On the Convergence of Alternating Least Squares Optimisation in Tensor Format Representations

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The problem considered is formally a quadratic minimization problem on the tensor product space $V = \bigotimes_{\nu=1}^{d} \mathbb{R}^{n_{\nu}}$. The optimization problem is described the quadratic functional $	ilde{f} : V \rightarrow \mathbb{R}$

$$
\tilde{f}(u) = \frac{1}{2} \langle A(u - u^*), (u - u^*) \rangle_{V}, \quad (1)
$$

where $A \in \mathbb{R}^{n_{1} \cdots n_{d} \times n_{1} \cdots n_{d}}$ is a positive definite matrix ($A > 0$, $A^{t} = A$) and $u^* \in V$. In practice only the tensor $b := Au^* \in V$ is given and $u^*$ is unknown. The minimisation of $\tilde{f}$ is equivalent to the minimisation of $f : V \rightarrow \mathbb{R}$

$$
f(u) := \frac{1}{2} \langle Au, u \rangle_{V} - \langle b, u \rangle_{V},
$$

since the term $\langle Au^*, u^* \rangle_{V}$ in (1) is constant. Further, a tensor $u$ is represented in a tensor format $U$. A tensor format $U : P := \bigotimes_{\mu=1}^{L} P_{\mu} \rightarrow V$ is a multilinear map from the cartesian product of parameter spaces $P_{\mu}$ ($d \leq L$) into the tensor space $V$. The solution $u^* = \arg\min_{u \in V} f(u)$ is approximated respect to the tensor format representation $U$, i.e. we are looking for a representation system $(p_{1}^{*}, \ldots, p_{L}^{*})^{t} \in P$ such that for $F := f \circ U : P \rightarrow \mathbb{R}$ we have

$$
F(p_{1}^{*}, \ldots, p_{L}^{*}) \leq F(p_{1}, \ldots, p_{L}), \quad \text{for all} \ (p_{1}, \ldots, p_{L})^{t} \in P.
$$

The alternating least squares (ALS) algorithm is defined recursively. Suppose that the $k$-th iterate $p_{k} = (p_{1}^{k}, \ldots, p_{L}^{k})^{t}$ and the first $\mu - 1$ components $p_{1}^{k+1}, \ldots, p_{\mu-1}^{k+1}$ of the $(k+1)$-th iterate $p_{k+1}^{k}$ have been determined. The basic step of the ALS algorithm is to compute

$$
p_{\mu}^{k+1} := \arg\min_{q_{\mu} \in P_{\mu}} F(p_{1}^{k+1}, \ldots, p_{\mu-1}^{k+1}, q_{\mu}, p_{\mu+1}^{k}, \ldots, p_{L}^{k}).
$$

Thus, in order to obtain $p_{k+1}$ from $p_{k}$, we have to solve successively $L$ ordinary least squares problems.

In our talk, the convergence of the ALS method is analysed.
References