$ by Pinsky (see [Pi86]) and was left open there for $p \geq 2$. However, this result was generalized by König and Mörters (see [KM05]) where the existence of exponential moments for the intersection local time was investigated and it was proved that , if $\phi_1, \ldots, \phi_n$ are bounded non-negative Borel functions with compact support in B and if

$$\Theta(\phi_1, \ldots, \phi_n) = \inf\{\frac{p}{2}\|\nabla\psi\|_2^2 \; : \; \psi \in \mathcal{D}(B), \; \Sigma_{i=1}^n \|\phi_i \psi\|_{2p}^2 = 1\}.$$

Then,

$$\mathbb{E}\left[\exp\left(\Sigma_{i=1}^n \langle \phi_i^{2p}, \ell \rangle^{\frac{1}{p}}\right)\right] \begin{cases} < \infty & \text{if } \Theta(\phi) > 1 \\ = \infty & \text{if } \Theta(\phi) < 1 \end{cases} \tag{5.5}$$

Furthermore, (5.2) was extended to show that

$$\lim_{a\to\infty} \frac{1}{a} \log \mathbb{P}\left[\Sigma_{i=1}^n \langle \phi_i^{2p}, \ell \rangle^{\frac{1}{p}} > a\right] = -\Theta(\phi_1, \ldots, \phi_n) \tag{5.6}$$

where for a function f and a measure μ, $\langle f, \mu \rangle$ denotes the integral $\int f \, d\mu$.

- **Dimension spectrum for thick points** : As proposed at the beginning of this section, the aymptotic limit in the Theorem 5.1 determines the behavior of the Brownian sample paths where they produce a large amount of intersection and this is carried out by determining the size of the "thick points" of the space. More explicitly, the idea is to focus on those points which have a neighbourhood around which the mass of ℓ is untypically large. The key argument leading to this result is is to find a gauge function ϕ such that the upper Hausdorff density of the intersecion local time is bounded. Formally :

$$0 < \sup_{x\in s} \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{\phi(r)} < \infty.$$

Having found such a function ϕ, a point $x \in S$ is called **thick** , if

$$\limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{\phi(r)} > 0.$$

Now the question concerning the size of the set of thick points in answered neatly by the **Hausdorff dimension spectrum** of the set which is defined as the function :

$$f(a) = dim\{x \in S : \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{\phi(r)} = a\}.$$

for each $a > 0$. It was shown by the authors (see [KM02]) that for $d = 3$ and $p = 2$,
  (i)

$$\sup_{x\in\mathbb{R}^3} \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{r(\log\frac{1}{r})^2} = \frac{1}{\Theta(U)^2}$$

(ii)

$$dim\{x \in S : \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{r(\log\frac{1}{r})^2} = a\} = 1 - \sqrt{a}\Theta(U).$$

6. Upper tail asymptotics from moment asymptotics

The goal of this section is to provide a sketchy proof of Theorem 5.1 via several steps. Needless to say, we shall not go into mounds of technical details of the stuff. Instead, we focus our attention on the main idea.

It turns out that the integer moments of the random variables $\ell(U)$ are more feasible to work with rather than the probability of the event $\{\ell(U) > a\}$. This statement provokes a natural appeal to Le Gall's moment formula (3.19). We restate it here in a different form which would turn out to be more useful for our purpose.

Lemma 6.1.

$$\mathbb{E}\left[\ell(U)^k\right] = \int_U dy_1 \ldots \int_U dy_k \prod_{j=1}^k \Sigma_{\sigma\in S_K} G(x^j, y_{\sigma(1)}) \prod_{i=2}^k G(y_{\sigma(i-1)}, y_{\sigma(i)}).$$

Although we have provided a rigorous proof of the above lemma, it is possible to derive it from the symbolical formula (2) on a heuristical level as follows:

$$\mathbb{E}\left[\ell(U)^k\right] = \mathbb{E}\left[\left(\int_U dy \prod_{j=1}^p \int_0^{T_j} ds\ \delta_y(W_j(s))\right)^k\right]$$

$$= \int_U dy_1 \ldots \int_U dy_k \prod_{j=1}^p \mathbb{E}_{x_j}\left[\sum_{\sigma\in S_k} \int_{0\le s_1\le\cdots\le s_k\le T_j} \cdots \int \prod_{i=1}^k \delta_{y_\sigma(i)}(W_j(s_i))\, ds_i\right]$$

$$= \int_U dy_1 \ldots \int_U dy_k \prod_{j=1}^p \sum_{\sigma\in S_k} p_{s_1}(x^j, y_{\sigma(1)}) \prod_{i=2}^k p_{s_k}(y_{\sigma(k-1)}, y_{\sigma(k)})\, ds_i \ldots ds_1$$

$$= \int_U dy_1 \ldots \int_U dy_k \prod_{j=1}^k \sum_{\sigma\in S_K} G(x^j, y_{\sigma(1)}) \prod_{i=2}^k G(y_{\sigma(i-1)}, y_{\sigma(i)}).$$

6.1 Green's Operator Recall that in course of describing the Wiener sausage construction of the intersection local time, we have already introduced the Green's function together with the exponential random time ζ. Here we write it for the Brownian motion killed at the exit time from $B(0, R)$:

$$G(x, y) = \int_0^\infty p_s(x, y) \, ds$$

Note that this function depends on the dimension d of the ambient space and on the way we stop the Brownian motion on it's exit from the fixed open ball. But it does neither depend on the domain U nor on the number p of motions. This function can also be computed explicitly as follows (see [PS78,p.114]), if $d = 2$

$$G(x, y) = \begin{cases} \frac{1}{\pi} \left[\log \left| \frac{x}{|x|} R - |x| \frac{y}{R} \right| - \log |x - y| \right] & \text{if } x \neq 0, \\ \log R - \log |y| & \text{if } x = 0. \end{cases}$$

and if $d \geq 3$ and $R < \infty$, then

$$G(x, y) = \begin{cases} c_d \left[|x - y|^{2-d} - |\frac{x}{|x|} R - |x| \frac{y}{R}|^{2-d} \right] & \text{if } x \neq 0, \\ c_d \left[|y|^{2-d} - R^{2-d} \right] & \text{if } x = 0. \end{cases}$$

and if $d \geq 3$ and $R = \infty$, then

$$G(x, y) = \frac{c_d}{|x - y|^{d-2}}. \tag{6.1}$$

We note that

$$G(x, y) = G(y, x) \tag{6.2}$$

and if $p < \frac{d}{d-2}$, then the function $G^p(0, .)$ is integrable in a neighbourhood of the origin.

Now, let U be an open, bounded set in $\mathbb{R}^d$ such that the closure of U is contained in $B(0, R)$ .Then we define the **Green's operator** $\mathcal{T} : L^{\frac{2p}{2p-1}}(U) \to L^{2p}(U)$ such that

$$\mathcal{T}(f(x)) = \int_U G(x, y) f(y) \, dy. \tag{6.3}$$

We note that $\mathcal{T}$ is a symmetric operator (which follows from Fubini's theorem and the fact that G is symmetric) and also $\mathcal{T}$ is continuous (which follows from Hölder's inequality). Moreover, its restriction $\mathcal{T} : L^\infty(U) \to L^\infty(U)$ is symmetric, positive and compact with norm $\int_U \int_U G(x, y) \, dx \, dy$

6.2 Variational Characterization In this subsection, we introduce the variational formula which turns out to feature the moment asymptotics of the $\ell(U)$ and describe the relationship of upper tail asymptotics and moment asymptotics on the platform of variational representation. First, we need to introduce some notation.

For any probability measure μ and a finite measure $\tilde{\mu}$ given on the same measurable space, the **relative entropy** is defined as

$$H(\mu|\tilde{\mu}) = \begin{cases} \int \log(\frac{d\mu}{d\tilde{\mu}}(x)) \, d(\mu(x)) & \text{if } \mu \ll \tilde{\mu} \\ \infty & \text{else.} \end{cases} \tag{6.4}$$

We note that, if $\tilde{\mu}$ is also a probability measure, then by Jensen's inequality we always have $H(\mu|\tilde{\mu}) \geq 0$ and the equality holds if and only if $\mu = \tilde{\mu}$.

More specifically, we fix $U \subset \mathbb{R}^d$ an open bounded set and let $\mathcal{M}_1(U)$ denote the space of all probability measures. Naturally, $\mathcal{M}_1(U)$ is equipped with the weak topology. The ambient space

U being a separable subspace of $\mathbb{R}^d$, $\mathcal{M}_1(U)$ is a weakly separable and metrizable space (an easy consequence of Urysohn's metrization theorem). For any $\mu \in \mathcal{M}_1(U)$, we define

$$I(\mu) = H(\mu|\lambda) \tag{6.5}$$

to be the relative entropy of μ with respect to the Lebesgue measure λ on U. Note that $I(\mu) \geq -\log \lambda(U)$ and the equality holds if and only if μ is the normalized Lebesgue measure on U. Moreover, I is a convex and lower semicontinuos function. Now we denote by

$$\mathcal{M}_1^*(U) = \{\nu \in \mathcal{M}_1(U^2) : \nu(A \times U) = \nu(U \times A) \text{ for all Borel set} A \subset U\}$$

the set of all probability measures ν in U^2 which has the same marginals $\nu_1(A) = \nu(A \times U)$ and $\nu_2(A) = \nu(U \times A)$. Now, for $\nu \in \mathcal{M}_1^*(U)$, we know that the function

$$I_\mu^2(\nu) = \begin{cases} H(\nu|\nu_1 \otimes \mu) & \nu \in \mathcal{M}_1^*(U) \\ \infty & \text{else} \end{cases}$$

is the large deviation rate function for the pair empirical measures of an i.i.d sequence with marginal distribution μ (see Appendix, Theorem 10.2). In particular, I_μ^2 is lower semicontinuos and convex.

Next, we define a function $\mathcal{G} : \mathcal{M}_1(U) \to \mathbb{R}$ by

$$\mathcal{G}(\mu) = \inf_{\nu \in \mathcal{M}_1(U^2)} \{I_\mu^2(\nu) - \langle \nu, \log G \rangle\} \tag{6.6}$$

where we extend the notation $\langle , \rangle$ to integrals on U^2. Observe that it suffices to take the infimum over measures ν satisfying $\nu \ll \mu \otimes \mu$. We can replace $I_\mu^2(\nu)$ in the definition of $\mathcal{G}$ by either the relative entropy $H(\nu|\mu \otimes \mu)$ or the mutual information $H(\nu|\nu_1 \otimes \nu_2)$. We write:

$$k^* = \inf_{\mu \in \mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\}. \tag{6.7}$$

Having introduced the necessary notations, we are all set to state the main result which is in some sense the cornerstone of the link between a maximizing problem of the Green's operator and the large deviation rate function we described above.

Proposition 6.2. *For every postive integer $p < \frac{d}{d-2}$, we have:*

$$\sup\{\langle g^{2p-1}, \mathcal{T}g^{2p-1} \rangle : g \in L^{2p}(U) \text{ with } ||g||_{2p} = 1\} = \exp\left(-\frac{1}{p} \inf_{\mu \in \mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\}\right). \tag{6.8}$$

Moreover, g is a maximizer of the left hand side if and only if the measure $d(\mu x) = g^{2p}(x)dx$ is a minimizer on the right hand side of (6.2). Every minimizing sequence of the variational problem on the right hand side of (6.2) has a subsequence converging weakly to a minimizer.

We shall come back with a sketchy proof of the above proposition a bit later.

6.3 Tails from moments

As we asserted before, we deal with the moment asymptotics of the intersection local time instead of the tail asymptotics. The following result opens up the gate for us:

Proposition 6.3.

$$\lim_{k \uparrow \infty} \frac{1}{k} \log \mathbb{E}\left[\frac{\ell(U)^k}{(k!)^p}\right] = - \inf_{\mu \in \mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\}. \tag{6.9}$$

In order to derive the upper tail asymptotics from the above proposition, we need the following Tauberian theorem.

Lemma 6.4. *Let X be any non-negative random variable and fix $p \in \mathbb{N}$. Then for any $x \in \mathbb{R}$, the following two implications hold:*

(i)

$$\limsup_{k \uparrow \infty} \frac{1}{k} \log \mathbb{E}\left[\frac{X^k}{k!^p}\right] \leq -x \;\Rightarrow\; \limsup_{a \uparrow \infty} a^{-\frac{1}{p}} \log \mathbb{P}\left[X > a\right] \leq -p\mathrm{e}^{\frac{x}{p}}. \tag{6.10}$$

(ii)

$$\limsup_{k \uparrow \infty} \frac{1}{k} \log \mathbb{E}\left[\frac{X^k}{k!^p}\right] = -x \;\Rightarrow\; \limsup_{a \uparrow \infty} a^{-\frac{1}{p}} \log \mathbb{P}\left[X > a\right] = -p\mathrm{e}^{\frac{x}{p}} \tag{6.11}$$

PROOF : The proof of (i) is easy and based on the substitution $a_k = e^{-\kappa}k^p$, Markov's inequality, and Stirling's formula as follows.

$$\limsup_{k \uparrow \infty} a_k^{-1/p} \log P\{X > a_k\} = e^{\kappa/p} \limsup_{k \uparrow \infty} \frac{1}{k} \log P\left\{X^k > e^{-k\kappa}k^{kp}\right\}$$

$$\leq e^{\kappa/p} \limsup_{k \uparrow \infty} \frac{1}{k} \log E\left[\frac{X^k}{e^{-k\kappa}k^{kp}}\right] = e^{\kappa/p} \limsup_{k \uparrow \infty} \left(\frac{1}{k} \log E\left[\frac{X^k}{(k!)^p}\right] + \kappa - p\right) \tag{6.12}$$

$$\leq -pe^{\kappa/p}.$$

Since $a_{k+1}^{-1/p}/a_k^{-1/p} \to 1$ as $k \uparrow \infty$, we see that it is sufficient to consider the subsequence a_k rather than an arbitrary sequence tending to infinity.

The proof of (ii) is based on the construction of the transformed measure

$$d\widehat{P}^k(X) = \frac{X^k}{E[X^k]} \, dP(X), \qquad \text{for } k \in \mathbb{N},$$

and the fact that the random variable

$$Y_k = \log\left(\frac{X}{e^{-\kappa}k^p}\right)$$

satisfies, for every $\varepsilon > 0$,

$$\lim_{k \uparrow \infty} \widehat{P}^k\{|Y_k| \leq \varepsilon\} = 1. \tag{6.13}$$

To prove this, fix an arbitrary $\varepsilon > 0$ and pick some small number $\alpha > 0$. Then, by the Markov inequality, we may estimate

$$\widehat{P}^k\{Y_k \geq \varepsilon\} = \widehat{P}^k\{X^{k\alpha} \geq e^{(\varepsilon-\kappa)k\alpha}k^{pk\alpha}\} \leq e^{-\varepsilon k\alpha}e^{\kappa k\alpha}k^{-pk\alpha}\widehat{E}^k[X^{k\alpha}],$$

where $\widehat{E}^k$ denotes expectation with respect to $\widehat{P}^k$. Note that $\widehat{E}^k[X^{k\alpha}] = E[X^{k(1+\alpha)}]/E[X^k]$. Using our assumption and Stirling's formula, we see that the quotient has the asymptotic behaviour $e^{-(\kappa+p)k\alpha}k^{pk\alpha}(1+\alpha)^{kp(1+\alpha)}e^{o(k)}$ as $k \uparrow \infty$. Inserting this in the right hand side above, we get

$$\widehat{P}^k\{Y_k \geq \varepsilon\} \leq \exp\left(pk\alpha\left(-\frac{\varepsilon}{p} - 1 + \frac{1+\alpha}{\alpha}\log(1+\alpha) + o(1)\right)\right), \qquad \text{as } k \uparrow \infty.$$

If $\alpha > 0$ is chosen small enough, then the expression between the inner brackets is negative and bounded away from zero, such that we obtain that $\lim_{k \uparrow \infty} \widehat{P}^k\{Y_k \geq \varepsilon\} = 0$. Analogously one shows that $\lim_{k \uparrow \infty} \widehat{P}^k\{Y_k \leq -\varepsilon\} = 0$, and this implies (6.13).

In order to finish the proof of the lower bound, we keep $\varepsilon > 0$ arbitrarily fixed and substitute this time $a = e^{-\kappa} k^p e^{-\varepsilon}$. It is again clear that the consideration of this subsequence suffices. Note that $\{X > a\} = \{Y_k > -\varepsilon\} \supset \{|Y_k| \leq \varepsilon\}$. This implies that

$$a^{-1/p} \log P\{X > a\} \geq e^{\kappa/p} e^{\varepsilon/p} \frac{1}{k} \log P\{|Y_k| \leq \varepsilon\}.$$

Note that $P\{|Y_k| \leq \varepsilon\} = \widehat{E}^k[X^{-k} \mathbf{1}_{\{|Y_k| \leq \varepsilon\}}] E[X^k]$ and that we may estimate $X^{-k} \geq e^{k(-\varepsilon+\kappa)} k^{-pk}$ on $\{|Y_k| \leq \varepsilon\}$. Using this estimate, our assumption on the asymptotics of $E[X^k]$ and Stirling's formula, we obtain

$$\liminf_{a \uparrow \infty} a^{-1/p} \log P\{X > a\} \geq e^{\kappa/p} e^{\varepsilon/p} \left(-\varepsilon - p + \liminf_{k \uparrow \infty} \frac{1}{k} \log \widehat{P}^k\{|Y_k| \leq \varepsilon\} \right).$$

Because of (6.13), the latter limit inferior is equal to zero. After letting $\varepsilon \downarrow 0$, we get the assertion.
□

Proposition 6.2 combined with Proposition 6.3 and the second part of Lemma 6.4 gives the following result which is not far from Theorem 5.1 :

Theorem 6.5.

$$\lim_{a \to \infty} \frac{1}{a^{\frac{1}{p}}} \log \mathbb{P}\left[\ell(U) > a\right] = -\frac{p}{\rho^*} \tag{6.14}$$

where

$$\rho^* = \sup\{\langle g^{2p-1}, \mathcal{T} g^{2p-1} \rangle \ : \ g \in L^{2p}(U) \text{ with } \|g\|_{2p} = 1\} \tag{6.15}$$

Remark : In the special case $p = 1$, the famous Rayleigh-Ritz formula describes the right hand side in the formula (6.15) as the principal eigenvalue of the compact symmetric operator $\mathcal{T}$ on $L^2(U)$. Hence, existence, uniqueness and many more properties of the maximizer are known. However, for the general case $p \geq 2$, alredy the uniqueness seems to be an open problem, apart from the special case of the unit ball on $\mathbb{R}^3$ which has been carried out by König an Mörters (see Theorem 1.3 in [KM02]).

6.4 Large moment asymptotics In this subsection, we present a sketchy proof of the Proposition 6.3. The proof of this result obtained by the authors in [KM02] involve rather hard combinatorial tools using discretization of the integrals by finite partitioning of the domain U. Instead of going through the intricate technical details, we appeal to a result obtained by Trashorras (see [Tr06]) concerning the large deviation rate function for the symmetrised empirical pair measures. Thus, Le Gall's moment formula coupled with the above mentioned result, Varadhan's lemma and Sanov's theorem filters out the desired large deviation rate function.

Without entailing any loss of generality, we assume that the Brownian motions start from the same point (since different starting points make no difference other than complicating our calculations). Thus Le Gall's moment formula Lemma 6.1 boils down to the following form:

$$\mathbb{E}\left[\ell(U)^k\right] = \int_U dy_1 \dots \int_U dy_k \left[\sum_{\sigma \in S_k} \prod_{i=1}^k G(y_{\sigma(i-1)}, y_{\sigma(i)})\right]^p. \tag{6.16}$$

Let

$$\Phi_k(y) = \sum_{\sigma \in S_k} \frac{1}{k!} \prod_{i=1}^k G(y_{\sigma(i-1)}, y_{\sigma(i)}) \text{ for } y = (y_1, \dots, y_k) \in U^k.$$

Now, note that, $\Phi_k(y)$ does not depend on the vector $y = (y_1, \dots, y_k)$, but only on the set $\{y_1, \dots, y_k\}$. In other words, $\Phi_k(y)$ is permutation invariant. Formally, $\Phi_k(y) = \Phi_k(y_\sigma)$ for any $\sigma \in S_k$, where we

put $y_\sigma = (y_{\sigma(1)}, \ldots, y_{\sigma(k)})$. Therefore, to prove Proposition 6.3, it suffices to show that

$$\lim_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \; (\Phi_k(y))^p \;=\; - \inf_{\mu\in\mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\} \tag{6.17}$$

Now, note that

$$\Phi_k(y) = \mathbb{E}_{\tilde{\sigma}}\left[e^{k\cdot\frac{1}{k}\sum_{i=1}^k \log \; G(y_{\sigma(i-1)}, y_{\sigma(i)})}\right]$$

where $\tilde{\sigma}$ is the uniform probability measure on the symmetric group S_k. In other words,

$$\tilde{\sigma}(\sigma) = \frac{1}{k!} \; \forall \sigma \in S_k.$$

Let $\mu_k^y = \frac{1}{k}\sum_{i=1}^k \delta_{y_i} \in \mathcal{M}_1(U)$ be the empirical measures. Formally, for every $A \subset U$,

$$\mu_k^y(A) \;=\; \frac{1}{k}\sum_{i=1}^k \; \delta_{y_i}(A)$$

To this end, we emphasize that Φ_k^y is not a function of the vector $y = (y_1, \ldots, y_k)$, but the empirical measures μ_k^y.

Again, the symmetrised empirical pair measures are defined as

$$\mathcal{L}_{k,\sigma}^y \;=\; \frac{1}{k}\sum_{i=1}^k \; \delta_{(y_{\sigma(i-1)}, y_{\sigma(i)})} \in \mathcal{M}_1(U^2).$$

Note that the map $\mathcal{L}_{k,\cdot}^y : (S_k, 2^{S_k}, \tilde{\sigma}) \to \mathcal{M}_1(U^2)$ defined by $\sigma \mapsto \mathcal{L}_{k,\sigma}^y$ is a random variable.
Now, we can write:

$$\frac{1}{k}\sum_{i=1}^k \log \; G(y_{\sigma(i-1)}, y_{\sigma(i)}) \;=\; \langle \log G, \mathcal{L}_{k,\sigma}^y \rangle$$

where $\langle f, \mu \rangle$ stands for the integral $\int f \, d\mu$.
Now (6.16) can be further simplified as :

$$\mathbb{E}\left[\ell(U)^k\right] \;=\; \lambda(U)^k \, \mathbb{E}_U\left[\left(\mathbb{E}_{\tilde{\sigma}}(e^{k\langle \log G, \mathcal{L}_{k,\sigma}^y\rangle})\right)^p\right] \tag{6.18}$$

where $y = (y_1, \ldots, y_k)$ is random vector consisting of independent and uniformly distributed random variables on U (in principle, we should have introduced a different notation for the random y to distinguish it from the non-random y we had used before. But it is conceptually clear and harmlessly convenient to use the same notation as before).

Now an obstacle pops up from the unboundedness of the Green's function. However, this technical issue can be taken care of by cutting off the function at a large level and restoring the exponential rate of $\Phi_k(y)$ asymptotically as the truncation level approaches infinity. This result already exists in the literature (see Lemma 3.3 in [KM02]) and we only recall the statement without stepping into the details.

For $M > 0$, the cut-off Green's function is defined as $G_M = G \wedge M$ and we denote

$$\Phi_{k,M}(y) = \frac{1}{k!}\sum_{\sigma\in S_k}\prod_{i=1}^k G_M(y_{\sigma(i-1)}, y_{\sigma(i)}).$$

Then we have

Lemma 6.6. *There is $C_0 > 0$ and, for all sufficiently large $M > 1$ and small $\eta \in (0, 1)$, there are constants $C_M > 0$ and $\varepsilon_\eta > 0$ such that, for any $k \in \mathbb{N}$,*

$$\int_{U^k} dy\, (\Phi_k^j(y))^p \leq 2^p pk(2C_0)^k C_M^{\eta k} + 2^p(1+\varepsilon_\eta)^k p \sum_{m=\lceil k(1-p\eta)\rceil}^{k} \int_{U^m} dy\, \left(\Phi_{m,M}^j(y)\right)^p, \qquad (6.19)$$

where $\lim_{M\uparrow\infty} C_M = \lim_{\eta\downarrow 0} \varepsilon_\eta = 0$.

Now we come back to the main proof. Using the notation introduced above, we can write

$$\int_{U^k} dy\, (\Phi_{k,M}(y))^p = \mathbb{E}_U\left[\left(\mathbb{E}_{\tilde{\sigma}}(e^{k\langle \log G_M, \mathcal{L}_{k,\sigma}^y\rangle})\right)^p\right] \lambda(U)^k.$$

To this end, we write:

$$\mathbb{E}_{\tilde{\sigma}}(e^{k\langle \log G_M, \mathcal{L}_{k,\sigma}^y\rangle})^p = e^{p.k.\frac{1}{k}\log \mathbb{E}_{\tilde{\sigma}}(e^{k\langle \log G_M, \mathcal{L}_{k,\sigma}^y\rangle})}$$

Let us denote $\frac{1}{k}\log \mathbb{E}_{\tilde{\sigma}}(e^{k\langle \log G_M, \mathcal{L}_{k,\sigma}^y\rangle})$ by $\mathcal{F}_k(\mu_k^y)$ (we again point out that $\frac{1}{k}\log \mathbb{E}_{\tilde{\sigma}}(e^{k\langle \log G_M, \mathcal{L}_{k,\sigma}^y\rangle})$ depends on the set $\{y_1, \ldots, y_k\}$ via the empirical measures μ_k^y).

Now, the result due to Trashorras (see Theorem 1, [Tr06]) says if $\mu_k^y \Rightarrow \mu$, the sequence $\mathcal{L}_{k,\sigma}^y$ satisfies a large deviation principle on $\mathcal{M}_1(U^2)$ (more precisely, the pull-back measures $\tilde{\sigma}(\mathcal{L}_{k,\cdot}^y)^{-1}$ satisfies a LDP in $\mathcal{M}_1(U^2)$) with a good rate function

$$\mathcal{J}(\nu) = \begin{cases} H(\nu | \mu \otimes \mu) & \text{if } \nu \in \mathcal{M}_1^*(U^2) \\ \infty & \text{else} \end{cases}$$

where H stands for the relative entropy defined in subsection 6.2 and "$\Rightarrow$" denotes the weak convergence on the space of probability measures $\mathcal{M}_1(U)$. Now we want to apply Varadhan's lemma (see Appendix, Theorem 10.4) to the above large deviation rate function H for the continuous bounded mapping

$$\psi : \mathcal{M}_1(U^2) \to \mathbb{R} \text{ with } \nu \mapsto \int_{U^2} \log G_M\, d\nu$$

in the weak topology (note that G_M is bounded and bounded away from 0 and this is the reason why we introduced the cut-off function). But a new problem pops up here from the fact that the measures $\mathcal{L}_{k,\cdot}^y$ depends also on the set $\{y_1, y_2, \ldots, y_k\}$ in terms of the point measures and hence the usual statement of Varadhan's lemma can not be applied directly. However, it follows from the above result of Trashorras and usual Varadhan's lemma that if $\mu_k^y \Rightarrow \mu$, then

$$\mathcal{F}_k(\mu_k^y) \to -\mathcal{G}_M(\mu) \qquad (6.20)$$

where $\mathcal{G}_M$ is defined by (6.6) with G being replaced by G_M. In order to get rid of the above mentioned obstacle, we define a map $\mathcal{H}_k : \mathcal{M}_1(U) \to [-\infty, \infty)$ by

$$\mathcal{H}_k(\mu) = \begin{cases} \mathcal{F}_k(\mu_k^y) & \text{if there exist } y_1, \ldots, y_k \text{ in } U \text{ with } \mu_k^y = \mu \\ -\infty & \text{else} \end{cases}$$

In other words, $\mathcal{H}_k$ and $\mathcal{F}_k$ coincide on the subset of the point measures. As we said, we need some small technical refinement in the proof of the version of Vardhan's lemma we are interested in and this gets us rid of the y-dependence of the measures $\mathcal{L}_{k,\cdot}^y$. We highlight the particular step in the upper bound part which needs to be taken care of. The rest is the standard continuation of the proof available in the literature.

To this end, note that for *any* sequence of measures $(\mu_k)_{k\in\mathbb{N}}$ (which are not necessarily empirical measures) in $\mathcal{M}_1(U)$, with $\mu_k \Rightarrow \mu$, it follows from the dichotomous definition of $\mathcal{H}_k$ and (6.20) that

$$\limsup_{k\to\infty} \mathcal{H}_k(\mu_k) \le -\mathcal{G}_M(\mu).$$

This means $\forall \epsilon > 0$ and every $\mu \in \mathcal{M}_1(U)$, there is a $\delta > 0$ and $K \in \mathbb{N}$ such that $\forall k > K$ and $\forall \nu \in \{\nu \in \mathcal{M}_1(U) : d(\nu,\mu) < \delta\}$, we have

$$\mathcal{H}_k(\nu) \le -\mathcal{G}_M(\mu) + \epsilon \tag{6.21}$$

where d is a metric which induces the weak topology in $\mathcal{M}_1(U)$. Then one can continue with the standard upper bound proof of Varadhan's lemma (see page 138-139 in [DZ98]) to infer that as $k \to \infty$,

$$\int_{U^k} \Phi_{k,M}(y)^p \, dy \le \lambda(U)^k \mathbb{E}_U\left[e^{-k[p\mathcal{G}_M(\mu_k^y)+o(1)]} \right].$$

Now by Sanov's theorem (see Appendix, Theorem 10.1) we know that the empirical measures μ_k^y satisfy a large deviation principle in $\mathcal{M}_1(U)$ with good rate function I (which is the relative entropy with respect to the Lebesgue measure, look back at (6.5)). We again appeal to Varadhan's lemma for the rate function I and the continuous bounded function $\mathcal{G}_M : \mathcal{M}_1(U) \to \mathbb{R}$ (infact, upper semicontinuity (which is clear from the definition (6.6) with G replaced by G_M) is enough for the large deviation upper bound and upper boundedness of $\mathcal{G}_M$ essentially follows from (6.21)) to conclude that as $k \to \infty$

$$\int_{U^k} dy \, [\Phi_{k,M}(y)]^p \le \lambda(U)^k \, e^{-k[\inf_{\mu\in\mathcal{M}_1(U)}\{I(\mu)+p\mathcal{G}_M(\mu)\}+o(1)]}. \tag{6.22}$$

Now we complete the proof by sending M to infinity and we only prove the upper bound part (for the lower bound, see [KM02]). For this, we use the cutting argument as follows. In (6.19), first we take logarithms in both sides, divide by k and then let $k \to \infty$. According to Lemma 6.6, after letting $M \to \infty$, we have

$$\limsup_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \, \Phi_k(y)^p \tag{6.23}$$

$$\le \quad \log(1 + \varepsilon_\eta) + \lim_{M\uparrow\infty} \limsup_{k\uparrow\infty} \frac{1}{k} \log \sum_{m=\lceil k(1-p\eta)\rceil}^{k} \int_{U^m} dy \, \Phi_{m,M}(y)^p.$$

Here we have used the simple fact that for any fixed positive integer N and $a_k^i > 0$,

$$\limsup_{k\uparrow\infty} \frac{1}{k} \log \left(\sum_{i=1}^{N} a_k^i \right) = \max_{i=1}^{N} \limsup_{k\uparrow\infty} \frac{1}{k} \log a_k^i.$$

We again use this fact to estimate the last integral on the right hand side from above by the maximum on $m \in \{\lceil k(1-p\eta)\rceil, \dots, k\}$ and then by 6.22, we have

$$\limsup_{m\uparrow\infty} \frac{1}{m} \log \int_{U^m} dy \, \Phi_{m,M}(y)^p \le - \inf_{\mu\in\mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}_M(\mu)\}. \tag{6.24}$$

Now we first send $M \uparrow \infty$ and then $\eta \downarrow 0$, to obtain

$$\limsup_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \, \Phi_k(y)^p \le \lim_{\eta\downarrow 0}\log(1 + \varepsilon_\eta) - \lim_{\eta\downarrow 0}(1-p\eta) \inf_{\mu\in\mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\}$$

$$= - \inf_{\mu\in\mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\}.$$

This finishes the proof of 6.17.

7. Analysis of the variational formulae

The goal of this section is to go through a sketchy proof of the Proposition 6.2. It turns out that the variational formula (6.15) is much easier to deal with rather than (6.7). In fact, in contrast to (6.7), the continuity and compactness properties of the Green's operator makes the analysis much easier by getting hold of existence of maximizers, their positivity and Euler-Lagrange equations.

Before we go to the actual proof, it is necessary to discuss some general continuity properties of the Green's operator.

Lemma 7.1. (i) *If $d = 2$ and $q > 1$, then $\mathcal{T}$ is a bounded linear map from $L^q(U)$ into $L^\infty(U)$.*

(ii) *If $d \geq 3$ and $1 < q < \frac{d}{2}$, then $\mathcal{T}$ is a bounded linear map from $L^q(U)$ into $L^{\frac{dq}{d-2q}}(U)$.*

Proof. The first statement follows easily from Hölder's inequality using that in $d = 2$ we have $\sup_x \int G(x,y)^q \, dy < \infty$ for all $q > 1$.

If $d \geq 3$ we recall, e.g. from [LL01], the Hardy-Littlewood-Sobolev inequality. For all $s, r > 1$, $0 < \lambda < d$ with $1/r + \lambda/d + 1/s = 2$ there is a constant $C > 0$ with

$$\left| \int_U \int_U f(x) \, |x-y|^{-\lambda} h(y) \, dx \, dy \right| \leq C \, \|f\|_s \|h\|_r . \tag{7.1}$$

Recall that $G(x,y) \leq c_d |x-y|^{2-d}$ and use the Hardy-Littlewood-Sobolev inequality with $\lambda = d - 2$ and $s = q$, $r = dq/(dq + 2q - d)$, which yields $\langle h, \mathcal{T}f \rangle \leq C\|f\|_q \|h\|_r$ for any $f \in L^q(U)$ and $h \in L^r(U)$. Hence $\mathcal{T}$ maps f continuously into the dual of $L^r(U)$, which is $L^{\frac{dq}{d-2q}}(U)$. This proves (ii). □

For our purposes, it is convenient to rewrite (6.15) as

$$\varrho^* = \sup\{ \langle f, \mathcal{T}f \rangle \; : \; f \in L^{2p/(2p-1)}(U) \text{ and } \|f\|_{2p/(2p-1)} \leq 1 \}. \tag{7.2}$$

It is clear that the supremum in (7.2) may be restricted to positive normalized functions $f \in L^{2p/(2p-1)}(U)$. We start by showing that the operator $\mathcal{T} \colon L^{2p/(2p-1)}(U) \to L^{2p}(U)$ is continuous, and establish (6.2).

Lemma 7.2. *Suppose p is a positive integer with $p < d/(d-2)$.*

(i) *$\mathcal{T}$ is a bounded linear map from $L^{2p/(2p-1)}(U)$ into $L^{2p}(U)$. In particular, $\varrho^* \leq \|\mathfrak{A}\|$.*

(ii) *For all $\mu \in \mathcal{M}_1(U)$ with $g^{2p}(x) \, dx = \mu(dx)$ we have*

$$\exp\left(-\frac{1}{p}\big(I(\mu) + p\mathcal{G}(\mu)\big) \right) \leq \langle g^{2p-1}, \mathcal{T}g^{2p-1} \rangle . \tag{7.3}$$

(iii) *Equality in (7.3) holds if and only if there is $\varrho > 0$ with the property*

$$\mathcal{T}g^{2p-1}(x) = \varrho g(x) \;\; \text{for μ-almost every } x \in U. \tag{7.4}$$

Moreover, in this case $I(\mu) + p\mathcal{G}(\mu) = -p \log \varrho$ and $\mathcal{G}(\mu) = -\log \varrho - 2\langle \mu, \log g \rangle$.

Sketch of Proof : To prove the first part, recall that for a finite measure μ, and $0 < s < s' \leq \infty$, $L^{s'}(\mu)$ is continuously embedded into $L^s(\mu)$ and apply the first part of Lemma 7.1 for $d = 2$. For $d \geq 3$, we can choose q in such a way that

- $1 < q < \frac{d}{2}$
- $\frac{dq}{d-2q} \geq 2p$
- $q \leq \frac{2p}{2p-1}$

Then we apply the second part of Lemma 7.1 to finish the first part of this lemma.

Now we show (ii). We assume that $I(\mu) + p\mathcal{G}(\mu) < \infty$ (otherwise, there is nothing to prove). We fix $\delta > 0$. By definition of $\mathcal{G}(\mu)$, we can choose $\nu \in \mathcal{M}_1^*(U)$ with both marginals being equal to μ such that

$$\mathcal{G}(\mu) > I_\mu^2(\nu) - \langle \log G, \nu \rangle - \delta$$

We claim that $\langle \log G, \nu \rangle$ is finite. For this, let us take f to be the Lebesgue density of ν. In other words,

$$f(x,y)\, dx\, dy = \nu(\, dx\, dy)$$

We take $\epsilon > 0$ small enough so that

$$C = \log \sup_{x \in U} \int_U G^{p+\epsilon}(x,y)\, dy \ < \infty.$$

The above statement is true since $G^p(0,.)$ is locally intergrable around a neighborhood of the origin. Then, estimating $pI_\mu^2(\nu) \geq I_\mu^2(\nu)$, we obtain

$$\infty > I(\mu) + I_\mu^2(\nu) + \varepsilon\langle \log G, \nu \rangle - \langle \log G^{p+\varepsilon}, \nu \rangle = \varepsilon\langle \log G, \nu \rangle - \left\langle f, \log \frac{(g^{2p} \otimes g^{2p})^{1/2} G^{p+\varepsilon}}{f} \right\rangle. \quad (7.5)$$

Now, the second term can be bounded above with a tricky application of Jensen's inequality as follows:

$$\left\langle f, \log \frac{(g^{2p} \otimes g^{2p})^{1/2} G^{p+\varepsilon}}{f} \right\rangle = \int_U dx\, g^{2p}(x) \int_U dy\, \frac{f(x,y)}{g^{2p}(x)} \log \frac{g^{2p}(x) G^{p+\varepsilon}(x,y)}{f(x,y)}$$

$$\leq \int_U dx\, g^{2p}(x) \log \int_U dy\, \frac{f(x,y)}{g^{2p}(x)} \frac{g^{2p}(x) G^{p+\epsilon}(x,y)}{f(x,y)}$$

$$= \int_U dx\, g^{2p}(x) \log \left(\int_U G^{p+\varepsilon}(x,y)\, dy \right) \leq C.$$

Note that we could use Jensen's inequality since $\frac{f(x,y)}{g^{2p}(x)}\, dy$ is a probability measure on U.

From this, we conclude that $\langle \log G, \nu \rangle$ is finite. Now we use the lower bound in a variational principle for I_μ^2, which we recall from [DZ98, 6.5.10]. For all measurable $u : U^2 \to (0, \infty)$ that are bounded from 0 and infinity,

$$I_\mu^2(\nu) \geq \int_{U^2} \nu(dx\, dy) \log \left(\frac{u(x,y)}{\int u(y,z)\, d\mu(z)} \right). \quad (7.6)$$

For arbitrarily large $M > 0$ and small $\varepsilon > 0$ the function

$$u(x,y) = \frac{G(x,y)}{(g \wedge M)(y) \vee \varepsilon G(x,y)},$$

is admissible in (7.6), which yields

$$I_\mu^2(\nu) \geq \int \nu(dxdy) \log \left(\frac{G(x,y)}{(g \wedge M)(y) \vee \varepsilon G(x,y)} \right) - \int \mu(dy) \log \left(\int \frac{G(y,z)\, \mu(dz)}{(g \wedge M)(z) \vee \varepsilon G(y,z)} \right). \quad (7.7)$$

Now we use the fact $\langle \log G, \nu \rangle < \infty$ and $\int \nu(dxdy) \log g(y) = \langle \mu, \log g \rangle < \infty$ and use dominated convergence theorem for the first term in the right hand side of (7.6) as $\epsilon \downarrow 0$ and bounded convergence theorem for the second term as $M \uparrow \infty$. Therefore taking limits in (7.6) we obtain:

$$I_\mu^2(\nu) \geq \langle \nu, \log G \rangle - \left\langle \mu, \log(g \cdot \mathfrak{A}g^{2p-1}) \right\rangle.$$

Recalling the choice of ν and letting $\delta \downarrow 0$, we obtain

$$\mathcal{G}(\mu) \geq -\langle \mu, \log(g.\mathcal{T}g^{2p-1}) \rangle \quad (7.8)$$

We apply this and again a clever application of Jensen's inequality implies

$$\exp\Big(-\frac{1}{p}\big(I(\mu)+p\mathcal{G}(\mu)\big)\Big) \le \exp\Big(-\frac{1}{p}\langle\mu,\log g^{2p}\rangle + \langle\mu,\log(g\mathcal{T}g^{2p-1})\rangle\Big)$$

$$= \exp\Big(\Big\langle\mu,\log\frac{\mathcal{T}g^{2p-1}}{g}\Big\rangle\Big) \le \Big\langle\mu,\frac{\mathfrak{A}g^{2p-1}}{g}\Big\rangle \tag{7.9}$$

$$= \langle g^{2p-1}, \mathcal{T}g^{2p-1}\rangle.$$

Finally, to prove (iii), assume that we have equality everywhere in (7.9). By strict convexity of the logarithm, equality in the second line implies that, for some constant $\varrho > 0$, we have $\mathfrak{A}g^{2p-1} = \varrho g$, for μ-almost every $x \in U$. Together with equality in the first line of (7.9), which is equality in (7.8), this yields that

$$\mathcal{G}(\mu) = -\big\langle\mu,\log(g\cdot\mathfrak{A}g^{2p-1})\big\rangle = -\log\varrho - 2\langle\mu,\log g\rangle.$$

Conversely, if (7.4) holds, we have equality in the second line of (7.9). To check equality in the first line, we define a probability measure $\nu \in \mathcal{M}^*(U)$ by

$$\nu(dxdy) = h(x,y)\,dx\,dy = \frac{1}{\varrho}g^{2p-1}(x)g^{2p-1}(y)G(x,y)\,dx\,dy.$$

The measure ν is well-defined by (7.4) and plugging this into (6.6) yields

$$\mathcal{G}(\mu) \le -\langle\mu,\log(\varrho g^2)\rangle.$$

This means that equality holds in (7.8) and hence also in the first line of (7.9), completing the proof of (iii). $\square$

Completion of the proof

Now we give the main steps of the rest of the proof of Proposition 6.2 and hence we also prove Proposition 5.1. The details can be found in [KM02].
First one needs to show the existence of maximizers in 7.2. We go through several steps.

Step 1 As a first step, we need to get convergence along subsequences with the help of a standard Banach-Alaoglu argument. More precisely, we need to show that every maximizing sequence for the variational problem in (7.2) has a subsequence which converges weakly in $L^1(U)$ towards some maximizer of this problem. For this we consider the set K of all non-negative L^1 functions f on U such that $\|f\|_{\frac{2p}{2p-1}} \le 1$. We show that

    (i) K is weakly compact in $L^1(U)$
    (ii) the mapping $f \mapsto \langle f, \mathcal{T}f\rangle$ is upper semicontinuous in the weak topology on $L^1(U)$.

To show the first part, we note that K being a uniform integrable family (see [DZ98, C7]), it is weakly relatively compact in $L^1(U)$. Therefore it suffices to show that K is weakly closed in $\{f \in L^1(U) \,:\, F \ge 0\}$. We fix $F \ge 0$ and $\|f\| > 1$ and set:

$$\phi_n = \Big(\frac{f\wedge n}{\|f\wedge n\|}\Big)^{\frac{1}{2p-1}} \in L^\infty(U)$$

and show that

    • $\liminf \int \phi_n f \ge \|f\| = 1$
    • $\int \phi_n g \le 1 \ \forall g \in K$.

The above two statements imply that for n large enough, the function $\phi = \phi_n \in L^\infty(U) = (L^1(U))^*$, satisfies

$$\limsup_{k \uparrow \infty} \langle \phi, g_k \rangle \leq 1 \leq \langle \phi, f \rangle$$

for all sequences (g_k) in K. This means that f is not in the weak L^1 closure of K. Hence K must be weakly closed.

For the second part, we show that $\limsup_{n \to \infty} \langle f_n, \mathcal{T} f_n \rangle \leq \langle f, \mathcal{T} f \rangle$ for every sequnece $\{f_n\}$ and f in K such that $f_n \to f$ weakly in $L^1)(U)$. In order to encounter the unboundedness of the Green's function, we cut off the Green's operator as $\mathcal{T} = (\mathcal{T} - \mathcal{T}_M) + \mathcal{T}_M$ where $\mathcal{T}_M$ is the same operator as $\mathcal{T}$ with G being replaced by $G \mathbb{1}_{G \leq M}$ for large $M > 0$. Next we show that $\lim_{M \to \infty} \sup_{n \in \mathbb{N}} \langle f_n, (\mathcal{T} - \mathcal{T}_M) f_n \rangle = 0$. This is easy to see by Hölder's inequality in $d = 2$. In $d \geq 3$ we use Hardy-Littlewood-Sobolev inequality to see that it is enough to prove that as $n \to \infty$, $\langle f_n, \mathcal{T}_M f_n \rangle \to \langle f, \mathcal{T}_M f \rangle$ and then take the limit as $M \to \infty$. A standard monotone class arguemnt finishes the proof.

Step 2 Now we derive the Euler-Lagrange equation for the maximizer of the (7.2). More precisely, we show that,

- any non-negative maximizer of the variational problem (7.2) is essentially bounded away from 0.
- If this maximizer is written as g^{2p-1}, the function $g \in L^{2p}(U)$ satisfies the Euler-Lagrange equation (7.4) with soem constant $\rho > 0$.

For the first assertion, we let $f \in L^{\frac{2p}{2p-1}}(U)$ be a non-negative and normalized maximizer of the variational problem (7.4) and we assume that f is not bounded away away from 0. In other words, the set $\{f \leq \epsilon\}$ has positive measure. We fix some constant $c > 0$ such that $\lambda\{f \geq c\} > 0$. We define a function $\tilde{f} \colon U \to [0, \infty)$ as follows

$$\tilde{f}(x) = \begin{cases} f(x) + a, & \text{if } f(x) \leq \varepsilon, \\ f(x) - b, & \text{if } f(x) \geq c, \\ f(x), & \text{otherwise.} \end{cases}$$

Now, we can play around with $a, b > 0$ and choose $\epsilon > 0$ so small that $\|\tilde{f}\|_{\frac{2p}{2p-1}} = 1$, but $\langle \tilde{f}, \mathcal{T}\tilde{f} \rangle > \langle f, \mathcal{T} f \rangle$, contradicting the maximality of f.

For the second part, we choose a perturbation $\phi : U \to \mathbb{R}$ bounded with $\int \phi \, dx = 0$. Note that, in the Hilbert space $L^2(U)$, the set of all such ϕ is the orthogonal complement of the subspace spanned by all constant functions. For small $\epsilon > 0$, we have $\|f + \epsilon\phi\|_1 = 1$ and $f + \epsilon\phi \geq 0$ on U. Now, f being a maximizer, we have

$$0 = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \langle (f + \varepsilon\phi)^{\frac{2p-1}{2p}}, \mathcal{T}(f + \epsilon\phi)^{\frac{2p-1}{2p}} \rangle = \frac{2p-1}{p} \langle \varphi, f^{\frac{-1}{2p}} \mathcal{T} f^{\frac{2p-1}{2p}} \rangle.$$

We conclude that, in $L^2(U)$, the function $f^{\frac{-1}{2p}} \mathcal{T}\left(f^{\frac{2p-1}{2p}}\right)$ is orthogonal to the orthogonal complement of the span of constants. Hence, there is a constant ρ, such that

$$\rho f^{\frac{1}{2p}}(x) = \mathcal{T} f^{\frac{2p-1}{2p}}(x) \ \text{ for } \lambda - almost \ every \ x \in U$$

Now, f being essentially positive, we have $\rho > 0$. Writing now g^{2p-1} for f, we have verified (7.4).

Step 3 In this final step, we obtain the existence of the minimizers in (6.7) and the convergence of minimizing sequences. We show that every minimizing sequence for this variational problem has a subsequence converging weakly to some minimizer of this and moreover, if g^{2p} denotes the

Lebesgue density of the limiting minimizing measure, then $g \in L^{2p}(U)$. This is proved by combining Lemma 7.2 with the previous two steps as follows. Let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}_1(U)$ be a minimizing sequence for the variational problem (6.7). We can assume that for large n, the measures μ_n have Lebesgue densities (otherwise all $I(\mu_n) = \infty$ and hence there is nothing to prove). Let $g_n^{2p} \in L^{\frac{2p}{2p-1}}$ be the corresponding densities. By the second part of Lemma 7.2, we can say that (g_n^{2p-1}) is a maximizing sequence of the variational problem in (7.2). By Step 1, we can extract a subsequence converging weakly to some $g^{2p-1} \in L^{\frac{2p}{2p-1}}(U)$ in $L^1(U)$. We define a new probability measure $\mu(dx) = g^{2p}(x) \, dx$ which is a minimizer of (6.7). Clearly, μ is the weak limit of the corresponding subsequence of (μ_n). Thus, we have a minimizer $\mu(dx) = g^{2p}(x)$ for (6.7) which satisfies the Euler-Lagrange equation in (7.4) with $\rho = \rho^*$ defined via (6.15). Again g is a maximizer of (6.15) if and only if μ is a minimizer of (6.7). Hence, we have proved Proposition 6.2 and hence of Proposition 6.14.

8. IDENTIFICATION OF THE MAIN VARIATIONAL FORMULA

In this section, we relate the two variational characterizations (6.15) and (5.3) with the help of a simple formula. As usual, we do not spell out all the details. This is carried out via a two-step mechanism. First, we prove that the minimizers exist in (5.3) and give their variational equations. In the second step, we write down the explicit formula which bridges the gap between the two variational representations.

Recall the definition of the function space $\mathcal{D}(B)$ in (5.1).

Lemma 8.1. *There exists a function $\psi \in \mathcal{D}(B)$ which satisfies*

$$\Theta(U) = \frac{p}{2} \|\nabla \psi\|_2^2 \quad and \|\psi \, 1_U\|_{2p}^2 = 1 \tag{8.1}$$

and with $h = \|1_U \psi\|_{2p}^{2-2p}$, we have

$$\frac{p}{\Theta(U)} \psi = \mathcal{T}\left(\psi^{2p-1} h\right) \quad -\frac{p}{2} \triangle \psi = \Theta(U) \psi^{2p-1} h \tag{8.2}$$

Proof. As a first step, we derive the existence of a minimiser in (5.3). Let $(\psi_k : k \in \mathbb{N})$ be a minimising sequence, that is, the functions $\psi_k \in \mathcal{D}(B)$ are nonnegative and satisfy $\|1_U \psi_k\|_{2p}^2 = 1$ for any $k \in \mathbb{N}$, and $\lim_{k \uparrow \infty} \frac{1}{2} \|\nabla \psi_k\|_2^2 = \Theta(\phi)$.

Let $\psi_* \in \mathcal{D}(B)$ denote the weak limit of a subsequence in accordance with *Lemma* 10.6 By local strong convergence in $L^{2p}(B)$, we also have $\|1_U \psi_*\|_{2p}^2 = 1$. By weak lower semicontinuity of the map $\psi \mapsto \|\nabla \psi\|_2^2$ (apply [LL01, Theorem 2.11]), we have that $\frac{1}{2} \|\nabla \psi_*\|_2^2 \le \liminf_{k \uparrow \infty} \frac{1}{2} \|\nabla \psi_k\|_2^2 = \Theta(\phi)$. Since ψ_* is certainly nonnegative, it is a minimiser in (5.3).

The second step is the derivation of the variational equation in (8.2) for any minimiser ψ_* in (5.3). Since $\|\nabla|\psi|\|_2^2 = \|\nabla \psi\|_2^2$ for any $\psi \in \mathcal{D}(B)$ (see [LL01, Theorem 6.17]), and since $\|1_U \psi\|_{2p}^2$ is positive homogeneous of order two in ψ, ψ_* is also a minimiser in the variational problem

$$\Theta(\phi) = \inf_{\psi \in \mathcal{D}(B)} \frac{\frac{p}{2} \|\nabla \psi\|_2^2}{\|1_U \psi\|_{2p}^2}. \tag{8.3}$$

Denote the quotient on the right hand side of (8.3) by $F(\psi)$. Let $\varphi \in C_c^\infty(B)$ be a smooth test function, then the map $\varepsilon \mapsto F(\psi_* + \varepsilon\varphi)$ can easily be differentiated at $\varepsilon = 0$. By minimality of ψ_* for F, this derivative is equal to zero. Recalling that $\|1_U\psi_*\|_{2p}^2 = 1$, this implies that

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\big\|\nabla(\psi_* + \varepsilon\varphi)\big\|_2^2 - \sum_{i=1}^{n}\|\nabla\psi_*\|_2^2\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\|\phi_i(\psi_* + \varepsilon\varphi)\|_{2p}^2 \\
&= 2\int_B \nabla\psi_* \cdot \nabla\varphi - 4\frac{\Theta(U)}{p}\|1_U\psi_*\|_{2p}^{2-2p}\langle\varphi, 1_U\psi_*^{2p-1}\rangle \\
&= -2\Big\langle\varphi, \Delta\psi_* + 2\frac{\Theta(U)}{p}\psi_*^{2p-1}h\Big\rangle,
\end{aligned}
\tag{8.4}
$$

where we used the definition of the distributional Laplacian in the last step, and $h = \|1_U\psi_*\|_{2p}^{2-2p}$ as in (8.2). As (8.4) holds for any smooth test function φ, we infer that the function in the right argument of the brackets on the right hand side is equal to zero, i.e., $-\frac{1}{2}\Delta\psi_* = \frac{\Theta(U)}{p}\psi_*^{2p-1}h$, which is the second identity in (8.2). By [LL01, Th. 6.21], the function $\psi = \frac{\Theta(U)}{p}\mathcal{T}(\psi_*^{2p-1}h)$ satisfies $-\frac{1}{2}\Delta\psi = \frac{\Theta(U)}{p}\psi_*^{2p-1}h$. Hence, by [LL01, Th. 9.3], ψ differs from ψ_* by a harmonic function in $\mathcal{D}(B)$, which therefore vanishes. This ends the proof of (8.1). □

As proposed, now we give the explicit formula which relates the two variational problems (6.15) and (5.3). Furthermore, the proof gives a one-to-one correspondence between the maximizers of (6.15) and minimizers of (5.3).

Proposition 8.2.

$$
\max\Big\{\langle g^{2p-1}, \mathcal{T}g^{2p-1}\rangle \ : \ g \in L^{2p}(U), \|g\|_{2p} = 1\Big\}^{-1}
$$
$$
= \min\Big\{\frac{1}{2}\|\nabla\psi\|_2^2 \ : \ \psi \in \mathcal{D}(B), \|1_U\psi\|_{2p}^2 = 1\Big\}
\tag{8.5}
$$

Proof The idea of the proof goes as follows. For the proofs of both '$\geq$' and '$\leq$' in (8.5), we pick the maximiser resp. the minimiser in one variational formula, construct admissible objects for the other one, and show that the other functional attains the inverse of the value of the maximum resp. minimum.

Let us begin with the proof of '$\geq$'. Recall that, in view of the results of the previous section, we know that the maximizers of the variational problem for ρ^* exists and staisfy the corresponding Euler-Lagrange equations. Theorefore, we can choose a maximizer $g \in L^{2p}(U)$ of the left hand side of (8.5). We define

$$
\psi = \frac{1}{\rho^*}\mathcal{T}\big(g^{2p-1}\big).
\tag{8.6}
$$

Since g satisfies the Euler-Lagrange equation $\rho^*g = \mathcal{T}\big(g^{2p-1}\big)$, we have $\psi(x) = g(x)$ on U. Hence $\|1_U\psi\|_{2p}^2 = \big(\int_U g^{2p}(x)\,dx\big)^2 = 1$. Then, clearly,

$$
\langle\frac{1}{\rho^*}g^{2p-1}, \mathcal{T}\Big(\frac{1}{\rho^*}g^{2p-1}\Big)\rangle = \frac{1}{\rho^*}.
$$

Now by Lemma 10.7, we have $\psi \in \mathcal{D}(B)$ and $\frac{1}{2}\|\nabla\psi\|_2^2 = \frac{1}{\rho^*}$. This implies '$\geq$'.

To prove '$\leq$', we choose ψ as the minimiser of the problem on the right hand side of in (8.5), by

Lemma 8.1. We define

$$g = \frac{\psi}{\|\psi\|_{2p}} \tag{8.7}$$

and note that $\|\psi\|_{2p}^2 = 1$ and $\|g\|_{2p} = 1$. We plug this g into the left hand side of (8.5) and we use the first equlity in (8.2) to obtain

$$\langle g^{2p-1}, \mathcal{T} g^{2p-1} \rangle = \langle \|\psi\|_{2p}^{2-2p} \psi^{2p-1}, \mathcal{T}\left(\|\psi\|_{2p}^{2-2p} \psi^{2p-1}\right)\rangle = \frac{p}{\Theta(U)} \langle \psi^{2p-1} \|\psi\|_{2p}^{2-2p}, \psi \rangle = \frac{p}{\Theta(U)}.$$

This completes the proof of this proposition.

9. Outlook

We want to ask if the minimiser ψ in the variational formula for $\Theta(U)$ admits a probabilistic interpretation. The answer to this question turns out to be affirmative with ψ^{2p} roughly being the density of the normalised version $\frac{\ell}{\ell(U)}$. This can be heuristically explained as follows. Recall that Donsker-Varadhan large deviation principle says that the term $\frac{1}{2}\|\nabla\psi\|_2^2$ describes the asymptotics of the normalised occupation measure of a single Brownian trajectory. In other words, the term ψ^2 roughly describes the path "density" of each of the motions. Hence, the p-fold product, ψ^{2p} should describe the density of the normalised intersection local time.

To make the above discussion precise, we define arandom probability measure L on U as

$$L(A) = \frac{\ell(A)}{\ell(U)} \quad \text{for} A \subset U \text{ Borel}$$

Now we want to know how the above measure L distributes unit mass over the set U if we condition the Brownian paths to have a large occupation measure $\ell(U)$. This has been partially answered by the following result, due to König/Mörters, see [KM05, Theorem 1.4].

Theorem 9.1 (Law of large masses). *Let d be a metric on U which induces weak topology on $\mathcal{M}_1(U)$. Let $\mathcal{M} \subset \mathcal{M}_1(U)$ be the set of probability measures*

$$\mu(dx) = \psi^{2p}(x)\,dx \tag{9.1}$$

on U with ψ a maximizer in the formula for $\Theta(U)$. Then for all $\epsilon > 0$,

$$\lim_{a \to \infty} \mathbb{P}[d(L, \mathcal{M}) > \epsilon | \ell(U) > a] = 0. \tag{9.2}$$

This makes precise the interpretation of ψ^{2p} as the asymptotic density of L under $\{\ell(U) > a\}$ as $a \to \infty$. It is natural to conjecture that the above convergence is exponentially fast in $a^{\frac{1}{p}}$. But it remains an open problem to determine the precise rate function in terms of a large deviation principle. Deeper questions concern the joint behavior of the p single paths under $\mathbb{P}[\cdot | \ell(U) > a]$ in terms of their occupation measures (more precisely, their restriction to the set S). Is it possible to understand L jointly together with these p measures in terms of a large deviation principle? Furthermore, when the domain of the motions is bounded, can we also put U equal to B and look at $\ell/\ell(B)$ on the entire set B? These problems are still open and may very well open up a new direction of research.

10. Appendix

In this section, we state some rudimentary facts about the needed background material for readers who have not been introduced to basic large deviation theory, some elementary notions of Hausdorff measures and few relevant facts about the theory of Sobolev spaces.

10.1 An excursion to the theory of Large Deviations

Definition. Suppose we have a family of probability measures $(\mu_\epsilon)_{\epsilon>0}$ on a topological space X and suppose $I : X \to [0,\infty]$ is a function such that I is lower-semicontinuous (which means that, for every $\alpha \geq 0$, the level sets $\{I \leq \alpha\}$ are closed). Then, we say that, the family (μ_ϵ) satisfies a **Large Deviation Principle** (abbreviated as **LDP**) with **rate function** I if for every measurable set A in X, we have

$$- \inf_{x \in A^o} I(x) \leq \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(A) \leq \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(A) \leq - \inf_{x \in \overline{A}} I(x)$$

We say that I is a **good rate function** if the level sets of I are compact.

Theorem 10.1 (Sanov's theorem). *Suppose Σ be a complete separable metric space and $Y_1, y_2, \ldots, Y_n$ be a sequence of independent random variables, identically distributed according to $\mu \in \mathcal{M}_1(\Sigma)$. With δ_y denoting the probability measure concentrated at $y \in \Sigma$, the **empirical law** of $Y_1, Y_2, \ldots Y_n$ is defined as*

$$\mathbf{L}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i} \in \mathcal{M}_1(\Sigma) \tag{10.1}$$

Then, the emperical measures $\mathbf{L}_n$ satisfy a large deviation principle in $\mathcal{M}_1(\Sigma)$ equipped with the weak topology, with good, convex rate function $H(.|\mu)$, which is the relative entropy defined in (6.4).

Now we state a large deviation principle for pair emperical measures of Markov chains with finite state space.

Theorem 10.2. *Let Σ be a finite set and $(Y_n)_{n \in \mathbb{N}}$ be a Markov chain with state space Σ such that transition matrix $\Pi = (\pi(i,j)_{i,j \in \Sigma})$ is strictly positive. Then the **pair empirical measures**, which is defined as*

$$\mathbf{L}_{n,2}(y) = \frac{1}{n} \sum_{i=1}^{n} 1_y(Y_{i-1}, Y_i), \;\; y \in \Sigma^2 \tag{10.2}$$

satisfies a large deviation principle with rate function

$$I_2(q) = \begin{cases} \sum_{i=1}^{|\Sigma|} q_1(i) \sum_{j=1}^{|\Sigma|} \frac{q(i,j)}{q_1(i)} \log \frac{\frac{q(i,j)}{q_1(i)}}{\pi(i,j)} & \text{if } q_1 = q_2 \\ \infty & \text{else} \end{cases} \tag{10.3}$$

for every $q \in \mathcal{M}_1(\Sigma^2)$, where q_1 and q_2 are the marginals of q, namely

$$q_1(i) = \sum_{j=1}^{|\Sigma|} q(i,j) \text{ and } q_2(i) = \sum_{j=1}^{|\Sigma|} q(j,i).$$

Now we state a very elementary but extremely powerful theorem regarding transformation of LDPs. It says that the large deviation principle is preserved under continuous mappings.

Theorem 10.3 (Contraction principle). *Let X and Y be two Hausdorff topological spaces and f : $X \to Y$ is a continuous map. Suppose $(\mu_\epsilon)_{\epsilon > 0}$ satisfies LDP in X with good rate function I, then $(\mu_\epsilon \circ f^{-1})$ satisfies LDP in Y with good rate function*

$$J(y) = \inf\{I(x)\colon x \in f^{-1}(y) \quad for\ y \in Y\} \tag{10.4}$$

Remark As a corollary to the contraction principle, one can actually get hold of a large deviation rate function for the emperical measures of a Markov chain from the one for pair emperical measures. Indeed, for $q \in \mathcal{M}_1(\Sigma^2)$, the mapping $q \mapsto q_1$ is continuous and hence by the contraction principle, $J(\nu) = \inf\{I_2(q)\colon q \in \mathcal{M}_1(\Sigma^2), q_1 = \nu\}$ gives the large deviation rate function for the emperical measure in (10.1) for a Markov chain Y_n with an irreducible transition matrix.

We end our discussion on large deviations with an extremely powerful tool which could be used as a cornerstone for the development of the main theory.

Theorem 10.4 (Varadhan's integral lemma). *Let $\{Z_\epsilon\}$ be a family of random variables taking values in a regular topological space X and let μ_ϵ be the corresponding laws. Suppose (μ_ϵ) satisfies LDP in X with a good rate function I. Let $\phi\colon X \to \mathbb{R}$ be any continuous function such that either of the following conditions is satisfied.*

$$\lim_{M \to \infty} \limsup_{\epsilon \to 0} \epsilon \log \mathbb{E}\left[\mathrm{e}^{\frac{\phi(Z_\epsilon)}{\epsilon}} 1_{\{\phi(Z_\epsilon) \geq M\}}\right] = -\infty. \tag{10.5}$$

In other words, we have to blow up the expectation along the tail of the random variable $\phi(Z_\epsilon)$. Otherwise, we have to have an apriori moment condition saying that for some $\gamma > 1$,

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{E}\left[\mathrm{e}^{\gamma \frac{\phi(Z_\epsilon)}{\epsilon}}\right] < \infty. \tag{10.6}$$

Then, we have

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\left[e^{\frac{\phi(Z_\epsilon)}{\epsilon}}\right] = \sup_{x \in X}\{\phi(x) - I(x)\} \tag{10.7}$$

10.2 Hausdorff measures

In this section, we give the definitions and very basic examples of Hausdorff measures and Hausdorff dimensions.

Definition Let $h : [0, a] \to [0, \infty)$ be a right-continuous and non-decreasing function such that $h(0) = 0$. Let X be a metric space and A be a measurable subset of X. For each $\delta > 0$ with $\delta < a$, we set

$$\mu_h^\delta(A) = \inf\{\sum_{j=1}^{\infty} h(r_j)\colon\ A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j),\ r_j < \delta\ \forall j\}$$

Note that as a function of δ, $\mu_h^\delta(A)$ is non-decreasing. Hence, we set

$$\mu_h(A) = \lim_{\delta \downarrow 0} \mu_h^\delta(A)\ = \sup_{\delta > 0} \mu_h^\delta(A).$$

One can check that μ_h is a metric outer measure. The measure corresponding to μ_h is called the h-**Hausdorff measure** on X.

Examples :

- In the previous definition, take $X = \mathbb{R}$ and

$$h(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

  Then, μ_h is the counting measure on $\mathbb{R}$.
- Again if $X = \mathbb{R}$, and $h(t) = t$, then μ_h is the one dimensional Lebesgue measure.
- Take $X = \mathbb{R}^2$ and $h(t) = \pi t^2$. Then μ_h is the two dimensional Lebesgue measure on $\mathbb{R}^2$. It is obvious that the π needs to be introduced since it also appears in the area of the 2-dimensional balls.

Now, we look at a particular choice of h. Let, for $d \geq 0$,

$$h_d(t) = t^d.$$

We write μ_d for μ_{h_d} for notational convenience. Then the **Hausdorff dimension** of X is defined as

$$\dim(X) = \inf\{d \geq 0 \colon \mu_d(X) = 0\}$$

We end the discussion on Hausdorff measures with an interesting example.

Example Let C be the cantor's ternary set. Then, with a little bit of covering argument, it is possible to show that

$$\mu_\alpha(C) = 0 \text{ if } \alpha > \frac{\log 2}{\log 3}$$

$$\mu_\alpha(C) = \infty \text{ if } \alpha < \frac{\log 2}{\log 3}$$

The above two statements imply that $\dim(C) = \frac{\log 2}{\log 3}$.

10.3 Some technical facts about Sobolev spaces

We now recall the definition of the function space $\mathcal{D}(B)$ and state some properties of this space.
In the case of B bounded, $\mathcal{D}(B)$ is the classical Sobolev space $H_0^1(B)$ which is defined as the closure of $\mathcal{C}_c^\infty(B)$ in the sense of the Sobolev norm $\psi \mapsto (\|\nabla\psi\|_2^2 + \|\psi\|_2^2)^{1/2}$ in the Sobolev space $H^1(B)$. We first give a relation between $H_0^1(B)$ and $H^1(\mathbb{R}^d)$ in the case of a C^1-boundary.

Lemma 10.5. *Let $B \subset \mathbb{R}^d$ be an open bounded set with C^1-boundary. Let $\psi \in H^1(\mathbb{R}^d)$ such that $\psi = 0$ a.e. on B^c. Then the restriction of ψ to B lies in $H_0^1(B)$.*

In the case that $B = \mathbb{R}^d$, the space $\mathcal{D}(\mathbb{R}^d) = D^1(\mathbb{R}^d)$ is the space of functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, which vanish at infinity, i.e., $\{x \in \mathbb{R}^d \colon |f(x)| > a\}$ has finite Lebesgue measure for any $a > 0$, and whose distributional gradient is in $L^2(\mathbb{R}^d)$. Now we collect some sequential compactness properties of the space $\mathcal{D}(B)$.

Lemma 10.6. *Suppose $(\psi_k)_{k \in \mathbb{N}}$ is a sequence in $\mathcal{D}(B)$ such that $(\|\nabla\psi_k\|_2)_{k \in \mathbb{N}}$ is bounded. Fix any $q \in (1, 2d/(d-2))$ for $d \geq 3$ and any $q > 1$ for $d \leq 2$. Then there exists $\psi \in \mathcal{D}(B)$ and a subsequence $(\psi_{k_j})_{j \in \mathbb{N}}$ such that $\nabla\psi_{k_j} \to \nabla\psi$ weakly in $L^2(B)$ and $\psi_{k_j} \to \psi$ locally strongly in $L^q(B)$.*

The following connection between the energy of functions in $\mathcal{D}(B)$ and the energy of measures is important.

Lemma 10.7. *For any (positive) absolutely continuous measure $\mu \in \mathcal{M}(B)$ whose support is a compact subset of B and whose energy $\|\mu\|_{\mathrm{E}}^2$ is finite, the function $\psi = \mathfrak{A}(\mu)$ lies in $\mathcal{D}(B)$ and satisfies $\frac{1}{2}\|\nabla\psi\|_2^2 = \|\mu\|_{\mathrm{E}}^2$.*

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